Precautionary Saving and Capital Risk: Saving vs Asset Reallocation

Larry Selden and Xiao Wei*

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Abstract

In finance and macro models, increased capital risk results in higher risk free asset prices often attributed to precautionary saving. However at the demand level, even assuming the same preferences as in the equilibrium analysis, precautionary saving need not always hold. Assuming CES time and CRRA risk preferences, we derive conditions such that the consumer exhibits precautionary saving. Absent these conditions, a concrete example demonstrates that the consumer fails to exhibit precautionary saving. Key results are explained in terms of novel competing portfolio reallocation and saving components and portrayed by a natural extension of the canonical certainty Fisherian diagrammatic analysis.

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*Selden: Columbia University, Uris Hall, 3022 Broadway, New York, NY 10027 (e-mail: ls49@columbia.edu); Wei: School of Economics, Fudan University, 600 Guoquan Road, Shanghai 200433, P.R. China, Shanghai Institute of International Finance and Economics (e-mail: weixiao@fudan.edu.cn). An earlier version of this paper was circulated with the title "Precautionary with Capital Risk". We thank Pierre-Andre Chiappori, Darrell Duffie, Federico Echenique, Louis Eeckhoudt, Chistian Gollier, Fabio AR Gomes, Simon Grant, Thorsten Hens, Felix Kubler, Rajnish Mehra, John Quah, Pawel Zabczyk and especially Philippe Weil for their very helpful comments and suggestions. Wei acknowledges financial support from Innovative Research Groups Project of the National Natural Science Foundation of China (Grant No. 72121002).

1 Introduction

In recent years, a number of authors working on asset pricing and macro economic issues have characterized their findings in terms of precautionary saving. (Examples include Barsky 1989, Campbell and Cochrane 1999, Yi and Choi 2006, Reis 2009, Wachter 2013, Gomes and Ribeiro 2015, Cochrane 2017, Fernandez-Villaverde and Levintal 2018, Pflueger, Siriwardane and Sunderam 2020 and Ermolov 2022.) These papers assume consumption fluctuations over time which can be associated, for instance, with innovation shocks, rare disasters and production shocks. The asset return distributions are impacted by the assumed consumption dynamics. Since the first order conditions do not include terms associated with (labor) income risk, it seems reasonable to conclude that the individual agent intertemporal consumption optimization underlying these models reflects capital risk and not income risk. Thus, when these papers seek to explain changes in risk free asset prices in terms of precautionary saving, it should be based on capital risk rather than income risk. However, when going beyond just referencing precautionary saving to discuss its relevance, some researchers reference the classic Leland (1968) and Kimball (1990) definitions and characterization of precautionary saving based on income risk.[1] This is not surprising since, to our knowledge, there does not exist in the literature a formal definition of precautionary saving and hence a corresponding set of restrictions on preferences implying precautionary saving in the presence of capital risk.[2] Indeed, Sandmo (1970) stressed that for capital versus income risk, the consumer's saving behavior is different and more complicated.

In order to make progress in filling these seeming voids in the literature, we address three questions:

Q1 What is the appropriate definition of precautionary saving in the presence of capital risk?

[1] Barsky (1989) focuses on the prices of risky and risk free assets where risky labor income is **not** present in the model. And yet, Barsky (1989, p.1136) explicitly references Leland (1968) to justify conclusions based on the assumed preferences always implying that precautionary saving holds. Assuming the same preferences as employed by Barsky, we show that in the presence of capital risk, as opposed to income risk, precautionary saving need not always hold. Similar issues also arrises in Weil (1990) and Gomes and Ribeiro (2015, p. 110). Also, see footnote 17 below in Subsection 4.2 for an example in the literature which explicitly rationalizes increased equilibrium risk free asset prices by increased precautionary saving by consumers.

[2] Although Dreze and Modigliani (1972) consider the consumption-portfolio problem with risk free and risky assets, they never address the question of precautionary saving.

Q2 What restrictions on consumer preferences result in precautionary saving for capital risk and do they differ from the case of income risk?

Q3 Is the assertion that the equilibrium risk free asset price increases with capital risk as a result of precautionary saving consistent with the corresponding micro foundations partial equilibrium demand behavior?

The consumption-saving analysis based on (labor) income risk in Leland (1968) and Kimball (1990), as well as much of the ensuing literature, assumes two time periods and derives conditions on EU (Expected Utility) preferences such that the consumer increases saving via a risk free asset in response to an increase in income risk. Kimball and Weil (2009), assuming the more general form of KPS (Kreps and Porteus 1978 and Selden 1978) preferences, derive restrictions on the underlying time and risk preferences that give rise to precautionary saving for income risk. We also assume KPS preferences where time preferences are defined over certain first period and certainty equivalent second period consumption. The KPS utility is based on CES (constant elasticity of substitution) time preferences and CRRA (constant relative risk aversion) risk preferences.[3,4] In contrast to the consumption-saving problem typically assumed in the income risk analysis, we assume a consumption-portfolio problem characterized by the ability to invest in both a risk free and risky asset. The latter problem can be decomposed into a consumption-saving problem and a conditional portfolio problem. The canonical certainty Fisherian diagrammatic analysis extends naturally to the risky setting based on certain first period and certainty equivalent second period consumption where CRRA risk preferences ensure that the budget constraint is linear as in the certainty case. An increase in capital risk results in a reduction in the portfolio certainty equivalent rate of return which is fully analogous to a reduction in the risk free rate in the Fisherian certainty diagram. Moreover, the portfolio optimization can be represented in a dual risky and risk free asset space.[5]

In seeking to extend the notion of precautionary saving to a setting in which the consumer faces capital rather than income risk, it is useful to remember Leland's (1968) basic idea that an individual will save more in the form of a risk free asset

[3]The popular EZW (Epstein and Zin 1989 and Weil 1990) recursive preference model is also based on CES time and CRRA risk preference building block utilities and converges to the KPS model employed in this paper in the case of two time periods.

[4]The results in this paper based on CES-CRRA KPS preferences can be extended to more than two periods building on Kubler, Selden and Wei (2023).

[5]See Subsection 4.3 and the discussion of Figure 4.

when exogenous (labor) income becomes more risky. The increase in saving is precautionary in the sense that it creates a buffer stock of certain period two income to offset the possible occurrence of a bad outcome for risky income (e.g., Carroll, Hall and Zeldes 1992). However, the standard capital risk setting (e.g., Sandmo 1970) differs in that the consumer faces a consumption-portfolio problem in which she can invest in both a risk free and risky asset. In the corresponding intertemporal optimization, the level of capital risk is determined by the holdings of the risky asset and is endogenous whereas in the income risk consumption-saving problem the level of (income) risk is exogenous.[6] In the consumption-portfolio optimization, the individual's own saving decision to buy more of the risky asset can create risk rather than just buffering against risk and thus it does not seem appropriate, at least to us, to refer to increased saving in this case as being "precautionary".

Then how should precautionary saving be defined in the classic consumption-portfolio optimization? In response to an increase in the risk of the risky asset (i.e., capital risk), the consumer can be viewed as potentially making two decisions – determining her new level of savings and altering the composition of her portfolio of risky and risk free assets. The corresponding changes in the demand for the risk free asset are, respectively, referred to as the **saving component** and **portfolio reallocation component**. Since the increased saving in the form of the risk free asset can be viewed as increasing the buffer against bad outcomes for the simultaneously chosen risky asset, we refer to this component as **precautionary saving**. The reallocation component is based on the assumption that saving is not changed (from the pre-capital risk change level) and reflects the natural rebalancing of the portfolio due to the increase in capital risk.[7] The combination of these two components defines the total change in the demand for the risk free asset, but only the saving component should be associated with precautionary saving. This answers question Q1.

To address Q2, we consider a mean preserving spread of the risky asset payoffs in the consumption-portfolio problem.[8] As noted above, we assume preferences take the two period CES-CRRA KPS form. Then savings increase with capital risk if and only if the consumer's EIS (elasticity of intertemporal substitution) < 1 and she has a preference for intertemporal smoothing. Based on our definition for the case of capital risk, the consumer exhibits precautionary saving. This differs in two

[6] Gollier (2001, Chapter 19)

[7] See Subsection 4.2 for a detailed discussion of the two effects.

[8] Our results are sensitive to the particular definition of a mean preserving spread in risk assumed (see Section 3).

important ways from precautionary saving associated with income risk. First, in the income risk case, the risk preference property prudence plays a key role. Second, as shown by Kimball and Weil (2009), for CES-CRRA KPS preferences, the consumer always exhibits precautionary saving.

How do our results for consumption-portfolio demand behavior relate to the observation in Q3 that in a number of asset pricing and macro papers, an increase in the price of the risk free asset is explained by precautionary saving? We show that when the $EIS > 1$, the change in total risk free asset demand (the sum of the saving component and the reallocation component) can be negative. Under the specific mean preserving increase in risk we consider, the reallocation effect is always positive but the saving effect can be negative and the total demand for the risk free asset can fall.[9] At first blush, the fact that for the same increase in capital risk, risk free asset demand falls but the corresponding equilibrium price increases seems counterintuitive and "probably" incorrect. However, it is well-known (e.g., Mas-Colell, Whinston and Green, 1995, Chapter 15.E) that for more than two goods, such as the case we are considering, partial equilibrium demand and exchange equilibrium price comparative statics will in general not agree. This can happen due to market interactions between the non-numeraire goods. Indeed for agreement of the partial and exchange equilibrium comparative statics, rather strong restrictions on preferences that are not satisfied in the present setting must hold (Mukherji, 1975). This suggests that in general, researchers should be cautious when suggesting that equilibrium price changes can be explained by partial equilibrium comparative static analysis such as we consider in this paper.

In the next section, we present an important certainty, Fisherian consumption-saving result. Section 3 introduces the assumed KPS preferences and the two stage consumption-portfolio problem used to analyze saving and risk free asset demand in the presence of capital risk. Section 4 formalizes our new definition of precautionary saving and derives conditions such that the consumer exhibits precautionary saving and the total demand for the risk free asset increases. We provide a concrete example where the equilibrium price of the risk free asset increases with capital risk, but at the corresponding micro foundation partial equilibrium level the consumer fails to exhibit precautionary saving and the demand for the risk free asset falls. The final section offers concluding comments.

[9] See Example 1 in Subsection 4.2 and the Fisherian diagrammatic analysis in Subsection 4.3.

2 A Certainty Consumption-Saving Comparative Static Result

The following proposition based on the classic Fisherian certainty setting will play an important role in our analysis of precautionary saving in the presence of capital risk, since for the latter case the consumption-saving budget constraint will be shown to be linear as in the certainty case.

Proposition 1 *Assume the certainty optimization problem*[10]

$$\max_{c_1, c_2} U(c_1, c_2) = u_1(c_1) + u_2(c_2) \quad S.T. \;\; c_2 = (I - c_1) R_f, \tag{1}$$

where $u_i' > 0$, $u_i'' < 0$ $(i = 1, 2)$. *Then*

$$\frac{\partial c_1}{\partial R_f} \gtreqless 0 \Leftrightarrow \frac{\partial s_1}{\partial R_f} \lesseqgtr 0 \Leftrightarrow -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \gtreqless 1. \tag{2}$$

Remark 1 *The result in Proposition 1 stated in the context of gross substitutes and complements was given by Wald (1936) and a modern proof was provided by Varian (1985). However, the simple geometric interpretation given below seems to be new.*

To provide geometric intuition for Proposition 1, define the intertemporal marginal rate of substitution and minus the slope of the constraint in eqn. (1), respectively, by

$$m_1 =_{def} \frac{u_1'(c_1)}{u_2'(c_2)} \qquad \text{and} \qquad m_2 =_{def} R_f = -\frac{c_2}{c_1 - I}.$$

In Figure 1, consider the two constraint lines anchored at a common point. At the tangency between the lower constraint and indifference curve, $m_1 = m_2$. Increasing R_f in eqn. (1) corresponds to a rotation of the lower constraint line northeast and can be viewed as changing c_2 for a fixed c_1. The elasticities of the two slope changes with respect to c_2 are given by

$$\frac{\partial \ln m_1}{\partial \ln c_2} = -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \qquad \text{and} \qquad \frac{\partial \ln m_2}{\partial \ln c_2} = 1, \tag{3}$$

[10] We generalize the commonly assumed discounted utility

$$U(c_1, c_2) = u(c_1) + \beta u(c_2)$$

by allowing the period one and two utilities to differ. If there is a discount function, it is embedded in the function u_2. Also, whenever we refer to U being additively separable, that is meant to hold up to an increasing transformation.

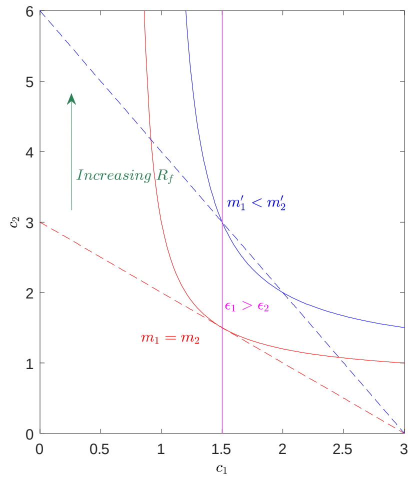


Figure 1:

where $\partial \ln m_1 / \partial \ln c_2$ will be referred to as the $EIMRS$ (elasticity of intertemporal marginal rate of substitution) with respect to period two consumption. Thus, the conclusion of Proposition 1 can be rewritten as

$$\frac{\partial c_1}{\partial R_f} \gtreqqless 0 \Leftrightarrow \frac{\partial s_1}{\partial R_f} \lesseqqgtr 0 \Leftrightarrow EIMRS \gtreqqless 1. \tag{4}$$

Returning to the case in Figure 1, because the $EIMRS > 1$ in response to an increase in R_f, the higher indifference curve intersects the shifted constraint at the initial optimal c_1, implying that the tangent to the indifference curve is steeper than the shifted constraint (i.e., $-m_1' > -m_2'$ or $m_1' < m_2'$). Therefore, the new optimal c_1 is to the right of the initial c_1-value, implying that c_1 increases and s_1 decreases with R_f.

Two important observations should be made relating to the $EIMRS$ condition (4). First, the $EIMRS$ condition depends only on u_2 and is independent of u_1. Second, the $EIMRS$ is in general distinct from the familiar EIS, where the reciprocal of the latter is defined by

$$\frac{1}{EIS} = \frac{d \ln m_1}{d \ln(\frac{c_2}{c_1})}.$$

The quantity $1/EIS$ is often interpreted as an aversion to intertemporal substitution. One special case where the $EIMRS$ and EIS are closely related is when U takes the following popular CES form

$$U(c_1, c_2) = -\frac{c_1^{-\delta_1}}{\delta_1} - \beta \frac{c_2^{-\delta_1}}{\delta_1} \quad (\delta_1 > -1), \tag{5}$$

where for this utility, $EIMRS = 1/EIS = 1 + \delta_1$ and the condition in Proposition 1 can be expressed as

$$\frac{\partial s_1}{\partial R_f} \lesseqqgtr 0 \Leftrightarrow \frac{1}{EIS} \gtreqqless 1 \Leftrightarrow \delta_1 \gtreqqless 0. \tag{6}$$

Thus if $\delta_1 > 0$, the consumer exhibits an aversion to intertemporal substitution which is greater than the benchmark log utility ($\delta_1 = 0$) resulting in c_1 increasing and s_1 decreasing with an increase in R_f.

3 Consumption-Portfolio Problem

As in Sandmo (1970), the consumer's savings problem in the presence of capital risk can be formulated in terms of the classic consumption-portfolio problem. We assume preferences are defined over certain first period and random second period consumption pairs, $(c_1, \widetilde{c}_2)$, and are represented by the following KPS utility function[11]

$$U(c_1, \widehat{c}_2) = U\left(c_1, V^{-1}EV(\widetilde{c}_2)\right), \tag{7}$$

where U represents time preferences over certain (c_1, c_2)-pairs satisfying $(c_1, c_2) \in C \subseteq \mathbb{R}^2_+$, $EV(\widetilde{c}_2)$ is a standard one period EU representation and $\widehat{c}_2$ denotes certainty equivalent second period consumption. Throughout the rest of this paper, the time and risk preference utilities, respectively, will take the standard CES form in eqn. (5) and the following CRRA form

$$V(c_2) = -\frac{c_2^{-\delta_2}}{\delta_2} \; (\delta_2 > -1). \tag{8}$$

We assume that there is one risky asset and one risk free asset when analyzing the consumption-portfolio problem.[12] At the beginning of period one, the consumer chooses a level of certain first period consumption c_1 and a set of asset holdings, where the returns on the latter fund consumption in period two. In the portfolio setting, markets will generally be incomplete with more states than assets. The random variable $\widetilde{\xi} > 0$ denotes the period two payoff on the risky asset and ξ_f is the

[11] The form (7) was axiomatized by Selden (1978). Kreps and Porteus (1978) introduced an alternative representation, which in a two period setting, is ordinally equivalent to (7).

[12] The results in this paper can be extended to the case of multiple risky assets. For a definition of an increase in risk used in the case of multiple risky assets, see, for instance, Meyer and Ormiston (1994).

payoff for the risk free asset. The prices of the risky and risk free assets are denoted respectively by p and p_f. To ensure that there is no arbitrage,

$$\frac{\min\left(\widetilde{\xi}\right)}{p} < \frac{\xi_f}{p_f} < \frac{\max\left(\widetilde{\xi}\right)}{p}.$$

The condition that the risk free (gross) rate of return R_f is less than the expected (gross) rate of return for the risky asset $E\widetilde{R}$

$$R_f = \frac{\xi_f}{p_f} < \frac{E\widetilde{\xi}}{p} = E\widetilde{R} \tag{9}$$

guarantees a positive demand for the risky asset. The number of units of the risky and risk free assets are denoted by n and n_f, respectively. Then random period two consumption is defined by $\widetilde{c}_2 = \widetilde{\xi}n + \xi_f n_f$.

To consider a pure increase in capital risk, it is common to define a mean preserving increase in risk on the payoffs of the risky asset as

$$\widetilde{\xi}(\lambda) = \widetilde{\xi} + \lambda\widetilde{\epsilon}, \tag{10}$$

where $E\left[\widetilde{\epsilon}|\widetilde{\xi}\right] = 0$ ensures that the asset's expected payoff remains fixed and $\lambda \geq 0$ is the increasing risk shift parameter. However RS (Rothschild and Stiglitz 1971) show that in a simple portfolio setting with one risky and one risk free asset, the mean preserving spread (10) can result in the counterintuitive conclusion that a risk averse consumer with single period EU preferences increases rather than decreases her demand for the risky asset unless one assumes very strong conditions such as the measure of relative risk aversion is less than 1. As Cohen (1995, p.77) notes, the RS definition of increasing risk (10) corresponds to a notion of risk aversion that is too strong. To avoid this problem, we instead employ the following MPMS (mean preserving monotone spread) notion of an increase in risk.

Definition 1 *(Quiggin 1993, p.14)* $\widetilde{\xi}(\lambda) = \widetilde{\xi} + \lambda\widetilde{\epsilon}$ *is a MPMS of* $\widetilde{\xi}$ *if and only if* $E\left[\widetilde{\epsilon}|\widetilde{\xi}\right] = 0$ *and* $\widetilde{\xi}$ *and* $\widetilde{\epsilon}$ *are comonotonic in the sense that for any two states of nature* (θ_1, θ_2),

$$\left(\xi_{\theta_1} - \xi_{\theta_2}\right)\left(\epsilon_{\theta_1} - \epsilon_{\theta_2}\right) \geq 0.$$

In the rest of this paper, the discussion of the derivatives with respect to λ or the change in the λ-value will always be based on Definition 1 since a decrease in the demand for the risky asset and an increase in the demand for the risk free asset in response to an increase in capital risk play a key role in the reallocation component of risk free asset demand discussed in Subsection 4.2 below.

To provide a natural extension of the seminal certainty Fisherian consumption-saving analysis considered in the prior section to the case of capital risk, it will prove extremely useful to express the classic consumption-portfolio problem in the form of a two-stage optimization. The first stage portfolio problem conditional on c_1 is defined by

$$(n(c_1), n_f(c_1)) = \arg\max_{n,n_f} EV\left(\widetilde{\xi}n + \xi_f n_f\right) \tag{11}$$

subject to

$$pn + p_f n_f \leq I - c_1, \tag{12}$$

where I denotes initial period one income. Period one consumption is the numeraire with price $p_1 \equiv 1$. Period one saving is defined by $s_1 = I - c_1$. The second stage consumption-saving problem corresponds to[13]

$$c_1 = \arg\max_{c_1} u_1(c_1) + u_2(\widehat{c}_2(c_1)), \tag{13}$$

where

$$\widehat{c}_2(c_1) = V^{-1}\left(EV\left(\widetilde{\xi}n(c_1) + \xi_f n_f(c_1)\right)\right). \tag{14}$$

If the $\widehat{c}_2(c_1)$-constraint for the second stage consumption-saving optimization is affine in c_1, the analysis can be significantly simplified and the certainty Proposition 1 can be employed. As is shown below, this is the case if the period two conditional NM index V takes the CRRA form in eqn. (8).[14]

Proposition 2 *If V takes the CRRA form in eqn. (8), then for any distribution $\widetilde{\xi}$, the $\widehat{c}_2$ constraint takes the form $\widehat{c}_2 = \widehat{R}_p(I - c_1)$, where $\widehat{R}_p > R_f$.*

Proof. The first order condition implies

$$E\left[\left(\widetilde{\xi} - \frac{p}{p_f}\xi_f\right)\left(\widetilde{\xi}n + \xi_f n_f\right)^{-1-\delta_2}\right] = 0.$$

[13] Since the consumer's consumption-portfolio constraint is linear (Proposition 2 below), a unique solution is guaranteed since the CES utility (5) is strictly quasiconcave.

[14] Selden (1980) argues that the $\widehat{c}_2(c_1)$-constraint can be rewritten in a linear form when V is a member of HARA (hyperbolic absolute risk aversion) class. However, the following Proposition gives, for the assumed CRRA risk preferences, the specific form of the constraint and proves that $\widehat{R}_p > R_f$.

Following the covariance inequality,

$$\begin{aligned} 0 &= E\left[\left(\widetilde{\xi}-\frac{p}{p_f}\xi_f\right)\left(\widetilde{\xi}n+\xi_f n_f\right)^{-1-\delta_2}\right] \\ &\gtreqless E\left[\widetilde{\xi}-\frac{p}{p_f}\xi_f\right]E\left[\left(\widetilde{\xi}n+\xi_f n_f\right)^{-1-\delta_2}\right] \Leftrightarrow n \lesseqgtr 0. \end{aligned}$$

Since

$$E\left[\widetilde{\xi}-\frac{p}{p_f}\xi_f\right]>0 \quad \text{and} \quad E\left[\left(\widetilde{\xi}n+\xi_f n_f\right)^{-1-\delta_2}\right]>0,$$

optimal demand denoted by n^* is positive. Define

$$\widehat{R}_p=\frac{\widehat{c}_2}{I-c_1}=\frac{\left(E\left[\left(\widetilde{\xi}n+\xi_f n_f\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}}}{I-c_1}.$$

If $n=0$, then

$$\widehat{R}_p=\frac{\xi_f n_f}{I-c_1}=\frac{\xi_f\left(I-c_1\right)}{\left(I-c_1\right)p_f}=R_f.$$

In the first stage optimization, we consider the problem

$$\max_{n,n_f}\left(E\left[\left(\widetilde{\xi}n+\xi_f n_f\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}} \quad S.T.\ \ pn+p_f n_f \leq I-c_1.$$

Since the preferences are homothetic, optimal demand $\left(n^*, n_f^*\right)$ must satisfy

$$n^*=\zeta_1\left(I-c_1\right)>0 \quad \text{and} \quad n_f^*=\zeta_2\left(I-c_1\right),$$

where ζ_1 and ζ_2 are constants. Because $\left(n^*, n_f^*\right)$ is the optimal demand where $n^*>0$, the resulting $\widehat{c}_2^*$ must be larger than the case with $n=0$. Therefore,

$$\widehat{R}_p=\frac{\widehat{c}_2^*}{I-c_1}=const>\frac{\widehat{c}_2\big|_{n=0}}{I-c_1}=R_f.$$

■

Because the CRRA utility is homothetic, the portfolio certainty equivalent return $\widehat{R}_p$ is independent of saving s_1.

4 Capital Risk Saving and Asset Demand Analysis

In this section, we derive conditions, utilizing the time preference $EIMRS$ (EIS), such that saving increases with capital risk. We then formalize our new definition for precautionary saving based on an increase in the demand for the risk free asset that is better suited for the case of capital risk characterizing the consumption portfolio problem. Sufficient conditions are derived such that the consumer exhibits precautionary saving. The change in total risk free asset demand is decomposed into a savings component and an asset reallocation component.

4.1 Increased Risk and Saving Behavior

We next give conditions such that saving increases with a capital risk MPMS by combining Propositions 1 and 2.

Proposition 3 *Consider the consumption-portfolio optimization problem (11)-(14) where the KPS time and risk preference utility building blocks take the forms (5) and (8). Then*

$$\frac{\partial s_1}{\partial \lambda} \gtreqqless 0 \Leftrightarrow EIMRS \gtreqqless 1 \Leftrightarrow EIS \lesseqqgtr 1 \Leftrightarrow \delta_1 \gtreqqless 0.$$

Proof. We only need to prove that when $\lambda_2 > \lambda_1$, we have $\widehat{R}_p(\lambda_1) > \widehat{R}_p(\lambda_2)$. Note that for any given (n, n_f), since the MPMS is a special form of second order stochastic dominance, we have

$$V^{-1}E\left[V\left(\widetilde{\xi}(\lambda_1)\,n + \xi_f n_f\right)\right] \geq V^{-1}E\left[V\left(\widetilde{\xi}(\lambda_2)\,n + \xi_f n_f\right)\right].$$

Denoting the optimal asset demands associated with $\widetilde{\xi}(\lambda_1)$ as $\left(n^*, n_f^*\right)$ and associated with $\widetilde{\xi}(\lambda_2)$ as $\left(n^{**}, n_f^{**}\right)$, then

$$\begin{aligned} V^{-1}E\left[V\left(\widetilde{\xi}(\lambda_1)\,n^* + \xi_f n_f^*\right)\right] &> V^{-1}E\left[V\left(\widetilde{\xi}(\lambda_1)\,n^{**} + \xi_f n_f^{**}\right)\right] \\ &\geq V^{-1}E\left[V\left(\widetilde{\xi}(\lambda_2)\,n^{**} + \xi_f n_f^{**}\right)\right], \end{aligned}$$

implying that $\widehat{R}_p(\lambda_1) > \widehat{R}_p(\lambda_2)$. ∎

It is natural to wonder why for the CES-CRRA building block utilities, the increase in s_1 in response to an increase in capital risk is determined by the same condition as the certainty case when changing R_f. This can be seen if we decompose $\partial c_1/\partial\lambda$ into income and substitution effects. For the CRRA risk preferences,

$$\left(\frac{\partial c_1}{\partial \lambda}\right)_{income} = \frac{\widehat{c}_2}{\widehat{R}_p^2}\frac{\partial c_1}{\partial I}\frac{\partial \widehat{R}_p}{\partial \lambda} < 0.$$

If time preferences are representable by the CES utility, then

$$c_1 = \frac{I}{1 + \widehat{R}_p^{-\frac{\delta_1}{1+\delta_1}}} \quad \text{and} \quad \widehat{c}_2 = \frac{\widehat{R}_p^{\frac{1}{1+\delta_1}} I}{1 + \widehat{R}_p^{-\frac{\delta_1}{1+\delta_1}}},$$

implying that

$$\left(\frac{\partial c_1}{\partial \lambda}\right)_{subsitution} = -\frac{1}{1+\delta_1} \frac{\widehat{c}_2}{\widehat{R}_p^2} \frac{\partial c_1}{\partial I} \frac{\partial \widehat{R}_p}{\partial \lambda} > 0.$$

Therefore, the sign of $\partial c_1/\partial \lambda$ is completely determined by the comparison between δ_1 and 0,[15] or between the negative income effect and the positive substitution effect as in the certainty Fisherian case of Proposition 1.

4.2 Precautionary Saving and Increased Risk Free Asset Demand

The total change in demand for the risk free asset in response to an increase in capital risk is comprised of two separate components. First as discussed in the prior subsection, savings can change. This in turn can result in a larger demand for the risk free asset, which we refer to as the saving component and denote by $\left(\frac{\partial n_f}{\partial \lambda}\right)_{saving}$. The second component corresponding to a reallocation of the consumer's asset portfolio is denoted by $\left(\frac{\partial n_f}{\partial \lambda}\right)_{reallocation}$.[16] In order to emphasize that this reallocation is based on fixing savings at the pre-change risk level, it will sometimes be convenient to alternatively express the component as $\left(\frac{\partial n_f}{\partial \lambda}\right)_{s_1=const}$. The size of this reallocation component depends on the relative attractiveness of the risky asset and the risk preferences of the consumer. It is essential to distinguish between these two components since their sign and size play key roles in determining whether the demand for the risk free asset actually increases with capital risk as has been suggested in intuitive justifications for why the equilibrium risk free rate decreases with risk.[17]

[15] The geometric intuition is the same as in Figure 1.

[16] As earlier, to ensure that the demand for the risk free asset always increases with an increase in capital risk, we assume that the portfolio reallocation is based on a Quiggin MPMS.

[17] Campbell and Cochrane (1999, p. 212) make explicit the intuitive argument for why the risk free rate of interest decreases with increases in risk. They assume external habit formation EU preferences and a dynamic consumption process and argue that risky and risk free asset prices are determined endogenously in the model. Risk is associated with consumption being low relative to the external habit level of consumption in a bad state. They refer to the last term in their formula

For the consumption-portfolio problem (11)-(14), we can express $\partial n_f/\partial\lambda$ as follows

$$\frac{\partial n_f}{\partial\lambda} = \left(\frac{\partial n_f}{\partial\lambda}\right)_{saving} + \left(\frac{\partial n_f}{\partial\lambda}\right)_{reallocation}, \tag{15}$$

where

$$\left(\frac{\partial n_f}{\partial\lambda}\right)_{saving} = \frac{\partial n_f}{\partial s_1}\frac{\partial s_1}{\partial\lambda} \quad \text{and} \quad \left(\frac{\partial n_f}{\partial\lambda}\right)_{reallocation} = \left(\frac{\partial n_f}{\partial\lambda}\right)_{s_1=const}. \tag{16}$$

It follows from Quggin (1993, Proposition 7.2) that for CRRA risk preferences, we always have

$$\left(\frac{\partial n_f}{\partial\lambda}\right)_{reallocation} \geq 0.$$

If risk free asset demand satisfies $\partial n_f/\partial s_1 > (<)0$, then

$$\left(\frac{\partial n_f}{\partial\lambda}\right)_{saving} > 0 \Leftrightarrow \frac{\partial s_1}{\partial\lambda} > (<)0. \tag{17}$$

Since the time preference utility is additively separable in $(c_1, \widehat{c}_2)$ and the budget constraint is linear in $(c_1, \widehat{c}_2)$, c_1 and $\widehat{c}_2$ are both normal goods, which implies that

$$0 < \frac{\partial c_1}{\partial I} < 1.$$

Since

$$\frac{\partial n_f}{\partial I} = \frac{\partial n_f}{\partial s_1}\frac{\partial(I - c_1)}{\partial I} = \frac{\partial n_f}{\partial s_1}\left(1 - \frac{\partial c_1}{\partial I}\right),$$

$\partial n_f/\partial s_1$ has the same sign as $\partial n_f/\partial I$ and $\partial n_f/\partial s_1 > (<)0$ corresponds to the risk free asset being a normal good if $n_f > (<)0$.[18] Then based on the saving component and defining n_f as the optimal demand for the risk free asset, we have the following definition.

Definition 2 *A consumer is said to exhibit precautionary saving if and only if* $n_f \geq 0$ *and*

$$\left(\frac{\partial n_f}{\partial\lambda}\right)_{saving} > 0.$$

(8), for the equilibrium risk free rate as the precautionary savings term and state "As uncertainty increases, consumers are more willing to save, and this willingness drives down the equilibrium risk-free interest rate..[and increases the risk free asset price]".

[18] For the definition of a risk free asset being a normal good, see Kubler, Selden and Wei (2013, p. 1038). Applying a similar argument to the risky asset and following Theorem 1 in Kubler, Selden and Wei (2013), it follows that for CRRA risk preferences, optimal risky asset demand is always positive and $\partial n/\partial s_1 > 0$.

Definition 2 is a natural extension of the intuition of the income risk case where increased risk free asset demand is equivalent to an increase in saving and provides a buffer against increased income risk. Suppose the consumer holds the risk free asset long in response to an increase in capital risk. If saving increases and this results in larger holdings of the risk free asset, the latter can be viewed as providing a buffer to the risk of the simultaneously increased holdings of the risky asset. In our view, the reallocation component in the capital risk case is totally distinct from the buffer resulting from the increased saving in the risk free asset.

Remark 2 *We require $n_f \geq 0$ in our definition for precautionary saving, since if the consumer shorts the risk free asset, reducing the short position of the risk free asset is not a buffer for the increased capital risk. However, $n_f \geq 0$ imposes a restriction on the CRRA risk preference parameter δ_2. It follows from Kubler, Selden and Wei (2013) that*

$$n_f \geq 0 \Leftrightarrow \delta_2 \geq \delta_2^{critical},$$

where $\delta_2^{critical}$ corresponds to a threshold level of the risk preference parameter. To the extent we seek to provide microfoundation underpinnings for asset pricing and macro equilibrium comparative statics, it seems quite reasonable to rule out $n_f < 0$ as we do in Definition 1. When $n_f = 0$, we need to distinguish whether we have a δ_2 value such that $n_f = 0$ always (the Engel curve for n_f is a horizontal line) or the Engel curve just starts from $n_f = 0$ and can be positive. For the first case, we do not have precautionary saving since $\partial n_f/\partial s_1 = 0$. This problem can be avoided if we instead assume $\delta_2 > \delta_2^{critical}$.

In the asset pricing and macro papers referenced in Section 1, it is argued that the risk free asset price (risk free rate) increases (decreases) with non-income risk due to the precautionary motive. For CES-CRRA KPS preferences, a consistent microeconomics foundation exists for changes in capital risk based on Definition 2. A consumer exhibits precautionary saving, if and only if $\delta_1 > 0$ assuming δ_2 is adequately large. Also, the change in total demand for the risk free asset which includes the asset reallocation component satisfies $\partial n_f/\partial\lambda > 0$. This follows immediately from Proposition 5 in Kubler, Selden and Wei (2014) and Proposition 3 above and is summarized in the following.

Corollary 1 *Consider the consumption-portfolio optimization problem (11)-(14). If the KPS time and risk preference utilities take the forms (5) and (8) and assuming*

$n_f > 0$, *then*

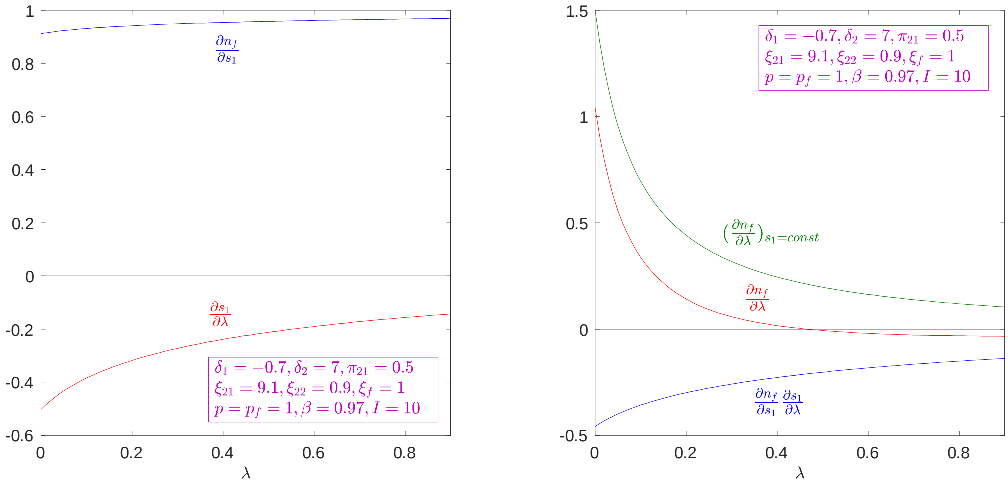
$$\left(\frac{\partial n_f}{\partial \lambda}\right)_{saving} \Leftrightarrow \delta_1 > 0 \quad \textit{and} \quad \frac{\partial n_f}{\partial \lambda} > 0 \textit{ if } \delta_1 > 0.$$

Remark 3 *It should be noted that if instead of assuming a MPMS increase in risk in Corollary 1, one were to assume the more general RS mean preserving spread, the sufficient conditions for a positive reallocation effect would become the very restrictive assumption that risk preferences exhibit relative risk aversion of less than or equal to* 1*, or* $\delta_2 \leq 0$*. Importantly, for the EU special case of KPS preferences based on CES time and CRRA risk preferences where* $\delta_1 = \delta_2$*, one can not have* $\delta_1 > 0$ *if* $\delta_2 \leq 0$*. Then to ensure that overall demand for the risk free asset increases with risk, it is sufficient for the saving component to be weakly positive. If* $\partial n_f / \partial s_1 > 0$*, the only two period EU case which can satisfy this requirement is the Cobb-Douglas utility where* $\delta_1 = \delta_2 = 0$*.*

Remark 4 *Based on calibration estimates, Bansal and Yaron (2004) and Barro (2009) suggest that the* $EIS > 1$ ($\delta_1 < 0$)*. This implies that in consumption-portfolio microfoundation applications based on CES-CRRA KPS preferences, the consumer will not exhibit precautionary saving as we define it. However as discussed in Thimme (2017), there exists alternative evidence suggesting that the* EIS *can be significantly less than unity corresponding to* $\delta_1 > 0$*. If one requires the risk free asset to be held long, i.e.,* $n_f \geq 0$*, then* $\partial n_f / \partial \lambda > 0$ *can be guaranteed only when* $\delta_1 > 0$*. Indeed, this is illustrated in the example below where the assumed value of* EIS *is not that different from the values assumed in Bansal and Yaron (2004) and Barro (2009) and* n_f *can be decreasing with capital risk.*

We next consider an example in which the sufficient conditions in Corollary 1 do not hold and it is possible for risk free asset demand to decrease with capital risk. We show that this can be inconsistent with equilibrium risk free price and rate behavior independent of whether one uses the precautionary motive as an interpretation.

Example 1 *Assume a consumer has CES-CRRA KPS preferences corresponding (5) and (8) and solves the consumption-portfolio problem (11)-(14), where for simplicity*



(a) (b)

Figure 2:

there are only two states of nature. Further assume the following parameter values[19]

$$
\begin{aligned}
\delta_1 &= -0.7, \delta_2 = 7, \pi_{21} = 0.5, \xi_{21} = 9.1, \xi_{22} = 0.9, \xi_f = 1, \\
\epsilon_{21} &= 1, \epsilon_{22} = -1, p = p_f = 1, \beta = 0.97, I = 10. \qquad (18)
\end{aligned}
$$

Given that $\delta_1 < 0$ (EIS > 1), it follows from Proposition 3 that $\partial s_1/\partial\lambda < 0$. For this example, $\partial n_f/\partial s_1 > 0$. Both derivatives are plotted in Figure 2(a). As shown in Figure 2(b), the asset reallocation component

$$\left(\frac{\partial n_f}{\partial \lambda}\right)_{reallocation} > 0,$$

for all values of the MPMS shift parameter λ. However, the saving component

$$\left(\frac{\partial n_f}{\partial \lambda}\right)_{saving} = \frac{\partial n_f}{\partial s_1}\frac{\partial s_1}{\partial \lambda} < 0.$$

Figure 2(b) illustrates that the overall effect corresponding to $\partial n_f/\partial\lambda$ can be negative when $\lambda > 0.47$. Thus if one assumes that the EIS > 1 ($\delta_1 < 0$) (see Remark 4),

[19] In computing the modified distribution of risky asset payoffs (ξ_{21}, ξ_{22}), we begin with the values in (18). Then to compute the shifted payoff values, we apply the spread in eqn. (10) where we assume $E\left[\widetilde{\epsilon}|\widetilde{\xi}\right] = 0$ and $var\left[\widetilde{\epsilon}|\widetilde{\xi}\right] = 1$.

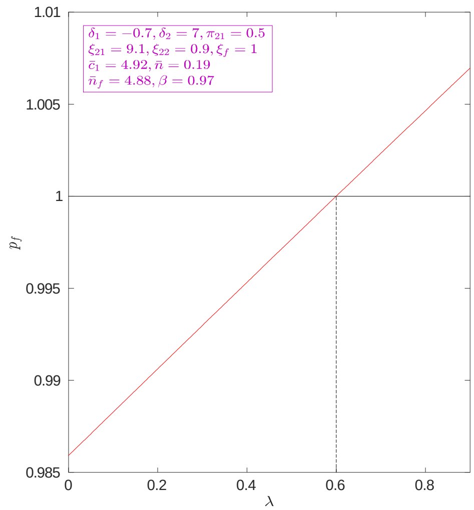


Figure 3:

saving will decrease which can result in a decrease in the saving component of risk free asset demand which can exceed the positive reallocation component. Clearly in this case, the consumer cannot be said to exhibit precautionary saving.[20] *Finally, to validate the possible inconsistency between the partial equilibrium demand and exchange equilibrium price comparative static changes with respect to capital risk discussed in Section 1, assume $\lambda = 0.6$.*[21] *In Figure 2(b) at this value of λ, $\partial n_f / \partial \lambda < 0$. If we assume endowment values corresponding to the optimal demands*

$$\bar{c}_1 = 4.92, \bar{n} = 0.19, \bar{n}_f = 4.88, \tag{19}$$

and keep the other parameter values in (18) unchanged, then the equilibrium price $p_f = 1$ when $\lambda = 0.6$. The risk free asset price p_f is plotted versus λ in Figure 3. When $\lambda = 0.6$, p_f is increasing with λ, which is inconsistent with $\partial n_f / \partial \lambda < 0$.[22]

[20] A decrease in n_f can also hold even when the saving effect is positive, if one considers a Rothschild and Stiglitz (1971) mean preserving spread rather than the MPMS.

[21] This specific value of the MPMS capital risk shift parameter implies the following modified values of the risky asset state payoffs

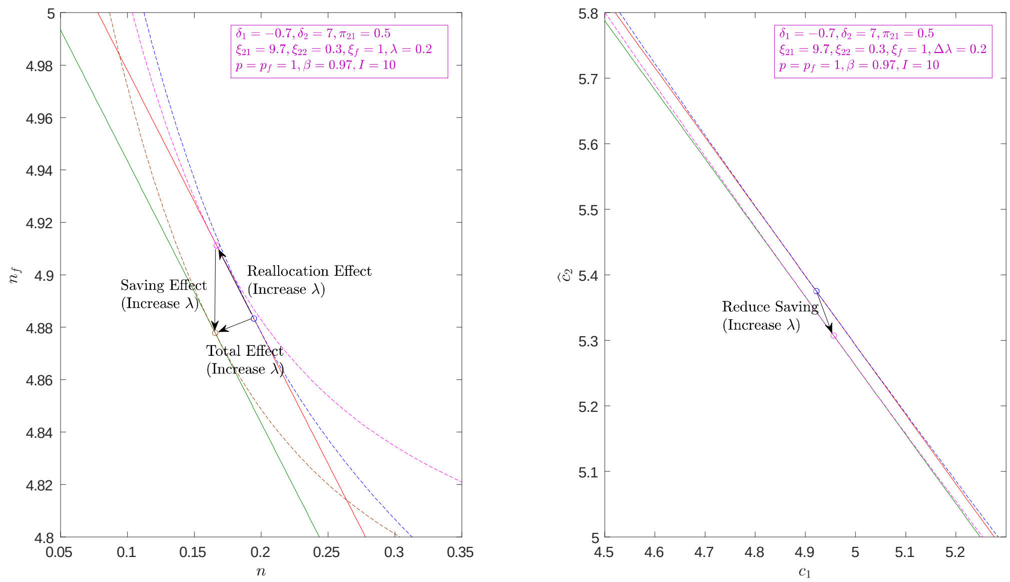
$$\xi_{21}(\lambda) = \xi_{21} + \lambda = 9.7 \quad \text{and} \quad \xi_{22}(\lambda) = \xi_{22} - \lambda = 0.3.$$

[22] It should be emphasized that the partial equilibrium demand and exchange equilibrium price comparative statics with respect to λ can only be compared at the endowments in (19) which are

4.3 A Fisherian Analysis of Capital Risk

Given that the consumer is maximizing CES-CRRA KPS preferences, it is possible to naturally extend the canonical certainty Fisherian consumption-saving diagrammatic analysis (e.g., Figure 1) to the case of the consumption-portfolio optimization. Utilizing the two stage optimization, introduced in Subsection 3, one can separate the consumption-portfolio problem into (i) a consumption-saving problem involving c_1 and $\widehat{c}_2$ and certainty time preferences as in the Fisherian diagram and (ii) a separate portfolio optimization conditional on the level of saving. As shown in Proposition 2, the assumption of CRRA risk preferences ensures that the $\widehat{c}_2$-constraint is affine as in the certainty case. The slope of the constraint is the portfolio certainty equivalent return $\widehat{R}_p$ which is constant for all levels of saving. The portfolio optimization is portrayed in a second graph which plots the maximization of the risk preference $EV(\widetilde{c}_2)$ subject to the portfolio budget constraint. These two diagrams based on the data in Example 1 are given in Figure 4 and are used to illustrate how an increase in capital risk can lead to an overall decrease in n_f rather than an increase corresponding to precautionary saving. In Figure 4(a), we consider the asset optimization in $n - n_f$ space. When capital risk increases, the $EV(n\widetilde{\xi} + n_f\xi_f)$ indifference curves shift and assuming period one consumption if fixed, the optimal point moves northwest along the budget constraint corresponding to an increase in n_f and a reduction n. This is the reallocation component. To investigate the saving component, see the consumption-saving optimization in $c_1 - \widehat{c}_2$ space in Figure 4(b). An increase in capital risk results in a decrease in $\widehat{R}_p$ and the budget constraint anchored at the point $(0, I)$ rotates southwest. This parallels the shift in the certainty constraint in the standard Fisherian diagram in Figure 1 corresponding to a reduction in R_f. (Since the changes in our numerical example are quite small, we only give a magnified region of the figure.) Given that the time preference parameter $\delta_1 = -0.7 < 0$, saving decreases with risk, resulting in less investment $I - c_1$ in the portfolio. The impact of this reduction in saving results in a parallel southwestern shift of the asset budget constraint in Figure 4(a). As a result, the new optimum corresponds to a decrease in the demand for the risk free asset. This negative saving component can be seen to dominate the positive reallocation component.

consistent with the equilibrium $p_f = 1$. To compare the demand and equilibrium price changes at different equilibrium (p_f, p)-values, one must derive different sets of endowments.



$\delta_1 = -0.7, \delta_2 = 7, \pi_{21} = 0.5$
$\xi_{21} = 9.7, \xi_{22} = 0.3, \xi_f = 1, \lambda = 0.2$
$p = p_f = 1, \beta = 0.97, I = 10$
Reallocation Effect (Increase λ)
Saving Effect (Increase λ)
Total Effect (Increase λ)
n_f
n
$\delta_1 = -0.7, \delta_2 = 7, \pi_{21} = 0.5$
$\xi_{21} = 9.7, \xi_{22} = 0.3, \xi_f = 1, \Delta\lambda = 0.2$
$p = p_f = 1, \beta = 0.97, I = 10$
Reduce Saving (Increase λ)
$\hat{c}_2$
c_1

(a) (b)

Figure 4:

5 Conclusion

In this paper, the CES-CRRA KPS preference model is used to analyze the change in saving and risk free asset demand in response to an increase in capital risk in the classic consumption-portfolio setting. A new definition of precautionary saving appropriate for capital risk is proposed based on the saving component (and not incorporating the change in risk free asset demand associated with the reallocation component). Conditions on preferences are derived such that the consumer exhibits precautionary saving. We also show that the combined change in demand for the risk free asset resulting from the saving and reallocation components can be positive or negative. As a result, the exchange equilibrium and partial equilibrium demand comparative statics can diverge despite comments in the literature implying that this is not the case.

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