Advertising in the Circular Model of Differentiated Products

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Abstract

We introduce advertising in the circular model of differentiated products. We analyse two games. The three-stage game has entry in the first stage, location in the second, and advertising and price choice in the third. The four-stage game replaces the advertising and price subgame of the three-stage game with two stages, advertising choice followed by price choice. We find that advertising commitment in the four-stage game allows firms to support the same prices (as in the no-commitment three-stage game) with lower advertising expenditure. This induces entry so that in the perfect equilibrium there is less advertising per firm and in total in the four-stage pre-commitment game than in the three-stage one. This is despite the fact that there is a larger number of firms (brands) at the equi-librium of the four-stage game. In relation to optimality both games result in higher diversity and underprovision of advertising with bigger divergence from optimality observed in the four-stage pre-commitment game.

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Advertising in the Circular Model of Differentiated Products*

We augment the circular model of differentiated products [Salop (1979), Novshek (1980), Economides (1985)] to include advertising. Advertising enhances the (perceived) value of a product. There are two different interpretations of the formal model we are going to present. In the first interpretation of the model, advertising attaches to differentiated commodities images that consumers value. The attached images have inherent value, i.e. are inherently desirable, but they need not have any immediate connection with the physical attributes of the product before attachment.[1] The attached mental image is assumed to be equally desirable to all consumers. It is sufficient to consider that advertising a attaches an image of value a to the advertised differentiated product. In a second interpretation of the model a can be thought of as a <u>quality</u> feature of the differentiated product such that marginal production costs are independent of it. In this interpretation an example of a higher a may be a better design of a computer program which leaves variable production costs unaffected and also does not change the position of the product in the spectrum as it does not affect the kind of the task it performs.

As in the standard circular model of differentiated products, there are n products, $x_1, \ldots, x_n$, each produced by one firm located on a circumference of length 1. The value to consumer z of one unit of product x_j offered at price p_j is $V_j - p_j - |z - x_j|$, a decreasing function of the distance between x_j and z. Thus consumer z can be thought of as residing at z and travelling distance $|z - x_j|$.[2] The maximal value of product x_j, V_j, realizable only by the consumer residing at x_j, is composed of its inherent value k and of the value added by advertising a_j, i.e. $V_j = k + a_j$. When a_j is interpreted as quality, k represents the value of the lowest possible quality level which is normalized at $a_j = 0$.

Consumers are assumed to be uniformly distributed with density normalized to 1 on the circle according to their most preferred commodity. The demand faced by firm j which is in direct competition with firms $j-1$ and $j+1$ is

$$D_j = (x_{j+1} - x_{j-1} + p_{j+1} + p_{j-1} - 2p_j - a_{j+1} - a_{j-1} + 2a_j)/2.$$

Assuming zero marginal costs, a fixed cost F, and convex advertising costs $C(a) = ca^2/2$, firm j has profit function $\Pi_j = p_j D_j - C(a_j) - F$. Fixed costs associated with the advertising technology can be lumped together with the fixed production costs in F without any problem.

Traditionally the circular model has been analysed as a sequential game where firms enter in stage 1, locate in stage 2 and choose prices in stage 3. Here we analyse two separate games that differ in the position of the advertising move. First we analyse a game where advertising and price decisions are taken simultaneously in the last stage of the game. As in the traditional model this stage is preceded by the location stage, which is itself preceded by the entry stage. We seek subgame-perfect equilibria.

The use of sequential games can be justified by the fact that all strategic variables are not equally flexible. In the very short run only prices and maybe advertising are flexible and therefore available as strategic variables. In the long run product specification (location) is flexible, and in the very long run firms have the ability to enter and exit.

I. The Entry → Location → (Advertising and Price) Game

We start by analysing the last stage of the game. In this discussion we will confine ourselves to strategies that do not involve "undercutting" where one or more firms is left with zero demand through the actions of opponents. This is because we want to focus on the effects of advertising rather than on problems of existence. Let it be noted, however, that, for symmetric and many not-too-asymmetric locations, price equilibria have been established in Salop (1979) and Economides (1985) even when "undercutting" strategies are allowed.

First order conditions for maximization with respect to price are

$$p_{j-1}/4 + p_j - p_{j+1}/4 = (x_{j+1} - x_{j-1})/4 + (2a_j - a_{j+1} - a_{j-1})/4 \equiv e_j$$
$$j = 1, \ldots, n.$$

They can be summarized as

$$\mathbf{A}p = y + \mathbf{H}a \equiv e \tag{1}$$

where

$\mathbf{A}_j = (0, \ldots, 0, -1/4, 1, -1/4, 0, \ldots, 0)$ is the j^{th} row of $\mathbf{A}$,

$y_j = (x_{j+1} - x_{j-1})/4$ is the j^{th} row of y,

$\mathbf{H}_j = (0, \ldots, 0, -1/4, 1/2, -1/4, 0, \ldots, 0)$ is the j^{th} row of $\mathbf{H}$.

Note that both $\mathbf{A}$ and $\mathbf{H}$ are circulant[3], symmetric matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & -1/4 & & & & & -1/4 \\ -1/4 & 1 & -1/4 & & & & \\ & & & & & \bigcirc & \\ & & & -1/4 \;\; 1 \;\; -1/4 & & & \\ & \bigcirc & & & & & \\ & & & & -1/4 & 1 & -1/4 \\ -1/4 & & & & & -1/4 & 1 \end{bmatrix}, \quad \mathbf{H} = \mathbf{A} - I/2,$$

where I is the identity matrix.

First order conditions for maximization with respect to advertising are $p_j^* = ca_j^*$, $j = 1, \ldots, n$. Together with the conditions from maximization with respect to prices, they imply $c\mathbf{A}a^* = y + \mathbf{H}a^*$ $\Leftrightarrow$ $(c\mathbf{A} - \mathbf{H})a^* = y$ $\Leftrightarrow$ $a^* = (c\mathbf{A} - \mathbf{H})^{-1}y$. Since both $\mathbf{A}$ and $\mathbf{H}$ are circulant symmetric matrices so is $(c\mathbf{A} - \mathbf{H})$. Its inverse $Q \equiv (c\mathbf{A} - \mathbf{H})^{-1}$ exists provided that $c \neq 2$.[4] Q is circulant and symmetric, as the inverse of a circulant, symmetric matrix. Thus all elements of the main diagonal of Q have the same value q_1, the value of all elements of the upper and lower diagonals is q_2, etc. These numbers can be directly calculated as solutions to the system of equations $Q(c\mathbf{A} - \mathbf{H}) = I$. Thus, the equilibrium of the last stage of the game is $a^* = (c\mathbf{A} - \mathbf{H})^{-1}y$, $p^* = ca^*$. Equilibrium profits are $\Pi_j^* = c^2(a_j^*)^2(1 - 1/(2c)) - F$.

In stage 2 firms choose locations expecting the equilibrium profits of the advertising and price subgame. The rate of change of profits with respect to location can be directly calculated as

$$d\Pi_j/dx_j = 2c^2 a_j^*[1 - 1/(2c)]da_j^*/dx_j,$$

and

$$da_j^*/dx_j = \sum_{i=-n/2}^{n/2} q_{1+|i|}(dy_{j+i}/dx_j) = (q_2 - q_2)/4 = 0,$$

because $dy_{j-1}/dx_j = q_2/4$, $dy_{j+1}/dx_j = -q_2/4$, $dy_i/dx_j = 0$, for $i \neq j+1, j-1$. Therefore a symmetric equilibrium of the location game exists. Denoting by $d = 1/n$ the distance between consecutive firms, the equilibrium values of advertising, prices and profits are $a^{3s}(d) = d/c$, $p^{3s}(d) = d$, $\Pi^{3s}(d) = d^2[1 - 1/(2c)] - F$.

Finally, in stage 1 firms choose to enter or not in the industry. Assuming zero profits outside the industry, the free entry equilibrium number of firms will be the integer part of $n^{3s*} = \{[1 - 1/(2c)]/F\}^{1/2}$ which makes profits zero. The full equilibrium includes $p^{3s*} = 1/n^{3s*}$ and $a^{3s*} = 1/(cn^{3s*})$.

Theorem 1: The three-stage game with entry choice in the first stage, location choice in the second stage and advertising and price choice in the third stage has the following perfect equilibrium: number of active firms $n^{3s*} = \{[1 - 1/(2c)]/F\}^{1/2}$, symmetric locations $d = 1/n^{3s*}$, advertising level per firm $a^{3s*} = 1/(cn^{3s*})$, and price level $p^{3s*} = 1/n^{3s*}$.

It is instructive to compare this equilibrium with the surplus maximizing one, which we analyse in the next section.

II. Optimality

The surplus generated by one firm d apart from its immediate neighbors using advertising a is $(k + a)d - (d/2)^2 - C(a) - F$, so that total surplus when there are n firms in the industry $(d = 1/n)$ is

$$S(n, a) = k + a - nC(a) - nF - 1/(4n).$$

Maximization with respect to a yields $a^o = 1/(nc)$. After substitution of a^o, total surplus, $S(n) = k - nF + (1/c - 1/2)/(2n)$ is a concave function of n for $c > 2$. Maximization with respect to n yields

$$n^o = \{[1/4 - 1/(2c)]/F\}^{1/2}.$$

Theorem 2: The surplus maximizing market structure consists of $n^o = \{[1/4 - 1/(2c)]/F\}^{1/2}$ differentiated products located symmetrically $1/n^o$ apart with advertising levels $a^o = 1/(cn^o)$ per product.

The optimal number of differentiated products is smaller than the equilibrium number. In fact, the ratio of the equilibrium number of products to the optimal number is larger than 2, $n^{3s}/n^o = [1 - 1/(2c)]/[1/4 - 1/(2c)]\}^{1/2} > 2$, and it is declining in c with $\lim_{c \to \infty} n^{3s}/n^o = 2$. In other words, as advertising becomes more expensive, and is used less, the distortion in the number of products at equilibrium compared to optimality is reduced. In the limit, as advertising costs go to infinity, the distortion in product diversity approaches the distortion of the circular model without advertising, as analysed by Salop (1979).

Corollary 1: The optimal number of products is smaller than at the equilibrium of the three-stage game.

Corollary 2: The advertising level per firm is higher at optimality than at the three-stage equilibrium, but the total amount of advertising is the same.

Proof: It is immediate that $a^o n^o = a^{3s*} n^{3s*} = 1/c$, and since $n^o < n^{3s*}$, it follows that $a^o > a^{3s*}$.

In the second interpretation of the model with a representing quality level, corollary 2 says that the quality level of products at the three-stage equilibrium will be lower than optimal.

III. The Entry → Location → Advertising → Price Game

Now we analyse a four-stage game where entry, location, advertising expenditure and price are chosen sequentially. Again we seek a non-cooperative subgame-perfect equilibrium. Here advertising can be thought of as a commitment because it allows players to reveal how aggressively they are willing to compete before prices are chosen.

The first order conditions on prices are as in the earlier game

$$\mathbf{A}p = y + \mathbf{H}a \qquad (1)$$

and can be solved to determine equilibrium prices

$$p^{*} = \mathbf{A}^{-1}(y + \mathbf{H}a) \qquad (2).$$

The inverse of $\mathbf{A}$ exists and it is a circulant matrix $\mathbf{B} \equiv \mathbf{A}^{-1}$. Since $\mathbf{B}$ is circulant, all elements in a diagonal of $\mathbf{B}$ are equal. Let the element of the diagonal of distance i from the main diagonal be b_{1+i}.

In the advertising subgame the objective function is the equilibrium profits of the price subgame, $\Pi^{a}_{j} = (p^{*}_{j})^{2} - c(a_{j})^{2}/2$. First order conditions for profit maximization in the advertising subgame are

$$ca^{*}_{j} = 2p^{*}_{j}(dp^{*}_{j}/da_{j}), \quad j = 1, \ldots, n \qquad (3).$$

From (1) and (2)

$$p^{*}_{j} = \sum_{i=-n/2}^{n/2} b_{1+|i|}e_{j+i}$$

and $de_{j}/da_{j} = 1/2$, $de_{j-1}/da_{j} = -1/4$, $de_{j+1}/da_{j} = -1/4$, while $de_{j+i-1}/da_{j} = 0$ for $i \neq 0, 1, 2,$ so that

$$dp^{*}_{j}/da_{j} = (b_{1} - b_{2})/2.$$

It follows then from (3) that $a^{*}_{j} = (b_{1} - b_{2})p^{*}_{j}/c$, so that

$$a^{*} = (b_{1} - b_{2})\mathbf{A}^{-1}(y + \mathbf{H}a^{*})/c \iff \mathbf{A}a^{*} = (b_{1} - b_{2})(y + \mathbf{H}a^{*})/c$$

$$\iff (\mathbf{A} - (b_{1} - b_{2})\mathbf{H})a^{*} = (b_{1} - b_{2})y \iff$$

$$a^{*} = (b_{1} - b_{2})(\mathbf{A} - (b_{1} - b_{2})\mathbf{H}/c)^{-1}y \qquad (4).$$

$(\mathbf{A} - (b_{1} - b_{2})\mathbf{H}/c)$ is a symmetric circulant matrix invertible if $c \neq b_{1} - b_{2}$.[5]

In the location game, the objective function is the equilibrium profits of the advertising subgame $\Pi^{\ell}_{j} = c[c/(b_{1} - b_{2})^{2} - 1/2](a^{*}_{j})^{2}$. It is clear that the derivative of profits with respect to location is proportional to da^{*}_{j}/dx_{j}. From (4), since $(\mathbf{A} - (b_{1} - b_{2})\mathbf{H}/c)^{-1}$ is symmetric and circulant, $da^{*}_{j}/dx_{j} = 0$, using the same reasoning as in Section I. Therefore a symmetric equilibrium exists in the location game.

The full symmetric equilibrium with firms d apart is $p^{4s}(d) = d$, $a^{4s}(d) = (b_1 - b_2)d/c$, $\Pi^{4s}(d) = d^2[1 - (b_1 - b_2)^2/(2c)] - F$. In comparing this equilibrium with the symmetric location equilibrium of the three-stage game, we see that for the same number of firms (same distance between firms) the prices are the same, advertising is lower and profits are higher in the present game.[6] Thus pre-commitment on advertising in the present game allows firms to achieve the same prices and revenues while reducing advertising levels and costs. Therefore overall profits improve when pre-commitment is allowed.

In the first stage of the game, which is interpreted as the very long run, firms decide on entry. The number of firms that makes profits zero is $n^{4s*} = \{[1 - (b_1 - b_2)^2/(2c)]/F\}^{1/2}$, and the equilibrium number of firms is the integer part of n^{4s*}. Typically n^{4s*} provides a very good approximation of its integer part. The full equilibrium also includes $p^{4s*} = 1/n^{4s*}$, $a^{4s*} = (b_1 - b_2)/(cn^{4s*})$.

Theorem 3: The four-stage game with entry in the first stage, location choice in the second stage, advertising level choice in the third stage, and price choice in the fourth stage, has the following perfect equilibrium: $n^{4s*} = \{[1 - (b_1 - b_2)^2/(2c)]/F\}^{1/2}$, symmetric locations at distance $d = 1/n^{4s*}$, advertising level per firm $a^{4s*} = (b_1 - b_2)/(cn^{4s*})$, and price level $p^{4s*} = 1/n^{4s*}$.

In comparison with the three-stage game, we see that the four-stage game has a larger number of products at equilibrium, $n^{4s*} > n^{3s*} > n^o$, and therefore diverges even more from optimal diversity. As a result of the availability of pre-commitment in advertising, profits are higher for any given number of firms and therefore at equilibrium there are even more products than in the three-stage game.

Corollary 3: The long run equilibrium number of firms in the four-stage game with advertising pre-commitment is larger than in the three-stage game without pre-commitment.

The opportunity of competition in advertising (or quality), as well as the availability of precommmitment in this strategic variable, creates a distortion in the number of varieties which far exceeds the distortion in the game without advertising.

Since total surplus is concave in n and maximized at n^o, it immediately follows:

Corollary 4: Social Welfare is lower when the possibility of pre-commitment exists -- in the four-stage game.

Comparing the overall equilibrium of the four-stage game with the three-stage one we see that in the four-stage game prices and advertising levels are lower. The availability of pre-commitment on advertising gives firms a strategic advantage in the advertising and subsequent stages of the game over a game of no pre-commitment. As we saw above, for fixed n, firms in the four-stage game are able to achieve the same revenues as in the three-stage game, but with lower advertising and costs thereof. However, in the overall game the strategic advantage mechanism has further consequences. More firms enter when pre-commitment is available. As a result prices and revenues fall in the subgames that are now played by a larger number of firms. Advertising falls further as it has to support lower prices.

Corollary 5: The advertising level per firm in the four-stage game is lower than in the three-stage game. Further, total advertising expenditure is also lower at the four-stage game: $a^{4s*} < a^{3s*} < a^o$, and $n^{4s*}a^{4s*} < n^{3s*}a^{3s*} = n^o a^o$.[7]

It is also useful to see these results from the point of view of the second interpretation of the model in which a is a quality feature with no effects on marginal production costs. Costs of quality a, given by $C(a)$ are considered fixed, as far as production is concerned. All the results can be re-read by substituting "quality level" for "advertising level". In particular, when firms have the ability to pre-commit (in the four-stage game) they end up at an equilibrium of lower quality levels. Precommitment allows each firm to support prices equal to the ones of the three-stage no pre-commitment game with lower investment in quality. This causes higher profits at the pre-commitment game for any fixed number of firms, n. As a result, at the free entry equilibrium there are more firms at the four-stage rather than the three-stage game. Crowding of firms in the commodity space lowers prices and results in even lower quality levels in the overall perfect equilibrium with precommitment. Thus the ability of pre-commitment reduces the quality levels while increasing the number of brands. Both of these effects affect total surplus adversely, and thus pre-commitment reduces social welfare.

IV. Conclusion

We have shown that the overcrowding of brands in the traditional circular model of differentiated products is accentuated when the possibility of advertising (or varying quality level) is introduced. Further, quality is underprovided. Both these adverse influences on total welfare are intensified when firms have the ability to precommit themselves on advertising or quality level. Salop (1979) pointed to the fact that in a model of differentiated products, profits did not signal correctly, and as a result there is excess diversity at the free entry equilibrium. We have shown here that the addition of another dimension of competition, be it advertising or quality competition, makes things even worse, with a larger distortion in the number of varieties and in total surplus. A limitation of competition through the introduction of precommitments has further adverse effects on product diversity and total surplus.

Footnotes

*I thank Bruce Ackerman and Kelvin Lancaster for helpful discussions and the participants of the Columbia Industrial Organization Workshop for their comments.

1. We assume that the brands are trademarked so that it is possible to attach to the product a desired mental image which is irrelevant to its physical characteristics. Without a trademark the desired mental image would have to be attached to features of the product that can be imitated. Thus, it would be nearly impossible to compete in "perception advertising" in a meaningful way.

2. The cost per unit of distance has been normalized to 1.

3. In a circulant matrix each row is a shift to the right of the row above it.

4. A circulant symmetric matrix is invertible if the sum of the elements of a row is different than zero. For $(c\mathbf{A} - \mathbf{H})$ the sum of a row is $c - c/2 - 1/2 = c/2 - 1/2$. Therefore $(c\mathbf{A} - \mathbf{H})^{-1}$ exists for $c \neq 1$.

5. The sum of a row of $(\mathbf{A} - (b_1 - b_2)\mathbf{H}/c)$ is $1/2 - (b_1 - b_2)/(2c)$ and differs from zero if $c \neq b_1 - b_2$. Thus $(\mathbf{A} - (b_1 - b_2)\mathbf{H}/c)^{-1}$ exists if $c \neq b_1 - b_2$.

6. $a^{3s}(d) < a^{4s}(d) \iff (b_1 - b_2)d/c < d/c \iff b_1 - b_2 < 1$, which is true because $b_1 - b_2$ varies between .25 and .4225 as n varies between 2 and ∞.

7. $n^{4s*}a^{4s*} < n^{3s*}a^{3s*} \iff (b_1 - b_2)/c < 1/c \iff b_1 - b_2 < 1$, which is true as seen in footnote 6.

References

D'Aspremont, Claude, Jean Jaskold-Gabszewicz & Jacques-Francois Thisse (1979), "On Hotelling's 'Stability in Competition'," *Econometrica* 47, 1145-1150.

Economides Nicholas (1985), "The Circular Model of Differentiated Products, mimeo.

Fudenberg, Drew, and Jean Tirole (1984), "The Fat-Cat Effect, The Puppy-Dog Ploy, and the Lean and Hungry Look," *American Economic Review* 74 (2), 361-66.

Grossman, Gene, and Carl Shapiro (1984), "Informative Advertising with Differentiated Products," *Review of Economic Studies* 51, 63-82.

Hotelling, Harold (1929), "Stability in Competition," *Economic Journal* 39, 41-57.

Lancaster, Kelvin (1979), *Variety, Equity and Efficiency*, Columbia University Press, New York.

Novshek, William (1980), "Equilibrium in Simple Spatial (or Differentiated Product Markets," *Journal of Economic Theory* 22 (2), 313-326.

Salop, Steven (1979), "Monopolistic Competition with Outside Goods," *Bell Journal of Economics* 10 (1), 141-156.

Schmalensee, Richard, (1982), "Product Differentiation Advantages of Pioneering Brands," *American Economic Review* 72, 349-65.