The Max-Min Principle of Product Differentiation

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Abstract

We analyze two- and three-dimensional variants of Hotelling's model of differentiated products. In our setup, consumers can place different importance on each product attribute; this is measured by a weight in the disutility of distance in each dimension. Two firms play a two-stage game; they choose locations in stage 1 and prices in stage 2. We seek subgame-perfect equilibria. We find that all such equilibria have maximal differentiation in one dimension only; in all other dimensions, they have minimum differentiation. An equilibrium with maximal differentiation in a certain dimension occurs when consumers place sufficient importance (weight) on that attribute. Thus, depending on the importance consumers place on each attribute, in two dimensions there is a "max-min" equilibrium, a "min-max" equilibrium, or both. In three dimensions, depending on the weights, there can be a "max-min-min" equilibrium, a "min-max-min" equilibrium, a "min-min-max" equilibrium, any two of them, or all three.

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The Max-Min Principle of Product Differentiation

1. Introduction

The theory of product differentiation has as its goal the determination of market structure and conduct of firms that can determine the specifications of their products besides choosing output and price. Traditional models of product differentiation and marketing have focused on products that are defined by one characteristic only.[1] One-characteristic models are sufficient for the understanding of the interaction between product specification and price. The main question in this setting is the degree of product differentiation at equilibrium -- does the acclaimed "Principle of Minimum Differentiation" (stating that product specifications will be very similar at equilibrium) hold? Intensive research on this question has conclusively determined that the Principle of Minimum Differentiation does not hold for any well-behaved model.[2] Thus as long as we confine product differentiation to one dimension, there will be significant differences in the equilibrium product specifications. However, most goods are defined by a long vector of product attributes, and *a priori*, the failure of the Principle of Minimum Differentiation is not clear in multi-attribute competition. Furthermore, if it does fail, questions characterizing the failure will naturally arise.

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[1] See Hotelling (1929), Vickrey (1964), D'Aspremont, Gabszewicz and Thisse (1979), Salop (1979), Economides (1984), Anderson, de Palma, and Thisse (1992), among others in economics and Hauser and Shugan (1983), Moorthy (1988) and Kumar and Sudarshan (1988) in marketing.

[2] See Neven (1985) for a discussion of the necessary conditions for minimal differentiation. Also note that the failure of minimal differentiation does not necessarily imply maximal differentiation. D'Aspremont *et al.* (1979) establish a maximal differentiation equilibrium in a one-dimensional variant of Hotelling (1929) by assuming a quadratic disutility of distance (transportation cost) function. Economides (1986b) establishes intermediate (neither minimum nor maximal) differentiation equilibria for a disutility of distance (transportation cost) function of the form d^a, $5/3 < a < 1.26$. Economides (1984) establishes intermediate differentiation equilibria by allowing for a finite maximal utility (reservation price) for a differentiated good in the original linear disutility of distance function of Hotelling (1929).

The Principle of Minimum Differentiation fails in one-dimensional models because product similarity increases competition, and reduces prices and profits. When we add a second dimension, two possibilities emerge: products can be significantly differentiated in both dimensions (maximum-maximum differentiation or max-max) or products may have quite different degrees of product differentiation in different dimensions (for example minimum differentiation in one dimension and maximum differentiation in another or max-min). The logic of the results of the one-dimensional models is not sufficient to show which of the two configurations will arise in a two- dimensional model.

The present paper determines the equilibrium configuration in a standard two-dimensional model as max-min. That is, we establish that firms will try to maximally differentiate in one dimension and minimally differentiate in another. We call this the ***Principle of Maximum-Minimum Differentiation***. We further show that when products can be differentiated in three dimensions, firms differentiate maximally in one dimension and minimally in the remaining two. We call this the ***Principle of Max-Min-Min Differentiation***.

In our setup, the disutility of distance function has different weights in each dimension. These weights measure the importance that consumers place on each attribute of the product. We find that the nature and number of equilibria depend crucially on these weights. For example, when consumers care a lot about the attribute of the first dimension (and therefore place a high weight on it), the "max-min" equilibrium exists, where firms maximally differentiate in the first dimension only. Similarly, when the consumers place a high weight on the second attribute, the "min-max" equilibrium exists, where firms maximally differentiate in the second dimension only. When the weights are roughly comparable, both equilibria exist. The same pattern hold in the three-characteristics model. The "max-min-min" equilibrium, where firms maximally differentiate in the first dimension only, occurs when the weight of the first attribute is large. When, in addition, the weight of the second attribute is significant as well, the "min-max-min" equilibrium occurs as well. When all weights are comparable, the "min-min-max" equilibrium occurs in addition to the previous two.

The qualitative relationship between weights and type and number of equilibria is very important because it can used to show a seamless transition from Hotelling's one-characteristic paradigm to models of two and three characteristics. Start with the original one-dimensional model of Hotelling. It can be embedded in a two dimensional model, where the weight placed by the consumers in the second attribute is negligible. We show, that if this second weight is small, the equilibrium of two-dimensional model will have maximal differentiation in the first dimension, and no differentiation in the second dimension ("max-min"). Adding a third attribute that the consumers do not consider important preserves the equilibrium pattern, which now becomes "max-min-min". Only when the second weight is significant, a second equilibrium ("min-max") appears.

In three dimensions, the equilibria show minimal differentiation in two dimensions and maximal in one. Thus, if the maximal differentiation in one dimension remains unobserved, the equilibrium may seem to be one of minimal differentiation. However, if the only dimension observed is the one about which consumers care the most, then the previous discussion shows that maximal differentiation will be observed.

All our results are established in a framework of a two-stage game, in the first stage of which firms simultaneously choose locations, while in the second stage they simultaneously choose prices. Thus, the equilibria we describe are subgame perfect, and firms anticipate the effects of changes in their locations to the equilibrium prices. Intuitively, this game structure captures the fact that prices are more flexible (easier to change) in the short run, while product specifications are not; pricing decisions often are made when product specifications cannot be changed.[3]

[3] See Salop (1979), Economides (1989), and Rao and Steckel (forthcoming).

In the existing literature, few papers have allowed determination of product specifications in two characteristics, notably Economides (1989a, 1993), Neven and Thisse (1990) and Vandenbosch and Weinberg (1994).[4] Neven and Thisse (1990) investigate product quality and variety decisions of two firms in a two dimensional product space. They combine the "horizontal" differentiation (ideal point) and "vertical" differentiation (vector attribute) paradigms, and investigate subgame-perfect equilibria for product and price decisions in a duopoly. Vandenbosch and Weinberg (1994) analyze a model of two-dimensional vertical (quality) differentiation. We analyze a model of two-dimensional variety differentiation.[5]

The remainder of this paper is organized as follows. In Section 2, we present the market environment. In Section 3, we analyze the two dimensional market and derive the price and position equilibria. We extend the model to three dimensions in Section 4. Finally in Section 5, we conclude with a discussion of our results and provide directions for future research.

2. The Model

We describe the model in general terms that are relevant for markets of either two or three attributes. We assume that there are two firms, labeled 1 and 2, and each offers a single n-attribute product. The position of a product i can be represented in n-dimensional attribute space by an n-tuple, $\theta_i \in [0, 1]^n$. The elements of θ_i give the position of the product on each of the n attributes. Each consumer is represented by an ideal point which gives the coordinates of the product which the consumer would prefer to all others if all were sold at the same price.

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[4] This is in contrast with analysis on the interaction of price and location competition in multidimensional settings without explicit locational determination as in Economides (1986a), or two-dimensional models that can be reduced to one-dimensional competition as Lane (1980), Hauser and Shugan (1983), Hauser (1988), and Ansari, Economides, and Ghosh (1994).

[5] After a working paper version of this article had been circulating, we discovered that Tabuchi (1994) had independently derived similar results for a two-dimensional variety model with equal weights on attributes.

A consumer j can therefore be represented by the vector of coordinates of his ideal point, $A_j \in [0, 1]^n$.

Each consumer's utility is a decreasing function of the square of the Euclidean distance between the product specifications and the consumer's ideal point.[6] Formally, a consumer of type A_j derives the following utility from buying one unit of product i at price p_i:

$$U(A_j; \theta_i; p_i) = Y - \mathbf{w}\cdot\|\theta_i - A_j\|^2 - p_i. \tag{1}$$

Y is a positive constant, the same for all consumers and assumed to be high enough so that all consumers buy a differentiated product. **w** is a vector of weights that the consumers attach to attributes. *We assume that the* **w** *vector is same across all individuals.*

Consumers' ideal points are distributed uniformly over the attribute space; consumers also possess perfect information about brand positions and prices in the market. Firms maximize profits and have zero marginal costs of production.[7] Firms compete by following a two-stage process. In the first stage they simultaneously choose product positions. Once these are determined, they simultaneously choose prices in the second stage. We seek subgame-perfect equilibria of the game implied by this framework. Thus, firms anticipate the impact of location decisions on equilibrium prices. Given this basic model structure, we analyze next the two-dimensional market in detail.[8]

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[6] Models of product differentiation involving a quadratic utility loss function include D'Aspremont *et al.* (1979), Neven (1985), and Economides (1989b). Also ideal point models in marketing assume that preferences are negatively related to the square of the Euclidean distance between the product and the consumer's ideal point (see, e.g. Green and Srinivasan (1978)).

[7] Positive constant marginal costs lead to formally equivalent results.

[8] The geometric structure of our two dimensional model parallels that in Neven and Thisse (1990). However, important differences do arise in the structure of the positioning stage within the two stage game.

3. The Two Dimensional Model

3.1 Demand Formulation

In two dimensions, the joint space of consumers and products is a unit square. A product i is represented by the vector $V_i = (x_i, y_i)$, whereas an arbitrary consumer can be identified by the address (a, b). Without loss of generality, we assume that $y_2 \geq y_1$ and $x_2 \geq x_1$.[9] A consumer's utility for product i takes the form

$$U_i(a, b; x_i, y_i, p_i) = I - w_1(a - x_i)^2 - w_2(b - y_i)^2 - p_i \quad \text{for} \quad i = 1, 2. \tag{2}$$

The demand for product i is generated by consumers who obtain greater utility from it than from the other product. To characterize the market area of firm 1, consisting of consumers who buy product 1, we need to derive the locus of consumers who are indifferent between brands 1 and 2. Their ideal points satisfy $U_1(a, b; x_1, y_1, p_1) = U_2(a, b; x_2, y_2, p_2)$, which is equivalent to

$$b(a) = [(p_2 - p_1) + S - 2aw_1X]/[2w_2Y]$$

where $S = w_1(x_2^2 - x_1^2) + w_2(y_2^2 - y_1^2)$, $X = x_2 - x_1$, and $Y = y_2 - y_1$.

This locus is a straight line and partitions the total market (the unit square) into two demand areas for the firms (see Figure 1). Given our assumptions regarding the product positions, the area below the separating line represents firm 1's demand and the area above it represents firm 2's demand. The slope of the separating line (b-line) is independent of the prices, but the intercept is not. The location of the line within the unit square depends upon the price difference, $p_1 - p_2$, between the two firms. When firm 1 increases its price (or firm 2 decreases its price), the separating line shifts down reducing the market area for firm 1.

The demand for firm 1, D_1, is obtained by integrating the b(a) line over the appropriate range of a. Since consumers always buy one product or the other, $D_2 = 1 - D_1$. Assuming zero

[9] The other situation, in which $x_1 > x_2$, can be dealt with in a symmetric way.

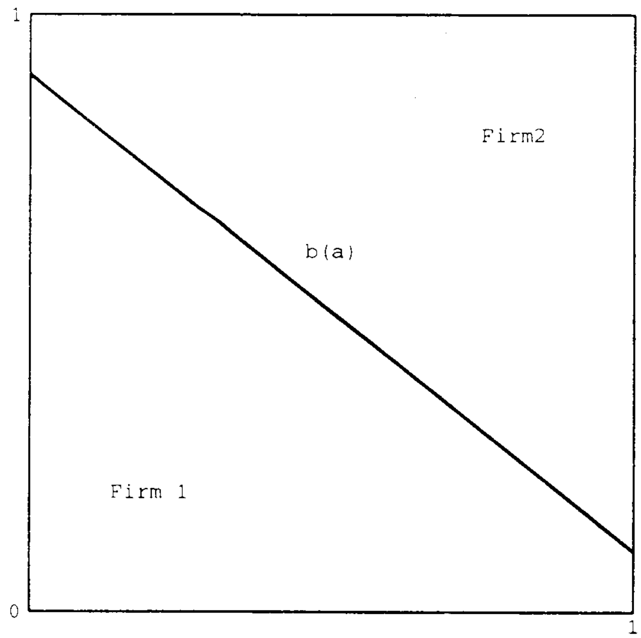


Figure 1: Market Areas for the Two Firms

costs, profits are $\Pi_1 = p_1D_1$, $\Pi_2 = p_2D_2$. Since the domain of integration depends upon the slope and location of the indifference line within the unit square, the demand expressions change whenever the indifference line shifts its locations and passes through a corner of the market. See Figure 2. To capture the dependence of the demand expressions on the relationship between prices and product positions, we first distinguish between two scenarios that depend only upon the positions of product 1 and product 2 and the relative importance consumers give to the attributes, and do not depend on prices. In scenario A, shown in the left column in Figure 2, product positions for the two firms satisfy condition A: $|\partial b/\partial a| < 1 \Leftrightarrow w_1X \leq w_2Y$, i.e., the weighted difference in positions along attribute 2 is greater than the weighted difference in positions along attribute 1. In scenario B, shown in the right column in Figure 2, product positions satisfy condition B: $w_2Y < w_1X$, which is just the negation of condition A. Once product positions are fixed, the relevant scenario can be identified, and the nature of dependence of the demand and profits on the price difference of the two firms can be studied within each scenario. Within each scenario, we identify three cases that are distinguished by the difference in the prices $p_2 - p_1$.

3.1.1 Scenario A

We first analyze scenario A and focus on the dependence of the demand expressions and profit functions on the relative price difference between the firms. We fix the positions of both brands and the price of firm 2, p_2. When p_1 is high, the b line that separates the market areas is shown in Case 1A, Figure 2. As firm 1 reduces its price, the relative price difference decreases and the b line shifts up and crosses the vertical sides of the square, as shown in Case 2A in Figure 2. Finally, when firm 1's price is even lower, the b line cuts the top and right of the unit square, a situation shown in Case 3A, Figure 2. The demand expressions for each of these cases are different because the domain of integration differs across the cases; the demand expressions are summarized below. We label the demand expressions for firm i in case k as D_i^k.

Case 1A: When $0 \leq (p_2 - p_1 + S) \leq 2w_1X$, the demand of firm 1 is $D_1^{1A} \equiv (p_2 - p_1 + S)^2/(8w_1w_2XY)$.

Case 2A: When $2w_1X \leq (p_2 - p_1 + S) \leq 2w_2Y$, the demand of firm 1 is $D_1^{2A} \equiv (p_2 - p_1 + S - w_1X)/(2w_2Y)$.

Case 3A: When $2w_2Y \leq (p_2 - p_1 + S) \leq 2(w_1X + w_2Y)$, the demand of firm 1 is $D_1^{3A} \equiv (p_2 - p_1 + S - w_1X - w_2Y)^2/(8w_1w_2XY)$.

3.3 Scenario B

In Scenario B, the product positions satisfy $w_1Y < w_2X$; we again have three cases for the demand. When p_1 is high, the b line has the orientation shown in case 1B of Figure 2. Notice that even though the orientation of the indifference line is different from that in case 1A,

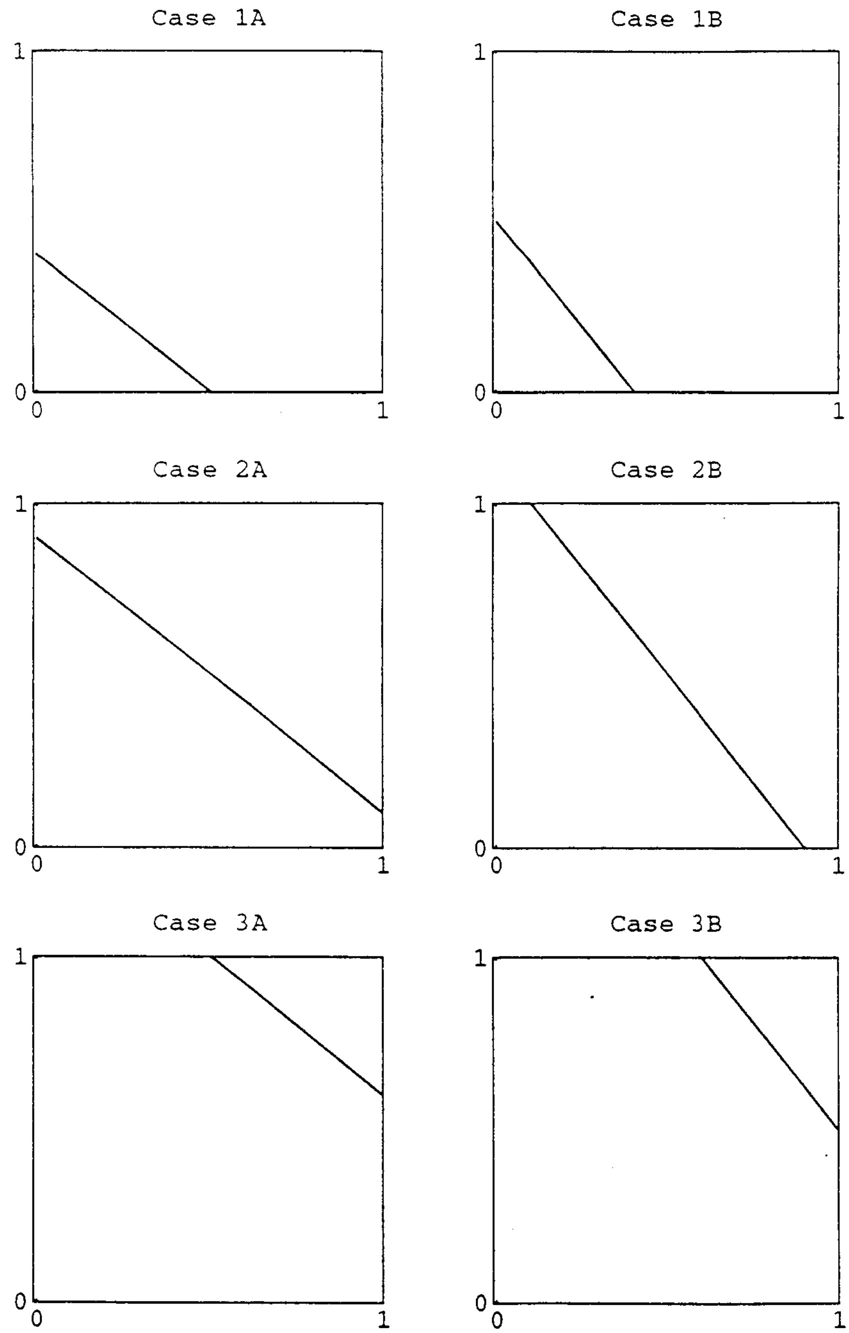


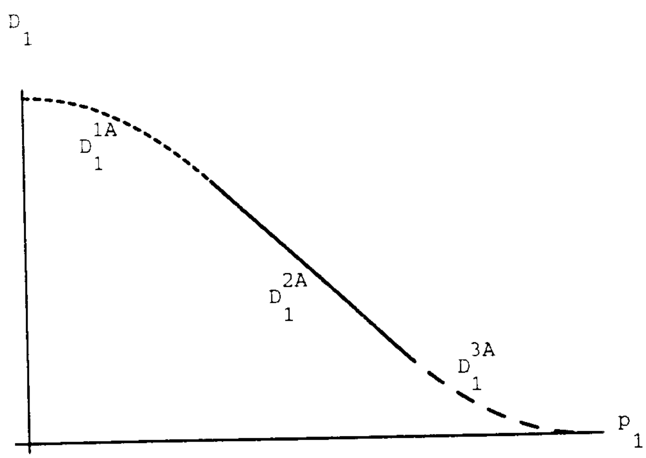
Figure 2: Market Areas for the Various Cases Under Scenarios A and B

the demand expression in case 1B is the same as in case 1A because the domain of integration remains the same.[10] When p_1 decreases, the market areas are as in case 2B of Figure 2. In case 2B, the demand expression differs from that of case 2A because the orientation of the separating line changes the domain of integration. Finally, when p_1 is low, in case 3B of Figure 2, the demand expressions are the same as in case 3A. These demand expressions and price domains are summarized below.

<u>**Case 1B**</u>: When $0 \leq (p_2 - p_1 + S) \leq 2w_2Y$, the demand of firm 1 is $D_1^{1B} = D_1^{1A}$.

<u>**Case 2B**</u>: When $2w_2Y \leq p_2 - p_1 + S \leq 2w_1X$, the demand of firm 1 is $D_1^{2B} \equiv (p_2 - p_1 + S - w_2Y)/(2w_1X)$.

<u>**Case 3B**</u>: When $2w_1X \leq (p_2 - p_1 + S) \leq 2(w_1X + w_2Y)$, the demand of firm 1 is $D_1^{3B} = D_1^{3A}$.



<u>**Figure 3**</u>**: Demand for Firm 1**

[10] However, the ranges of price p_1 for which these demand expressions hold are different across the two scenarios.

Figure 3, shows a typical demand curve for firm 1 under Scenario A. The demand equations show that the demand is continuous across the different price domains. The different segments of D_2 can also be derived in a manner analogous to that for D_1.

3.4 Price Equilibrium

In this section, we show that a unique non-cooperative price equilibrium exists for any pair of product positions (chosen by the two firms in the first stage), and we calculate the equilibrium prices.

The main step in proving existence is in establishing that each firms' profit function is quasi-concave in its own price. The concavity properties of the profit function depend upon the choice of the utility function and the distribution of consumer preferences. Caplin and Nalebuff (1991) have established twin restrictions on utility functions and preference distributions that guarantee existence of price equilibria for a number of firms with n-dimensional product specifications. Our utility function (2) is a special case of the general utility function of Caplin and Nalebuff (Assumption A1, p. 29). In addition, the uniform distribution of consumer preferences is concave and confirms with the ρ-concavity conditions employed in Caplin and Nalebuff. Hence our model satisfies assumptions A1 and A2, of Caplin and Nalebuff. Then from their Theorems 1 and 2, a price equilibrium exists in our model, for any pair of positions.

Since the profit function is twice differentiable and the distribution of preferences is concave (and therefore log-concave), we apply the uniqueness result (Proposition 6, p. 42) of Caplin and Nalebuff to ensure that the price equilibrium is unique for each pair of positions. What remains is to calculate the equilibrium prices for each pair of locations. This has to be done for each case in each scenario. The equilibrium price functions are obtained by solving the first order conditions of the profit functions $\partial\Pi_1/\partial p_1 = \partial\Pi_2/\partial p_2 = 0$.[11] We start with case 1A.

[11] Second order conditions also hold.

<u>Case 1A:</u> The demand for firm 1 is given by D_1^{1A} and for firm 2 is given by $D_2^{1A} = 1 - D_1^{1A}$. The first order conditions yield two solutions. We eliminate the one that gives negative prices. The remaining solution gives equilibrium prices,

$$p_1^{1A*} = [S + (S^2 + 32w_1w_2XY)^{1/2}]/8, \;\; p_2^{1A*} = [3(S^2 + 32w_1w_2XY)^{1/2} - 5S]/8,$$

The above equilibrium prices apply to product positions which satisfy $0 \leq p_2^{1A*} - p_1^{1A*} + S \leq 2w_1X$. Positions that result in case 1A also satisfy $w_1X \leq w_2Y$. We define the pairs of locations that satisfy these conditions (and therefore result in case 1A) as R_{1A}. R_{1A} is a subset of the four-dimensional hypercube $[0, 1]^4$. The condition $p_2^{1A*} - p_1^{1A*} + S \geq 0$ is always true, whereas $p_2^{1A*} - p_1^{1A*} + S \leq 2w_1X$ is satisfied if

(G1) $$4w_1X \geq 2w_2Y + S.$$

<u>Case 2A</u>: Equilibrium prices are:

$$p_1^{2A*} = (2w_2Y + S - w_1X)/3, \;\; p_2^{2A*} = (4w_2Y - S + w_1X)/3,$$

Substitution of equilibrium prices in the defining condition of case 2A, results in $2w_1X \leq p_2^{2A*} - p_1^{2A*} + S \leq 2w_2Y$. The LHS of this inequality is satisfied when condition (G1) fails, i.e., when $(4w_1X \leq 2w_2Y + S)$, whereas the RHS of the inequality is satisfied when

(G2) $$4w_2Y \geq 2w_1X + S.$$

These two conditions, in conjunction with $w_1X \leq w_2Y$, define the region R_{2A}, of location pairs for which p_1^{2A*} and p_2^{2A*} define an equilibrium.

<u>Case 3A:</u> Equilibrium prices are:

$$p_1^{3A*} = [5S - 10(w_1X + w_2Y) + 3N]/8, \;\; p_2^{3A*} = [2(w_1X + w_2Y) - S + N]/8$$

$$\text{where } N = [(S - 2(w_1X + w_2Y))^2 + 32w_1w_2XY]^{1/2}.$$

These equilibrium expressions apply for product positions that satisfy $2w_2Y \leq p_2^{3A*} - p_1^{3A*} + S \leq 2(w_1X + w_2Y)$. While the RHS is always true, the other condition LHS is true if condition (G2) fails. This condition, together with $w_1X \leq w_2Y$, defines the region R_{3A}.

__Case 1B:__ In this case the demand and profit expressions are the same as in case 1A. Therefore the equilibrium price expressions are identical to those of case 1A. The region of locations for which these prices form an equilibrium is now given by R_{1B} and corresponds to the region represented by (G2) in conjuction with $w_1X > w_2Y$.

__Case 2B:__ The equilibrium prices are:

$$p_1^{2B*} = (2w_1X + S - w_2Y)/3, \quad p_2^{2B*} = (4w_1X - S + w_2Y)/3$$

These form the equilibrium price pair when the product positions satisfy $2w_2Y \leq p_2^{2B*} - p_1^{2B*} + S \leq 2w_1X$. The LHS is satisfied if (G2) fails whereas the RHS is satisfied if (G1) holds.[12]

__Case 3B:__ The demand and equilibrium price and profit expressions are the same as in case 3A. The region of the hypercube for which these expressions apply is named R_{3B} and is defined by pairs of locations where (G1) fails and $w_1X > w_2Y$ holds.

It is clear from the above discussion that the six demand regions R_{1A}, R_{2A}, R_{3A}, R_{1B}, R_{2B}, R_{3B}, partition the four dimensional hypercube $[0, 1]^4$. When we fix the product location of one firm (say firm 2), these regions can be represented in two-dimensional product space. Figure 4 shows the regions of the locations of (x_1, y_1) when firm 2 is located at $(x_2, y_2) = (1/2, 1)$. The regions now show the locus of firm 1 locations that satisfy the inequalities pertaining to the six demand cases. Figure 4 shows that as firm 1 changes its location, different demand regions are encountered. Moreover, the nature of these regions, and even the existence of some of them, depends upon the ratio of weights $w = w_2/w_1$. For instance, in Figure 4, when $w = 1$, Regions 1A and 1B are never encountered. In contrast when $w = 0.6$, all six regions are encountered.

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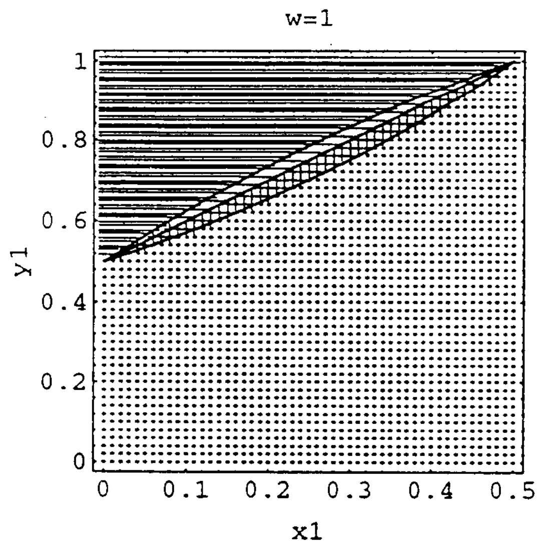
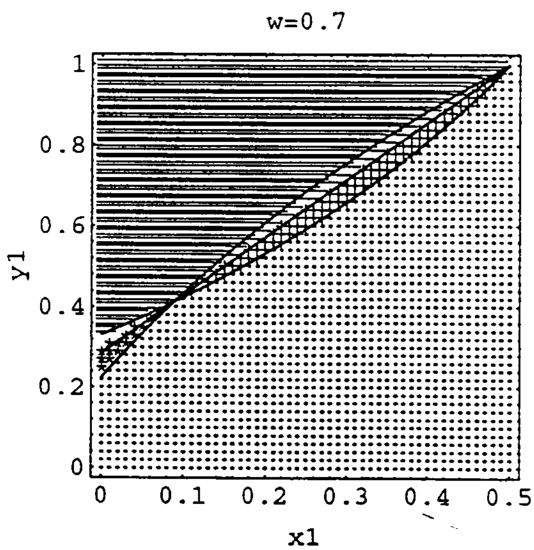
[12] Note the symmetry between the price expressions in the two scenarios.





Legend: R_{2B} is = ; R_{3B} is - ; R_{3A} is + ;
R_{2A} is . ; R_{1A} is * and R_{1B} is blank.

Figure 4: Regions of Location of Firm 1 when Firm 2 is Located at (1/2, 1).

3.5 Product Equilibria

We now establish the subgame-perfect equilibrium positions of firms. With subgame-perfection, firms anticipate the equilibrium prices in the subgames. We can write profits in the locations stage as

$$\Pi_i(x_1, y_1, x_2, y_2) \equiv \Pi_i(p_1^*(x_1, y_1, x_2, y_2), p_2^*(x_1, y_1, x_2, y_2), x_1, y_1, x_2, y_2), \quad i = 1, 2.$$

Thus, a change in location has two effects on profits: a direct effect, and an indirect effect through prices.[13]

Depending on the ratio of the weights $w \equiv w_2/w_1$, there are either one or two location equilibria (and their mirror images). At all equilibria, there is minimum differentiation in one dimension and maximum differentiation in the other. The first candidate equilibrium is $(x_1^*, y_1^*) = (1/2, 0)$, $(x_2^*, y_2^*) = (1/2, 1)$, i.e., firms are located in the middle of the horizontal segments of the box, implying minimum differentiation in x and maximum differentiation in y. We call this the "min-max" equilibrium. The second candidate equilibrium is $(x_1^{**}, y_1^{**}) = (0, 1/2)$, $(x_2^{**}, y_2^{**}) = (1, 1/2)$, i.e., firms are located at the middle points of the vertical segments of the box, implying minimum differentiation in y and maximum differentiation in x. We call this the "max-min" equilibrium. We show that, for $w < 0.406$, only the "max-min" (second) equilibrium exists; for $0.406 \leq w \leq 2.463 = 1/0.406$ both equilibria exist; and, for $2.463 < w$, only the "min-max" (first) equilibrium exists.

The method of our proof is as follows. Suppose that firm 2 is located at $(x_2^*, y_2^*) = (1/2, 1)$. We identify the direction in which profits of firm 1 increase as its location changes by calculating the (vector) gradient of profits, $\mathbf{D}\Pi_1^t$. We do this by evaluating analytic expressions

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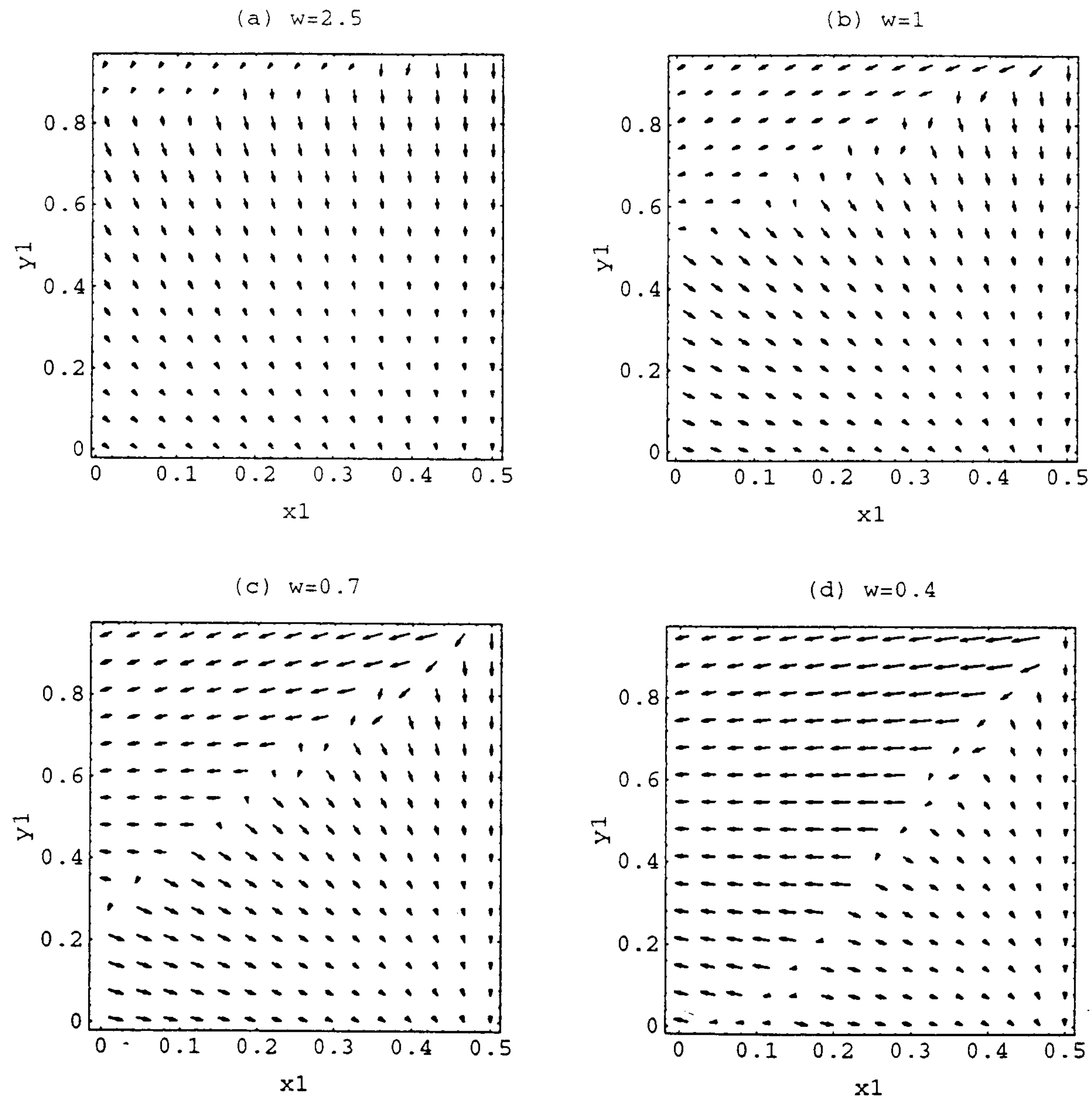
[13] Essentially the indirect effect is through the price of the opponent: $d\Pi_i^t/dx_i = \partial\Pi_i/\partial x_i + (\partial\Pi_i/\partial p_i)(dp_i^*/dx_i) + (\partial\Pi_i/\partial p_j)(dp_j^*/dx_i) = \partial\Pi_i/\partial x_i + (\partial\Pi_i/\partial p_j)(dp_j^*/dx_i)$, since $\partial\Pi_i/\partial p_i = 0$ at the price subgame.



<u>Figure 5</u>: Gradient of Profits for Firm 1

for $\mathbf{D}\Pi_1^{\ell}$.[14] We identify locations for firm 1 where the profit gradient is zero (critical points of the gradient), and determine if each point is a local maximum, minimum, or saddle point (see Figure 5). We find that, for $w \geq 1$, there is only one local maximum, at (1/2, 0); therefore it is also a ***global*** maximum (in x_1 and y_1) of the profit function Π_1^{ℓ}. It follows that, for $w \geq 1$, (1/2, 0) is the best response to (1/2, 1). Therefore, for this range of w, "min-max" is an equilibrium.

Given that firm 2 is located at $(x_2^*, y_2^*) = (1/2, 1)$, for $w < 1$, there are two local maxima of firm 1's profits, at (1/2, 0) and (0, 1/2). Let the "middle location" profits of firm 1 be $\Pi_1^{\ell}(M) \equiv \Pi_1(1/2, 0, 1/2, 1)$, and the "left" profits be $\Pi_1(L) \equiv \Pi_1(0, 1/2, 1/2, 1)$. We show the comparison of $\Pi_1(M)$ and $\Pi_1(L)$ in Figure 5. For $1 > w > 0.406$, $\Pi_1(M) > \Pi_1(L)$;[15] therefore $(x_1^*, y_1^*) = (1/2, 0)$ is the (global) best reply of firm 1 to $(x_2^*, y_2^*) = (1/2, 1)$. For the same w range, by symmetry with respect to the horizontal axis through (1/2, 1/2), $(x_2^*, y_2^*) = (1/2, 1)$ is the global best reply to $(x_1^*, y_1^*) = (1/2, 0)$; therefore $(x_1^*, y_1^*) = (1/2, 0)$, $(x_2^*, y_2^*) = (1/2, 1)$, i.e., "min-max", is a subgame-perfect equilibrium.

For $w < 0.406$, $\Pi_1^{\ell}(M) < \Pi_1^{\ell}(L)$; therefore (0, 1/2) is the best reply to (1/2, 1). However, (1/2, 1) is ***not*** the best reply (0, 1/2). This is established as follows. Let firm 1 be at (0, 1/2). The problem for the choice of location seen from the point of view of firm 2 is symmetric to the problem of firm 1 (that has been discussed so far) with axis of symmetry the negative diagonal of the square. Further, from the point of view of firm 2, the relative weights are $w' = w_1/w_2 = 1/w$. Thus, in this case, $w < 0.406 \Leftrightarrow w' > 2.463 > 1$. It follows from the previous arguments that (1, 1/2) is a global best reply to (0, 1/2). Since (1/2, 1) is not the best reply to (0, 1/2), the first candidate equilibrium (min-max") is ***not*** an equilibrium. Using

[14] These expressions differ across the various R regions of location pairs. The expressions as well as the Mathematica evaluation and plotting routines are available from the authors.

[15] For $w < 1$, $\Pi_1(M) = ww_1/2$, and $\Pi_1(L) = (5 + w)^2 w_1/144$, so that $\Pi_1(M) = \Pi_1(L) \Leftrightarrow w = 31 - 6\sqrt{26} \approx 0.406$.

the same arguments as earlier, and appealing to symmetry, we have that, for $w' = w_1/w_2 = 1/w > 0.406$, the second candidate equilibrium ("max-min") is an equilibrium.

Putting the conditions together, it follows that **for $w < 0.406$ only the "max-min" equilibrium $[(x_1^{**}, y_1^{**}) = (0, 1/2), (x_2^{**}, y_2^{**}) = (1, 1/2)]$ exists; for $w > 1/0.406 = 2.463$, only the "min-max" equilibrium $[(x_1^{*}, y_1^{*}) = (1/2, 0), (x_2^{*}, y_2^{*}) = (1/2, 1)]$** exists; and **for $0.406 < w < 2.463$, both the "max-min" and the "min-max" equilibria exist.** Remembering the definition of w, $w = w_1/w_2$, note that the "min-max" equilibrium exists when w_2 is relatively large, and similarly, the "max-min" equilibrium exists when w_1 is relatively large. When w_1 and w_2 are roughly of similar magnitude, both equilibria exist. When one weight is much larger than the other, there is only one equilibrium where maximal differentiation occurs in the dimension that corresponds to the high weight.

At both equilibria, the firms share the market equally. At the first equilibrium, prices for both brands are $p_1 = p_2 = w_2$ and profits are $\Pi_1 = \Pi_2 = w_2/2$; at the second equilibrium, prices are $p_1 = p_2 = w_1$ while profits are $\Pi_1 = \Pi_2 = w_1/2$.

Interestingly, positions implying maximal differentiation on both attributes ("max-max") are ***not*** equilibrium positions. Even though both firms have profits equal to the equilibrium profits when they are maximally differentiated on both attributes, such a pair of positions is not an equilibrium. Given that its opponent has located at the corner of the square, a firm has a unilateral incentive to deviate from such a position and move inward.[16]

At this stage it is natural to ask whether this pattern of equilibria generalizes to higher dimensions. In higher dimensions, will competitors differentiate on more than one attribute or continue to differentiate only on one attribute? Will we continue to get just two equilibria or will

[16] In a way, this is similar to locational incentives in the one-dimensional linear transportation cost model of Hotelling (1929). In that model, profits were equal for any symmetric locations, but each firm had a unilateral incentive to move toward the other.

the number of equilibria depend on the dimensionality of the product space? To answer these questions, we analyze a three-dimensional market in the next section.

4. Three Dimensional Model

4.1 Demand Formulation

In three dimensions, the space of attributes is a unit cube. A product position is denoted by a triple $\theta_i = (x_i, y_i, z_i)$ and an ideal point is denoted by $A_j = (a, b, c)$. We continue to assume that $x_2 \geq x_1$, $y_2 \geq y_1$, and $z_2 \geq z_1$, and that the attribute weights w_1, w_2, w_3, are constant across consumers. Thus the utility of consumer A_j when he buys one unit of product θ_i is

$$U_i(a, b, c; x_i, y_i, z_i) = I - w_1(a - x_i)^2 - w_2(b - y_i)^2 - w_3(c - z_i)^2 - p_i; \quad i = 1, 2.$$

The market areas are given by three-dimensional regions of the cube separated by a plane, rather than two-dimensional regions of a square separated by a line as in the two-dimensional case we discussed earlier. The locus of consumers on this plane, who are indifferent between buying from either firm, is given by

$$c(a, b) = [p_2 - p_1 + S - 2aw_1X - 2bw_2Y]/(2w_3Z)$$

where $S = w_1(x_2^2 - x_1^2) + w_2(y_2^2 - y_1^2) + w_3(z_2^2 - z_1^2)$, $X = x_2 - x_1$, $Y = y_2 - y_1$, $Z = z_2 - z_1$, and $(a, b) \in [0, 1] \times [0, 1]$ are the coordinates of the consumer in the first two dimensions. The region below (above) the plane is composed of customers of product 1 (2). As before, when firm 1 decreases its price p_1 (or firm 2 increases p_2), the plane shifts upwards, thereby increasing firm 1's demand D_1. As in the two dimensional model, the expression for demand depends upon the position and orientation of the plane.

4.2 Price Equilibria

The remainder of our analysis parallels the two-dimensional model. The only real difference is the level of complexity. Because of this complexity, we only outline the analysis here. Complete details are available from the authors upon request.

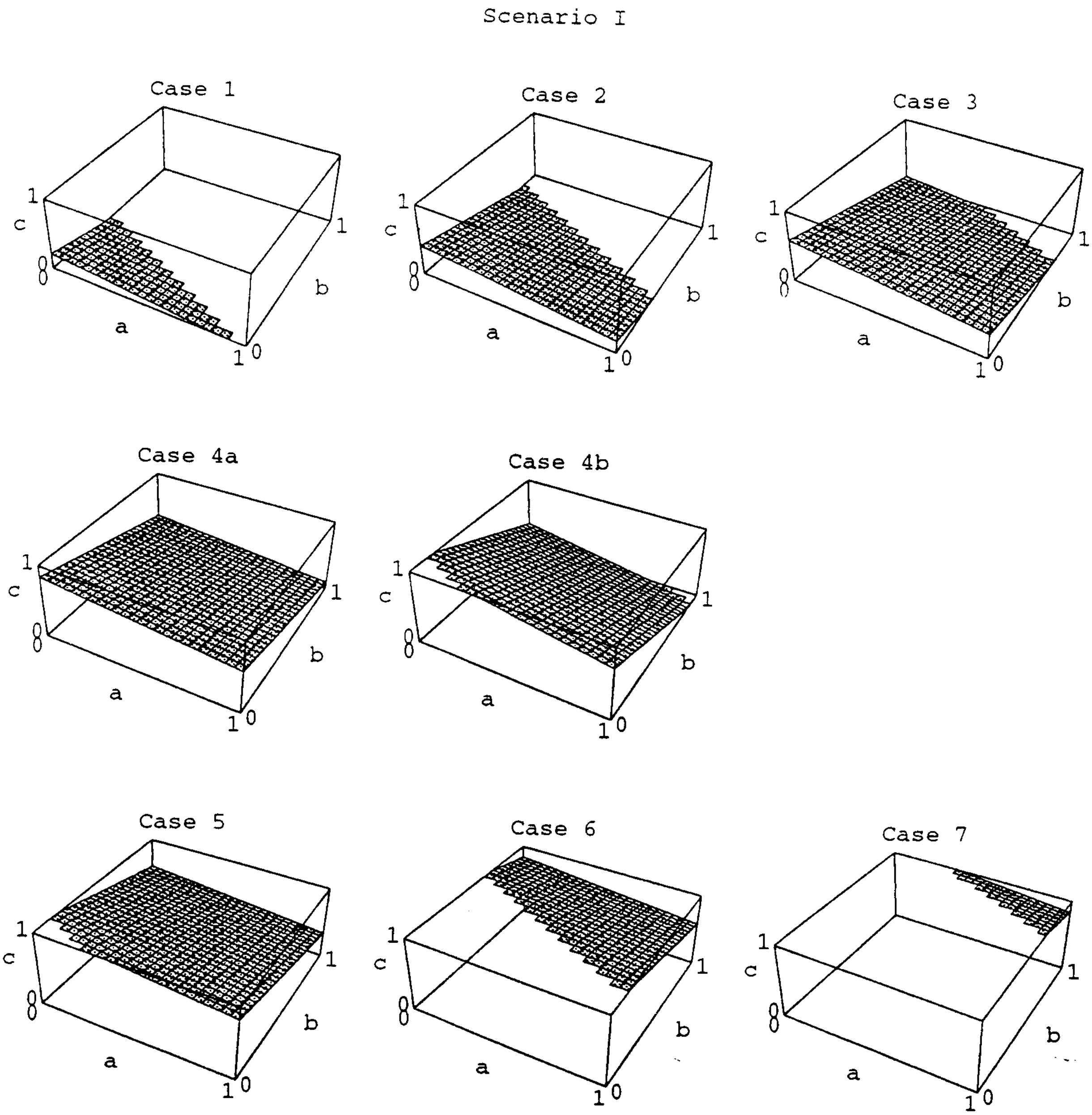
Whereas there were only two scenarios (A and B) in the two-dimensional model, there are twelve distinct scenarios in three dimensions. Different positions of the two brands define different orientations of the plane and intersections of the cube, thus defining the various scenarios. The twelve scenarios are defined in detail in Appendix 1. For each scenario, there are seven cases (instead of three) and corresponding demand expressions. Typical cases are shown in Figure 6. As firm 1 decreases its price p_1, the plane that separates the market areas (more precisely the market volumes) moves up. A new case arises whenever the plane cuts a corner of the cube. In each case, price equilibria are computed for each potential pair of product positions. In some of the cases, this is impossible to do analytically. Hence, we resort to numerical methods. In particular, we use iterative root-finding algorithms to find the roots of the systems of pairs of cubic equations that emerge from the first order conditions.[17]

4.3 Product Equilibria

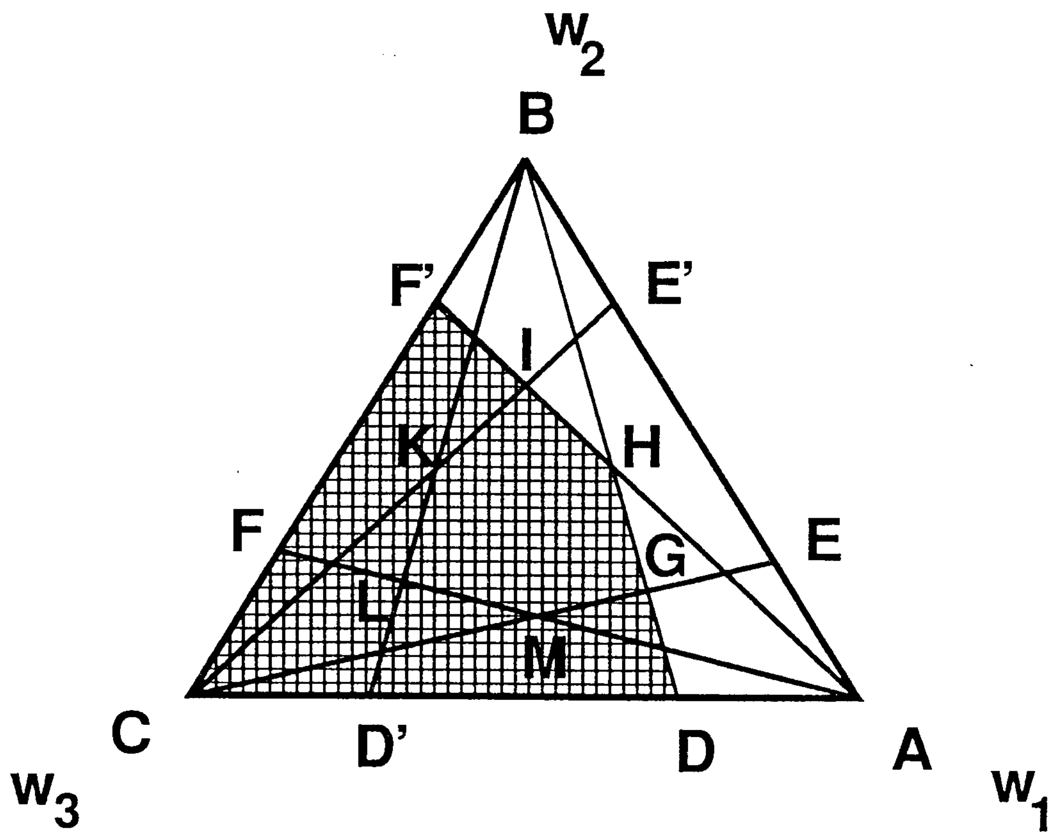
Using a procedure similar to that used in analyzing the two-dimensional model (outlined in detail in Appendix 2), we identified three position equilibria. In each equilibrium configuration, the two brands are differentiated on only one attribute and occupy central positions on the other two attributes. Specifically,

1. When $w_3/w_1 \geq 0.406$ and $w_3/w_2 \geq 0.406$, the equilibrium positions are $\theta_1^* = (1/2, 1/2, 0)$, $\theta_1^* = (1/2, 1/2, 1)$, i.e., "min-min-max".

[17] The entire procedure was programmed in *Mathematica*.



<u>Figure 6</u>: Market Areas (Volumes) in Three Dimensions



<u>Figure 7</u>: **Regions of Existence of Equilibria in the Three-Dimensional Simplex**

2. When $w_2/w_1 \geq 0.406$ and $w_2/w_3 \geq 0.406$, , the equilibrium positions are $\theta_1^* = (1/2, 0, 1/2)$, $\theta_1^* = (1/2, 1, 1/2)$, i.e., "min-max-min".

3. When $w_1/w_2 \geq 0.406$ and $w_1/w_3 \geq 0.406$, , the equilibrium positions are $\theta_1^* = (0, 1/2, 1/2)$, $\theta_1^* = (1, 1/2, 1/2)$, i.e., "max-min-min".

The regions of existence of these equilibria can easily illustrated on the three-dimensional simplex in Figure 7, where $w_1 + w_2 + w_3 = 1$, $w_1, w_2, w_3 \geq 0$.[18] On segment AC define the points D and D′ such that (AD)/(DC) = (CD′)/(AD′) = 0.406, with similar definitions of E, E′, F, and F′ on segments AB and BC. The region of the weights $\mathbf{w} = (w_1, w_2, w_3) \in$ (CDHF′) that leads to a "min-min-max" equilibrium is shaded. Similarly, $\mathbf{w} \in$ (AE′KD′) leads to a "max-min-min" equilibrium, and $\mathbf{w} \in$ (BFME) leads to a "min-max-min" equilibrium. Notice that, roughly speaking, each equilibrium has maximal differentiation in the dimension that corresponds to the highest weight. Further, when the weights are roughly similar and fall in the central hexagon (MGHIKL), all three equilibria exist. In regions where two weights are high but the third weight is low, two equilibria exist, each with maximal differentiation in the dimension that corresponds one of the two high weights. For example, for $\mathbf{w} \in$ (DGMLD′), "min-min-max" and "max-min-min" are both equilibria. In regions where only one weight is large (close to the vertices) only one equilibrium exists -- the one that differentiates maximally in the dimension the large weight. For example, for $\mathbf{w} \in$ (CD′LF) only the "max-min-min" equilibrium exists.

At each equilibrium, both firms charge equal prices and share the market equally. This pattern of equilibrium positions confirms our understanding that in multidimensional spaces, firms seek to differentiate their offerings on one dimension only in order to reduce the impact of price

[18] The weights are, in general unrestricted. For a point $\mathbf{w}$ where the weights do not add up to 1, draw the ray from the origin to that point and project it to $\mathbf{w}'$ on the simplex. It is clear that the equilibrium existence properties of $\mathbf{w}$ and $\mathbf{w}'$ are identical. Thus, the description of the existence areas on the simplex is sufficient. Of course, the equivalence of the existence properties of $\mathbf{w}$ and $\mathbf{w}'$ arises fundamentally from the fact that the utility function allows for a normalization of its parameters, and the linear restriction $w_1 + w_2 + w_3 = 1$ is a permissible normalization.

competition.[19] Once products are differentiated maximally in one dimension, firms assume identical (central) positions on the other attributes.[20]

5. Conclusion

In this paper we have examined product positioning and pricing in a multi-attribute framework. We derived subgame-perfect equilibrium positions and associated prices for a duopoly. In one dimension, maximal differentiation holds as shown in D'Aspremont *et al.* (1979). We find that, in two dimensions, there are two equilibria when all consumers consider the two attributes as equally important. In each of these equilibria, firms are maximally differentiated on one attribute and minimally differentiated on other. Moreover, when firms are minimally differentiated on one attribute, they occupy central positions on that attribute. We also find that when attributes are differentially weighted by the consumers, so that one attribute has significantly greater importance than the other, only a single equilibrium remains. In this equilibrium firms maximally differentiate on the more important attribute and occupy central positions on the other attribute.

In moving from two to three dimensions we showed that the essential character of the equilibrium does not change. In particular, at the three-dimensional equilibrium, firms are maximally differentiated on one dimension only. In three dimensions, depending on the importance that consumers place in each attribute, there is one, two, or three equilibria. In each equilibrium, firms are maximally differentiated on one attribute and minimally differentiated on the other two. An equilibrium with maximal differentiation in a certain dimension occurs when consumers place sufficient importance to the corresponding attribute. Thus, if consumers place importance only on the first attribute, the equilibrium is "max-min-min," i.e., it has maximal

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[19] This idea was suggested to us independently as a conjecture by Jacques Thisse.

[20] Finally, we must note that we have not shown that these are the only locational equilibria. However, we were unable to locate any other equilibrium despite extensive search.

differentiation in the first dimension only. When consumers place importance on the second attribute as well, the "min-max-min" equilibrium occurs too. Further, when consumers place importance on the third attribute as well, the "min-min-max" equilibrium occurs in addition to the other two. Thus, for example, when all attributes are weighted equally, all three equilibria ("max-min-min," "min-max-min," and "min-min-max") exist.

An important aspect of our results is the multiplicity of equilibria in both the two and the three-dimensional models. When consumers value all attributes roughly equally, all locational n-tuples with maximal differentiation in one dimension and minimal differentiation in all others, are equilibria. As more weight is put on a particular dimension, equilibria get eliminated one by one until we reach a unique equilibrium. This shows that advertising can have a very important role in eliminating certain equilibria, even if the effect of advertising on the underlying preferences is marginal.

There are a number of directions in which these results can be extended. First, there is the obvious extension to higher dimensional spaces. Are the equilibrium locations of a n-dimensional attribute spaces only differentiated in one dimension? Second, how do the locational results fare when there are more than two competitors? Third, what for what classes of distributions can we extend our duopoly positioning results? All these are very interesting questions that we leave for further research.

References

Anderson, Simon P., Andre de Palma, and Jacques-Francois Thisse, (1993) *Discrete Models of Product Differentiation*, MIT Press.

Ansari, Asim, Nicholas Economides, and Avijit Ghosh (1994), "The Defender Model with Non-uniform Preferences," *Marketing Science*.

Caplin, Andrew, and Barry Nalebuff (1991a), "Aggregation and Imperfect Competition: On the Existence of Equilibrium," *Econometrica*, vol. 59, no. 1, pp. 25-59.

D'Aspremont, Claude, Jaskold-Gabszewicz, Jean, and Thisse, Jacques-Francois, (1979), "On Hotelling's 'Stability in Competition'," *Econometrica*, vol. 47, pp. 1145-1150.

Economides, Nicholas, (1984), "The Principle of Minimum Differentiation Revisited," *European Economic Review*, vol. 24, pp. 345-368.

Economides, Nicholas, (1986a), "Nash Equilibrium Existence in Duopoly with Products Defined by Two Characteristics," *Rand Journal of Economics*, vol. 17, no. 3, pp. 431-439.

Economides, Nicholas, (1986b), "Minimal and Maximal Product Differentiation in Hotelling"s Duopoly," *Economics Letters*, vol 21, pp. 67-71.

Economides, Nicholas, (1989a), "Quality Variations and Maximal Variety Differentiation," *Regional Science and Urban Economics*, vol. 19, pp. 21-29.

Economides, Nicholas, (1989b), "Symmetric Equilibrium Existence and Optimality in Differentiated Products Markets," *Journal of Economic Theory*, vol. 47. no. 1, pp. 178-194.

Economides, Nicholas, (1993), "Quality Variations in the Circular Model of Variety Differentiated Products," *Regional Science and Urban Economics*, vol. 23, no. 2, pp. 235-257.

Green, Paul E. and V. Srinivasan, (1978), "Conjoint Analysis in Consumer Research: Issues and Outlook," *Journal of Consumer Research*, vol. 5, pp. 103-123.

Hauser, John R. (1988), "Competitive Price and Positioning Strategies," *Marketing Science*, vol. 7, pp. 76-91.

Hauser, John R., and M. Shugan, (1983), "Defensive Marketing Strategy," *Marketing Science*, vol. 2, no. 4, pp. 319-360.

Hotelling, Harold, (1929), "Stability in Competition," *Economic Journal*, vol. 39, pp. 41-57.

Kumar, R. V., and D. Sudarshan (1988), "Defensive Marketing Strategies: An Equilibrium Analysis Based on Decoupled Response Function Models," *Management Science*, vol. 34, 7, pp. 805-815.

Lane, W. J., (1980), "Product Differentiation in a Market with Endogenous Sequential Entry," *Bell Journal of Economics*, vol. 11, no. 1, pp. 237-260.

Moorthy, Sridhar K., (1986), "Product and Price Competition in a Duopoly," *Marketing Science*, vol. 5, 3, pp. 141-168.

Neven, Damien, (1985), "Two-stage (Perfect) Equilibrium in Hotelling"s Model," *Journal of Industrial Economics*, vol. 33, 317-326.

Neven, Damien, and Jacques-Francois Thisse, "On Quality and Variety Competition," in *Economic Decision-Making: Games, Econometrics and Optimization*, eds. J.J. Gabszewicz, J.-F. Richard and L.A. Wolsey, Amsterdam: North-Holland, 1990, pp. 175-199.

Rao, Vithala R. and Joel H. Steckel (1994), "A Cross-Cultural Analysis of Price Response to Environmental Changes," *Marketing Letters*, forthcoming.

Salop, Steven, C., (1979b), "Monopolistic Competition with Outside Goods," *Bell Journal of Economics*, vol. 10, pp. 141-156.

Tabuchi, Takatoshi, (1994), "Two-Stage Two Dimensional Spatial Competition between Two Firms," *Regional Science and Urban Economics*, vol. 24, pp. 207-227.

Vandenbosch, Mark B., and Charles B. Weinberg, (1994), "Product and Price Competition in a Two-Dimensional Vertical Differentiation Model," forthcoming, *Marketing Science*

Vickrey, William S. (1964), *Microstatics*. New York: Harcourt, Brace and World.

Appendix 1: Demand Definitions for the Three-Dimensional Model

We first document all twelve scenarios. The defining features are:

Scenario 1A: $w_1X \le w_2Y \le (w_1X + w_2Y) \le w_3Z$

Scenario 1B: $w_1X \le w_2Y \le w_3Z \le (w_1X + w_2Y)$

Scenario 2A: $w_1X \le w_3Z \le (w_1X + w_3Z) \le w_2Y$

Scenario 2B: $w_1X \le w_3Z \le w_2Y \le (w_1X + w_3Z)$

Scenario 3A: $w_2Y \le w_1X \le (w_1X + w_2Y) \le w_3Z$

Scenario 3B: $w_2Y \le w_1X \le w_3Z \le (w_1X + w_2Y)$

Scenario 4A: $w_2Y \le w_3Z \le (w_2Y + w_3Z) \le w_1X$

Scenario 4B: $w_2Y \le w_3Z \le w_1X \le (w_2Y + w_3Z)$

Scenario 5A: $w_3Z \le w_1X \le (w_1X + w_3Z) \le w_2Y$

Scenario 5B: $w_3Z \le w_1X \le w_2Y \le (w_1X + w_3Z)$

Scenario 6A: $w_3Z \le w_2Y \le (w_2Y + w_3Z) \le w_1X$

Scenario 6B: $w_3Z \le w_2Y \le w_1X \le (w_2Y + w_3Z)$

While each scenario in the two attributes model generates three cases, each of the above twelve scenarios generates seven cases. In naming the resulting demand expressions, we follow the conventions developed in the two dimensional model.

We start with Scenario IA. For any fixed price p_2, if p_1, is such that $p_2 - p_1 + S \le 0$, then firm 1 has no demand. In order to calculate the demand segment D_1^{1A}, we need to define the region of integration. The separating plane intersects the (a, b) plane in a straight line given by *blimit* = $(p_2 - p_1 + S - 2aw_1X)/(2w_2Y)$. This straight line, *blimit*, further intersects the *a* axis at point *aint* = $(p_2 - p_1 + S)/(2w_1X)$. Now, when p_1 is reduced so that $0 \le L \le 2X$, where L = $(p_2 - p_1 + S)$, the separating plane intersects all three axes as in case 1, Figure 6, and we have

Case 1A: when $0 \le L \le 2w_1X$, demand for firm 1 is

$$D_1^{1A} = \int_0^{aint} \int_0^{blimit} c(a, b)\, db\, da = L^3/H,$$

where $H = 48w_1w_2w_3XYZ$.

When p_1 is further decreased, the separating plane passes through the corner (1, 0, 0) of the product space and as shown in case 2, Figure 6, and we have

Case 2A: when $2w_1X \leq L \leq 2w_2Y$, demand is

$$D_1^{2A} = \int_0^1 \int_0^{blimit} c(a, b)\, db\, da = D_1^{1A} - (L - w_1X)^3/H$$

When p_1 is further reduced, the plane while moving up crosses the corner (0, 1, 0) and we have

Case 3A: when $2w_2Y \leq L \leq 2(w_1X + w_2Y)$,

$$D_1^{3A} = \int_0^{aint} \int_0^1 c(b, a)\, db\, da + \int_{aint1}^1 \int_0^{blimit} c(b, a)\, db\, da = D_1^{2A} - (L - 2w_2Y)^3/H,$$

where $aint1 = (L - 2w_2Y)/(2w_1X)$ is the intercept of the *blimit* line with the line $b = 1$.

On further reduction in p_1, we have case 4a, Figure 7, where the indifference plane intersects the vertical faces of the unit cube.

Case 4A: when $2(w_1X + w_2Y) \leq L \leq 2w_3Z$, we have

$$D_1^{4A} = \int_0^1 \int_0^1 c(b, a)\, db\, da = D_1^{3A} + (L - 2(w_1X + w_2Y))^3/H,$$

which simplifies further to a linear function in p_1 given by

$$D_1^{4A} = (L - w_1X - w_2Y)/(2w_3Z).$$

When p_1 is further reduced so that the separating plane moves past the (0, 0, 1) corner, we have

Case 5A: when $2w_3Z \leq L \leq 2(w_1X+w_3Z)$,

$$D_1^{5A} = \int_0^{aint2} \int_0^{blimit1} db\, da + \int_0^{aint2} \int_{blimit1}^1 c(b, a)\, db\, da + \int_{aint2}^1 \int_0^1 c(b, a)\, db\, da$$

where *blimit1* = $(L - 2aw_1X - 2w_3Z)/(2w_2Y)$ is the line of intersection of $c(a, b)$ with the plane $c = 1$. *aint2* is obtained by substituting $b = 0$ in *blimit1*. Hence, *aint2* = $(L - 2w_3Z)/(2w_1X)$, whereas, *bint2* = $(L - 2w_3Z)/(2w_2Y)$, is obtained by setting $a = 0$ in *blimit1*. The demand expression then is

$$D_1^{5A} = D_1^{4A} - (L - 2w_3Z)^3/H$$

Next, on further reduction in p_1, the plane passes past (1, 0, 1) and we get:

Case 6A: when p_1 satisfies $2(w_1X + w_3Z) \le L \le 2(w_2Y + w_3Z)$, we have

$$D_1^{6A} = \int_0^1 \int_0^{blimit1} db\, da + \int_0^1 \int_{blimit1}^1 c(b, a)\, db\, da$$

which reduces to

$$D_1^{6A} = D_1^{5A} + (L - 2(w_1X + w_2Y))^3/H.$$

Finally, we calculate the point of intersection of *blimit1* with the line $c = 1$ and $b = 1$ to get *aint3* = $(L - 2(w_2Y + w_3Z))/(2w_1X)$. Now, as shown in case 7, Figure 6, we have

Case 7A: when $2(w_2Y + w_3Z) \le L \le 2(w_1X + w_2Y + w_3Z)$, the demand is given by

$$D_1^{7A} = \int_0^{aint3} \int_0^1 db\, da + \int_{aint3}^1 \int_0^{blimit1} db\, da + \int_{aint3}^1 \int_{blimit1}^1 c(b, a)\, db\, da.$$

This reduces to

$$D_1^{7A} = D_1^{6A} + (L - 2(w_2Y + w_3Z))^3/H.$$

This completes Scenario 1A. We next describe the demand expressions and regions pertaining to Scenario 1B.

Scenario 1B.

Case 1B: When $0 \le L \le 2w_1X$, demand for firm 1 is $D_1^{1B} = D_1^{1A}$

Case 2B: When $2w_1X \le L \le 2w_2Y$, demand for firm 1 is $D_1^{2B} = D_1^{2A}$

Case 3B: When $2w_2Y \le L \le 2w_3Z$, demand is $D_1^{3B} = D_1^{3A}$.

Case 4B: When $2w_3Z \leq L \leq 2(w_1X + w_2Y)$, $D_1^{4B} = D_1^{3A} - (L - 2w_3Z))^3/H$.

Case 5B: When $2(w_1X + w_2Y) \leq L \leq 2(w_1X + w_3Z)$, $D_1^{5B} = D_1^{5A}$

Case 6B: When $2(w_1X + w_3Z) \leq L \leq 2(w_2Y + w_3Z)$, we have $D_1^{6B} = D_1^{6A}$, and

Case 7B: When $2(w_2Y + w_3Z) \leq L \leq 2(w_1X + w_2Y + w_3Z)$, $D_1^{7B} = D_1^{7A}$.

As is evident from the above, the two scenarios differ across only one demand expression. However, the regions of the product space associated with cases 3, 4, and 5, are different across the two scenarios.

We now show how the remaining scenarios can be obtained from the two that were analyzed above. We define transformation rules that we use on the above derived demand expressions and price inequalities so as to obtain the corresponding expressions in the other scenarios. The transformation rules are as follows:

rep2 = $(w_1X \to w_1X, w_2Y \to w_3Z, w_3Z \to w_2Y)$; rep3 = $(w_1X \to w_2Y, w_2Y \to w_1X, w_3Z \to w_3Z)$; rep4 = $(w_1X \to w_2Y, w_2Y \to w_3Z, w_3Z \to w_1X)$; rep5 = $(w_1X \to w_3Z, w_2Y \to w_1X, w_3Z \to w_2Y)$; and rep6 = $(w_1X \to w_3Z, w_2Y \to w_2Y, w_3Z \to w_1X)$.

These rules work as follows. In order to obtain the seven cases of demand and the associated price domains for Scenario 2A, we apply rep2 on the corresponding demand expressions and price domains of Scenario 1A. For example, the demand expression for case 1 of Scenario 2A can be obtained by simultaneously substituting in D_1^{1A} above, w_1X in place of w_1X, w_3Z in place of w_2Y, and w_2Y in place of w_3Z. These replacement rules follow from the geometric symmetry associated with the sides of the unit cube. Similarly, Scenario 2B can be analyzed by applying rep2 on the corresponding expressions of Scenario 1B. The other scenarios can be analyzed in an analogous manner.

Appendix 2: Numerical Determination of Product Equilibrium in Three Dimensions

Step 1: Initialize $j = 1$

Step 2: Fix $\theta_2 = (x_2, y_2, z_2)$, the position of brand 2 to some initial triple $\theta_2^{(j)} \in [0, 1]^3$

Step 3: Vary the position of brand 1 from $(0, 0, 0)$ to $\theta_2 - \varepsilon$, on a three dimensional grid.

Step 4: For each pair of positions,

a. Identify the scenario that is true.

b. Determine the price solutions for each of the seven cases associated with the scenario.

c. Check the boundary conditions of each case and identify the case that yields the equilibrium prices.

d. Calculate profits based on the equilibrium prices.

Step 5: Identify the position of brand 1 that maximizes profits for that brand. Call this position $\theta_1 = \theta_1^j$.

Step 6: Fix θ_1 at θ_1^j. Vary the position of brand 2 from $\theta_1^j + \varepsilon$ to $(1, 1, 1)$ on a three dimensional grid and invoke step 4.

Step 7: Identify the position for brand 2 that maximizes profit for that brand. Call this $\theta_2 = \theta_2^{j+1}$, and fix the position of brand 2 at this point.

Step 8: Increment j by 1 and repeat the steps 3-7. Stop if $\theta_1^{j+1} = \theta_1^j$ and $\theta_2^{j+1} = \theta_2^j$. Otherwise, return to Step 3.

The entire algorithm was programmed in *Mathematica*.