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RESEARCH ARTICLE

Identification of smooth ambiguity

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Abstract
Individuals behave differently when they know the objective probability of events and when they do not. The smooth ambiguity model accommodates both ambiguity (uncertainty) and risk. We consider an individual who trades financial assets to maximize a smooth ambiguity utility over two dates. For an incomplete, competitive asset market, we give sufficient conditions for consumption and asset demand functions generated by smooth ambiguity preferences to identify the ambiguity and risk indices as well as the ambiguity probability measure. Restrictions imposed on asset payoffs play an important role in separating risk and ambiguity preferences, and linear independence of indirect marginal utility functions over assets pins down the ambiguity beliefs. The identification procedure can determine whether the individual possesses smooth ambiguity, Kreps-Porteus-Selden or expected utility preferences. Also, our argument applies even if the conditional probability distributions in the support of the ambiguity probability measure are not observed.

Keywords Risk · Uncertainty · Identification

JEL Classification Numbers D11 · D80 · D81

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1 Introduction

Ambiguity preferences distinguish between uncertainty, where an individual cannot assign unambiguous probabilities to specific events, and risk, where such an assignment is possible.[1] Indeed, over the years following the critical contributions of Ellsberg (1961) in response to von Neumann and Morgenstern (1947) and Savage (1954), laboratory data have demonstrated that choices of individuals often do not conform to expected utility that does not distinguish between risk and uncertainty, and there has been continued interest in experimental tests that focus on this and related questions.[2] A number of utility models have been proposed to distinguish between uncertainty and risk: the seminal formulation of multiple priors and maxmin preferences of Gilboa and Schmeidler (1989), multiplier preferences of Anderson et al. (2003), smooth ambiguity preferences of Klibanoff et al. (2005), variational preferences of Maccheroni et al. (2006) and uncertainty averse preferences of Cerreia-Vioglio et al. (2011).

The identification of ambiguity preferences from observable data is essential for drawing either positive or normative conclusions in an environment with ambiguity. The above mentioned decision theory papers give unique utility representations, hence identification, by putting axiomatic restrictions on hypothetically observed pairwise rankings of Savage or Anscombe-Aumann acts. In this paper we consider an alternative approach based on identification of ambiguity preferences from asset demands. This identification is useful. As argued by Mas-Colell (1977), demand functions are a valuable source of information about the consumer, and identification from demand functions ensures that the assumed underlying utility representation is unique.

We focus on the model of smooth ambiguity preferences of Klibanoff et al. (2005).[3] In the smooth ambiguity model, an individual has a set of probabilities over asset payoffs, and is uncertain which probability distribution is the true one. The individual has a subjective ambiguity belief over these distributions. Each probability measure can be regarded as a conditional probability distribution, conditional on realizations of some randomizing device (e.g., a horse race). The subjectively relevant conditional probabilities are the support of the subjective ambiguity probability. In the application of Klibanoff et al. (2005) in portfolio choice, the expected utility of a portfolio is computed based on the set of conditional probabilities, and the overall ambiguity utility of the portfolio is the expected utility of these expected utilities based on the subjective ambiguity probability. As Klibanoff et al. (2005) note, their model achieves a separation of ambiguity as characterized by their uncertainty beliefs and their aversion to uncertainty. The smooth ambiguity model has been used in asset demand analysis, such as Gollier (2011) and Iwaki and Osaki (2014).[4]

[1] Ghirardato (2004), p. 36.

[2] Camerer and Weber (1992) and Attanasi et al. (2014) and the references cited therein.

[3] See also Seo (2009) and Nascimento and Riella (2013).

[4] As one referee pointed out, smoothness as promoted by Klibanoff et al. (2005) and adopted here is not an innocuous technical assumption. Epstein and Schneider (2010, p. 316) illustrate the limitations of the smooth ambiguity model.

1.1 Main results

In this paper, we assume that the consumption and asset demand functions are observable and generated by some two-period smooth ambiguity preferences, and we derive sufficient conditions for the identification of an individual's distinct preferences over uncertainty and risk as well as the individual's subjective ambiguity beliefs. Importantly, the argument for identification applies to incomplete asset markets.

We consider three utility representations of two-period smooth ambiguity preferences: the first representation axiomatized by Klibanoff et al. (2009), also called recursive smooth ambiguity utility, separates risk aversion from ambiguity aversion, henceforth referred to as KMM; the second representation accommodates KPS (Kreps and Porteus 1978; Selden 1978) preferences as a special case which separates risk aversion from intertemporal substitution, henceforth referred to as KPS*; the third representation proposed by Hayashi and Miao (2011) is a generalized recursive smooth ambiguity utility with intertemporal substitution, risk aversion and ambiguity aversion disentangled, henceforth referred to as HM.[5] Both KMM and KPS* representations are special cases of the HM utility representation.

In Section 3, we first show the challenge of identifying the smooth ambiguity preferences through examples in which even when some of the preference parameters are known, the identification may still fail since the sufficient conditions are not satisfied.

Our main results are the sufficient conditions for unique identification of the three smooth ambiguity utility representations, and the explicit procedure to recover the preferences and beliefs. In identifying the KPS* utility representation, one key innovation in the extension of prior results under pure risk is the introduction of an ambiguity free asset, with payoff distributions that coincide across ambiguity states. The presence of an ambiguity free asset separates the risk attitudes from the ambiguity attitudes, since the composition of a portfolio involving only the risk free asset and the ambiguity free asset is determined by the risk attitudes only. In identifying the KMM utility representation, a quasi-ambiguous asset is introduced, with conditional mean returns and variances differing in only one ambiguity state. Here again, we exploit the payoff structure to disentangle the risk attitudes from the ambiguity attitudes. Identification of the HM utility representation can be achieved with the existence of both an ambiguity free asset and a quasi-ambiguous asset. If the indirect marginal utilities over assets (i.e., marginal utilities as functions of asset quantities) across ambiguity states are linearly independent, then the ambiguity belief can be uniquely identified. We give a constructive recovery procedure when the linear independence condition is strengthened to differentiable linear independence.

For the identification, we only require that the actual conditional probabilities, i.e., the support of the ambiguity probability measure, lie in a set of candidate conditional probabilities. Indeed, once the ambiguity probability measure is uniquely identified, it will determine which conditional probability distributions are employed by the individual. As a result, the inability to observe the specific conditional probabilities an individual uses in his demand optimization does not interfere with our recovery process.

[5] The KMM, KPS* and HM representations are formally defined by Equations (3)-(5).

1.2 Related literature

The identification of fundamentals from observable demand functions can be posed, most simply, in the context of certainty. There, Mas-Colell (1977) demonstrates that the demand function or correspondence identifies the preferences of the consumer. Importantly, the argument for identification is local: if prices are restricted to an open neighborhood, they identify fundamentals in an associated neighborhood. Evidently, the arguments extend to economies under pure risk, but with a complete system of markets in elementary securities. Identification becomes problematic, and more interesting, when the set of observations is restricted. Under pure risk, this arises when the asset market is incomplete and the payoffs to investors are restricted to a subspace of possible payoffs. Nevertheless, Green et al. (1979), Dybvig and Polemarchakis (1981) and Geanakoplos and Polemarchakis (1990) demonstrate that identification is possible as long as the utility function has an expected utility representation with a state-independent cardinal utility index, and the distribution of asset payoffs is known. Polemarchakis (1983) extends the argument to the joint identification of tastes and beliefs. The argument relies crucially on the presence of a risk free asset and, more importantly, does not allow risk due to future endowments.[6] Polemarchakis and Selden (1984) identify the risk and time indices in the context of Selden (1978) preferences.

This paper contributes to the identification literature by extending the identification of expected utility preferences to the case of ambiguity preferences. First, we introduce an ambiguity free asset and a quasi-ambiguous asset to separate risk attitudes from ambiguity attitudes, which enables us to utilize the techniques developed in Kübler and Polemarchakis (2017) for identifying expected utility preferences with income risk to achieve identification of smooth ambiguity preferences with capital risk. Without disentangling risk preferences from ambiguity preferences, the techniques in Kübler and Polemarchakis (2017) could not be used to identify beliefs for the smooth ambiguity preferences since the indirect marginal utilities over assets would be unknown and the condition of linear independence would be indeterminate. Furthermore, our identification argument is constructive in the sense that we start with explicit consumption and asset demand functions and give a procedure to construct the utility indices and beliefs that rationalize the demands. We show that our differentiable linear independence assumption is sufficient for this construction. This differs from Kübler and Polemarchakis (2017) which only gives sufficient conditions for the uniqueness of beliefs rather than providing a constructive procedure. Finally, our analysis allows the individual to have smooth ambiguity, KPS or expected utility preferences and our identification argument can determine which of the three preferences the individual possesses without having to know this in advance of the identification process, whereas Kübler and Polemarchakis (2017) restrict their analysis to expected utility.

The rest of the paper is organized as follows. Section 2 introduces the setup and the three two-period smooth ambiguity models. Section 3 starts with examples to show the challenge of identification and then presents the main identification results for the three smooth ambiguity utility models. Concluding comments are given in

[6] Kübler and Polemarchakis (2017) derive conditions that guarantee identification with no knowledge either of the cardinal utility index (attitudes towards risk) or of the distribution of future endowments or payoffs of assets. The argument applies even if the asset market is incomplete and demand is observed only locally.

Section 4. Appendix A presents an example to illustrate our recovery procedure for KPS* representation and demonstrate that our argument applies even if the conditional probability distributions in the support of the ambiguity probability measure are not observed. This example also shows that whether the set of candidate conditional probabilities contains the actual ones can be tested. Appendix B provides an example to illustrate the recovery process for the KMM representation. Appendix C shows how KPS preferences can be identified using our recovery procedure.

2 Setup

There are two dates: date 0 and date 1, and uncertainty at date 1 is represented by states of the world. States of the world are $\omega \in \boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is a finite set and has the following product structure: $\boldsymbol{\Omega} = \boldsymbol{A} \times \boldsymbol{S}$, where $a \in \boldsymbol{A}$ are ambiguity states, and $s \in \boldsymbol{S}$ are risk states. $\boldsymbol{\Omega}$ can be interpreted as a set of possible outcomes of two-stage lotteries. In this case, elements in $\boldsymbol{A}$ and $\boldsymbol{S}$ are, respectively, outcomes of first and second stage lotteries.[7] We assume risk states are contingent on which asset payoffs and consumption values are realized.[8] A probability measure on the set of states of the world, $\boldsymbol{\pi} \in \Delta(\boldsymbol{\Omega})$, can be expressed as $\boldsymbol{\pi} = \boldsymbol{\mu} \otimes \boldsymbol{\nu}$, where $\boldsymbol{\mu} \in \Delta(\mathbf{A})$ is a probability measure over states of uncertainty, $\boldsymbol{\nu} : \boldsymbol{A} \to \Delta(\boldsymbol{S})$ is a family of conditional probability measures over states of risk, and $\pi_{as} = \mu_a \nu_{as}$. The ambiguity probability measure $\boldsymbol{\mu}$ represents the uncertainty about which conditional probability distribution in $\{\boldsymbol{\nu}_a\}_{a=1}^{A}$ is the true one. The support of $\boldsymbol{\mu}$ is the set of conditional probability measures to which $\boldsymbol{\mu}$ assigns strictly positive probabilities, and it is the conditional probability measures in the support of $\boldsymbol{\mu}$ that are actually used by the individual when making decision.

A distribution of wealth across risk states at date 1 is

$$\boldsymbol{x} = (..., x_s, ...) \in \mathbb{R}^S_{++}.$$

In a static setting, Klibanoff et al. (2005) provide a set of axioms that are necessary and sufficient for the existence of a risk index, an ambiguity index, and a probability measure,

$$u : \mathbb{R}_{++} \to \mathbb{R}, \quad \tilde{\phi} : u(\mathbb{R}_{++}) \to \mathbb{R} \quad \text{and} \quad \boldsymbol{\mu},$$

respectively, such that

$$U(\boldsymbol{x}) = E_{\boldsymbol{\mu}} \tilde{\phi}\big(E_{\boldsymbol{\nu}_a} u(x_s)\big) \tag{1}$$

[7] A product state space is also used in Nau (2006) and Ergin and Gul (2009), and is similar to a two-stage lottery in Anscombe and Aumann (1963), though as a referee pointed out that our terminology is not typical. Segal (1990) gives arguments for the potential superiority of the Anscombe and Aumann (1963) over the Savage (1954) setup in the analysis of ambiguity attitudes.

[8] If asset payoffs and consumption are contingent on both ambiguity states and risk states, results here go through with some notational change. The assumption made here makes the argument stronger, and it is consistent with observations in practice.

represents ambiguity preferences.[9] Alternatively, if

$$\phi = \widetilde{\phi} \circ u, \quad \phi : \mathbb{R}_{++} \to \mathbb{R},$$

then (1) takes the form

$$U(\boldsymbol{x}) = E_{\boldsymbol{\mu}} \phi\Big(u^{-1}\big(E_{\boldsymbol{\nu}_a} u(x_s)\big)\Big). \tag{2}$$

For the representation (1), a positive affine transformation of the risk index u does not change preferences if and only if a compensating transformation is applied to $\tilde{\phi}$. In contrast, the preferences corresponding to (2) are invariant to a positive affine transformation of the risk index.[10] Under the formulation (2), an individual is strictly ambiguity averse if ϕ is strictly more concave than u.

When considering a two-period model, we consider three representations.

The KMM utility representation is a recursive smooth ambiguity model based on equation (1) proposed by Klibanoff et al. (2009):

$$U(x_0, \boldsymbol{x}) = u(x_0) + \beta \tilde{\phi}^{-1}\Big(E_{\boldsymbol{\mu}} \tilde{\phi}\big(E_{\boldsymbol{\nu}_a} u(x_s)\big)\Big). \tag{3}$$

In this representation, preference under certainty and risk is determined by u, and preference under uncertainty is determined by ϕ. If there is no ambiguity, i.e., only one probability measure $\boldsymbol{\nu}$ is in the support of $\boldsymbol{\mu}$ or $\tilde{\phi}$ is a linear function, the KMM representation becomes the expected utility:

$$U(x_0, \boldsymbol{x}) = u(x_0) + \beta E_{\boldsymbol{\nu}} u(x_s).$$

The KPS* utility representation is a two-period smooth ambiguity model based on equation (2):

$$U(x_0, \boldsymbol{x}) = \phi(x_0) + \beta E_{\boldsymbol{\mu}} \phi\Big(u^{-1}\big(E_{\boldsymbol{\nu}_a} u(x_s)\big)\Big). \tag{4}$$

In this representation, preference under certainty and uncertainty is determined by ϕ, and preference under risk is determined by u.

If there is no ambiguity, the KPS* representation becomes the KPS utility:

$$U(x_0, \boldsymbol{x}) = \phi(x_0) + \beta \phi\Big(u^{-1}\big(E_{\boldsymbol{\nu}} u(x_s)\big)\Big).$$

[9] As defined earlier, index a in the representation refers to a typical ambiguity state, the probability measure μ is over ambiguity states, and $\boldsymbol{\nu}_a$ is the probability measure conditional on each ambiguity state a.

[10] This point is discussed in Klibanoff, Marinacci, and Mukerji (2005, p. 1858).

Furthermore, if $\phi = u$, then the KPS utility specializes to the expected utility form. Therefore, both KPS and expected utilities become special cases of KPS* representation.[11]

The HM utility representation is a generalized recursive smooth ambiguity model proposed by Hayashi and Miao (2011):

$$U(x_0, \boldsymbol{x}) = u_1(x_0) + u_2\Big((\tilde{\phi} \circ u)^{-1}\big(E_{\boldsymbol{\mu}}\tilde{\phi}\big(E_{\boldsymbol{v}_a}u(x_s)\big)\big)\Big). \tag{5}$$

This representation allows for a separation among intertemporal substitution u_2, risk aversion u, and ambiguity aversion ϕ. If $u_1 = u_2 = u$, then the above utility function becomes KMM representation (3); if $u_1 = u_2 = \phi = \tilde{\phi} \circ u$, then the above utility function becomes KPS* representation (4).

The utility function over date 0 consumption and distributions of date 1 wealth

$$U(x_0, \boldsymbol{x}) : \mathbb{R}^{S+1}_{++} \to \mathbb{R}$$

is twice differentiable, strictly monotonically increasing and strictly quasi-concave in x_0 and $\boldsymbol{x}$, and satisfies a boundary condition: the closure of the indifference curve through any strictly positive distribution is contained in the strictly positive orthant or

$$(\bar{x}_0, \bar{\boldsymbol{x}}) \in \mathbb{R}^{S+1}_{++} \;\Rightarrow\; Cl\,\{(x_0, \boldsymbol{x}) : U(x_0, \boldsymbol{x}) = U(\bar{x}_0, \bar{\boldsymbol{x}})\} \subseteq \mathbb{R}^{S+1}_{++}.$$

At date 0, the consumption good and financial assets are traded. Assets are $j \in \boldsymbol{J}$. Payoffs of asset j at date 1 defined across risk states are

$$\boldsymbol{r}_j = (..., r_{sj}, ...)',$$

a column vector, conditional on risk state s. Payoffs of the set of assets are $\mathbf{R}_s = (..., r_{sj}, ...)$, a row vector, and the matrix of asset is

$$\boldsymbol{R} = (..., \boldsymbol{r}_j, ...) = (..., \boldsymbol{R}_s, ...)'$$

that has full column rank or, equivalently, payoffs of assets, $\{\boldsymbol{r}_j\}$, are linearly independent.[12]

A portfolio of assets is $\boldsymbol{y} = (..., y_j, ...)$ and it generates the distributions of wealth across risk states $\boldsymbol{x} = \boldsymbol{R}\boldsymbol{y}$. The set of portfolios that generate strictly positive $\boldsymbol{x}$ is non-empty,

$$\boldsymbol{Y} = \{\boldsymbol{y} : \boldsymbol{R}\boldsymbol{y} \gg \boldsymbol{0}\} \neq \emptyset,$$

that is open. The domain of asset prices not allowing for arbitrage is

$$\boldsymbol{P} = \{\boldsymbol{p} : \boldsymbol{R}\boldsymbol{y} > 0 \Rightarrow \boldsymbol{p}\boldsymbol{y} > 0\} = \{\boldsymbol{p} = \boldsymbol{q}\boldsymbol{R}, \boldsymbol{q} \gg \boldsymbol{0}\}.$$

[11] Equation (4) has similarities to KPS utility, but it can't separate intertemporal substitution from ambiguity version; in contrast, KPS utility separates intertemporal substitution from risk aversion.

[12] For notational ease, we assume asset payoffs do not change across observations. However, all our results hold if asset payoffs vary, but are observable.

Given the consumption price p_0, asset price vector $\boldsymbol{p} = (p_1, ..., p_J)$, asset payoffs $\boldsymbol{R}$ and conditional probability measures $\{\boldsymbol{\nu}_a\}_{a=1}^{A}$, the optimization problem of the individual is

$$\max_{x_0>0, \boldsymbol{y}\in \boldsymbol{Y}} U(x_0, \boldsymbol{R}\boldsymbol{y}), \quad s.t. \; p_0 \cdot x_0 + \boldsymbol{p} \cdot \boldsymbol{y} \leq 1. \tag{6}$$

A solution to the optimization problem, $x_0(p_0, \boldsymbol{p})$ and $\boldsymbol{y}(p_0, \boldsymbol{p})$, exists, satisfies $x_0(p_0, \boldsymbol{p}) > 0$ and $\boldsymbol{R}\boldsymbol{y}(p_0, \boldsymbol{p}) \gg 0$, and is unique. It defines the demand function for consumption and assets,

$$x_0 : (p_0, \boldsymbol{p}) \to \mathbb{R}_{++}, \quad \text{and} \quad \boldsymbol{y} : (p_0, \boldsymbol{p}) \to \boldsymbol{Y}.$$

Importantly, the demand functions are invertible under our assumptions.

The following provides a simple example of the consumption-portfolio choice problem using our notation and setup. Assume there are four risk states, and three assets with payoffs: $\boldsymbol{r}_1 = (1, 1, 1, 1)$, $\boldsymbol{r}_2 = (1, 2, 0, 0)$, and $\boldsymbol{r}_3 = (0, 0, 0, 1)$, i.e.,

$$\boldsymbol{R} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

There are two ambiguity states with conditional probabilities $\boldsymbol{\nu}_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ and $\boldsymbol{\nu}_2 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{4}, \frac{1}{12})$.

If we know the individual's discount factor $\beta = \frac{9}{10}$, risk index $u(x) = x^{\frac{1}{2}}$, ambiguity index $\phi(x) = x^{\frac{1}{4}}$ and ambiguity probability measure $\boldsymbol{\mu} = (\frac{1}{3}, \frac{2}{3})$ with $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ being its support, then the individual utility according to KMM representation is

$$\begin{aligned} U(x_0, \boldsymbol{R}\boldsymbol{y}) = x_0^{\frac{1}{2}} + & \\ \frac{9}{10}\Bigg\{ & \frac{1}{3}[\frac{1}{2}(y_1 + y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1 + 2y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1)^{\frac{1}{2}} + \frac{1}{6}(y_1 + y_3)^{\frac{1}{2}}]^{\frac{1}{4}} \\ & + \frac{2}{3}[\frac{1}{2}(y_1 + y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1 + 2y_2)^{\frac{1}{2}} + \frac{1}{4}(y_1)^{\frac{1}{2}} + \frac{1}{12}(y_1 + y_3)^{\frac{1}{2}}]^{\frac{1}{4}} \Bigg\}^4; \end{aligned}$$

and the individual utility according to KPS* representation is

$$\begin{aligned} U(x_0, \boldsymbol{R}\boldsymbol{y}) = x_0^{\frac{1}{4}} + & \\ \frac{9}{10}\Bigg\{ & \frac{1}{3}[\frac{1}{2}(y_1 + y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1 + 2y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1)^{\frac{1}{2}} + \frac{1}{6}(y_1 + y_3)^{\frac{1}{2}}]^{\frac{1}{2}} \\ & + \frac{2}{3}[\frac{1}{2}(y_1 + y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1 + 2y_2)^{\frac{1}{2}} + \frac{1}{4}(y_1)^{\frac{1}{2}} + \frac{1}{12}(y_1 + y_3)^{\frac{1}{2}}]^{\frac{1}{2}} \Bigg\}. \end{aligned}$$

3 Identification

In reality, an individual's ambiguity preferences and beliefs are not observable; instead, it is the consumption and portfolio choices that are observable in principle.

Assume we have a set

$$\mathfrak{O} = \left\{x_0(p_0, \boldsymbol{p}), \boldsymbol{y}(p_0, \boldsymbol{p}), \{\boldsymbol{\nu}_a\}_{a=1}^{A}, \boldsymbol{R}\right\}$$

of observations of consumption and asset (inverse) demand functions $x_0(p_0, \boldsymbol{p})$ and $\boldsymbol{y}(p_0, \boldsymbol{p})$, families of finite conditional probability distributions $\{\boldsymbol{\nu}_a\}_{a=1}^{A}$, and the asset payoff matrix $\boldsymbol{R}$.

We address the following question: suppose that our observations $\mathfrak{O}$ are consistent with the existence of KMM, KPS* or HM representation of two-period smooth ambiguity preferences.[13] Can the underlying unobservable preferences and beliefs be uniquely identified?

The identification literature since Mas-Colell (1977) assumes the existence of some preferences underlying observable demand functions, and try to determine the uniqueness from the demand functions. Notice that assuming the existence of smooth ambiguity preferences does not trivialize the research question we address here, since there could exist multiple smooth ambiguity preferences which rationalize the observed consumption and asset demand functions. The following example, which modifies an example in Dybvig and Polemarchakis (1981) in the context of static smooth ambiguity preferences (1), shows that two smooth ambiguity preferences generate the same asset demand.

Example 1 Suppose the asset payoff matrix is

$$\boldsymbol{R} = \begin{bmatrix} c & d \\ e^{\pi} c & e^{\pi} d \\ d & c \\ e^{\pi} d & e^{\pi} c \end{bmatrix}.$$

There are two ambiguity states with conditional probability measures $\boldsymbol{\nu}_1 = (\pi_{11}, \pi_{12}, \pi_{13}, \pi_{14})$ and $\boldsymbol{\nu}_2 = (\pi_{21}, \pi_{22}, \pi_{23}, \pi_{24})$, respectively.

Then it can be verified that the smooth ambiguity preferences with $\tilde{\phi}(x) = \ln(x)$ and $u(x) = x^{\frac{l}{2}}$ and the the smooth ambiguity preferences with $\tilde{\phi}(x) = \ln(x)$ and $u(x) = x^{\frac{l}{2}} + x^{\frac{l}{2}} \sin(\ln(x))$ will generate the same asset demand if the two preferences share the same probability measures $\boldsymbol{\mu}$, $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ with $\pi_{11} = \pi_{12}(e^{\pi})^{\frac{l}{2}}$, $\pi_{13} = \pi_{14}(e^{\pi})^{\frac{l}{2}}$, $\pi_{21} = \pi_{22}(e^{\pi})^{\frac{l}{2}}$ and $\pi_{23} = \pi_{24}(e^{\pi})^{\frac{l}{2}}$.

The assumption that finitely many conditional probability measures $\{\boldsymbol{\nu}_a\}_{a=1}^{A}$ are observable does not require that we know the conditional probability measures which the individual actually uses when making consumption-portfolio choice; it only

[13] When markets are incomplete, there may exist non-smooth ambiguity preferences to rationalize the consumption and asset demand.

requires that the actual conditional probabilities lie in some finite probability set $\{\boldsymbol{\nu}_a\}_{a=1}^{A}$.[14] Indeed, once the ambiguity probability measure $\boldsymbol{\mu}$ is identified, the conditional probability measures in its support will also be identified. From a theoretical point of view, whether the candidate probability set $\{\boldsymbol{\nu}_a\}_{a=1}^{A}$ contains the actual conditional probability measures can be tested. When the smooth ambiguity utility is identified from the observed demand functions, the newly derived demand functions from maximizing the identified utility can be compared with the observed demand functions. If the actual conditional probability measures are not in the candidate set, then the newly derived demand functions will be different from the observed demand functions. Therefore, if the identified ambiguity belief is inconsistent with the observed demand functions, we can try a different finite set and verify whether it contains actual conditional probabilities.

Before presenting the identification result, we give two simple examples in which the identification of $\boldsymbol{\mu}$ fails even if the risk index u, the ambiguity index ϕ and the discount factor β are known. Note that we are concerned with the more demanding case: the joint identification of u, ϕ, β and $\boldsymbol{\mu}$.

Example 2 Suppose $\phi(x) = x^{\frac{1}{3}}$, $u(x) = \ln x$, $\beta = 1$, there is only one ambiguous asset, and its payoffs satisfy $\sum_s \pi_{1s} \ln r_s = \sum_s \pi_{2s} \ln r_s$.

With KPS* utility, the investor solves the maximization problem

$$\max_{x_0, y} \; (x_0)^{\frac{1}{3}} + \sum_a \mu_a \left(e^{\sum_s \pi_{as} \ln(r_s y)}\right)^{\frac{1}{3}},$$

$$s.t. \; p_0 \cdot x_0 + p \cdot y = 1.$$

The following first order condition characterizes demand for the asset,

$$\sum_a \mu_a \left(e^{\sum_s \pi_{as} \ln r_s + \ln y}\right)^{\frac{1}{3}} = \frac{py}{(x_0)^{\frac{2}{3}} p_0}.$$

Since $\sum_s \pi_{1s} \ln r_s = \sum_s \pi_{2s} \ln r_s$, μ_1 and μ_2 cannot be separately identified.

Note that the functional form of the ambiguity index is immaterial.

Example 3 Suppose $\phi(x) = x^{\frac{1}{3}}$, $u(x) = \ln x$, $\beta = 1$, there are two ambiguous assets with $r_{11} = r_{21} = r_{12} = r_{22}$, $\pi_{11} + \pi_{12} = \pi_{21} + \pi_{22}$, $\pi_{1s} = \pi_{2s}$ for $s = 3, ..., S$.

With KPS* utility, the investor solves the maximization problem

$$\max_{x_0, y_1, y_2} \; (x_0)^{\frac{1}{3}} + \sum_a \mu_a \left(e^{\sum_s \pi_{as} \ln(r_{1s} y_1 + r_{2s} y_2)}\right)^{\frac{1}{3}},$$

$$s.t. \; p_0 \cdot x_0 + p_1 \cdot y_1 + p_2 \cdot y_2 = 1.$$

[14] The candidate probability set $\{\boldsymbol{\nu}_a\}_{a=1}^{A}$ could be arbitrarily large as long as the conditional probability measures in this set generate one ambiguity free asset or quasi-ambiguous asset as defined in Definitions 2 and 5 below.

The following first order conditions characterize demand for the two assets,

$$\sum_a \mu_a e^{(\pi_{a1}+\pi_{a2})\ln r_{11}(y_1+y_2)+\sum_{s=3}^S \pi_{as}\ln(r_{1s}y_1+r_{2s}y_2)} \left((\pi_{a1}+\pi_{a2})\frac{1}{y_1+y_2}+\sum_{s=3}^S \pi_{as}\frac{r_{1s}}{r_{1s}y_1+r_{2s}y_2}\right) \left(e^{(\pi_{a1}+\pi_{a2})\ln r_{11}(y_1+y_2)+\sum_{s=3}^S \pi_{as}\ln(r_{1s}y_1+r_{2s}y_2)}\right)^{-\frac{2}{3}} = \frac{p_1}{(x_0)^{\frac{2}{3}}p_0},$$

and

$$\sum_a \mu_a e^{(\pi_{a1}+\pi_{a2})\ln r_{11}(y_1+y_2)+\sum_{s=3}^S \pi_{as}\ln(r_{1s}y_1+r_{2s}y_2)} \left((\pi_{a1}+\pi_{a2})\frac{1}{y_1+y_2}+\sum_{s=3}^S \pi_{as}\frac{r_{2s}}{r_{1s}y_1+r_{2s}y_2}\right) \left(e^{(\pi_{a1}+\pi_{a2})\ln r_{11}(y_1+y_2)+\sum_{s=3}^S \pi_{as}\ln(r_{1s}y_1+r_{2s}y_2)}\right)^{-\frac{2}{3}} = \frac{p_2}{(x_0)^{\frac{2}{3}}p_0}.$$

Since $\pi_{11}+\pi_{12}=\pi_{21}+\pi_{22}$, $\pi_{1s}=\pi_{2s}$ for $s=3,...,S$, μ_1 and μ_2 cannot be separately identified.

Note that the functional forms of the ambiguity or the risk index is immaterial. It is the distribution of asset payoffs that drives the argument.

3.1 Identification of KPS* representation

To present our identification results in logical flow, we start with the identification of KPS* representation. The demand for consumption and assets satisfies the necessary and sufficient first order conditions for the optimization problem (6),

$$\begin{gathered} \boldsymbol{DU}(x_0,\boldsymbol{Ry})=\lambda(p_0,\boldsymbol{p}),\ \lambda>0, \\ p_0\cdot x_0+\boldsymbol{p}\cdot\boldsymbol{y}=1. \end{gathered} \tag{7}$$

Under KPS* representation, these conditions generate the family of marginal rates of substitution of consumption and assets defined by

$$m_{j0}(x_0,\boldsymbol{y}) = \frac{\frac{\partial U(x_0,\boldsymbol{Ry})}{\partial y_j}}{\frac{\partial U(x_0,\boldsymbol{Ry})}{\partial x_0}} = \frac{\beta E_{\boldsymbol{\mu}}\phi'(u^{-1}(E_{\boldsymbol{v}_a}u(\boldsymbol{Ry})))\frac{E_{\boldsymbol{v}_a}u'(\boldsymbol{Ry})\boldsymbol{r}_j}{u'(u^{-1}(E_{\boldsymbol{v}_a}u(\boldsymbol{Ry})))}}{\phi'(x_0)} > 0 \tag{8}$$

and

$$m_{jk}(x_0, \boldsymbol{y}) =$$

$$\frac{\frac{\partial U(x_0, \boldsymbol{R}\boldsymbol{y})}{\partial y_j}}{\frac{\partial U(x_0, \boldsymbol{R}\boldsymbol{y})}{\partial y_k}} = \frac{E_{\boldsymbol{\mu}}\phi^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{R}\boldsymbol{y})))\frac{E_{\boldsymbol{\nu}_a}u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{R}\boldsymbol{y})))}}{E_{\boldsymbol{\mu}}\phi^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{R}\boldsymbol{y})))\frac{E_{\boldsymbol{\nu}_a}u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{R}\boldsymbol{y})))}} > 0, \tag{9}$$

where $\boldsymbol{\mu}$ is the probability measure over ambiguity states, and $\boldsymbol{\nu}_a$ is the probability measure conditional on each ambiguity state associated with the distribution of returns for each asset.

To obtain positive identification results, we introduce one risk free asset and one ambiguity free asset.

Definition 1 An asset is *risk free* if it generates the same payoff across risk states.

Note that in Example 3, even if one asset is risk free, the identification of $\boldsymbol{\mu}$ is still not possible.

Definition 2 An asset is *ambiguity free* if, conditional on each ambiguity state, it generates the same distribution of returns.

Example 4 There are 3 risk states and 2 ambiguity states. The probability distributions conditional on ambiguity states are $(\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, 0)$. An asset that pays $(1, 1, 1)$ across risk states is risk free. An asset that pays $(1, a, a)$ across risk states is ambiguity free, even if $a \neq 1$ and the asset is not risk free.

The introduction of an ambiguity free asset makes identification possible because a portfolio that involves only the risk free asset and the ambiguity free asset allows for a clear distinction between risk aversion and ambiguity aversion. Since the existence of an ambiguity free asset plays a crucial role in the identification argument, it deserves attention.[15] Evidently, for arbitrary asset payoffs and conditional probabilities, an ambiguity free asset need not exist. Being ambiguity free is a joint restriction on asset payoffs $\boldsymbol{r} = (r_1, ..., r_s, ..., r_S)$ and the conditional probability distributions $\{\boldsymbol{\nu}_a\}_{a=1}^{A}$ where $\boldsymbol{\nu}_a = (\nu_{a1}, ..., \nu_{as}, ..., \nu_{aS})$.[16] One extreme case is a risk free asset, with $r_s = r_{s^{'}}$ for all s and $s^{'}$, that is ambiguity free independently of conditional probabilities.[17] At the other extreme, an asset with $r_s \neq r_{s^{'}}$ for any s and $s^{'}$ cannot be ambiguity free for any conditional probabilities. To understand ambiguity free asset returns, we partition the set of risk states $S = \{1, ..., s, ..., S\}$ into N disjoint subsets $\{S^n\}_{n=1}^{N}$, such that

$$S^n = \{s, s^{'} \in S : r_s = r_{s^{'}}\}.$$

[15] An ambiguity free asset appears in Klibanoff, Marinacci, and Mukerji (2005, p.1876) where the effects of ambiguity and risk attitudes on portfolio choice are examined numerically. Note, however, that, different from our assumption, asset payoffs in their example depend on both ambiguity and risk states, which is atypical in the financial economics literature.

[16] For the analysis of an ambiguity free asset, we consider a single asset and omit its index.

[17] In the remainder of this paper when we refer to an ambiguity free asset, we will mean it is ambiguity free, but, risky, even though we do not emphasize the latter property.

That is, S^n is a subset of risk states on which the payoffs of the asset coincide. For the three risk states in Example 4, we partition them into two subsets: $S^1 = \{s = 1\}$, $S^2 = \{s = 2, s = 3\}$.

Remark 1 An asset with payoffs $\boldsymbol{r} = (r_1, ..., r_s, ..., r_S)$ is ambiguity free under conditional probability distributions $\{\boldsymbol{\nu}_a\}_{a=1}^A$ if and only if $\sum_{s\in S^n} \nu_{as} = \sum_{s\in S^n} \nu_{a's}$ for all n, a and a'.

The argument is immediate.

To state the identification theorem, define the $(n-1)$ dimensional unit sphere $\Lambda^{n-1} = \{\boldsymbol{\alpha} \in R^n : \sum_{i=1}^n \alpha_i^2 = 1\}$.

Definition 3 Functions $\{g_i\}_{i=1}^n$ with $g_i : D \subset \mathbb{R}^m \to \mathbb{R}$ are *linearly independent* if there does not exist $\boldsymbol{\alpha} \in \Lambda^{n-1}$, such that $\sum_{i=1}^n \alpha_i g_i(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in D$.

For identification, we use a result from Kübler and Polemarchakis (2017).

Lemma 1 *If functions $g_1,...,g_n$ are linearly independent on a set D, then there must exist finitely many points $\boldsymbol{x}_1,...,\boldsymbol{x}_N \in D$, such that, for no $\boldsymbol{\alpha} \in \Lambda^{n-1}$, is $\sum_{i=1}^n \alpha_i g_i(\boldsymbol{x}_j) = 0$ for all $j = 1, ..., N$.*

On the portfolio set $\boldsymbol{Y}$ which generate strictly positive consumption, we define the indirect marginal utility functions, i.e., the marginal utility over assets and not consumption, across ambiguity states

$$\left\{ f_a(\boldsymbol{y}) = \phi'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y}))) \frac{E_{\boldsymbol{\nu}_a} u'(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j}{u'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})))} \right\}_{a=1}^A .$$

The following theorem states that the utility indices ϕ and u could be explicitly computed from the observed consumption and asset demand, and the ambiguity probability measure $\boldsymbol{\mu}$ would then be uniquely determined if the indirect marginal utility functions $\left\{f_a(\boldsymbol{y})\right\}_{a=1}^A$ are linearly independent.

Theorem 1 *If*

1. *the KPS* representation of smooth ambiguity utility satisfies the condition that $\phi\left(u^{-1}(\cdot)\right)$ is strictly concave on $\mathbb{R}$, with the indices u and ϕ being twice differentiable, strictly increasing, and strictly concave on $\mathbb{R}_{++}$,*
2. *there is an asset, $j = 1$ that is risk free: $\boldsymbol{r}_1 = 1$ across states of the world,*
3. *there is an asset, $j = 2$ that is ambiguity free: its payoff distribution is invariant to the states of ambiguity,*
4. *the family of conditional probability measures over states of risk, $\boldsymbol{\nu} : \boldsymbol{A} \to \Delta(\boldsymbol{S})$ is known, and*
5. *the functions $\left\{f_a(\boldsymbol{y})\right\}_{a=1}^A$ are linearly independent on $\boldsymbol{Y}$,*

then, the demand for consumption and assets identifies the risk index u and the ambiguity index ϕ on $\mathbb{R}_{++}$, each up to a positive affine transformation, the discount factor β, as well as the ambiguity probability measure $\boldsymbol{\mu}$.

Proof We argue in a series of steps.

Step 1—identification of the discount factor β.

From the marginal rate of substitution between consumption and risk free asset in equation (8),

$$\phi^{'}(x_0) = \beta E_{\boldsymbol{\mu}}\phi^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{R}\boldsymbol{y})))\frac{E_{\boldsymbol{\nu}_a}u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_1}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{R}\boldsymbol{y})))}\frac{1}{m_{10}(x_0, \boldsymbol{y})}. \tag{10}$$

At $x_0 = \overline{x}_0$ and portfolio $\overline{\boldsymbol{y}} = (\overline{x}_0, 0, ..., 0)$

$$\beta = m_{10}(\overline{x}_0, \overline{\boldsymbol{y}}). \tag{11}$$

Since the marginal rate of substitution between consumption and risk free asset $m_{10}(x_0, \boldsymbol{y})$ can be derived from the observed demand functions, hence is observable, β is identified from $m_{10}(x_0, \boldsymbol{y})$ at $\overline{x}_0$ and $\overline{\boldsymbol{y}}$.

Step 2—identification of the ambiguity index ϕ.

From equation (10), if we normalize $\phi^{'}(\overline{x}_0) = 1$, we have

$$\phi^{'}(x_0) = \frac{m_{10}(\overline{x}_0, \boldsymbol{y})}{m_{10}(x_0, \boldsymbol{y})}. \tag{12}$$

Integrating both sides with respect to x_0 will identify the ambiguity index ϕ on $\mathbb{R}_{++}$ up to a positive affine transformation.

Step 3—identification of the risk index u.

We restrict attention to the portfolios $\boldsymbol{y} = (y_1, y_2, 0, ..., 0)$, and we let $\tilde{\boldsymbol{y}} = (y_1, y_2)$ be the associated truncated portfolio. Since the distribution of payoffs for assets 1 and 2 is invariant across states of ambiguity, there exists a probability measure, $\tilde{\boldsymbol{\nu}} \in \Delta(\boldsymbol{S})$, and a matrix of payoffs of assets over states of risk $\tilde{\boldsymbol{R}} = (\boldsymbol{1}_{\#\boldsymbol{S}}, \tilde{\boldsymbol{r}}_2)$,[18] such that, the distribution of payoffs of assets generated by $(\boldsymbol{\nu}_a, \boldsymbol{R}\boldsymbol{y})$, for any state of ambiguity, coincides with the distribution generated by $(\tilde{\boldsymbol{\nu}}, \tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})$. As a consequence, from equation (9),

$$m_{12}(x_0, \tilde{\boldsymbol{y}}) = \frac{E_{\tilde{\boldsymbol{\nu}}}u^{'}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})}{E_{\tilde{\boldsymbol{\nu}}}u^{'}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})\tilde{\boldsymbol{r}}_2} > 0. \tag{13}$$

Since the portfolio $\tilde{\boldsymbol{y}}$ involves asset 1 and asset 2 only, it is ambiguity-free. As a result, the composition of the portfolio $\tilde{\boldsymbol{y}}$ is determined by the risk attitude, independent of the ambiguity attitude. Therefore, we can separate the risk attitude from the ambiguity attitude in the presence of an ambiguity free asset. Following the pure risk analysis in

[18] $\boldsymbol{1}_{\#\boldsymbol{S}}$ is the vector of 1's of dimension $\#\boldsymbol{S}$, the cardinality of $\boldsymbol{S}$.

Dybvig and Polemarchakis (1981), we differentiate both sides of equation (13) with respect to y_2, and evaluate the resulting functional at $\bar{\mathbf{y}} = (x, 0, ..., 0)$,

$$-\frac{u^{''}(x)}{u^{'}(x)} = \frac{\partial m_{12}(x_0, \bar{\mathbf{y}})}{\partial y_2} \frac{(E_{\tilde{\mathbf{v}}} r_2)^2}{E_{\tilde{\mathbf{v}}} r_2^2 - (E_{\tilde{\mathbf{v}}} r_2)^2}.$$

Notice that $\frac{u^{''}(x)}{u^{'}(x)}$ is the derivative of function $\ln u^{'}(x)$, and the right-hand side can be derived from the observed demand functions. Then, integrating both sides twice with respect to x will identify the cardinal risk index u on $\mathbb{R}_{++}$ up to a positive affine transformation.

Step 4—identification of the probability measure $\boldsymbol{\mu}$.

From the marginal rate of substitution between consumption and ambiguous asset j in equation (8),

$$\beta E_{\boldsymbol{\mu}} \phi^{'}(u^{-1}(E_{\mathbf{v}_a} u(\mathbf{R}\mathbf{y}))) \frac{E_{\mathbf{v}_a} u^{'}(\mathbf{R}\mathbf{y})\mathbf{r}_j}{u^{'}(u^{-1}(E_{\mathbf{v}_a} u(\mathbf{R}\mathbf{y})))} = m_{j0}(x_0, \mathbf{y})\phi^{'}(x_0). \tag{14}$$

We define

$$f_a(\mathbf{y}) = \phi^{'}(u^{-1}(E_{\mathbf{v}_a} u(\mathbf{R}\mathbf{y}))) \frac{E_{\mathbf{v}_a} u^{'}(\mathbf{R}\mathbf{y})\mathbf{r}_j}{u^{'}(u^{-1}(E_{\mathbf{v}_a} u(\mathbf{R}\mathbf{y})))}.$$

Equation (14) can be rewritten as

$$\sum_{a=1}^{A} \beta \mu_a f_a(\mathbf{y}) = m_{j0}(x_0, \mathbf{y})\phi^{'}(x_0). \tag{15}$$

By Lemma 1, if the functions $\{f_a\}_{a=1}^{A}$ are linearly independent, we can find a positive integer N and points $\mathbf{y}_1, ..., \mathbf{y}_N$, such that the system of equations

$$\sum_{a=1}^{A} \beta \alpha_a f_a(\mathbf{y}_i) = 0, \quad i = 1, ..., N \tag{16}$$

has no solution with $\boldsymbol{\alpha} \neq 0$. Since the first-order conditions (15) holds on the open set $\mathbf{Y}$, for each y_i $(i = 1, ..., N) \in \mathbf{Y}$ satisfying equation (16), there exists $(x_{0i}, \mathbf{p}_i)$ such that

$$\sum_{a=1}^{A} \beta \mu_a f_a(\mathbf{y_i}) = m_{j0}(x_{0i}, \mathbf{y_i})\phi^{'}(x_{0i}). \tag{17}$$

This is a linear system in $(\mu_1, ..., \mu_a, ...\mu_A)$, and it must have a unique solution. Suppose not, that is, there exist at least two solutions $(\mu_1^{'}, ..., \mu_a^{'}, ...\mu_A^{'})$ and

$(\mu_1^{''}, ..., \mu_a^{''}, ...\mu_A^{''})$ satisfying equation system (17). Then the following system of equations

$$\sum_{a=1}^{A} \beta(\mu_a^{'} - \mu_a^{''}) f_a(\boldsymbol{y}_i) = 0, \quad i = 1, ..., N \tag{18}$$

has a solution with $\boldsymbol{\mu}' - \boldsymbol{\mu}'' \neq 0$, which contradicts the linear independence assumption, i.e., equation (16) has no nonzero solution. Therefore, the unique solution identifies the probability measure $\boldsymbol{\mu}$ and the identified $\boldsymbol{\mu}$ determines which conditional probability measures in $\{\boldsymbol{\nu}_a\}_{a=1}^A$ are in its support. □

Remark 2 The identification of ϕ and u does not use the linear independence of indirect marginal utilities across ambiguity states $\{f_a(\boldsymbol{y})\}_{a=1}^A$. Given both ϕ and u have been identified, and conditional distributions of the asset payoffs are known, the linear independence of $\{f_a(\boldsymbol{y})\}_{a=1}^A$ can be directly checked as illustrated in Example 6 in Appendix A. Actually, since both ϕ and u are identified from consumption and asset demand, the linear independence of functions $\{f_a(\boldsymbol{y})\}_{a=1}^A$ can be equivalently defined in terms of observable consumption and asset demand, and conditional probability distributions.

Remark 3 The identification in Theorem 1 assumes observation of conditional probability distributions $\{\boldsymbol{\nu}_a\}_{a=1}^A$. Here, $\{\boldsymbol{\nu}_a\}_{a=1}^A$ is a set of candidate conditional probabilities, which includes the actual conditional probabilities as a subset. We can allow for arbitrarily many conditional probabilities as long as these conditional probabilities in this set generate one ambiguity free asset. The $\boldsymbol{\mu}$ when determined assigns zero probability to the conditional probabilities that are not consistent with (or relevant for) the given consumption and asset demand. Therefore, the subset of $\{\boldsymbol{\nu}_a\}_{a=1}^A$ which is subjectively relevant is also identified: it is the subset that consists of the conditional probability distributions assigned nonzero probability by $\boldsymbol{\mu}$. This point is further illustrated in the example in Appendix A.

Remark 4 Although the continuum case is excluded, the number of ambiguity states can be arbitrary, and can be more or less than the number of risk states. For the identification, only one ambiguous asset is needed, and the markets can be very incomplete.

Remark 5 The theorem requires the existence of a risk free asset. As under pure risk in Green et al. (1979), we can show that, without a risk free asset, the marginal rate of substitution between two ambiguity free assets identifies the risk index u so long as the underlying risk index u is analytic at $x = 0$. Once the risk index u is identified, the identification of the ambiguity index and the ambiguity probability measure follows the same argument as in the above proof.

It should be noted that Theorem 1 provides conditions for there to be a unique β, u, ϕ and $\boldsymbol{\mu}$. Also, it provides algorithms for the recovery of β, u and ϕ, but not for $\boldsymbol{\mu}$. To identify $\boldsymbol{\mu}$, we need linear independence in Theorem 1 to be satisfied. Linear independence in Theorem 1 is satisfied if functions are differentiably linearly independent. To determine differentiable linear independence is an easier task, and it is this property that, as demonstrated in the examples in appendices, is used to recover $\boldsymbol{\mu}$.

Define a differential operator

$$\Delta_k = (\frac{\partial}{\partial x_1})^{j_1} \cdots (\frac{\partial}{\partial x_m})^{j_m}, \quad j_1 + ... + j_m \leq k.$$

Definition 4 The functions $g_1, ..., g_n$ are *differentiably linearly independent* on D if there is some $k \geq n-1$ and some $\bar{\boldsymbol{x}} \in D$, such that each g_i is at least C^k at $\bar{\boldsymbol{x}}$ and such that there are differential operators $\Delta_{k_1}, ...\Delta_{k_n}$, with $k_i \leq k$, for all $i = 1, ...n$, such that the matrix

$$\boldsymbol{W} = \begin{pmatrix} \Delta_{k_1}(g_1) & \cdots & \Delta_{k_1}(g_i) & \cdots & \Delta_{k_1}(g_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_{k_j}(g_1) & \cdots & \Delta_{k_j}(g_i) & \cdots & \Delta_{k_j}(g_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_{k_n}(g_1) & \cdots & \Delta_{k_n}(g_i) & \cdots & \Delta_{k_n}(g_n) \end{pmatrix}$$

is nonsingular.

It is easy to see that, if $g_1, ..., g_n$ are differentiably linearly independent on D, they are linearly independent since differentiable linear independence implies that there cannot be an open neighborhood of $\bar{\boldsymbol{x}}$ and some $\boldsymbol{\alpha} \in \Lambda^{n-1}$ such that $\sum_{i=1}^{n} \alpha_i g_i(\boldsymbol{x}) = 0$ for all x in the neighborhood. Although the converse is generally not true, linear independence and differentiable linear independence are equivalent if functions are analytic. In this case, we can take $\Delta_{k_i} = \Delta_{i-1}$ and the $\boldsymbol{W}$ matrix becomes the Wronskian matrix.

We show that, if the functions

$$\left\{ f_a(\boldsymbol{y}) = \phi^{'}(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y}))) \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})))} \right\}_{a=1}^{A}$$

are differentiably linearly independent, then the probability measure $\boldsymbol{\mu}$ can be computed explicitly.

Proposition 1 *Given β, u and ϕ, if the functions $\left\{ f_a(\boldsymbol{y}) \right\}_{a=1}^{A}$ are differentiably linearly independent on $\boldsymbol{Y}$, then, the probability measure $\boldsymbol{\mu}$ can be explicitly computed.*

Proof Let j be an ambiguous asset. Under our notation, equation (8) can be rewritten as

$$E_{\boldsymbol{\mu}} f_a(\boldsymbol{y}) = \frac{1}{\beta} m_{j0}(x_0, \boldsymbol{y}) \phi^{'}(x_0). \tag{19}$$

On both sides, we take successive differentiation of equation (19) with respect to y_j. Denote by f_a^n the nth derivative of function f_a, and $m_{j0}^{(n)}$ is the nth derivative of function m_{j0}.

Then, after the $(N-1)$th differentiation, these $N(\geq A)$ equations can be written in matrix form, with $(\mu_1, ..., \mu_A)$ unknown :

$$\begin{pmatrix} f_1^0 & \cdots & f_a^0 & \cdots & f_A^0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^n & \cdots & f_a^n & \cdots & f_A^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^{N-1} & \cdots & f_a^{N-1} & \cdots & f_A^{N-1} \end{pmatrix}_{N\times A} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_a \\ \vdots \\ \mu_A \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} m_{j0}^{(0)}(x_0, \boldsymbol{y})\phi^{'}(x_0) \\ \vdots \\ \frac{1}{\beta} m_{j0}^{(n)}(x_0, \boldsymbol{y})\phi^{'}(x_0) \\ \vdots \\ \frac{1}{\beta} m_{j0}^{(N-1)}(x_0, \boldsymbol{y})\phi^{'}(x_0) \end{pmatrix} \tag{20}$$

Given that the functions u and ϕ have been identified, and the conditional probability distributions $\{\boldsymbol{\nu}_a\}_{a=1}^A$ are known, the left hand side matrix is computable. The fact that the functions $\{f_a(\boldsymbol{y})\}_{a=1}^A$ are differentiably linearly independence implies that the left hand side matrix has rank A at some portfolio $\boldsymbol{y}$, then the probability measure $\boldsymbol{\mu}$ can be computed. □

Proposition 1 has an important corollary. As the KPS* representation of smooth ambiguity preferences is specialized to the KPS utility (or expected utility) special case whenever there is only one ambiguity state, i.e., the set of actual conditional probability measures is a singleton, our recovery process will recover the KPS utility (or expected utility) from the consumption and asset demand functions if the demand is generated by maximization of a KPS utility (or an expected utility).[19] This will be illustrated through the example in Appendix C.

3.2 Identification of KMM representation

The first order conditions (7) corresponding to KMM representation generate the family of marginal rates of substitution of consumption and assets defined by

$$m_{j0}(x_0, \boldsymbol{y}) = \frac{\frac{\partial U(x_0, \boldsymbol{R}\boldsymbol{y})}{\partial y_j}}{\frac{\partial U(x_0, \boldsymbol{R}\boldsymbol{y})}{\partial x_0}} = \frac{\frac{\beta E_{\boldsymbol{\mu}} \tilde{\phi}^{'}\left(E_{\boldsymbol{\nu}_a} u(x_s)\right) E_{\boldsymbol{\nu}_a} u^{'}(x_s) r_j}{\tilde{\phi}^{'}\left(\tilde{\phi}^{-1}\left(E_{\boldsymbol{\mu}}\tilde{\phi}\left(E_{\boldsymbol{\nu}_a} u(x_s)\right)\right)\right)}}{u^{'}(x_0)} > 0 \tag{21}$$

and

$$m_{jk}(x_0, \boldsymbol{y}) = \frac{\frac{\partial U(x_0, \boldsymbol{R}\boldsymbol{y})}{\partial y_j}}{\frac{\partial U(x_0, \boldsymbol{R}\boldsymbol{y})}{\partial y_k}} = \frac{E_{\boldsymbol{\mu}} \tilde{\phi}^{'}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})) E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{R}\boldsymbol{y}) \boldsymbol{r}_j}{E_{\boldsymbol{\mu}} \tilde{\phi}^{'}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})) E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{R}\boldsymbol{y}) \boldsymbol{r}_k} > 0. \tag{22}$$

[19] Kübler et al. (2020) derive necessary and sufficient conditions for the consumption and asset demand functions to be generated by a KPS utility.

Here, an ambiguity free asset does not help identify KMM representation. We introduce a quasi-ambiguous asset which imposes some restrictions on the first and second moments of an ambiguous asset's payoffs, which will be exploited to separate the risk attitudes from the ambiguity attitudes.

Definition 5 An asset is quasi-ambiguous in ambiguity state 1 if

1. the conditional first moments satisfy

$$E_{\boldsymbol{\nu}_1}\boldsymbol{r}_j \neq E_{\boldsymbol{\nu}_2}\boldsymbol{r}_j = ... = E_{\boldsymbol{\nu}_A}\boldsymbol{r}_j;\ and$$

2. the conditional second moments satisfy

$$E_{\boldsymbol{\nu}_1}(\boldsymbol{r}_j)^2 \neq E_{\boldsymbol{\nu}_2}(\boldsymbol{r}_j)^2 = ... = E_{\boldsymbol{\nu}_A}(\boldsymbol{r}_j)^2.$$

Example 5 There are 4 risk states and 3 ambiguity states. The three conditional probabilities are $\nu_1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, $\nu_2 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ and $\nu_3 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{4}, \frac{1}{12})$. An asset that pays $\boldsymbol{r} = (1, 1, 0, 0)$ across risk states is quasi-ambiguous in state 1 since $E_{\boldsymbol{\nu}_1}\boldsymbol{r} \neq E_{\boldsymbol{\nu}_2}\boldsymbol{r} = E_{\boldsymbol{\nu}_3}\boldsymbol{r}$ and $E_{\boldsymbol{\nu}_1}\boldsymbol{r}^2 \neq E_{\boldsymbol{\nu}_2}\boldsymbol{r}^2 = E_{\boldsymbol{\nu}_3}\boldsymbol{r}^2$.

Like being ambiguity free, being quasi-ambiguous is also a joint restriction on asset payoffs $\boldsymbol{r} = (r_1, ..., r_s, ..., r_S)$ and the conditional probability distributions $\{\boldsymbol{\nu}_a\}_{a=1}^A$. The identification of KMM representation can be achieved by introduction of a quasi-ambiguous asset because the first two moments of the quasi-ambiguous asset can be recovered from the marginal rate of substitution between the risk free asset and the quasi-ambiguous asset, which will help pin down $\frac{\tilde{\phi}''(x)}{\tilde{\phi}'(x)}$.

Define the indirect marginal utility functions on the portfolio set $\boldsymbol{Y}$

$$\left\{ f_a(\boldsymbol{y}) = \tilde{\phi}'\left(E_{\boldsymbol{\nu}_a}u(x_s)\right)E_{\boldsymbol{\nu}_a}u'(x_s)r_j - m_{j1}(x_0, \boldsymbol{y})\tilde{\phi}'\left(E_{\boldsymbol{\nu}_a}u(x_s)\right)E_{\boldsymbol{\nu}_a}u'(x_s)r_1 \right\}_{a=2}^A.$$

Theorem 2 *If*

1. *the KMM representation of smooth ambiguity utility satisfies the condition that the indices* $u : \mathbb{R}_{++} \to \mathbb{R}$ *and* $\tilde{\phi} : u(\mathbb{R}_{++}) \to \mathbb{R}$ *are twice differentiable, strictly increasing, and strictly concave,*
2. *there is an asset,* $j = 1$ *that is risk free:* $\boldsymbol{r}_1 = 1$ *across states of the world,*
3. *there is an asset,* $j = 2$ *that is quasi-ambiguous in some ambiguity state (without loss of generality in ambiguity state 1) with positive probability,*
4. *the family of conditional probability measures over states of risk,* $\boldsymbol{\nu} : \boldsymbol{A} \to \Delta(\boldsymbol{S})$ *is known, and*
5. *the functions* $\left\{f_a(\boldsymbol{y})\right\}_{a=2}^A$ *are linearly independent on* $\boldsymbol{Y}$,

then, the demand for consumption and assets identifies the risk index $u : \mathbb{R}_{++} \to \mathbb{R}$ *and the ambiguity index* $\tilde{\phi} : u(\mathbb{R}_{++}) \to \mathbb{R}$ *up to a positive affine transformation, the discount factor* β, *as well as the ambiguity probability measure* $\boldsymbol{\mu}$.

Proof The proof parallels with that of Theorem 1.

Step 1—identification of the discount factor β.

The discount factor can be identified from the marginal rate of substitution between consumption and risk free asset in equation (21):

$$u^{'}(x_0) = \frac{\beta E_{\boldsymbol{\mu}} \tilde{\phi}^{'}\big(E_{\boldsymbol{\nu}_a} u(x_s)\big) E_{\boldsymbol{\nu}_a} u^{'}(x_s) r_1}{\tilde{\phi}^{'}\Big(\tilde{\phi}^{-1}\big(E_{\boldsymbol{\mu}} \tilde{\phi}\big(E_{\boldsymbol{\nu}_a} u(x_s)\big)\big)\Big)} \frac{1}{m_{10}(x_0, \boldsymbol{y})}. \tag{23}$$

At $x_0 = \overline{x}_0$ and portfolio $\overline{\boldsymbol{y}} = (\overline{x}_0, 0, ..., 0)$, we get

$$\beta = m_{10}(\overline{x}_0, \overline{\boldsymbol{y}}). \tag{24}$$

Step 2—identification of the risk index u.

From equation (23), if we normalize $u^{'}(\bar{x}_0) = 1$, we have

$$u^{'}(x_0) = \frac{m_{10}(x_0, \boldsymbol{y})}{m_{10}(\bar{x}_0, \boldsymbol{y})}, \tag{25}$$

then the index $u : \mathbb{R}_{++} \to \mathbb{R}$ can be identified up to a positive affine transformation by integrating above equation with respect to x_0.

Step 3—identification of the ambiguity index $\tilde{\phi}$.

From equation (22), the marginal rate of substitution between asset 2 and asset 1 is

$$m_{21}(x_0, \boldsymbol{y}) = \frac{E_{\boldsymbol{\mu}} \tilde{\phi}^{'}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})) E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{R}\boldsymbol{y}) \boldsymbol{r}_2}{E_{\boldsymbol{\mu}} \tilde{\phi}^{'}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})) E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{R}\boldsymbol{y}) \boldsymbol{r}_1}. \tag{26}$$

At portfolio $\overline{\boldsymbol{y}} = (\overline{x}_0, 0, ..., 0)$, this gives us

$$\mu_1 E_{\boldsymbol{\nu}_1} \boldsymbol{r}_j + (1 - \mu_1) E_{\boldsymbol{\nu}_{a \neq 1}} \boldsymbol{r}_2 = m_{21}(x_0, \overline{\boldsymbol{y}}). \tag{27}$$

Since $E_{\boldsymbol{\nu}_1} \boldsymbol{r}_j$, $E_{\boldsymbol{\nu}_{a \neq 1}} \boldsymbol{r}_j$ and $m_{21}(x_0, \overline{\boldsymbol{y}})$ are known, equation (27) will give us the value of μ_1.

Then we can compute the values

$$E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} (\boldsymbol{r}_2)^2 = \mu_1 E_{\boldsymbol{\nu}_1} (\boldsymbol{r}_2)^2 + (1 - \mu_1) E_{\boldsymbol{\nu}_{a \neq 1}} (\boldsymbol{r}_2)^2,$$

and

$$E_{\boldsymbol{\mu}} (E_{\boldsymbol{\nu}_a} \boldsymbol{r}_2)^2 = \mu_1 (E_{\boldsymbol{\nu}_1} \boldsymbol{r}_2)^2 + (1 - \mu_1)(E_{\boldsymbol{\nu}_{a \neq 1}} \boldsymbol{r}_2)^2.$$

If we take derivative on both sides of equation (27) with respect to y_2 and evaluate the resulting functional at portfolio $\boldsymbol{y} = (x, 0, ..., 0)$, we have

$$[E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_2)^2 - (E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_2)^2]\frac{\tilde{\phi}''(x)}{\tilde{\phi}'(x)} =$$

$$[(E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_2)^2 - E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}(\boldsymbol{r}_2)^2]\frac{u''(x)}{(u'(x))^2} + \frac{\partial m_{21}(x_0, x, 0, ..., 0)}{\partial y_2}\frac{1}{u'(x)}. \tag{28}$$

Given the risk index u identified, the above equation in turn identifies the ambiguity index $\tilde{\phi} : u(\mathbb{R}_{++}) \to \mathbb{R}$ up to a positive affine transformation. Note that the identified ambiguity index $\tilde{\phi}$ depends on the normalization of risk index u, since for the KMM representation, a positive affine transformation of the risk index u does not change preferences if and only if a compensating transformation is applied to $\tilde{\phi}$.

Step 4—identification of the probability measure $\boldsymbol{\mu}$*.*

From the marginal rate of substitution between risk free asset and ambiguous asset j in equation (22),

$$m_{j1}(x_0, \boldsymbol{y}) = \frac{E_{\boldsymbol{\mu}}\tilde{\phi}'(E_{\boldsymbol{\nu}_a}u(\boldsymbol{R}\boldsymbol{y}))E_{\boldsymbol{\nu}_a}u'(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j}{E_{\boldsymbol{\mu}}\tilde{\phi}'(E_{\boldsymbol{\nu}_a}u(\boldsymbol{R}\boldsymbol{y}))E_{\boldsymbol{\nu}_a}u'(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_1}. \tag{29}$$

We define

$$f_a(\boldsymbol{y}) = \tilde{\phi}'\left(E_{\boldsymbol{\nu}_a}u(x_s)\right)E_{\boldsymbol{\nu}_a}u'(x_s)r_j - m_{j1}(x_0, \boldsymbol{y})\tilde{\phi}'\left(E_{\boldsymbol{\nu}_a}u(x_s)\right)E_{\boldsymbol{\nu}_a}u'(x_s)r_1.$$

Equation (29) can be rewritten as

$$\sum_{a=2}^{A}\mu_a f_a(\boldsymbol{y}) = \mu_1 m_{j1}(x_0, \boldsymbol{y})\tilde{\phi}'\left(E_{\boldsymbol{\nu}_1}u(x_s)\right)E_{\boldsymbol{\nu}_1}u'(x_s)r_1 - \mu_1\tilde{\phi}'\left(E_{\boldsymbol{\nu}_1}u(x_s)\right)E_{\boldsymbol{\nu}_1}u'(x_s)r_j.$$

As argued in Theorem 1, if the functions $\{f_a\}_{a=2}^A$ are linearly independent, then $(\mu_2, ..., \mu_a, ...\mu_A)$ must be unique. And the identified $\boldsymbol{\mu}$ determines which conditional probability measures in $\{\boldsymbol{\nu}_a\}_{a=1}^A$ are in its support. □

We can also show that, if the functions

$$\left\{f_a(\boldsymbol{y}) = \tilde{\phi}'\left(E_{\boldsymbol{\nu}_a}u(x_s)\right)E_{\boldsymbol{\nu}_a}u'(x_s)r_j - m_{j1}(x_0, \boldsymbol{y})\tilde{\phi}'\left(E_{\boldsymbol{\nu}_a}u(x_s)\right)E_{\boldsymbol{\nu}_a}u'(x_s)r_1\right\}_{a=2}^A$$

are differentiably linearly independent, then the probability measure $\boldsymbol{\mu}$ can be computed explicitly. We state the result, but omit the proof.

Proposition 2 *Given* β, u *and* ϕ, *if the functions* $\left\{f_a(\boldsymbol{y})\right\}_{a=2}^A$ *are differentiably linearly independent on* $\boldsymbol{Y}$, *then, the probability measure* $\boldsymbol{\mu}$ *can be explicitly computed.*

3.3 Identification of HM representation

Denote

$$x_1 = (\tilde{\phi} \circ u)^{-1}\big(E_{\boldsymbol{\mu}}\tilde{\phi}\big(E_{\boldsymbol{\nu}_a}u(x_s)\big)\big).$$

The first order conditions (7) corresponding to HM representation generate the family of marginal rates of substitution of consumption and assets defined by

$$m_{j0}(x_0, \boldsymbol{y}) = \frac{\frac{\partial U(x_0, \boldsymbol{R}\boldsymbol{y})}{\partial y_j}}{\frac{\partial U(x_0, \boldsymbol{R}\boldsymbol{y})}{\partial x_0}} = \frac{u_2'(x_1)\frac{E_{\boldsymbol{\mu}}\tilde{\phi}'\left(E_{\boldsymbol{\nu}_a}u(x_s)\right)E_{\boldsymbol{\nu}_a}u'(x_s)r_j}{(\tilde{\phi}\circ u)'(x_1)}}{u_1'(x_0)} > 0 \tag{30}$$

and

$$m_{jk}(x_0, \boldsymbol{y}) = \frac{\frac{\partial U(x_0, \boldsymbol{R}\boldsymbol{y})}{\partial y_j}}{\frac{\partial U(x_0, \boldsymbol{R}\boldsymbol{y})}{\partial y_k}} = \frac{E_{\boldsymbol{\mu}}\tilde{\phi}'(E_{\boldsymbol{\nu}_a}u(\boldsymbol{R}\boldsymbol{y}))E_{\boldsymbol{\nu}_a}u'(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j}{E_{\boldsymbol{\mu}}\tilde{\phi}'(E_{\boldsymbol{\nu}_a}u(\boldsymbol{R}\boldsymbol{y}))E_{\boldsymbol{\nu}_a}u'(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k} > 0. \tag{31}$$

It turns out that the identification of the generalized smooth ambiguity utility does not involve extra technical complexities. To identify this generalized smooth ambiguity utility, we require existence of both an ambiguity free asset and a quasi-ambiguous asset.

Define the indirect marginal utility functions on the portfolio set $\boldsymbol{Y}$

$$\left\{f_a(\boldsymbol{y}) = \tilde{\phi}'\left(E_{\boldsymbol{\nu}_a}u(x_s)\right)E_{\boldsymbol{\nu}_a}u'(x_s)r_j - m_{j1}(x_0, \boldsymbol{y})\tilde{\phi}'\left(E_{\boldsymbol{\nu}_a}u(x_s)\right)E_{\boldsymbol{\nu}_a}u'(x_s)r_1\right\}_{a=2}^{A}.$$

Theorem 3 *If*

1. *the HM representation of smooth ambiguity utility satisfies the condition that the indices* $u_1 : \mathbb{R}_{++} \to \mathbb{R}$, $u_2 : \mathbb{R}_{++} \to \mathbb{R}$, $u : \mathbb{R}_{++} \to \mathbb{R}$ *and* $\tilde{\phi} : u(\mathbb{R}_{++}) \to \mathbb{R}$ *are twice differentiable, strictly increasing, and strictly concave,*
2. *there is an asset,* $j = 1$ *that is risk free:* $\boldsymbol{r}_1 = 1$ *across states of the world,*
3. *there is an asset,* $j = 2$ *that is ambiguity free: its payoff distribution is invariant to the states of ambiguity,*
4. *there is an asset,* $j = 3$ *that is quasi-ambiguous in some ambiguity state (without loss of generality in ambiguity state 1) with positive probability,*
5. *the family of conditional probability measures over states of risk,* $\boldsymbol{\nu} : \boldsymbol{A} \to \Delta(\boldsymbol{S})$ *is known, and*
6. *the functions* $\left\{f_a(\boldsymbol{y})\right\}_{a=2}^{A}$ *are linearly independent on* $\boldsymbol{Y}$,

then, the demand for consumption and assets identifies $u_1 : \mathbb{R}_{++} \to \mathbb{R}$, $u_2 : \mathbb{R}_{++} \to \mathbb{R}$, $u : \mathbb{R}_{++} \to \mathbb{R}$ *and* $\tilde{\phi} : u(\mathbb{R}_{++}) \to \mathbb{R}$, *each up to a positive affine transformation, as well as the ambiguity probability measure* $\boldsymbol{\mu}$.

Proof The result can be proved by combining the proofs in Theorem 1 and 2, and we only sketch the argument without details.

From equation (30), the marginal rate of substitution between risk free asset and first period consumption is

$$m_{10}(x_0, \boldsymbol{y}) = \frac{u_2^{'}(x_1)\frac{E_{\boldsymbol{\mu}}\tilde{\phi}'\left(E_{\boldsymbol{v}_a}u(x_s)\right)E_{\boldsymbol{v}_a}u^{'}(x_s)r_1}{(\tilde{\phi}\circ u)^{'}(x_1)}}{u_1^{'}(x_0)}.$$

At portfolio $\boldsymbol{y} = (x, 0, 0...0)$, we have

$$u_1^{'}(x_0) = u_2^{'}(x)\frac{1}{m_{10}(x_0, x, 0, 0...0)}. \tag{32}$$

If we normalize $u_1^{'}(\bar{x}_0) = 1$ at $\bar{x}_0$ and $u_2^{'}(\bar{x}) = 1$ at $\bar{x}$, then

$$u_1^{'}(x_0) = \frac{1}{m_{10}(x_0, \bar{x}, 0, 0...0)}$$

and $u_1 : \mathbb{R}_{++} \to \mathbb{R}$ can be identified up to a positive affine transformation from integrating the above equation with respect to x_0;

$$u_2^{'}(x) = m_{10}(\bar{x}_0, x, 0, 0...0)$$

and $u_2 : \mathbb{R}_{++} \to \mathbb{R}$ can be identified up to a positive affine transformation from integrating the above equation with respect to x.

From equation (31), the marginal rate of substitution between risk free asset and ambiguity free asset is

$$m_{12}(x_0, \boldsymbol{y}) = \frac{E_{\boldsymbol{\mu}}\tilde{\phi}^{'}(E_{\boldsymbol{v}_a}u(\boldsymbol{R}\boldsymbol{y}))E_{\boldsymbol{v}_a}u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_1}{E_{\boldsymbol{\mu}}\tilde{\phi}^{'}(E_{\boldsymbol{v}_a}u(\boldsymbol{R}\boldsymbol{y}))E_{\boldsymbol{v}_a}u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_2}.$$

Following the proof in Theorem 1, the risk index $u : \mathbb{R}_{++} \to \mathbb{R}$ can be identified up to a positive affine transformation from $m_{12}(x_0, \boldsymbol{y})$.

The marginal rate of substitution between quasi-ambiguous asset and risk free asset is

$$m_{31}(x_0, \boldsymbol{y}) = \frac{E_{\boldsymbol{\mu}}\tilde{\phi}^{'}(E_{\boldsymbol{v}_a}u(\boldsymbol{R}\boldsymbol{y}))E_{\boldsymbol{v}_a}u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_3}{E_{\boldsymbol{\mu}}\tilde{\phi}^{'}(E_{\boldsymbol{v}_a}u(\boldsymbol{R}\boldsymbol{y}))E_{\boldsymbol{v}_a}u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_1}.$$

Following the proof in Theorem 2, the ambiguity index $\tilde{\phi} : u(\mathbb{R}_{++}) \to \mathbb{R}$ can be identified up to a positive affine transformation from $m_{31}(x_0, \boldsymbol{y})$.

Finally, the proof in Theorem 2 can be used to show that if the functions $\left\{f_a(\boldsymbol{y})\right\}_{a=2}^{A}$ are linearly independent, then the ambiguity probability measure μ will be uniquely identified. □

Furthermore, it can also be shown that if the functions $\left\{ f_a(\boldsymbol{y}) \right\}_{a=2}^{A}$ are differentiably linearly independent, then the probability measure $\boldsymbol{\mu}$ can be computed explicitly.

Proposition 3 *Given u_1, u_2, u and $\tilde{\phi}$, if the functions $\left\{ f_a(\boldsymbol{y}) \right\}_{a=2}^{A}$ are differentiably linearly independent on $\boldsymbol{Y}$, then, the probability measure $\boldsymbol{\mu}$ can be explicitly computed.*

As the HM representation of smooth ambiguity model accommodates KMM representation, KPS* representation, KPS utility as well as expected utility as special cases, an important corollary is that our recovery procedure will recover KMM representation or KPS* representation or KPS utility or expected utility from the consumption and asset demand, without assuming which utility holds ex ante.

4 Concluding remarks

In this paper, we give sufficient conditions and a constructive recovery procedure for identification of the smooth ambiguity preferences from observable consumption and asset demand functions in incomplete asset markets. The existence of an ambiguity free asset and a quasi-ambiguous asset separates the risk index from the ambiguity index. The (differentiable) linear independence of the indirect marginal utility functions uniquely pin down the ambiguity beliefs, and the identified ambiguity beliefs determine its support, i.e., the actual conditional probability measures in the candidate probability set.

This paper suggests several important questions for future research. The identification argument assumes the existence of underlying preferences or a utility function that satisfy particular properties, and ascertains their uniqueness given the demand functions. It would lead to erroneous conclusions if the underlying preferences are not smooth ambiguity preferences. One potential area of future research would be to derive necessary and sufficient conditions on consumption and asset demand functions which ensure that they are the result of the maximization of some smooth ambiguity utility, as the incomplete market demand test for KPS utility derived in Kübler et al. (2020).

Second, the assumption of (differentiable) linear independence of the indirect marginal utilities over assets across ambiguity states plays a crucial rule in identifying the ambiguity beliefs. Though this assumption can be checked based on observable demand functions, it is an important question to give a complete characterization of linear independence in terms of economic fundamentals, i.e., risk aversion, ambiguity aversion, conditional probability measures and asset payoffs as in Kübler and Polemarchakis (2017).

A Appendix A Example for identification of KPS* representation

We illustrate the recovery argument for the KPS* representation through the following example. We demonstrate that assuming observation of conditional probability measures only requires that the actual conditional probability measures lie in a set of candidate conditional probability measures and the actual conditional probability measures in the support of $\boldsymbol{\mu}$ will be determined once $\boldsymbol{\mu}$ is identified. We will consider two cases. In the first, we assume we exactly know the two ambiguity states and the corresponding conditional probability measures, which the individual actually uses in his portfolio optimization. For the second case, it is assumed that we only know a set of candidate ambiguity states and the corresponding conditional probability measures. This set includes the actual ambiguity states and conditional probabilities. The recovery process identifies the specific conditional probability measures employed by the individual in his optimization. We also show that whether the set of candidate conditional probabilities contains the actual ones can be tested.

Example 6 Assume there are three assets with payoffs: $\boldsymbol{r}_1 = (1, 1, 1, 1)$, $\boldsymbol{r}_2 = (1, 2, 0, 0)$, and $\boldsymbol{r}_3 = (0, 0, 0, 1)$. The demand for first period consumption and three assets is given by[20]

$$x_0 = \frac{\frac{p_1}{p_0}\frac{(\frac{1}{3}-4\frac{p_3}{p_1})(2\frac{p_3}{p_1}-\frac{1}{3})}{\frac{p_3}{p_1}-\frac{1}{10}}}{p_1\frac{(\frac{1}{3}-4\frac{p_3}{p_1})(2\frac{p_3}{p_1}-\frac{1}{3})}{\frac{p_3}{p_1}-\frac{1}{10}} + p_1 + p_3\frac{\frac{4}{5}(9\frac{p_3}{p_1}-1)}{\frac{1}{10}-\frac{p_3}{p_1}}}, \tag{A.1}$$

$$y_1 = \begin{cases} 0 & \text{if } \frac{p_2}{p_1} < \frac{5}{6} \\ \begin{cases} [0, \frac{1}{p_1\frac{(\frac{1}{3}-4\frac{p_3}{p_1})(2\frac{p_3}{p_1}-\frac{1}{3})}{\frac{p_3}{p_1}-\frac{1}{10}}+p_1+p_3\frac{\frac{4}{5}(9\frac{p_3}{p_1}-1)}{\frac{1}{10}-\frac{p_3}{p_1}}}] & \text{if } \frac{1}{10} < \frac{p_3}{p_1} < \frac{1}{6} \\ [\frac{1}{p_1\frac{(\frac{1}{3}-4\frac{p_3}{p_1})(2\frac{p_3}{p_1}-\frac{1}{3})}{\frac{p_3}{p_1}-\frac{1}{10}}+p_1+p_3\frac{\frac{4}{5}(9\frac{p_3}{p_1}-1)}{\frac{1}{10}-\frac{p_3}{p_1}}}, 0] & \text{if } \frac{1}{12} < \frac{p_3}{p_1} < \frac{1}{10} \end{cases} & \text{if } \frac{p_2}{p_1} = \frac{5}{6} \\ \frac{1}{p_1\frac{(\frac{1}{3}-4\frac{p_3}{p_1})(2\frac{p_3}{p_1}-\frac{1}{3})}{\frac{p_3}{p_1}-\frac{1}{10}}+p_1+p_3\frac{\frac{4}{5}(9\frac{p_3}{p_1}-1)}{\frac{1}{10}-\frac{p_3}{p_1}}} & \text{if } \frac{p_2}{p_1} > \frac{5}{6}, \end{cases} \tag{A.2}$$

$$y_2 = \begin{cases} \frac{1}{p_1\frac{(\frac{1}{3}-4\frac{p_3}{p_1})(2\frac{p_3}{p_1}-\frac{1}{3})}{\frac{p_3}{p_1}-\frac{1}{10}}+p_1+p_3\frac{\frac{4}{5}(9\frac{p_3}{p_1}-1)}{\frac{1}{10}-\frac{p_3}{p_1}}} & \text{if } \frac{p_2}{p_1} < \frac{5}{6} \\ \begin{cases} [0, \frac{1}{p_1\frac{(\frac{1}{3}-4\frac{p_3}{p_1})(2\frac{p_3}{p_1}-\frac{1}{3})}{\frac{p_3}{p_1}-\frac{1}{10}}+p_1+p_3\frac{\frac{4}{5}(9\frac{p_3}{p_1}-1)}{\frac{1}{10}-\frac{p_3}{p_1}}} - \frac{6}{5}y_1] & \text{if } \frac{1}{10} < \frac{p_3}{p_1} < \frac{1}{6} \\ [\frac{1}{p_1\frac{(\frac{1}{3}-4\frac{p_3}{p_1})(2\frac{p_3}{p_1}-\frac{1}{3})}{\frac{p_3}{p_1}-\frac{1}{10}}+p_1+p_3\frac{\frac{4}{5}(9\frac{p_3}{p_1}-1)}{\frac{1}{10}-\frac{p_3}{p_1}}} - \frac{6}{5}y_1, 0] & \text{if } \frac{1}{12} < \frac{p_3}{p_1} < \frac{1}{10} \end{cases} & \text{if } \frac{p_2}{p_1} = \frac{5}{6} \\ 0 & \text{if } \frac{p_2}{p_1} > \frac{5}{6}, \end{cases} \tag{A.3}$$

[20] All our main results are stated assuming knowledge of demand functions; however, as shown by this example, our identification argument works for demand correspondence.

$$y_3 = \frac{\frac{\frac{1}{5}(9\frac{p_3}{p_1}-1)}{\frac{1}{4}(\frac{1}{10}-\frac{p_3}{p_1})}}{p_1\frac{(\frac{1}{3}-4\frac{p_3}{p_1})(2\frac{p_3}{p_1}-\frac{1}{3})}{\frac{p_3}{p_1}-\frac{1}{10}} + p_1 + p_3\frac{\frac{4}{5}(9\frac{p_3}{p_1}-1)}{\frac{1}{10}-\frac{p_3}{p_1}}}. \tag{A.4}$$

We show that if the observed demand is generated by the KPS* representation, then it is possible to recover the smooth ambiguity utility and the ambiguity probabilities that rationalize the demand (A.1)- (A.4).[21] We consider two cases: in the first case, we exactly know the two ambiguity states with conditional probabilities $\boldsymbol{\nu}_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ and $\boldsymbol{\nu}_2 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{4}, \frac{1}{12})$, which are actually used by the individual; in the second case, we only know a set of candidate ambiguity states with conditional probabilities $\boldsymbol{\nu}_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, $\boldsymbol{\nu}_2 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{4}, \frac{1}{12})$ and $\boldsymbol{\nu}_3 = (\frac{1}{2}, \frac{1}{6}, \frac{5}{24}, \frac{1}{8})$, which includes the actual ones. Note that the first asset is risk free since its payoffs are constant across risk states; the second asset is ambiguity free in the sense that its payoffs vary across risk states, but its payoff distributions are constant across ambiguity states; the third asset is ambiguous since its payoff distributions differ across ambiguity states, and it is also risky since its payoffs vary across risk states.

To recover the generating utility functions ϕ and u and the discount factor β, we first invert the demand (A.1)- (A.4) and get the corresponding marginal rates of substitution.

From demand equations (A.2) and (A.3), at $\frac{p_2}{p_1} = \frac{5}{6}$ only the quantity $y_1 + \frac{5}{6}y_2$ is determined and the individual is indifferent between asset 1 and asset 2; at $\frac{p_2}{p_1} >$ (or $<$) $\frac{5}{6}$, the demand for asset 1 (or asset 2) is 0. We know that asset 1 and asset 2 are perfect substitutes. Therefore,

$$m_{12}(x_0, \boldsymbol{y}) = \frac{p_1(x_0, \boldsymbol{y})}{p_2(x_0, \boldsymbol{y})} = \frac{6}{5}. \tag{A.5}$$

Since $y_1 + \frac{5}{6}y_2 = \frac{1}{p_1\frac{(\frac{1}{3}-4\frac{p_3}{p_1})(2\frac{p_3}{p_1}-\frac{1}{3})}{\frac{p_3}{p_1}-\frac{1}{10}} + p_1 + p_3\frac{\frac{1}{5}(9\frac{p_3}{p_1}-1)}{\frac{1}{4}(\frac{1}{10}-\frac{p_3}{p_1})}}$, demand equation (A.4) implies

$$\frac{y_3}{y_1 + \frac{5}{6}y_2} = \frac{\frac{1}{5}(9\frac{p_3}{p_1} - 1)}{\frac{1}{4}(\frac{1}{10} - \frac{p_3}{p_1})},$$

from which we can solve for $\frac{p_3}{p_1}$:

$$m_{31}(x_0, \boldsymbol{y}) = \frac{p_3(x_0, \boldsymbol{y})}{p_1(x_0, \boldsymbol{y})} = \frac{\frac{1}{10}(y_1 + \frac{5}{6}y_2) + \frac{1}{80}y_3}{\frac{9}{10}(y_1 + \frac{5}{6}y_2) + \frac{1}{8}y_3}. \tag{A.6}$$

Substituting equation (A.6) into demand equation (A.1), we can solve for $\frac{p_1}{p_0}$:

$$m_{10}(x_0, \boldsymbol{y}) = \frac{p_1(x_0, \boldsymbol{y})}{p_0(x_0, \boldsymbol{y})} = x_0\frac{\frac{9}{10}(y_1 + \frac{5}{6}y_2) + \frac{1}{8}y_3}{(y_1 + \frac{5}{6}y_2 + \frac{1}{6}y_3)(y_1 + \frac{5}{6}y_2 + \frac{1}{12}y_3)}. \tag{A.7}$$

21 Note that KPS utility and expected utility are special cases of KPS* representation, and our recovery procedure can determine whether the true preferences are smooth ambiguity utility or KPS utility or expected utility as demonstrated in Appendix C.

From equations (A.6) and (A.7), we can get

$$m_{30}(x_0, \boldsymbol{y}) = \frac{p_3(x_0, \boldsymbol{y})}{p_0(x_0, \boldsymbol{y})} = x_0 \frac{\frac{1}{10}(y_1 + \frac{5}{6}y_2) + \frac{1}{80}y_3}{(y_1 + \frac{5}{6}y_2 + \frac{1}{6}y_3)(y_1 + \frac{5}{6}y_2 + \frac{1}{12}y_3)}. \tag{A.8}$$

Intuitively, the discount factor measures the value of one unit of consumption tomorrow in terms of one unit of consumption today. It will be recovered from the marginal rate of substitution between consumption and risk free asset $m_{10}(x_0, \boldsymbol{y})$ at $x_0 = \overline{x}_0$ and portfolio $\overline{\boldsymbol{y}} = (\overline{x}_0, 0, 0)$

$$\beta = m_{10}(\overline{x}_0, \overline{\boldsymbol{y}}) = \frac{9}{10}.$$

The marginal utility of consumption $\phi^{'}(x_0)$ can be traced out from the marginal rate of substitution between consumption and risk free asset when we vary x_0, since the variation of the marginal rate of substitution is attributed to the change of willingness to substitute for first period consumption. If we normalize $\phi^{'}(1) = 1$ at $\overline{x}_0 = 1$, we have

$$\phi^{'}(x_0) = \frac{m_{10}(1, \boldsymbol{y})}{m_{10}(x_0, \boldsymbol{y})} = x_0^{-1}.$$

Integrating both sides with respect to x_0 (and choosing the integration constant to be 0) gives

$$\phi(x_0) = \ln(x_0).$$

The risk index u can be integrated from $m_{12}(x_0, \boldsymbol{y})$ of equation (A.5), since at prices where only the risk free asset and the ambiguity free asset are demanded, the composition of such portfolio is determined solely by risk aversion. Differentiating $m_{12}(x_0, \boldsymbol{y})$ equation with respect to y_2 and evaluating the resulting functional at $\bar{\boldsymbol{y}} = (x, 0, 0)$, we have

$$-\frac{u^{''}(x)}{u^{'}(x)} = \frac{\partial m_{12}(x_0, \bar{\boldsymbol{y}})}{\partial y_2} \frac{(E_{\tilde{\boldsymbol{\nu}}} r_2)^2}{E_{\tilde{\boldsymbol{\nu}}} r_2^2 - (E_{\tilde{\boldsymbol{\nu}}} r_2)^2} = 0.$$

It can readily be verified that the integration and proper normalization yield

$$u(x) = x.$$

Case 1: knowing actual conditional probabilities

From the definition of $f_a(\boldsymbol{y})$, if we exactly know the two ambiguity states with conditional probabilities $\boldsymbol{\nu}_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ and $\boldsymbol{\nu}_2 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{4}, \frac{1}{12})$, which are actually used by the individual, we can define

$$f_1(\boldsymbol{y}) = \frac{1}{6}[y_1 + \frac{5}{6}y_2 + \frac{1}{6}y_3]^{-1},$$

and

$$f_2(\boldsymbol{y}) = \frac{1}{12}[y_1 + \frac{5}{6}y_2 + \frac{1}{12}y_3]^{-1}.$$

According to Theorem 1, if $f_1(\boldsymbol{y})$ and $f_2(\boldsymbol{y})$ are linearly independent, then the ambiguity probability measure $\boldsymbol{\mu}$ is unique. We show that $f_1(\boldsymbol{y})$ and $f_2(\boldsymbol{y})$ are actually differentiably linearly independent, then they must be linearly independent and μ can be explicitly computed by the procedure in Proposition 1.

At $\boldsymbol{y} = (1, 0, 0)$, we compute the matrix (with respect to asset 3) in the left hand side of equation (20) in Proposition 1

$$\begin{pmatrix} f_1(\boldsymbol{y}) & f_2(\boldsymbol{y}) \\ f_1^{'}(\boldsymbol{y}) & f_2^{'}(\boldsymbol{y}) \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{12} \\ \frac{-1}{36} & \frac{-1}{144} \end{pmatrix},$$

which has full rank. This means functions $f_1(\boldsymbol{y})$ and $f_2(\boldsymbol{y})$ are (differentiably) linearly independent.

We can also compute the column in the right hand side of equation (20)

$$\begin{pmatrix} \frac{1}{\beta} m_{30}^{(0)}(x_0, \boldsymbol{y})\phi^{'}(x_0) \\ \frac{1}{\beta} m_{30}^{(1)}(x_0, \boldsymbol{y})\phi^{'}(x_0) \end{pmatrix} = \begin{pmatrix} \frac{1}{9} \\ \frac{-1}{72} \end{pmatrix}.$$

Therefore, we can solve the equation (20) in Proposition 1

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{12} \\ \frac{-1}{72} & \frac{-1}{288} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{9} \\ \frac{-1}{144} \end{pmatrix},$$

and recover the ambiguity probability measure

$$\boldsymbol{\mu} = (\frac{1}{3}, \frac{2}{3}).$$

Case 2: knowing candidate conditional probabilities

If we only know a set of candidate ambiguity states with conditional probabilities $\boldsymbol{\nu}_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, $\boldsymbol{\nu}_2 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{4}, \frac{1}{12})$ and $\boldsymbol{\nu}_3 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{4})$, which includes the actual ones, we define $f_a(\boldsymbol{y})$ for all possible a,

$$f_1(\boldsymbol{y}) = \frac{1}{6}[y_1 + \frac{5}{6}y_2 + \frac{1}{6}y_3]^{-1},$$

$$f_2(\boldsymbol{y}) = \frac{1}{12}[y_1 + \frac{5}{6}y_2 + \frac{1}{12}y_3]^{-1},$$

and

$$f_3(\boldsymbol{y}) = \frac{1}{4}[y_1 + \frac{5}{6}y_2 + \frac{1}{4}y_3]^{-1}.$$

As in Case 1, we show that $f_1(\boldsymbol{y})$, $f_2(\boldsymbol{y})$ and $f_3(\boldsymbol{y})$ are actually differentiably linearly independent, and $\boldsymbol{\mu}$ can be explicitly computed by the procedure in Proposition 1.

At $\boldsymbol{y} = (1, 0, 0)$, we can compute the matrices in the left and right hand sides of equation (20) in Proposition 1 and get the following equation system

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{4} \\ \frac{-1}{36} & \frac{-1}{144} & \frac{-1}{16} \\ \frac{1}{108} & \frac{1}{864} & \frac{1}{32} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{9} \\ \frac{-1}{72} \\ \frac{10}{2592} \end{pmatrix}.$$

We can solve the equations and recover the ambiguity probability measure

$$\boldsymbol{\mu} = (\frac{1}{3}, \frac{2}{3}, 0).$$

The identified $\boldsymbol{\mu}$ determines that $\boldsymbol{\nu_1}$ and $\boldsymbol{\nu_2}$ are in its support and are the ones employed by the individual. Therefore, we can consider a large set of candidate conditional probabilities which includes the actual ones as a subset, and our recovery procedure will identify the actual ones in the support of $\boldsymbol{\mu}$.

Whether the set of candidate conditional probabilities includes the actual ones can be tested. Suppose the candidate set consists of just $\nu_2 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{4}, \frac{1}{12})$ and $\nu_3 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{4})$, and the actual conditional probability measure $\nu_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ is not in the candidate set. Using our identification procedure, we could recover the ambiguity probability measure $\boldsymbol{\mu} = (\frac{9}{10}, \frac{1}{10})$. It can be checked that if we maximize the identified smooth ambiguity utility subject to budget constraint, the newly derived demand functions differ from the observed demand functions. Hence this verifies that our candidate set of conditional probabilities does not include all of the actual conditional probabilities.

B Appendix B Example for identification of KMM representation

In this appendix, we provide an example to illustrate the recovery process for KMM representation. Here we assume the knowledge of the actual conditional probabilities. This assumption is just for simplicity, and can be relaxed as in Example 6.

Example 7 Assume there are three ambiguity states with conditional probabilities $\nu_1 = (0, 1, 0)$, $\nu_2 = (\frac{1}{2}, \frac{1}{2}, 0)$ and $\nu_3 = (\frac{1}{2}, 0, \frac{1}{2})$. There are three assets with payoffs: $r_1 = (1, 1, 1)$, $r_2 = (1, 0, 0)$ and $r_3 = (0, 0, 1)$. Note that the first asset is risk free since its payoffs are constant across risk states; the second asset is quasi-ambiguous in the sense that its payoffs vary across risk states, but its expected return and variance are constant across the second and third ambiguity states except the first ambiguity state. We use this complete market example to illustrate our argument for its simplicity, however, our argument works for incomplete markets. The demand functions for first period consumption and three assets are given by

$$x_0 = (0, \frac{1}{p_0}) \text{ at } \frac{p_1}{p_0} = \frac{3}{10}[1 + \frac{A}{2}]^{\frac{1}{3}}[(1 + \frac{A}{2})(\frac{p_2}{p_3} - 1)]^{\frac{1}{3}} \left\{ \frac{(1 + \frac{A}{2})(\frac{p_2}{p_3} - 1) + \frac{p_2}{p_3}}{(1 + \frac{A}{2})(\frac{p_2}{p_3} - 1)} \right\}, \tag{B.1}$$

$$y_1 = \frac{1 - p_0 x_0}{p_1 + p_2 A + p_3(2+A)(\frac{p_2}{p_3} - 2)}, \tag{B.2}$$

$$y_2 = \frac{A(1 - p_0 x_0)}{p_1 + p_2 A + p_3(2+A)(\frac{p_2}{p_3} - 2)}, \tag{B.3}$$

$$y_3 = \frac{(2+A)(\frac{p_2}{p_3} - 2)(1 - p_0 x_0)}{p_1 + p_2 A + p_3(2+A)(\frac{p_2}{p_3} - 2)}, \tag{B.4}$$

with $A = \frac{-(3p_2 - 2p_3 - \frac{1}{2}p_1) + \sqrt{(3p_2 - 2p_3 - \frac{1}{2}p_1)^2 - 2(p_2 - p_3)(4p_2 - 2p_3 - p_1)}}{p_2 - p_3}$.

We next show that if the observed demand is generated by KMM representation, then it is possible to recover the smooth ambiguity utility and the ambiguity probabilities that rationalize the demand functions (B.1)- (B.4).

To recover the generating utility functions $\tilde{\phi}$ and u and the discount factor β, we first invert the demand functions (B.1)- (B.4) and get the inverse demand functions $\{p_j(x_0, \boldsymbol{y})\}_{j=0}^{J}$. Denote $Z = \frac{1}{3}\ln[y_1] + \frac{1}{3}\ln[y_1 + \frac{1}{2}y_2] + \frac{1}{3}\ln[y_1 + \frac{1}{2}y_2 + \frac{1}{2}y_3]$. Utility maximization implies the following marginal rates of substitution

$$m_{10}(x_0, \boldsymbol{y}) = \frac{p_1(x_0, \boldsymbol{y})}{p_0(x_0, \boldsymbol{y})} = \frac{9}{10} e^Z \frac{1}{3} \frac{3y_1^2 + \frac{1}{4}y_2^2 + 2y_1y_2 + y_1y_3 + \frac{1}{4}y_2y_3}{y_1(y_1 + \frac{1}{2}y_2)(y_1 + \frac{1}{2}y_2 + \frac{1}{2}y_3)}, \tag{B.5}$$

$$m_{21}(x_0, \boldsymbol{y}) = \frac{p_2(x_0, \boldsymbol{y})}{p_1(x_0, \boldsymbol{y})} = \frac{1}{2} \frac{2y_1^2 + y_1y_2 + \frac{1}{2}y_1y_3}{3y_1^2 + \frac{1}{4}y_2^2 + 2y_1y_2 + y_1y_3 + \frac{1}{4}y_2y_3}. \tag{B.6}$$

As in Example 6, the discount factor can be recovered from the marginal rate of substitution between consumption and risk free asset $m_{10}(x_0, \boldsymbol{y})$ at $x_0 = \overline{x}_0$ and portfolio $\overline{\boldsymbol{y}} = (\overline{x}_0, 0, 0)$,

$$\beta = m_{10}(\overline{x}_0, \overline{\boldsymbol{y}}) = \frac{9}{10}.$$

We normalize $u^{'}(\bar{x}_0) = 1$, then

$$u^{'}(x_0) = \frac{m_{10}(x_0, \overline{\boldsymbol{y}})}{m_{10}(\bar{x}_0, \overline{\boldsymbol{y}})} = e^Z \frac{1}{3} \frac{3y_1^2 + \frac{1}{4}y_2^2 + 2y_1y_2 + y_1y_3 + \frac{1}{4}y_2y_3}{y_1(y_1 + \frac{1}{2}y_2)(y_1 + \frac{1}{2}y_2 + \frac{1}{2}y_3)}$$

which is a constant, independent of x_0. Integrating both sides with respect to x_0, we get

$$u(x_0) = x_0.$$

Since we know the three conditional probabilities, we can compute $E_{\boldsymbol{\nu}_1}\boldsymbol{r}_2 = 0$, $E_{\boldsymbol{\nu}_2}\boldsymbol{r}_2 = E_{\boldsymbol{\nu}_3}\boldsymbol{r}_2 = \frac{1}{2}$.

At portfolio $\boldsymbol{y}_0 = (x, 0, 0)$, we have

$$\mu_1 E_{\boldsymbol{v}_1} \boldsymbol{r}_2 + (1 - \mu_1) E_{\boldsymbol{v}_2} \boldsymbol{r}_2 = m_{21}(x_0, \boldsymbol{y}_0) = \frac{1}{3}. \tag{B.7}$$

From this equation, we get $\mu_1 = \frac{1}{3}$.

Then we can compute the values

$$E_{\boldsymbol{\mu}} E_{\boldsymbol{v}_a} (\boldsymbol{r}_2)^2 = \mu_1 E_{\boldsymbol{v}_1} (\boldsymbol{r}_2)^2 + (1 - \mu_1) E_{\boldsymbol{v}_2} (\boldsymbol{r}_2)^2 = \frac{1}{3},$$

and

$$E_{\boldsymbol{\mu}} (E_{\boldsymbol{v}_a} \boldsymbol{r}_2)^2 = \mu_1 (E_{\boldsymbol{v}_1} \boldsymbol{r}_2)^2 + (1 - \mu_1)(E_{\boldsymbol{v}_2} \boldsymbol{r}_2)^2 = \frac{1}{6}.$$

If we take derivative on both sides of equation (B.6) with respect to y_2 and evaluate the resulting functional at portfolio $\boldsymbol{y}_0 = (x, 0, 0)$, we have

$$[E_{\boldsymbol{\mu}} (E_{\boldsymbol{v}_a} \boldsymbol{r}_2)^2 - (E_{\boldsymbol{\mu}} E_{\boldsymbol{v}_a} \boldsymbol{r}_2)^2] \frac{\tilde{\phi}''(x)}{\tilde{\phi}'(x)} =$$

$$[(E_{\boldsymbol{\mu}} E_{\boldsymbol{v}_a} \boldsymbol{r}_2)^2 - E_{\boldsymbol{\mu}} E_{\boldsymbol{v}_a} (\boldsymbol{r}_2)^2] \frac{u''(x)}{(u'(x))^2} + \frac{\partial m_{21}(x_0, x, 0, 0)}{\partial y_2} \frac{1}{u'(x)}. \tag{B.8}$$

Rearrange the terms, we have

$$-\frac{\tilde{\phi}''(x)}{\tilde{\phi}'(x)} = x^{-1}.$$

Integrate both sides of the above equation twice with respect to x, we have

$$\tilde{\phi}(x) = \ln(x).$$

We define

$$\begin{aligned}
f_2(\boldsymbol{y}) &= \tilde{\phi}'\left(E_{\boldsymbol{v}_2} u(x_s)\right) E_{\boldsymbol{v}_2} u'(x_s) r_2 - m_{21}(x_0, \boldsymbol{y}) \tilde{\phi}'\left(E_{\boldsymbol{v}_2} u(x_s)\right) E_{\boldsymbol{v}_2} u'(x_s) r_1 \\
&= \frac{\frac{1}{2}}{y_1 + \frac{1}{2} y_2} - m_{21}(x_0, \boldsymbol{y}) \frac{1}{y_1 + \frac{1}{2} y_2}, \\
f_3(\boldsymbol{y}) &= \tilde{\phi}'\left(E_{\boldsymbol{v}_3} u(x_s)\right) E_{\boldsymbol{v}_3} u'(x_s) r_2 - m_{21}(x_0, \boldsymbol{y}) \tilde{\phi}'\left(E_{\boldsymbol{v}_3} u(x_s)\right) E_{\boldsymbol{v}_3} u'(x_s) r_1 \\
&= \frac{\frac{1}{2}}{y_1 + \frac{1}{2} y_2 + \frac{1}{2} y_3} - m_{21}(x_0, \boldsymbol{y}) \frac{1}{y_1 + \frac{1}{2} y_2 + \frac{1}{2} y_3}.
\end{aligned}$$

The first order condition can be rewritten as

$$\sum_{a=2}^{3} \mu_a f_a(\boldsymbol{y}) = \frac{1}{3} m_{31}(x_0, \boldsymbol{y}) \frac{1}{y_1}. \tag{B.9}$$

As in Example 6, we differentiate the above equation with respect to y_3, and evaluate the resulting equation at $\boldsymbol{y} = (1, 0, 0)$. Then we have

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{36} & \frac{-1}{18} \end{pmatrix} \begin{pmatrix} \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{9} \\ \frac{-1}{108} \end{pmatrix},$$

and recover the ambiguity probability measure

$$\boldsymbol{\mu} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

Here again, (differentiable) linear independence of $f_2(\boldsymbol{y})$ and $f_3(\boldsymbol{y})$ guarantees the recoverability of the ambiguity probability measure.

C Appendix C Example for identification of KPS preferences

In this appendix, we provide an example, in which the identified smooth ambiguity function is a KPS utility representation. Here, the inverse demand functions are given since closed form demand functions can't be derived.

Example 8 Assume there are three assets with payoffs: $\boldsymbol{r}_1 = (1, 1, 1, 1)$, $\boldsymbol{r}_2 = (1, 2, 0, 0)$, and $\boldsymbol{r}_3 = (0, 0, 0, 1)$. The inverse demand functions for first period consumption and three assets, i.e., equivalently the marginal rates of substitution are

$$m_{10}(x_0, \boldsymbol{y}) = \frac{9}{20}(x_0)^{\frac{3}{4}} \frac{(y_1 + y_2)^{-\frac{1}{2}} + \frac{1}{3}(y_1 + 2y_2)^{-\frac{1}{2}} + \frac{1}{3}(y_1)^{-\frac{1}{2}} + \frac{1}{3}(y_1 + y_3)^{-\frac{1}{2}}}{[\frac{1}{2}(y_1 + y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1 + 2y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1)^{\frac{1}{2}} + \frac{1}{6}(y_1 + y_3)^{\frac{1}{2}}]^{\frac{1}{2}}},$$

$$m_{12}(x_0, \boldsymbol{y}) = \frac{1}{2} \frac{(y_1 + y_2)^{-\frac{1}{2}} + \frac{1}{3}(y_1 + 2y_2)^{-\frac{1}{2}} + \frac{1}{3}(y_1)^{-\frac{1}{2}} + \frac{1}{3}(y_1 + y_3)^{-\frac{1}{2}}}{\frac{1}{2}(y_1 + y_2)^{-\frac{1}{2}} + \frac{1}{3}(y_1 + 2y_2)^{-\frac{1}{2}}},$$

$$m_{30}(x_0, \boldsymbol{y}) = \frac{3}{20}(x_0)^{\frac{3}{4}} \frac{(y_1 + y_3)^{-\frac{1}{2}}}{[\frac{1}{2}(y_1 + y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1 + 2y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1)^{\frac{1}{2}} + \frac{1}{6}(y_1 + y_3)^{\frac{1}{2}}]^{\frac{1}{2}}}.$$

Suppose these inverse demand functions are generated by KPS* representation. Assume we know a set of candidate ambiguity states with conditional probabilities $\boldsymbol{\nu}_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ and $\boldsymbol{\nu}_2 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{4}, \frac{1}{12})$, which includes the actual one. The first asset is risk free and the second asset is ambiguity free.

Follow the same steps as in Example 6, we can recover β, $\phi(x)$, and $u(x)$ respectively

$$\beta = \frac{9}{10}, \phi(x_0) = x_0^{\frac{1}{4}}, u(x) = x^{\frac{1}{2}}.$$

For the two candidate ambiguity states, we can define functions $f_1(\boldsymbol{y})$ and $f_2(\boldsymbol{y})$. At $\boldsymbol{y} = (1, 0, 0)$, we get equation (20) in Proposition 1

$$\begin{pmatrix} \frac{1}{12} & \frac{1}{24} \\ \frac{-13}{288} & \frac{-25}{1152} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ \frac{-13}{144} \end{pmatrix}.$$

Therefore, we can solve the equation system and recover the ambiguity probability measure

$$\boldsymbol{\mu} = (1, 0).$$

In this example, it is determined that only one conditional probability has been used. It follows that the individual is actually employing KPS utility rather than the more general ambiguity utility. Thus, our recovery process can recover the important KPS utility special class and associated probability measure from the observed consumption and asset demands of an individual.

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