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Revenue Management of Consumer Options for Sporting Events

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Abstract

Some event managers and ticket resellers offer call options under which a customer can pay a small amount now for the guaranteed option to attend a future sporting event by paying an additional amount later. We consider the case of *tournament options* in which the identity of the two teams playing in a tournament final such as the Super Bowl or the World Cup final are unknown at the time that options are sold. We develop an approach by which an event manager can determine the revenue maximizing prices and amounts of advance tickets and options to sell for a tournament final. We show that, under certain conditions, offering options will increase expected revenue for the event and can increase social welfare. We present a numerical application of our approach to the 2012 Super Bowl.

1 Introduction

The World Cup final, the Super Bowl, and the final game of the NCAA Basketball Tournament in the United States (a.k.a. "March Madness") are among the most popular sporting events in the world. Typically, demand exceeds supply for the tickets for these events, even when the tickets cost hundreds of dollars. However, since these events are the final games of a tournament, the identities of the two teams who will be facing each other are typically not known until shortly before the event. For example, the identity of the two teams who faced each other in the 2010 World Cup final was determined only after the completion of the two semi-final games, five days prior to the final. Yet, tickets for the World Cup Final are offered for sale many months in advance. While there may be many fans who are eager to attend the final game no matter who plays, many fans would only be interested in attending if their favored team – say Germany – were playing in the final. These fans face a dilemma. If they purchase an advanced ticket, and Germany does not advance to the final, then they have potentially wasted the price of the ticket. On the other hand, tickets are likely to be sold out well before it is known who will be playing in the finals, so if they wait, they may be unable to attend at all. In response to this dilemma, some sporting events have begun to offer "ticket options" in which a fan can pay a small amount up front for the right to purchase a seat later once the identity of the teams playing is known. Essentially, this is a call option by which the fan can limit her cost should her team not make the final while guaranteeing a seat if her team does make the final. In this paper, we address the revenue management problem faced by the event manager of a tournament final who has the opportunity to offer options for the final. We examine when it is most profitable to offer options to consumers and how the manager should set prices and availabilities for both the advance tickets and the options. We also address the social welfare implications of offering options.

Over the past five years, a number of events and third-parties have begun to offer call options for sporting event tickets. For example, the Rose Bowl is an annual post-season event in which two American college football teams are chosen to play against each other based on their records during the regular season. The identity of the teams playing is not known until a few weeks prior to the event, however, the Rose Bowl sells tickets many months in advance. In addition to general "advance tickets", the Rose Bowl also sells "Team Specific Reservations". As described on the Rose Bowl's web-site (http://teamreserve.tournamentofroses.com/markets/sports/college-fb/event/2011-rose-bowl):

> One Team Specific Ticket Reservation guarantees one face value ticket if your team makes it to the 2011 Rose Bowl. Face value cost is a charge over and above the price you pay for your Team Specific Ticket Reservation. If your team doesn't make it to the Game, there are no refunds for your purchased Team Specific Ticket Reservations, and tickets will not be provided.

Offering ticket options has become so popular that there is a software company, TTR that specializes in selling Internet platforms to teams and events that wish to offer options. In addition, at least one web site, www.OptionIT.com offers options for a variety of sporting events.

While options can be offered for any sporting event, in this paper, we consider the case of *tournament options*, which are sold for a future event in which the two opponents who will face each other are *ex ante* unknown. We assume that there are potential customers – "fans" – whose utility of attending the game is dependent upon whether or not their favored team is playing. In this case, the tournament option enables a fan to hedge against the possibility that her favored team is not selected to play in the game of interest – e.g. the World Cup final.

While we derive results that are applicable to more general tournament structures, we pay particular attention to *dyadic* tournaments. In a dyadic tournament, the teams can always be partitioned into two sets, $\mathcal{T}_1$ and $\mathcal{T}_2$ so that the final will feature a team from $\mathcal{T}_1$ facing a team in $\mathcal{T}_2$. The most common example of a dyadic tournament is a single-elimination tournament in which 2^n teams play each other at each stage with $1/2$ of the teams being eliminated until the last remaining two teams play in the final. Another example of a dyadic tournament occurs when the winners of two different leagues are chosen to play each other. As an example, prior to 1988, the Rose Bowl featured the Pac-10 Conference champion against the Big-10 Conference champion. There are also dyadic tournaments such as those used by the Super Bowl and the World Cup that combine round-robin and single-elimination structures.

1.1 Literature Review

Research on the use of options for sports events is very scarce. The first attempt to analyze such options was by Sainam, Balasubramanian and Bayus (2009). The authors devise a simple analytical model to evaluate the benefits of offering options to sports event organizers. They show that organizers can potentially increase their profits by offering options to consumers in addition to advance tickets. The authors also conduct a small numerical study to support their theoretical findings. However, they do not address the problem of pricing and selling such options.

This paper addresses a particular case of the classic revenue management problem of pricing and managing constrained capacity to maximize expected revenue in the face of uncertain demand. Overviews of revenue management can be found in Talluri and van Ryzin (2004a) and Phillips (2005). While the revenue management literature is vast, there has been relatively little research considering the specific application of revenue management to sporting events. Barlow (2000) discusses the application of revenue management to Birmingham FC, an English Premier League soccer team. Chapter 5 of Phillips (2005) discusses some pricing approaches used by baseball teams and Phillips et. al. (2006) describe a software system for revenue management applicable to sporting events. Duran and Swann (2007) and Drake et. al. (2008) consider the optimal time to switch from offering bundles (e.g. season tickets) to individual tickets for sports and entertainment industries. None of these works address the use of options.

In the absence of discounting, a consumer call option for a future service is equivalent to a partially refundable ticket. Gallego and Sahin (2010) show how such partially refundable tickets can increase revenue relative to either fully refundable or non-refundable tickets and that they can be used to allocate surplus between consumers and capacity providers. They show that offering an option wherein an initial payment of p gives the option of purchasing a service for an additional payment x at a later date can provide additional revenue for sellers. Gallego and Stefanescu (2011) discuss this as one of several “service engineering” approaches that sellers can use to increase profitability. The same result holds for a consumer call option in the case when the identity of the teams is known *ex ante*. Our work extends their work by incorporating the correlation structure on *ex post* customer utilities imposed by the structure of the tournament.

1.2 Overview

§2 describes how the tournament and consumer demand is modeled. To develop some intuition, §3 introduces the pricing problems for advance tickets and options. §3.1 analyzes the problem of pricing advance tickets when no options are offered. Afterwards, the options pricing problem is defined in §3.2. A dynamic programming formulation of the problem is given in §3.3 and the CDLP approximation of that formulation is given in §3.4. In §3.5 we introduce an equivalent formulation of the CDLP, specific to our setting, in which the number of variables and constraints increase linearly with respect to number of teams in our model. In §4 we give some important theoretical results when we have symmetry in the tournament, in other words, when the teams participating the event are identical. §5 explains how offering options to consumers affects their surplus. An important extension is given in §6 where we analyze how to inhibit a third party from taking advantage of prices to obtain a risk-free profit. Results of numerical experiments are given in §7. Finally, §8 concludes with some final remarks.

2 Model

We consider a tournament with $N (N \geq 3)$ teams, where there is uncertainty about the finalists. We address the problem of pricing and management of tickets and options for the final game. The event is held in a venue with a capacity of C seats of uniform quality. Sales are allowed during a finite horizon T that ends when the tournament starts. Notice that from the organizer's point of view seats are perishable assets, that is, unsold seats will not be of value after the tournament starts since they cannot be sold anymore.

$\mathcal{T}$ denotes the set of possible combinations of teams that might advance to the finals. For example, in the case where any combination of teams may play in the final game, we have $\mathcal{T} = \{\{i, j\} : 1 \leq i < j \leq N\}$. In the case of a single-elimination tournament, teams may be divided in two groups, denoted by $\mathcal{T}_1 = \{1, \ldots, N/2\}$ and $\mathcal{T}_2 = \{N/2+1, \ldots, N\}$, in such a way that the winner of each group advances to the final game. In this case the space of future outcomes is $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$.

The organizer offers $N + 1$ different products for the event: *advance tickets*, denoted by A and *options* for each team i, denoted by O^i. Advance tickets are paid in advance at a price p_a and guarantee a seat at the match. We assume that all advance tickets are priced uniformly. An option O^i for team i is purchased at a price p_o^i, and confers the buyer a right to exercise and purchase the underlying ticket at a strike price p_e^i.

The organizer is a monopolist and can influence demand by varying the price. The organizer faces the problem of pricing the products, and determining the number of products of each type to offer so as to maximize her expected revenue. A common practice in sporting events is that prices are announced in advance, and the promoter commits to those prices throughout

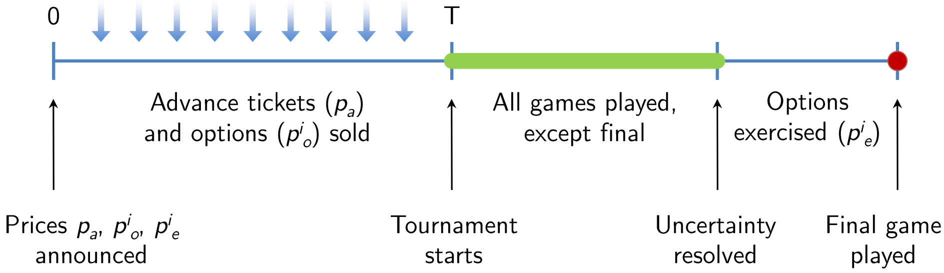

Figure 1: Sales horizon and actions involved in each period.

the horizon. We adhere to that static pricing practice in our model. However, the organizer does not commit in advance to allocate a fixed number of seats for each product. Thus, she can dynamically react to demand by changing the set of products offered at each point in time.

The organizer is assumed to be risk-neutral, and no discounting is performed. All costs incurred by the organizer are assumed to be sunk, so that there is no marginal cost for additional tickets sold. Hence, the organizer maximizes its expected revenue. Additionally, no overbooking is allowed.

The timing of the events is as follows. First, the organizer announces the advance ticket's price, and the options' premium and strike prices for each team. Then, the box office opens, and advance tickets are sold at a price p_a, and options at a price p_o^i for each team i. The sales horizon concludes when the tournament starts. At this point no more tickets are sold. Afterwards the tournament is played out, and the two teams playing in the final are revealed. At this point the holders of options for the finalists decide whether to exercise their rights and redeem their seats at their respective strike prices. Finally, the final game is played and the fans attend the event. Figure 1 illustrates the timing of the events.

A probability distribution $\{q^s\}_{s\in\mathcal{T}}$ for every possible set of outcome is assumed to be common knowledge and invariant throughout the sales horizon. The actual probability of team i advancing to the final is $q^i = \sum_{s\in\mathcal{T}:i\in s} q^s$. Since we sell tickets and options before the tournament starts, we only require the probabilities to be invariant before the games start. In addition, tournament participants' characteristics such as past performance and injury status are generally common knowledge. This information is used to calculate the betting odds which are also available to the public, and these in turn can be used to calculate $\{q^s\}_{s\in\mathcal{T}}$. Thus, our common knowledge assumption regarding probabilities is realistic.

A critical assumption of our model is that tickets and options are not transferrable. This can be enforced, for instance, by demanding some proof of identity at the entry gate. Non-transferability precludes the existence of a secondary market for tickets, that is, tickets cannot be resold and they can only be purchased from the organizer. This assumption, albeit unrealistic, simplifies the analysis. Later we discuss how this assumption can be relaxed.

Now, let us examine the consumers' choice behavior under the assumption that the fans are risk-neutral and utility maximizing. The market is segmented with respect to team preference, with N different segments corresponding to each team. We refer to consumers within segment i as *fans* of team i. In our model demand is stochastic and price sensitive, with customers arriving according to independent Poisson processes with intensity λ^i for segment i. Each consumer requests at most one available product.

A fan of team i has two sources of utility, (i) attending a final game with his favorite team playing, and (ii) attending the event with any other team playing. A fan is characterized by his private valuation for attending his preferred team's game, denoted by V. Valuations are random and drawn independently from a distribution with c.d.f $F_v^i(\cdot)$. We assume that $F_v^i(\cdot)$ is time-homogenous, and that fans do not update their valuations with time. As a result, expected utilities of the possible alternatives remain constant, and the fans do not switch decisions, so there are no cancelations or no-shows. When his preferred team is not playing, the fan obtains only a fraction $\ell^i \in [0,1]$ of his original valuation, and his total value for attending the event is $\ell^i V$. When $\ell^i = 1$, fans are indifferent to whether their preferred team is playing or not. In this case the valuation is mostly derived from attending a game. When ℓ^i is to close zero, fans have a strong preference towards their team, and are willing to attend the game only if their team is playing. We refer to ℓ^i as the "love of the game". All ℓ, $F_v(\cdot)$, and λ are common knowledge.

At the moment of purchase, a fan of team i has three choices, (i) buy an advance ticket, (ii) buy an option for his preferred team, or (iii) not purchase anything. The first choice, buying an advance ticket A, requires the payment of the advance ticket price p_a. Then, with probability q^i the fan expects to get a value of V from seeing his team in the final, and with probability $1-q^i$ she expects to get a value of $\ell^i V$ when his favorite team is not playing. Hence, the fan's expected utility for product A given a valuation of V, denoted by $U_a^i(V)$, is $U_a^i(V) = (q^i + (1-q^i)\ell^i)V - p_a$. The second choice, buying the option O^i, requires the payment of the premium price p_o^i at the moment of purchase. Notice that since valuations are not updated over time, once a fan buys an option he will always exercise if his team makes the final. Hence, with probability q^i his preferred team advances to the final, and she exercises by paying the strike price p_e^i and extracts a value V in return. The expected utility for product O^i given a valuation of V, denoted by $U_o^i(V)$, is $U_o^i(V) = q^i V - (p_o^i + q^i p_e^i)$. Finally, the utility of no purchase is set to $U_n = 0$. Table 1 summarizes the expenditures, values and expected utilities related to each decision.

Under our assumptions, a fan makes the decision that maximizes his expected utility. The actual decision, however, depends on the availability of advance tickets and options at the moment of arrival to the box office. For instance, assume that a fan arrives and finds both an advance ticket and an option available for purchase. If both of them provide positive utility, then the fan will choose the product that maximizes her expected utility. However, if the utility maximizing product was not available, then the fan will choose the other one.

We note that we are considering *team-specific* options, that is, a fan can exercise his options

Decision	Pays	Value	Ex. Utility
n: don't buy	0	0	0
a: buy A	p_a	V w.p. q^i $\ell^i V$ w.p. $1-q^i$	$(q^i + (1-q^i)\ell^i)V - p_a$
o: buy O^i	$p_o^i + p_e^i$ w.p. q^i p_o^i w.p. $1-q^i$	V w.p. q^i 0 w.p. $1-q^i$	$q^i V - (p_o^i + q^i p_e^i)$

Table 1: Expenditures, values and expected utilities related to each decision.

only if his team advances to the final. This is different from a *general option* that could be exercised no matter which teams advance to the final. Pricing and capacity management of general options are addressed in Gallego and Sahin (2010).

Let us now address the problem of characterizing the demand rate of every product subject to a given set of offered products. We partition the space of valuations for each market segment into five disjoint sets as shown in Table 2. I^i_{xyz} denotes the set of valuations for which decision x is the most preferred, y is the second most preferred, and z is the least preferred for segment i. For example, I^i_{aon} corresponds to the set of valuations where an advance ticket is the most highly preferred product, an option is the second most highly preferred product and buying nothing is the least preferred choice for segment i. The linearity of the expected utilities implies that these sets are intervals of $\mathbb{R}_+$. Figure 2 illustrates the expected utility for the three choices versus the realized value of V for the particular market segment i, and the corresponding valuation intervals. Observe that depending on prices and problem parameters, this graph can take two forms, either $I^i_n \cup I^i_{on} \cup I^i_{oan} \cup I^i_{aon} = \mathbb{R}_+$ (the first graph) or $I^i_n \cup I^i_{an} \cup I^i_{aon} = \mathbb{R}_+$ (the second graph).

Decision Priorities	Valuation Sets
n	$\{V : U^i_a(V) \leq 0, U^i_o(V) \leq 0\}$
on	$\{V : U^i_o(V) \geq 0 \geq U^i_a(V)\}$
an	$\{V : U^i_a(V) \geq 0 \geq U^i_o(V)\}$
oan	$\{V : U^i_a(V) \geq U^i_o(V) \geq 0\}$
aon	$\{V : U^i_o(V) \geq U^i_a(V) \geq 0\}$

Table 2: Decision priorities and corresponding valuation sets

Using the distribution of valuations in the population, the organizer can compute the probability that the private valuation of an arriving customer of team i belongs to a particular interval. The next proposition gives a simple characterization of these probabilities in terms of prices and the model primitives.

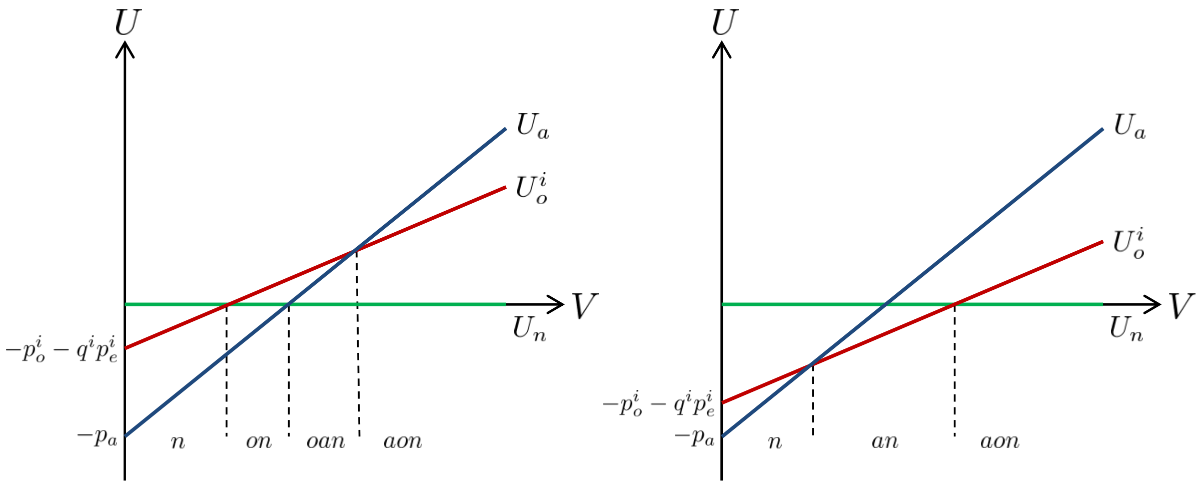

Figure 2: Graphs showing expected surplus for the three choices. The horizontal axis is divided in segments matching each decision. For instance, if V falls in the segment oan the fan would buy an option, and else she would buy an advance ticket.

Proposition 1. *The decision priority probabilities are*

$$\begin{aligned}
\pi_n^i &= \mathbb{P}\{V \in I_n^i\} = F_v^i(\min(c, b)),\\
\pi_{on}^i &= \mathbb{P}\{V \in I_{on}^i\} = (F_v^i(c) - F_v^i(b))^+,\\
\pi_{an}^i &= \mathbb{P}\{V \in I_{an}^i\} = (F_v^i(b) - F_v^i(c))^+,\\
\pi_{oan}^i &= \mathbb{P}\{V \in I_{oan}^i\} = (F_v^i(a) - F_v^i(c))^+,\\
\pi_{aon}^i &= \mathbb{P}\{V \in I_{aon}^i\} = 1 - F_v^i(\max(a, b)),
\end{aligned}$$

where $a = \frac{p_a - (p_o + qp_e)}{(1-q)\ell}$, $b = \frac{1}{q}(p_o + qp_e)$, *and* $c = \frac{p_a}{q + (1-q)\ell}$.

Now we turn to the problem of determining the demand rate for each product when the organizer offers only a subset $S \subseteq \mathcal{S} \equiv \{A, O^1, \ldots, O^N\}$ of the available products. Under our model the instantaneous arrival rate of fans of team i purchasing advance tickets when offering $S \subseteq \mathcal{S}$, denoted by $\lambda_a^i(S)$, is

$$\lambda_a^i(S) = \lambda^i \mathbf{1}_{\{A \in S\}}(\pi_{an}^i + \pi_{aon}^i + \mathbf{1}_{\{O^i \notin S\}}\pi_{oan}^i). \tag{1}$$

The arrival rate for advance ticket purchases is composed of three terms. The first term accounts for fans that are only willing to buy those tickets. The second term accounts for fans that are willing to buy the advance tickets, but when they are no longer available will buy the options as a second choice. Finally, the third term considers fans that prefer options as their first choice, but may end up buying advance tickets when they are not available. The aggregate arrival rate for advance ticket purchases when offering subset S is $\lambda_a(S) = \sum_{i=1}^{N} \lambda_a^i(S)$.

Similarly, the arrival rate of fans of team i buying options when offering S, denoted by $\lambda_o^i(S)$, is

$$\lambda_o^i(S) = \lambda^i \mathbf{1}_{\{O^i \in S\}}(\pi_{on}^i + \pi_{oan}^i + \mathbf{1}_{\{A \notin S\}}\pi_{aon}^i). \tag{2}$$

Finally, the arrival rate of fans that decide not to buy anything at the ticket office when offering S, denoted by $\lambda_n(S)$, is

$$\lambda_n(S) = \sum_{i=1}^{N} \lambda^i \big(\pi_n^i + \mathbf{1}_{\{A \notin S\}}\pi_{an}^i + \mathbf{1}_{\{O^i \notin S\}}\pi_{on}^i + \mathbf{1}_{\{A \notin S, O^i \notin S\}}(\pi_{aon}^i + \pi_{oan}^i)\big). \tag{3}$$

Notice that it must be the case that $\lambda_a(S) + \sum_{i=1}^{N} \lambda_o^i(S) + \lambda_n(S) = \lambda = \sum_{i=1}^{N} \lambda^i$ for all $S \subseteq \mathcal{S}$.

3 Pricing Problem

3.1 Advance Ticket Pricing Problem

To develop some insight, we first consider the problem where no options are offered, and only advance tickets are sold. In this case the organizer's problem is to find the price that maximizes the expected revenue under the constraint that at most C tickets can be sold. The maximum expected profit R_a^* for the organizer is:

$$R_a^* \equiv \max_{p_a} \mathbb{E}\left[p_a \min\{C, D_a(p_a)\}\right] \tag{4}$$

where $D_a(p_a)$ is the demand for advance tickets under price p_a. $D_a(p_a)$ is a Poisson random variable with mean $T\lambda_a(p_a)$ where $\lambda_a(p_a)$ denotes the arrival intensity of advance ticket purchase requests for all teams' fans under price p_a. In turn, using (1) the arrival intensity is

$$\lambda_a(p_a) \equiv \sum_{i=1}^{N} \lambda^i \mathbb{P}\{U_a^i(V) \geq 0\} = \sum_{i=1}^{N} \lambda^i \bar{F}_v \left(\frac{p_a}{q^i + (1 - q^i)\ell^i} \right) \tag{5}$$

The exact solution to problem (4) can be given in terms of the elasticity of demand with respect to price as inBitran and Caldentey (2003). However, we do not follow that path here. Instead, we approximate the exact solution using the deterministic version of the model (or certainty equivalent policy) where random variables are replaced by their means, and discrete quantities are assumed to be continuous. The resulting solution, which is often easier to compute, is asymptotically optimal to the exact problem.

The maximum revenue under the deterministic approximation, denoted R_a^D, is

$$R_a^D = \max_{p_a} \left\{p_a \min\{C, T\lambda_a(p_a)\}\right\}. \tag{6}$$

Let us make the following definitions before characterizing the optimal solution to the deterministic approximation. The *run-out rate*, given by $\lambda_a^0 = C/T$, is defined to be the rate of advance ticket sales at which the organizer sells all of its seats uniformly over the time horizon T. The corresponding *run-out price*, denoted by p_a^0, is the price which enables us to achieve the run-out sales rate and is obtained from $\lambda_a(p_a^0) = C/T$. Also, let λ_a^* be the least maximizer of the revenue rate function $\lambda_a p_a(\lambda_a)$. Using this notation, Gallego and van Ryzin (1995) showed that

$$R_a^D = T\min(p_a^*\lambda_a^*, p_a^0\lambda_a^0).$$

Hence, if the capacity of the stadium is large ($C > \lambda_a^* T$), the organizer ignores the problem of running out of seats and prices at the level that maximizes the revenue rate. In this case the organizer ends with $C - \lambda_a^* T$ unsold seats. If the seats are scarce ($C < \lambda_a^* T$), the organizer can afford to price higher, and it indeed prices at the highest level that still enables it to sell all the items. Notice that in the final game of a tournament it is likely that the number of seats will be scarce. Thus, in most sports events the second situation prevails. As advance ticket prices increase fans become more sensitive to the finalists. So, intuitively we expect options to be more attractive when seats are scarce. Indeed, we will later show that this is the case.

Let us now discuss the asymptotic optimality of the deterministic approximation using the following equation obtained from Gallego and van Ryzin (1995)

$$1 \geq \frac{R_a^*}{R_a^D} \geq 1 - \frac{1}{2\sqrt{\min(C, \lambda_a^* T)}}. \tag{7}$$

From equation (7) we see that the fluid model approximation is asymptotically optimal in two limiting cases: (i) the capacity of the stadium is large ($C \gg 1$) and there is plenty of time to sell them ($C < \lambda_a^* T$); or (ii) there is the potential for a large number of sales at the revenue maximizing price ($\lambda_a^* T \gg 1$), and there are enough seats to satisfy this potential demand ($C > \lambda_a^* T$). Thus, we see that if the volume of expected sales is large, the heuristics perform quite well.

3.2 Advance Ticket and Options Pricing Problem

We now consider both advance ticket and options' pricing problem faced by the organizer. Recall that prices are determined in advance, disclosed at the beginning, and remain constant during the sales horizon. However, the number of seats allocated to each product are not disclosed in advance, and thus may be used by the organizer to adjust her strategy as sales realize. The organizer may dynamically react to the demand by playing with the availability of the products, and thus control the number of tickets and options sold.

The sequential nature of the decisions involved suggests a partition of the problem into a two-stage optimization problem. Decision variables are prices in the first stage and product availability in the second stage. To elaborate, in the first stage the organizer looks for the set of prices $p = (p_a, p_o, p_e)$ that maximize the optimal value of the second-stage problem, which is the maximum expected revenue that can be extracted under fixed prices p. This partition is well-defined because prices are determined before the demand is realized, and are independent of the actual realization of the random data. The optimal value of the first-stage problem, denoted by R^*, is

$$R^* \equiv \max_{p_a, p_o, p_e} R^*(p)$$
$$\text{s.t. } p_a \geq 0, p_o \geq 0, p_e \geq 0,$$

where we denote by $R^*(p)$ the optimal value of the second-stage problem.

The second-stage problem takes prices as given, and optimizes the expected revenue by controlling the subset of products that is offered at each point in time. Notice that the second-stage decision variable is a control policy over the offer sets, which is determined as the demand realizes. We refer to this second-stage problem as the *Capacity Allocation Problem.* Next, we turn to the problem of determining the optimal value of the second-stage problem under fixed prices p.

Once prices are fixed, the organizer attempts to maximize her revenue by implementing adaptive non-anticipating policies that offer some subset $S \subseteq \mathcal{S} \equiv \{A, O^1, \ldots, O^N\}$ of the available products at each point in time. A control policy μ maps states of the system to control actions, i.e. the set of offered products. We denote by $S_\mu(t)$ the subset of products offered under policy μ at time t. Let $X_a(S_\mu(t))$ be the total number of advance tickets sold up to time t. Under our assumptions, $X_a(S_\mu(t))$ is a non-homogeneous Poisson process with arrival intensity $\lambda_a(S_\mu(t))$ as defined in (1). The organizer can thus affect the arrival intensity of purchase requests by controlling the offer set $S_\mu(t)$. An advance ticket is sold at time t if $\mathrm{d}X_a(S_\mu(t)) = 1$. Similarly, let $X_o^i(S_\mu(t))$ be the number of options sold for team i up to time t, and $\mathrm{d}X_o^i(S_\mu(t)) = 1$ when an option is sold at time t. Again, $X_o^i(S_\mu(t))$ is a non-homogeneous Poisson process with arrival intensity $\lambda_o^i(S_\mu(t))$ as defined in (2). With some abuse of notation, we define by $X_a = X_a(S_\mu(T))$ and $X_o^i = X_o^i(S_\mu(T))$ to be the total number of advance tickets and options sold respectively.

The organizer seeks to maximize her expected revenue, which is given by

$$\mathbb{E}\left[X_a p_a + \sum_{i=1}^{N} X_o^i(p_o^i + q^i p_e^i)\right],$$

where the first term accounts for the revenue from advance ticket sales and the second term accounts for the revenue from options under the assumption that *all options are exercised* which was previously discussed in §2. Notice that because prices remain constant during the time horizon, the revenue depends on the total number of tickets sold. Moreover, as a result of the linearity of expectation, the revenue depends only on the expected number of tickets sold.

The second-stage or Capacity Allocation Problem can be formalized as the following stochastic control problem similar to the one given in Liu and van Ryzin (2008):

$$
\begin{aligned}
R^*(p) = \max_{\mu \in \mathcal{M}} \mathbb{E} &\left[p_a X_a + \sum_{i=1}^{N} (p_o^i + q^i p_e^i) X_o^i \right] \\
\text{s.t. } X_a &= \int_0^T \mathrm{d}X_a(S_\mu(t)), \\
X_o^i &= \int_0^T \mathrm{d}X_o^i(S_\mu(t)), \qquad \forall i = 1, \ldots, N, \\
X_a + X_o^i + X_o^j &\le C, \qquad \text{(a.s.) } \forall \{i,j\} \in \mathcal{T}.
\end{aligned} \tag{8}
$$

where $\mathcal{M}$ is the set of all adaptive non-anticipating policies, and $R^*(p)$ is the expected revenue under the optimal policy μ^*. In the next section we reformulate (8) as a dynamic programming problem.

3.3 Dynamic Programming Formulation

We informally derive the Hamilton-Jacobi-Bellman equation for the second-stage problem by considering a small time interval δt. We define the value function $V(t, X_a, X_o)$ as the maximum expected revenue that can be extracted when t time units are remaining, and X_a number of advance tickets and X_o number of options have been sold. Hence, the goal is to find $R^*(p) = V(T, 0, 0)$. Applying the Principle of Optimality,

$$
\begin{aligned}
V(t, X_a, X_o) = \max_{S \subseteq \mathcal{S}} \Bigg\{ & \lambda_a(S) \delta t \left(p_a + V(t - \delta t, X_a + 1, X_o) \right) \\
& + \sum_{i \in S_o} \lambda_o^i(S) \delta t \left(p_o^i + q^i p_e^i + V(t - \delta t, X_a, X_o + e_i) \right) \\
& + (\lambda_n(S) \delta t + 1 - \lambda \delta t) V(t - \delta t, X_a, X_o) \Bigg\} + o(\delta t).
\end{aligned}
$$

Rearranging terms and letting $\delta t \to 0$ we obtain the Hamilton-Jacobi-Bellman (HJB) equation:

$$
\begin{aligned}
\frac{\partial V(t, X_a, X_o)}{\partial t} = \max_{S \subseteq \mathcal{S}} \Bigg\{ & \lambda_a(S) \left(p_a + V(t, X_a + 1, X_o) \right) \\
& + \sum_{i \in S_o} \lambda_o^i(S) \left(p_o^i + q^i p_e^i + V(t, X_a, X_o + e_i) \right) \\
& + (\lambda_n(S) - \lambda) V(t, X_a, X_o) \Bigg\},
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
V(0, X_a, X_o) &= 0 \qquad \text{for all } X_a, X_o, \\
V(t, X_a, X_o) &= -\infty \qquad \text{if } X_a + X_o^i + X_o^j > C \text{ for some outcome } \{i,j\} \in \mathcal{T}.
\end{aligned}
$$

Unfortunately, the resulting HJB equation is a partial differential equation that is in most cases very difficult to solve. The next section gives an asymptotically optimal deterministic approximation of (8) which is much easier to solve.

3.4 Choice-based Deterministic Linear Programming Model

In this section we follow Gallego et. al. (2004), and solve a deterministic approximation of (8) in which random variables are replaced by their means and quantities are assumed to be continuous. We denote by $r_a = p_a$ the expected revenue of selling an advance ticket, and by $r_o^i = p_o^i + q^i p_e^i$ the expected revenue of selling an option of team i. Under this approximation, when a subset of products S is offered, advanced tickets (resp. options for team i) are purchased at a rate of $\lambda_a(S)$ (resp. $\lambda_o^i(S)$). Since r_a (resp. r_o^i) is the expected revenue from the sale of an advance ticket (resp. option for team i), the rate of revenue generated from the sales of advance tickets is $r_a\lambda_a(S)$ (resp. $r_o^i\lambda_o^i(S)$ for options of team i). Additionally, because demand is deterministic and the choice probabilities are time homogeneous, we only care about the total amount of time each subset of products is offered and in the order in which they are offered. Thus, we only need to consider the amount of time each subset S is offered, denoted by $t(S)$, as the decision variables. Under this notation, the number of advance tickets sold is $\sum_{S\subseteq\mathcal{S}} t(S)\lambda_a(S)$, while the number options sold for team i is $\sum_{S\subseteq\mathcal{S}} t(S)\lambda_o^i(S)$. Finally, the total revenue of the organizer is $\sum_{S\subseteq\mathcal{S}} r(S)t(S)$, where $r(S) = r^T\lambda(S)$ is the revenue rate when subset S is offered, and $r = \left(r_a, r_o^1, \ldots, r_o^N\right)$ is the vector of expected revenues.

Thus, we obtain the following choice-based deterministic LP model (CDLP):

$$
\begin{aligned}
R^{CDLP}(p) \equiv \max_{t(S)} & \sum_{S\subseteq\mathcal{S}} r(S)t(S) && (9)\\
\text{s.t.} & \sum_{S\subseteq\mathcal{S}} t(S) = T, \\
& \sum_{S\subseteq\mathcal{S}} t(S)\left(\lambda_a(S) + \lambda_o^i(S) + \lambda_o^j(S)\right) \le C, \qquad \forall\{i,j\}\in\mathcal{T} && (10)\\
& t(S) \ge 0 \qquad \forall S\subseteq\mathcal{S}
\end{aligned}
$$

where $R^{CDLP}(p)$ denotes the maximum revenue of the CDLP under prices p. Notice that, no matter which teams advance to the final, the maximum number of options sold is $C - X_a$.

Both Liu and van Ryzin (2008) and Gallego et. al. (2004) showed that the CDLP provides an upper bound to the stochastic program (8) and that this bound is asymptotically tight as the capacity and time horizon are scaled, thus proving the asymptotical optimally of the CDLP. For completeness we state those results without proof.

Proposition 2. *The CDLP provides an upper bound on the stochastic problem:*

$$R^*(p) \le R^{CDLP}(p) \qquad \forall p \ge 0$$

Proposition 3. *Let $R_\theta^*(p)$ the optimal objective of the scaled stochastic problem in capacity θC and time horizon θT, then:*

$$\lim_{\theta\to\infty} \frac{1}{\theta} R_\theta^*(p) = R^{CDLP}(p) \qquad \forall p \ge 0$$

3.5 Efficient Formulation

Since linear program in (9) has one primal variable for each offer subset, it has 2^{N+1} primal variables in total. For instance, if the tournament has 32 teams the program would have more than 8 billion primal variables! Fortunately, by exploiting the structure of our choice model it is possible to derive an alternative formulation with a linear number of variables and constraints.

Recall that consumers are partitioned into N different market segments, each associated with a different team. To any given segment $i = 1, \ldots, N$ two different products are potentially offered: (i) advance tickets (A) and (ii) options for the associated team (O^i). We denote by $\mathcal{S}^i = \{A, O^i\}$ the set of products available for market segment i. Demands across segments are independent, and different segments are only linked through the capacity constraints. Since each segment has two products, only four offer sets need to be considered. Thus, for each market segment we only need the following decision variables: (i) the time both advance tickets and options are offered, denoted by $t^i\left(\{A, O^i\}\right)$, (ii) the time only advance tickets are offered, denoted by $t^i\left(\{A\}\right)$, (iii) the time only options are offered, denoted by $t^i\left(\{O^i\}\right)$, and (iv) the time no product is offered, denoted by $t^i\left(\emptyset\right)$. Given $t(S)$ $\forall S \subseteq \mathcal{S}$, the value of the new decision variables can be computed as follows:

$$
\begin{aligned}
t^i\left(\{A, O^i\}\right) &\equiv \sum_{S \subseteq \mathcal{S}: A \in S, O^i \in S} t(S), \qquad & t^i\left(\{A\}\right) &\equiv \sum_{S \subseteq \mathcal{S}: A \in S, O^i \notin S} t(S) \\
t^i\left(\{O^i\}\right) &\equiv \sum_{S \subseteq \mathcal{S}: A \notin S, O^i \in S} t(S), \qquad & t^i\left(\emptyset\right) &\equiv \sum_{S \subseteq \mathcal{S}: A \notin S, O^i \notin S} t(S)
\end{aligned}
\tag{11}
$$

Observe that for each segment offer times should sum up to length of the horizon, that is $\sum_{S \subseteq \mathcal{S}^i} t^i(S) = T$. An important observation is that by requiring $\sum_{S \subseteq \mathcal{S}^i \setminus \emptyset} t^i(S) \leq T$ we do not need to keep track of the time in which no product is offered for each segment. What this requirement tells us is for each segment if the time in which we sell at least one product adds up to at most the sales horizon T, then we do not need to keep track of the time in which no product is offered. Additionally, in order for the offer sets to be consistent across market segments, the total time advance tickets are offered in each segment should be equal, i.e. for some $T_a \geq 0$ it should be the case that $t^i\left(\{A, O^i\}\right) + t^i\left(\{A\}\right) = T_a$ for all $i = 1, \ldots, N$ where T_a denotes the total time advance tickets are offered throughout the sales horizon.

Thus, applying the aforementioned changes we obtain the following market-based deterministic LP (MDLP)

$$R^{MDLP}(p) \equiv \max_{t^i(S),T_a} \sum_{i=1}^{N} \sum_{S\subseteq\mathcal{S}^i} r^i(S)t^i(S) \tag{12}$$

$$\text{s.t.} \sum_{S\subseteq\mathcal{S}^i} t^i(S) \leq T \qquad \forall i = 1,\ldots,N \tag{13}$$

$$t^i\left(\{A, O^i\}\right) + t^i\left(\{A\}\right) = T_a \qquad \forall i = 1,\ldots,N \tag{14}$$

$$\sum_{k=1}^{N}\sum_{S\subseteq\mathcal{S}^k} t^k(S)\lambda_a^k(S) + \sum_{S\subseteq\mathcal{S}^i} t^i(S)\lambda_o^i(S) + \sum_{S\subseteq\mathcal{S}^j} t^j(S)\lambda_o^j(S) \leq C \qquad \forall\{i,j\} \in \mathcal{T} \tag{15}$$

$$T_a \geq 0, t^i(S) \geq 0 \qquad \forall S \subseteq \mathcal{S}^i, i = 1,\ldots,N,$$

where $r^i(S) = p_a\lambda_a^i(S) + r_o^i\lambda_o^i(S)$ is the revenue rate from market segment i when subset $S \subseteq \mathcal{S}^i$ is offered. Notice that the new optimization problem has $3N+1$ variables, which is much less than the original CDLP, and $O(N^2)$ constraints.

The following proposition proves the equivalence between the MDLP and the CDLP.

Proposition 4. *The MDLP is equivalent to the CDLP, i.e.* $R^{MDLP}(p) = R^{CDLP}(p)$ *for all prices* $p \geq 0$.

Proof. We first show that $R^{CDLP}(p) \leq R^{MDLP}(p)$ by showing that any solution of the CDLP can be used to construct a feasible solution to the MDLP with the same objective value. Let $\{t(S)\}_{S\subseteq\mathcal{S}}$ be a feasible solution to the CDLP. First, using the decision variables given by (11), the total number of advance tickets sold can be written as

$$\begin{aligned}
X_a &= \sum_{S\subseteq\mathcal{S}} t(S)\lambda_a(S) = \sum_{S\subseteq\mathcal{S}} t(S) \sum_{i=1}^{N} \lambda^i \mathbf{1}_{\{A\in S\}}(\pi_{an}^i + \pi_{aon}^i + \mathbf{1}_{\{O^i\notin S\}}\pi_{oan}^i) \\
&= \sum_{i=1}^{N} \Big(\sum_{S\ni A, S\ni O^i} t(S)\Big)\lambda^i(\pi_{an}^i + \pi_{aon}^i) + \Big(\sum_{S\ni A, S\not\ni O^i} t(S)\Big)\lambda^i(\pi_{an}^i + \pi_{aon}^i + \pi_{oan}^i) \\
&= \sum_{i=1}^{N} t^i\left(\{A,O^i\}\right)\lambda_a^i\left(\{A,O^i\}\right) + t^i\left(\{A\}\right)\lambda_a^i\left(\{A\}\right) = \sum_{i=1}^{N}\sum_{S\subseteq\mathcal{S}^i} t^i(S)\lambda_a^i(S),
\end{aligned} \tag{16}$$

where the second equality follows from (1), the third from exchanging summations, and the fourth from (1) again. Similarly, the number of option sold in market segment i can be written as

$$
\begin{aligned}
X_o^i &= \sum_{S\subseteq\mathcal{S}} t(S)\lambda_o^i(S) = \sum_{S\subseteq\mathcal{S}} t(S)\lambda^i \mathbf{1}_{\{O^i\in S\}}\left(\pi_{on}^i + \pi_{oan}^i + \mathbf{1}_{\{A\notin S\}}\pi_{aon}^i\right)\\
&= \Big(\sum_{S\ni A, S\ni O^i} t(S)\Big)\lambda^i(\pi_{on}^i + \pi_{oan}^i) + \Big(\sum_{S\not\ni A, S\ni O^i} t(S)\Big)\lambda^i(\pi_{on}^i + \pi_{oan}^i + \pi_{aon}^i)\\
&= t^i\left(\{A,O^i\}\right)\lambda_o^i\left(\{A,O^i\}\right) + t^i\left(\{A\}\right)\lambda_o^i\left(\{A\}\right) = \sum_{S\subseteq\mathcal{S}^i} t^i(S)\lambda_o^i(S), \qquad (17)
\end{aligned}
$$

where the second equality follows from (2), the third from exchanging summations, and the fourth from (2) again. Thus, the capacity constraint (15) is verified.

Second, the non-negativity constrains and the time-horizon length constraints (13) follow trivially. Next, for the advance selling market consistency constraints (14) notice that for all $i = 1, \dots, N$ we have that

$$
t^i\left(\{A,O^i\}\right) + t^i\left(\{A\}\right) = \sum_{S\subseteq\mathcal{S}:A\in S, O^i\in S} t(S) + \sum_{S\subseteq\mathcal{S}:A\in S, O^i\notin S} t(S) = \sum_{S\subseteq\mathcal{S}:A\in S} t(S) = T_a.
$$

Thus, advance tickets are offered the same amount of time in all markets.

Finally, the next string of equalities show that both solutions attain the same objective value

$$
\begin{aligned}
\sum_{S\subseteq\mathcal{S}} r(S)t(S) &= \sum_{S\subseteq\mathcal{S}} r^T\lambda(S)t(S) = \sum_{S\subseteq\mathcal{S}}\sum_{i=1}^N \left(r_a\lambda_a^i(S) + r_o^i\lambda_o^i(S)\right)t(S)\\
&= \sum_{i=1}^N\sum_{S\subseteq\mathcal{S}^i} r_a\lambda_a^i(S)t^i(S) + r_o^i\lambda_o^i(S)t^i(S) = \sum_{i=1}^N\sum_{S\subseteq\mathcal{S}^i} r^i(S)t^i(S),
\end{aligned}
$$

where the third equality follows from (16) and (17).

Next, we show that $R^{CDLP}(p) \geq R^{MDLP}(p)$ by showing that any solution of the MDLP can be used to construct a feasible solution to the CDLP with the same objective value. Let $\left\{t^i(S)\right\}_{S\subseteq\mathcal{S}^i, i=1,\dots,N}$ be a feasible solution to the MDLP. In the following we give a simple algorithm to compute a feasible solution $\{t(S)\}_{S\subseteq\mathcal{S}}$ for the CDLP.

First, we deal with offer sets containing advance tickets, and compute $t(S)$ for all $S\in\mathcal{S}$ such that $A\in S$. Let $[i]_{i=1,\dots,N}$ be the permutation in which teams are sorted in increasing order with respect to $t^i\left(\{A,O^i\}\right)$, i.e. $t^{[i]}\left(\{A,O^{[i]}\}\right) \leq t^{[i+1]}\left(\{A,O^{[i+1]}\}\right)$. Consider the following offer sets

$$
\begin{aligned}
S^{[i]} &= \left\{A, O^{[i]}, O^{[i+1]}, \dots, O^{[N]}\right\} \quad \forall i = 1,\dots,N\\
S^{[N+1]} &= \{A\}
\end{aligned}
$$

and associated times $t\left(S^{[i]}\right) = t^{[i]}\left(\{A,O^{[i]}\}\right) - t^{[i-1]}\left(\{A,O^{[i-1]}\}\right)$ for all $i = 1,\dots,N+1$, with $t^{[0]}\left(\{A,O^{[0]}\}\right) = 0$, and $t^{[N+1]}\left(\{A,O^{[N+1]}\}\right) = T_a$. Since teams are sorted with respect to $t^i\left(\{A,O^i\}\right)$, then $t\left(S^{[i]}\right) \geq 0$. Notice that this construction is valid because the market consistency constraints (14) guarantee that advance tickets are offered the same amount of time in all markets. Figure 3 sketches a graphical representation of the algorithm.

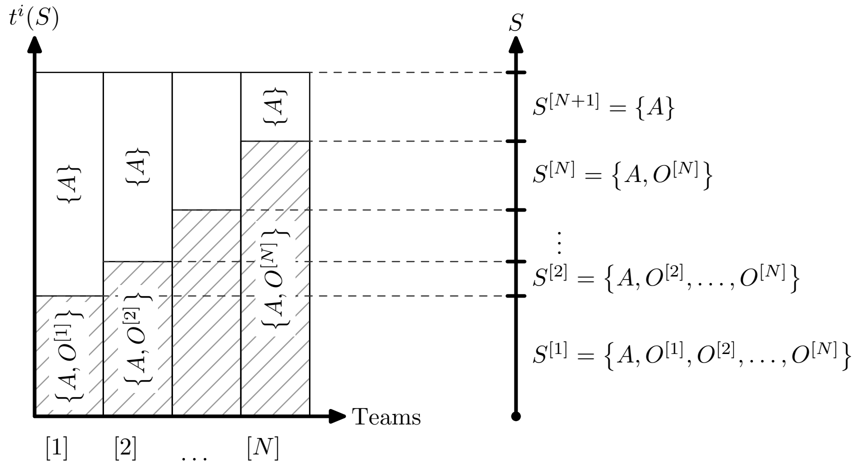

Figure 3: Computing a feasible solution for the CDLP (showed on the right) from a feasible solution from the MDLP (on the left) in the case of offer sets containing advance tickets.

Next, we look at the intuition behind this construction. Although the order is not important, consider a solution for the CDLP that offers the sets $S^{[i]}$ in sequential order; it starts with $S^{[1]}$, then $S^{[2]}$, and so forth until $S^{[N+1]}$. Hence, at first it offers all products, then team 1's options are removed, then team 2's options are removed, and so forth until the end when only advance tickets are offered. Hence, the optimal policy has a nested structure.

Finally, a similar argument holds for offer sets not containing advance tickets. □

4 The Symmetric Case

In this section we consider the symmetric problem, i.e. the case in which all teams have the same probability of advancing to the final and same arrival rates. Also, we assume that valuations are i.i.d. across teams and that the love of the game is constant throughout the population. These assumptions allow us to theoretically characterize the benefits of introducing options. First, we identify several conditions in which offering options is beneficial to the organizer. Second, we provide an asymptotic analysis for the case where the number of teams is large. These analysis will be based on the deterministic approximation of the problem.

Before we start our analysis, let us examine how the symmetry assumption affects the basic problem structure. In a symmetric problem with N teams, any given team has a probability $q = \frac{2}{N}$ of advancing to the final game. The arrival rate of fans of each team is $\frac{\lambda}{N}$, where λ denotes the aggregate arrival rate.

Let us first consider the advance ticket pricing problem under the symmetry assumption. Using (5) the total arrival intensity under price p_a is now

$$\lambda_a(p_a) = \lambda \bar{F}_v \left(\frac{p_a}{q + (1-q)\ell} \right).$$

Furthermore, we assume that the c.d.f. of V is continuous and strictly increasing. Thus, there is a one-to-one correspondence between prices and arrival rates, and the function $\lambda_a(p_a)$ has an inverse $p_a(\lambda_a)$ given by

$$p_a(\lambda_a) = (q + (1-q)\ell)\,\bar{F_v}^{-1}\left(\frac{\lambda_a}{\lambda}\right).$$

In this setting it is analytically convenient to work with intensities instead of prices as the decision variables. We assume the *value rate*

$$v(\lambda_a) = \lambda_a \bar{F_v}^{-1}\left(\frac{\lambda_a}{\lambda}\right)$$

to be regular and differentiable. Regularity implies that v is continuous, bounded, concave, satisfies $\lim_{\lambda_a \to 0} v(\lambda_a) = 0$, and has a least maximizer λ_a^*.

Let us give some definitions before we obtain a sufficient condition for the concavity of the value rate. We denote by $h(x) = f_v(x)/\bar{F_v}(x)$ the *failure rate*, and by $g(x) = xh(x)$ the *generalized failure rate* of V. The random variable V is said to have an *increasing failure rate* (IFR) if $h(x)$ is non-decreasing. Similarly, V has an *increasing generalized failure rate* (IGFR) if $g(x)$ is non-decreasing. IFR implies IGFR but the reverse does not hold. The next lemma shows that strict IFR guarantees the value rate to be strictly concave.

Lemma 1. *If the valuation random variable has IFR, then the value rate is strictly concave.*

Proof. The derivative of the value rate w.r.t. λ_a is

$$\frac{dv}{d\lambda_a}(\lambda_a) = \bar{F_v}^{-1}(\lambda_a/\lambda) - \frac{\lambda_a}{\lambda}\frac{1}{f_v\left(\bar{F_v}^{-1}(\lambda_a/\lambda)\right)}.$$

Composing the derivative with $\lambda_a(c) = \lambda\bar{F_v}(c)$ we get

$$\left(\frac{dv}{d\lambda_a} \circ \lambda_a\right)(c) = \frac{dv}{d\lambda_a}\left(\lambda\bar{F_v}(c)\right) = c - \frac{\bar{F_v}(c)}{f(c)} = c - \frac{1}{h(c)}.$$

IFR implies that the composite function is increasing in c. Since $\lambda_a(c)$ is decreasing, we conclude that original derivative is decreasing and v strictly concave. $\square$

We recast the problem with the arrival intensity as the decision variable; the promoter determines a target sales intensity λ_a and the market determines the price $p_a(\lambda_a)$ based on this quantity. So, the deterministic approximation of the advance ticket pricing problem (6) becomes

$$\begin{aligned} R_a^D = \max_{\lambda_a \geq 0} &\quad T\lambda_a p_a(\lambda_a) \\ \text{s.t.} &\quad T\lambda_a \leq C, \lambda_a \leq \lambda, \\ = \max_{\lambda_a \geq 0} &\quad T\,(q + (1-q)\ell)\,v(\lambda_a) \\ \text{s.t.} &\quad T\lambda_a \leq C, \lambda_a \leq \lambda, \end{aligned} \tag{18}$$

where we have written the objective in terms of the value rate. Thus, we obtain a concave maximization problem with linear inequality constraints. We denote by $R_a^D(\lambda_a)$ the objective function of (18).

The advance ticket pricing problem (18) is indeed equivalent to the problem of a monopolistic seller pricing a zero-cost product with limited capacity. Lariviere (2006) and van den Berg (2006) give sufficient conditions for the uniqueness of the optimal solution and the concavity of the objective. The weaker condition of strict IGFR is sufficient for all the results that follow in this section.

Next, we turn to the deterministic approximation of the advance ticket and options pricing problem. We look for symmetric solutions in which we charge the same expected price $r_o(p_o+qp_e)$ to all teams, and hence sell the same amount of options to all teams. At this point it should be noted that it is optimal to price products so that we never run out of tickets before the end of the sales horizon. Else, we leave some unsatisfied demand that can be captured by raising prices, and thus increase the revenue. As a consequence, we do not need to control the availability of the products anymore.

From Proposition 1 the aggregate arrival intensity under prices p_a and r_o can be computed as

$$\begin{aligned}\lambda_a(p_a, r_o) &= \lambda \bar{F}_v\left(\frac{p_a - r_o}{(1-q)\ell}\right), \\ \lambda_o^\Sigma(p_a, r_o) &= \lambda\left[\bar{F}_v\left(\frac{r_o}{q}\right) - \bar{F}_v\left(\frac{p_a - r_o}{(1-q)\ell}\right)\right],\end{aligned}$$

where we denote by λ_o^Σ the aggregate arrival intensity of all consumers buying options. Again, we work with arrival intensities as decision variables. Fortunately, there is a one-to-one correspondence between prices and arrival rates, and the inverse functions are given by

$$\begin{aligned}r_o(\lambda_a, \lambda_o^\Sigma) &= q\bar{F}_v^{-1}\left(\frac{\lambda_a + \lambda_o^\Sigma}{\lambda}\right), \\ p_a(\lambda_a, \lambda_o^\Sigma) &= q\bar{F}_v^{-1}\left(\frac{\lambda_a + \lambda_o^\Sigma}{\lambda}\right) + (1-q)\ell\bar{F}_v^{-1}\left(\frac{\lambda_a}{\lambda}\right).\end{aligned}$$

Let us now look at the advance ticket and options pricing problem. After the change of variables, the deterministic approximation of the advance ticket and options pricing problem

(18) becomes

$$
\begin{aligned}
R_o^D = &\max_{\lambda_a \geq 0, \lambda_o^\Sigma \geq 0} R_o^D(\lambda_a, \lambda_o^\Sigma) = T\lambda_a p_a(\lambda_a, \lambda_o^\Sigma) + T\lambda_o^\Sigma r_o(\lambda_a, \lambda_o^\Sigma) \\
&\quad \text{s.t. } T\lambda_a + T\frac{2}{N}\lambda_o^\Sigma \leq C, \\
&\quad \lambda_a + \lambda_o^\Sigma \leq \lambda, \\
= &\max_{\lambda_a \geq 0, \lambda_o^\Sigma \geq 0} T(1-q)\ell v(\lambda_a) + Tqv(\lambda_a + \lambda_o^\Sigma) \qquad (19) \\
&\quad \text{s.t. } T\lambda_a + T\frac{2}{N}\lambda_o^\Sigma \leq C, \qquad (20) \\
&\quad \lambda_a + \lambda_o^\Sigma \leq \lambda,
\end{aligned}
$$

where we have written the objective in terms of the value rate similar to the advance ticket pricing problem. Since the objective function is the linear combination of an affine function and a concave function, it is concave (see Boyd and Vandenberghe (2004)). Hence, (19) and (20) represent a concave maximization problem with linear inequality constraints.

We are now in a position to characterize some conditions under which options are beneficial to the organizer.

Proposition 5. *In the symmetric case, when the seats are scarce ($C < \lambda_a^* T$) and fans strictly prefer their own team over any other ($\ell < 1$) introducing options increases the revenue of the organizer ($R_o^D > R_a^D$). However, when the capacity of the stadium is large ($C \geq \lambda_a^* T$) or fans are indifferent between any team ($\ell = 1$) options do not increase the revenue ($R_a^D \geq R_o^D$).*

Proof. First, we look at the case where the seats are scarce ($C < \lambda_a^* T$). In the advance ticket pricing problem (18) the organizer can afford to price higher, and prices at the run-out rate $\lambda_a^0 = C/T$, i.e. the intensity at which all seats are sold over the time horizon. Note that λ_a^0 is a constrained global optimum of the advance selling problem, and $v'(\lambda_a^0) > 0$. Starting from $(\lambda_a^0, 0)$ in the options pricing problem, we will study the impact in the revenue of increasing the options' intensity.

Clearly, $(\lambda_a^0, 0)$ is a feasible solution of (19). Since capacity is binding, to compensate for an increase in λ_o^Σ the organizer needs to decrease the intensity of advance tickets. Thus, from (20) we obtain that $\frac{d\lambda_a}{d\lambda_o^\Sigma} = -\frac{2}{N} = -q$. The total derivative of the objective w.r.t. λ_o^Σ is then

$$
\frac{dR_o^D}{d\lambda_o^\Sigma} = \frac{\partial R_o^D}{\partial \lambda_o^\Sigma} + \frac{\partial R_o^D}{\partial \lambda_a}\frac{d\lambda_a}{d\lambda_o^\Sigma}. \qquad (21)
$$

Evaluating (21) at $(\lambda_a^0, 0)$ we obtain

$$
\begin{aligned}
\frac{dR_o^D}{d\lambda_o^\Sigma}(\lambda_a^0, 0) &= Tqv'(\lambda_a^0) - Tq\left((1-q)\ell + q\right)v'(\lambda_a^0) \\
&= Tq(1-q)(1-\ell)v'(\lambda_a^0) > 0.
\end{aligned}
$$

This implies that the current solution can be improved by introducing options.

Second, we consider the case where the capacity of the stadium is large ($C \geq \lambda_a^* T$). In the advance ticket pricing problem (18) the organizer ignores the problem of running out of seats and prices according to the revenue maximizer rate λ_a^*. Note that λ_a^* is an unconstrained global optimum, and thus $v'(\lambda_a^*) = 0$. We will show that $(\lambda_a^*, 0)$ is an optimal solution to the options pricing problem.

Clearly, $(\lambda_a^*, 0)$ is a feasible solution of (19). The gradient of the objective is

$$\begin{aligned}
\frac{\partial R_o^D}{\partial \lambda_a} &= T(1-q)\ell v'(\lambda_a) + Tqv'(\lambda_a + \lambda_o^\Sigma), \\
\frac{\partial R_o^D}{\partial \lambda_o^\Sigma} &= Tqv'(\lambda_a + \lambda_o^\Sigma).
\end{aligned}$$

Using the fact that $v'(\lambda_a^*) = 0$, we obtain that the gradient is zero at $(\lambda_a^*, 0)$. Hence, this solution is an unconstrained local optimum. Finally, concavity of the program implies that any local optimum is a global optimum. Thus, both problems attain the same objective value, and $R_o^D = R_a^D$.

Third, we consider the case where fans are indifferent between any team ($\ell = 1$). Note that the objective functions of the advance selling and options pricing problems become $R_a^D(\lambda_a) = Tv(\lambda_a)$, and $R_o^D(\lambda_a, \lambda_o^\Sigma) = T(1-q)v(\lambda_a) + Tqv(\lambda_a + \lambda_o^\Sigma)$ respectively. Let $(\lambda_a, \lambda_o^\Sigma)$ be any feasible solution to (19). We will show $\tilde{\lambda}_a = \lambda_a + q\lambda_o^\Sigma$ is a feasible solution for (18) with greater revenue.

Clearly, $\tilde{\lambda}_a$ is feasible. Regarding revenues

$$\begin{aligned}
R_a^D(\tilde{\lambda}_a) &= Tv\left(\lambda_a + q\lambda_o^\Sigma\right) \\
&= Tv\left((1-q)\lambda_a + q(\lambda_a + \lambda_o^\Sigma)\right) \\
&\geq T(1-q)v(\lambda_a) + Tqv(\lambda_a + \lambda_o^\Sigma) = R_o^D(\lambda_a, \lambda_o^\Sigma),
\end{aligned}$$

where the inequality follows from concavity of v. Thus, $\tilde{\lambda}_a$ always dominates the original solution. $\square$

Next, we study the asymptotic behavior of the different pricing schemes as the number of teams grows to infinity. Notice that this analysis is vacuous when capacity of the stadium is large, since according to Proposition 5 options do not increase the revenue. Hence, the following asymptotic analysis holds when seats are scarce.

We show that, when fans obtain a positive surplus from attending a game without their own team ($\ell > 0$), revenue under options pricing converges to the revenue under advance selling as N grows to infinity. Furthermore, the convergence rate is $O\left(\frac{1}{N}\right)$. The intuition behind this result is that, as the number of teams grows, fans are aware that the probability of their own team reaching the final event decreases. So, in order to keep options attractive for consumers, the organizer needs to set lower prices, and thus revenues generated by options subside. Because fans obtain a positive surplus from attending a game without their own team, as the number of teams grows, proportionally more consumers choose to buy advance tickets.

Proposition 6. *When seats are scarce ($C < \lambda_a^* T$) and fans obtain a positive surplus from attending a game without their own team ($\ell > 0$), the revenue under options pricing converges to the revenue under advance selling as N grows to infinity. Moreover, the convergence rate is given by*

$$1 \leq \frac{R_o^D}{R_a^D} \leq 1 + \frac{2}{N\ell} \frac{v(\lambda_a^*)}{v(\lambda_a^0)}.$$

Proof. First, observe that since capacity is scarce the optimal solution of the advance ticket pricing problem (18) is the run-out rate $\lambda_a^0 = C/T$, and it is independent of the number of teams. Let $\left\{ \left(\lambda_a^{(N)}, \lambda_o^{(N)} \right) \right\}_N$ be a sequence of optimal solutions to the advance ticket and options pricing problem (19) indexed by the number of teams. Scarcity of seats together with concavity guarantee that the capacity constraint (20) is binding at the optimal solution. Since intensities are bounded from above by λ, this guarantees that $\lim_{N \to \infty} \lambda_a^{(N)} = \lambda_a^0$. As a side note, it is not necessarily the case that $\lambda_o^{(N)}$ converges to zero as N goes to infinity.

Second, let us show that the following inequality holds

$$\lambda_a^{(N)} \leq \lambda_a^0 \leq \lambda_a^{(N)} + \lambda_o^{(N)} \leq \lambda_a^*. \tag{22}$$

The first inequality is a trivial consequence of the capacity constraint (20). For the second inequality observe that the capacity constraint (20) is binding, and thus $\lambda_a^0 = \lambda_a^{(N)} + \frac{2}{N} \lambda_o^{(N)} \leq \lambda_a^{(N)} + \lambda_o^{(N)}$. For the third inequality suppose that $\lambda_a^{(N)} + \lambda_o^{(N)} > \lambda_a^*$ for some N, and consider an alternate solution in which the options' intensity is decreased to $\tilde{\lambda}_o^{(N)} = \lambda_a^* - \lambda_a^{(N)}$. Clearly, $\tilde{\lambda}_o^{(N)} \geq 0$, the new solution satisfies the capacity constraint and the third inequality. Moreover,

$$\begin{aligned} R_o^D \left(\lambda_a^{(N)}, \lambda_o^{(N)} \right) &= T(1 - \frac{2}{N}) \ell v(\lambda_a^{(N)}) + T \frac{2}{N} v(\lambda_a^{(N)} + \lambda_o^{(N)}) \\ &\leq T(1 - \frac{2}{N}) \ell v(\lambda_a^{(N)}) + T \frac{2}{N} v(\lambda_a^*) = R_o^D \left(\lambda_a^{(N)}, \tilde{\lambda}_o^{(N)} \right), \end{aligned}$$

where the first inequality follows since λ_a^* is the least maximizer of v. Thus, the new solution is also optimal. This shows that if the third inequality does not hold for any N, we can construct a solution $\left(\lambda_a^{(N)}, \tilde{\lambda}_o^{(N)} \right)$ for which it holds. So, without loss of generality, we can conclude that the third inequality holds.

Finally, the ratio of optimal revenues can be written as

$$\begin{aligned} \frac{R_o^D \left(\lambda_a^{(N)}, \lambda_o^{(N)} \right)}{R_a^D(\lambda_a^0)} &= \frac{T(1 - \frac{2}{N}) \ell v(\lambda_a^{(N)}) + T \frac{2}{N} v(\lambda_a^{(N)} + \lambda_o^{(N)})}{T \left[(1 - \frac{2}{N}) \ell + \frac{2}{N} \right] v(\lambda_a^0)} \\ &= \frac{N\ell - 2\ell}{N\ell + 2(1 - \ell)} \frac{v(\lambda_a^{(N)})}{v(\lambda_a^0)} + \frac{2}{N\ell + 2(1 - \ell)} \frac{v(\lambda_a^{(N)} + \lambda_o^{(N)})}{v(\lambda_a^0)} \\ &\leq \frac{v(\lambda_a^{(N)})}{v(\lambda_a^0)} + \frac{2}{N\ell} \frac{v(\lambda_a^{(N)} + \lambda_o^{(N)})}{v(\lambda_a^0)} \leq 1 + \frac{2}{N\ell} \frac{v(\lambda_a^*)}{v(\lambda_a^0)}, \end{aligned}$$

where the second equation is obtained by algebraic manipulation, the first inequality follows from bounding the leading factor of the first term by 1 and the leading factor of the second term by $\frac{2}{N\ell}$, and the second inequality follows from (22) together with the fact that v is non-decreasing in $[0, \lambda_a^*]$. $\square$

In the case where fans obtain zero surplus from attending a game without their own team ($\ell = 0$), the previous discussion no longer holds. Now, options and advance tickets are equivalent to customers, and they are only interested in one outcome: their own team advancing to the final game. Because the probability of that outcome converges to zero, the number of sold tickets converges to zero as well. This observation, combined with the existence of the *null price* (or $\lim_{\lambda_a \to 0} v(\lambda_a) = 0$), causes the organizer's revenue to diminish to zero in all pricing schemes as the number of teams increase. Surprisingly, even though the revenues when only advance tickets are offered and, when both advance ticket and options are offered converge to zero, they do so at different rates. The rationale is that when the organizer offers only options each team has up to $C/2$ tickets available. Hence, the capacity of the stadium is extended, and for a suitable large N the organizer may price according to the revenue maximizer rate λ_a^*.

Proposition 7. *When seats are scarce ($C < \lambda_a^* T$) and fans obtain zero surplus from attending a game without their own team ($\ell = 0$), the revenue obtained when both advance tickets and options are offered strictly dominates the case when only advance tickets are offered. Moreover, their ratio is given by*

$$\frac{R_o^D}{R_a^D} = \frac{v\left(\min\left\{\lambda_a^*, \lambda_a^0 \frac{N}{2}\right\}\right)}{v(\lambda_a^0)}$$
$$= \frac{v(\lambda_a^*)}{v(\lambda_a^0)} > 1 \quad \textit{when } N \geq 2\left\lceil \frac{\lambda_a^*}{\lambda_a^0} \right\rceil.$$

Proof. If $\ell = 0$ options and advance tickets are equivalent to customers, and customers choose the product with the lowest price. Thus, we only need to consider the case where the organizer sells only options the whole time horizon. The options pricing problem is now

$$R_o^D = \max_{\lambda_o^\Sigma \geq 0} \quad T\frac{2}{N} v(\lambda_o^\Sigma)$$
$$\text{s.t. } T\lambda_o^\Sigma \leq \frac{N}{2}C, \quad \lambda_o^\Sigma \leq \lambda.$$

This problem is similar to the advance selling problem (18) except that capacity is scaled by $\frac{N}{2}$. Scarcity implies that $C < \lambda^* T$, and thus the optimal solution is $\lambda_o^{(N)} = \min\left\{\lambda_a^*, \lambda_a^0 \frac{N}{2}\right\}$. Then, the optimal value is $R_o^D = T\frac{2}{N} v(\min\left\{\lambda_a^*, \lambda_a^0 \frac{N}{2}\right\})$. Finally, observe that for $N \geq 2\left\lceil \frac{\lambda_a^*}{\lambda_a^0} \right\rceil$ the organizer may price according to the revenue maximizer rate λ_a^* and $R_o^D = T\frac{2}{N} v(\lambda_a^*)$. $\square$

5 Social Efficiency

5.1 General Case

How do the introduction of options affect customers' surplus? Options allow the fans to hedge against the risk of watching a team that it is not of their preference. As a consequence, a larger number of seats will be taken by fans of the teams that are playing in the final. So, intuitively we expect the introduction of options to increase the total surplus of the fans. In this section we show how to compute the total consumer surplus of an allocation. We will see that the surplus can conveniently be expressed in terms of the intersections of the utility lines as defined in Proposition 1 and the integrated tail of the valuations, which is defined as $\bar{G}_v^i(x) = \mathbb{E}\left[(V-x)^+\right] = \int_x^\infty \bar{F}_v^i(v)\mathrm{d}v$. In the next section we will characterize some conditions under which options are beneficial for the consumers in the symmetric case. We will see that these conditions coincide with the ones we developed previously in Section 4 regarding the revenue improvement resulting from offering options.

First, consider the *surplus rate*, which is the instantaneous rate at which surplus is generated, for team's i fans who purchase advance tickets. We distinguish whether options are offered simultaneously or not. In the case when options are not offered, fans purchase advance tickets when their utility $U_a^i(V)$ is non-negative. The arrival rate of such consumers is $\lambda_a^i(\{A\}) = \lambda^i \mathbb{P}\{U_a^i(V) \geq 0\}$, and their expected utility conditioned on them buying A is $\mathbb{E}\left[U_a^i(V) | U_a^i(V) \geq 0\right]$. Hence, the surplus rate is given by

$$\begin{aligned} s_a^i(\{A\}) &= \lambda^i \mathbb{E}\left[U_a^i(V)\mathbf{1}\{U_a^i(V) \geq 0\}\right] \\ &= \lambda^i(q^i + (1-q^i)\ell^i)\mathbb{E}\left[(V-c^i)\mathbf{1}\{V \geq c^i\}\right] = \lambda^i(q^i + (1-q^i)\ell^i)\bar{G}_v(c^i). \end{aligned} \tag{23}$$

When advance tickets are offered simultaneously with options, we need to take into account that fans purchase advance tickets when both $U_a^i(V) \geq 0$, and $U_a^i(V) \geq U_o^i(V)$. Now, the surplus rate is given by

$$\begin{aligned} s_a^i(\{A, O^i\}) &= \lambda^i \mathbb{E}\left[U_a^i(V)\mathbf{1}\{U_a^i(V) \geq 0, U_a^i(V) \geq U_o^i(V)\}\right] \\ &= \lambda^i(q^i + (1-q^i)\ell^i)\mathbb{E}\left[(V-c^i)\mathbf{1}\{V \geq \max\{a^i, c^i\}\}\right] \\ &= \lambda^i(q^i + (1-q^i)\ell^i)\left(\bar{G}_v(\max\{a^i, c^i\}) + (a^i - c^i)^+\bar{F}_v(a^i)\right). \end{aligned}$$

The surplus rate for team's i fans purchasing options can be obtained in a similar way. When advance tickets are not offered, fans purchase options when their utility $U_o^i(V)$ is non-negative, and the surplus rate is given by

$$s_o^i(\{O^i\}) = \lambda^i \mathbb{E}\left[U_o^i(V)\mathbf{1}\{U_o^i(V) \geq 0\}\right] = \lambda^i q^i \mathbb{E}\left[(V-b^i)\mathbf{1}\{V \geq b^i\}\right] = \lambda^i q^i \bar{G}_v(b^i).$$

The last case remaining is when options are offered simultaneously with advance tickets. Here, we need to take into account that fans purchase options when both $U_o^i(V) \geq 0$, and $U_o^i(V) \geq U_a^i(V)$.

The surplus rate is given by

$$\begin{aligned}
s_o^i(\{A, O^i\}) &= \lambda^i \mathbb{E}\left[U_o^i(V)\mathbf{1}\{U_o^i(V) \geq 0, U_o^i(V) \geq U_a^i(V)\}\right] \\
&= \lambda^i q^i \mathbb{E}\left[(V - b^i)\mathbf{1}\{b^i \leq V \leq a^i\}\right] \\
&= \lambda^i q^i \left(\bar{G}_v(\max\{a^i, b^i\}) - \bar{G}_v(a^i) + (a^i - b^i)^+ \bar{F}_v(a^i)\right).
\end{aligned}$$

As in the case of the revenue rates, we can compute the aggregate surplus rate when offering a subset S as $s(S) = \sum_{i=1}^N s_a^i(S) + s_o^i(S)$. Notice that the first summand is nonzero only if $A \in S$, while the second summand is nonzero if $O^i \in S$. Thus, given an allocation $\{t(S)\}_{S \subseteq \mathcal{S}}$ the total expected surplus is $\sum_{S \subseteq \mathcal{S}} s(S)t(S)$.

5.2 Symmetric Case

In the following we will show that, under some assumptions, the introduction of options increases the customers' surplus. Consequently, options can benefit both the promoter and the consumers. We proceed as in the case of the revenue. First, we obtain the surplus of the advance ticket pricing problem in terms of intensities as decision variables. Second, we move on to the advance ticket and options pricing problem, and give a simple expression for the consumer surplus. We conclude by identifying a sufficient assumption and proving our result.

From equation (23) the total surplus of consumers that will buy an advance ticket when the arrival intensity is λ_a, denoted by $S_a^D(\lambda_a)$, is

$$\begin{aligned}
S_a^D(\lambda_a) &= T\lambda(q + (1-q)\ell)\bar{G}_v\left(\frac{p_a(\lambda_a)}{q + (1-q)\ell}\right) \\
&= T(q + (1-q)\ell)s(\lambda_a)
\end{aligned}$$

where the surplus rate is defined as

$$s(\lambda_a) = \lambda \bar{G}_v\left(\frac{p_a(\lambda_a)}{q + (1-q)\ell}\right) = \lambda \bar{G}_v\left(\bar{F_v}^{-1}\left(\frac{\lambda_a}{\lambda}\right)\right) = \lambda \int_{\bar{F_v}^{-1}(\lambda_a/\lambda)}^{\infty} \bar{F}_v(v)\mathrm{d}v.$$

Notice that the surplus rate is defined on $[0, \lambda]$. Additionally, it is increasing, continuous, differentiable, non-negative, and bounded. The monotonicity stems from the fact that $\bar{F_v}^{-1}$ is decreasing and $\bar{G}_v$ is non-increasing. Moreover, it satisfies $\lim_{\lambda_a \to 0} s(\lambda_a) = 0$, and $\lim_{\lambda_a \to \lambda} = \lambda \mathbb{E}V$. In contrast to the revenue rate, the maximum is reached when the intensity is set to λ, or equivalently the price set to zero. Not surprisingly, the total consumer surplus is maximized when the tickets are given for free at the expense of the promoter's revenue.

In the advance ticket and options pricing problem two sources contribute to the total consumer surplus. The first source is consumers who choose advance tickets over options. The second source is consumers who chose options over advance tickets. Some algebra shows that the total consumer surplus in terms of the arrival intensities, denoted by $S_o^D(\lambda_a, \lambda_o^\Sigma)$, is

$$S_o^D(\lambda_a, \lambda_o^\Sigma) = T(1-q)\ell s(\lambda_a) + Tqs(\lambda_a + \lambda_o^\Sigma).$$

Observe that the formula for consumer surplus is similar to the organizer's revenue with the exception that the value rate is replaced by the surplus rate.

Next, we study how introducing options impacts the consumer surplus. Recall that, from Proposition 5, options are beneficial to the organizer only if capacity is scarce and fans strictly prefer their own team over any other. Hence, we only need to consider the consumer surplus under those assumptions, else the organizer has no incentive to sell options. The following proposition shows that if the surplus rate is convex, options do increase consumer surplus.

Proposition 8. *Suppose that the surplus rate is convex. When seats are scarce ($C < \lambda_a^* T$) and fans strictly prefer their own team over any other ($\ell < 1$) introducing options increase the consumer surplus ($S_o^D > S_a^D$).*

Proof. First, let $(\lambda_a, \lambda_o^\Sigma)$ be the optimal solution to the options pricing problem. Since seats are scarce, the capacity constraint (20) is binding in the optimal solution. Then $\lambda_a^0 = C/T = \lambda_a + q\lambda_o^\Sigma = (1-q)\lambda_a + q(\lambda_a + \lambda_o^\Sigma)$, where we have written λ_a^0 as a convex combination of λ_a and $\lambda_a + \lambda_o^\Sigma$. Consider the convex combination of the same points, denoted by $\hat{\lambda}_a$, in which we multiply the first weight by ℓ and re-normalize. Hence, $\hat{\lambda}_a$ is given by

$$\hat{\lambda}_a = \frac{(1-q)\ell}{q + (1-q)\ell}\lambda_a + \frac{q}{q + (1-q)\ell}(\lambda_a + \lambda_o^\Sigma).$$

Notice that $\hat{\lambda}_a > \lambda_a^0$. This follows from $\lambda_o^\Sigma > 0$ implying that the second point is strictly greater than the first, and the weight of this larger point being larger in $\hat{\lambda}_a$ than in λ_a^0.

Finally, we have that

$$\begin{aligned} S_o^D &= S_o^D(\lambda_a, \lambda_o^\Sigma) = T(1-q)\ell s(\lambda_a) + Tqs(\lambda_a + \lambda_o^\Sigma) \\ &\geq T(q + (1-q)\ell)s(\hat{\lambda}_a) > T(q + (1-q)\ell)s(\lambda_a^0) = S_a^D(\lambda_a^0) = S_a^D, \end{aligned}$$

where the first inequality follows from the convexity of the surplus rate, the second inequality from the fact that the surplus rate is increasing and $\hat{\lambda}_a > \lambda_a^0$, and the last equality from λ_a^0 being the optimal solution to the advance selling problem when seats are scarce. Thus, the introduction of options increases the consumer surplus. As a side note, any feasible solution to the options pricing problem in which the capacity constraint (20) is binding verifies that $S_o^D(\lambda_a, \lambda_o^\Sigma) \geq S_o^D$. □

Fortunately, the surplus rate turns out to be convex for many distributions. Lemma 2 gives one sufficient condition, namely the surplus rate is convex when the valuation random variable has IFR. Thus, options are beneficial for consumers whose valuations have IFR.

Lemma 2. *If the valuation random variable has (strict) IFR, then the surplus rate is (strictly) convex.*

Proof. The derivative of the surplus rate w.r.t. λ_a is

$$\frac{ds}{d\lambda_a}(\lambda_a) = \frac{\lambda_a}{\lambda} \frac{1}{f_v\left(\bar{F_v}^{-1}(\lambda_a/\lambda)\right)}.$$

Composing the derivative with $\lambda_a(c) = \lambda \bar{F_v}(c)$ we get

$$\left(\frac{ds}{d\lambda_a} \circ \lambda_a\right)(c) = \frac{ds}{d\lambda_a}\left(\lambda \bar{F_v}(c)\right) = \frac{\bar{F_v}(c)}{f(c)} = \frac{1}{h(c)}.$$

IFR implies that the composite function is non-increasing in c. Because $\lambda_a(c)$ is decreasing, we conclude that original derivative is non-decreasing and s is convex. The proof for the strict case follows similarly. □

6 Extensions

6.1 No-Arbitrage Pricing

We want to exclude the possibility of a third party, *the arbitrageur*, from taking advantage of differences in prices to obtain a risk-free profit. For instance, an arbitrageur may simultaneously offer options to fans and buy advance tickets to fulfill the obligations, or offer options for some teams while buying options from others.

In the following, we denote by $\theta = (\theta_a, \theta_o^1, \ldots, \theta_o^N) \in \mathbb{R}^{|\mathcal{S}|}$ a portfolio that assigns weight θ_i to product i. By convention, a positive value for θ_i indicates that the arbitrageur is buying product i from the organizer, while when θ_i is negative she is selling product i in the market. Using this notation, today's *market value* of the portfolio is given by

$$p^T\theta = \theta_a p_a + \sum_{i=1}^{N} \theta_o^i p_o^i.$$

Uncertainty is represented by the finite set $\mathcal{T}$ of *states*, one of which will be revealed as true. When state $\{i,j\} \in \mathcal{T}$ realizes, the *payoff* of the portfolio is $-\theta_o^i p_e^i - \theta_o^j p_e^j$. These can be written more compactly in matrix notation as $R\theta$, where $R \in \mathbb{R}^{|\mathcal{T}|\times|\mathcal{S}|}$ is the matrix of future payoffs. Notice that exploiting the structure of the problem we can the write payoff matrix as

$$R = \begin{pmatrix} \mathbf{0} & -\Lambda^T \operatorname{diag}(p_e) \end{pmatrix}, \tag{24}$$

where $\Lambda \in \mathbb{R}^{N\times|\mathcal{T}|}$ is such that $(\Lambda)_{is} = 1$ if in state s team i advances to the final and 0 otherwise.

Finally, in order to fulfill future obligations, the portfolio needs to satisfy $\theta_a + \theta_o^i + \theta_o^j \geq 0$ whenever state $(i,j) \in \mathcal{T}$ realizes. For instance, if the arbitrageur sells one option for team i and another for team j, then she needs to hold at least two advance tickets for the case that both teams advance in the final. Similarly, we write the obligation restriction in matrix notation as

$A\theta \geq 0$, where $A \in \mathbb{R}^{|\mathcal{T}|\times|\mathcal{S}|}$ is the obligation matrix. Again, we may exploit the structure of the problem, and write the obligation matrix as

$$A = \begin{pmatrix} \mathbf{1} & \Lambda^T \end{pmatrix}. \tag{25}$$

Textbook arbitrage requires no capital and entails no risk. Thus, an arbitrage opportunity is a transaction that involves no negative cash flow future state and a positive cash flow today. We formalize this statement in Definition 1.

Definition 1. *An arbitrage opportunity is a portfolio* $\theta \in \mathbb{R}^{|\mathcal{S}|}$ *with* $A\theta \geq 0$ *such that* $p^T\theta \leq 0$ *and* $R\theta \geq 0$ *with at least one strict inequality.*

The following theorem characterizes the set of arbitrage-free prices. The requirement that y is strictly positive for all outcomes is due to our definition of arbitrage. If instead we would employ a *strong arbitrage* definition as in LeRoy and Werner (2000), that is, we exclude portfolios with $A\theta \geq 0$ such that $p^T\theta < 0$ and $R\theta \geq 0$; then we would only require y to be non-negative.

Theorem 1. *Prices constitute an arbitrage-free market if and only if there exists* $z, y \in \mathbb{R}^{|\mathcal{T}|}$ *such that* $z \geq 0$*,* $y > 0$*, and*

$$\sum_{s\in\mathcal{T}} z_s = p_a, \sum_{s\in\mathcal{T}:i\in s} z_s = p_o^i + p_e^i \sum_{s\in\mathcal{T}:i\in s} y_s, \quad \forall i = 1,\ldots,N.$$

Proof. We want to show that there is no portfolio θ with $A\theta \geq 0$, $\begin{pmatrix} -p^T \\ R \end{pmatrix}\theta \geq 0$, and $\begin{pmatrix} -p^T \\ R \end{pmatrix}\theta \neq 0$. Equivalenty, from Tucker's Theorem of the Alternative Mangasarian (1987) there exists $z, y \in \mathbb{R}^{|\mathcal{T}|}$ such that $R^Ty + A^Tz = p$, $y > 0$, and $z \geq 0$. The result follows by exploiting the fact that $A^T = \begin{pmatrix} \mathbf{1}^T \\ \Lambda \end{pmatrix}$, and $R^T = \begin{pmatrix} \mathbf{0} \\ -\mathrm{diag}(p_e)\Lambda \end{pmatrix}$. $\square$

Notice that by normalizing in 1, we can interpret z as probability distributions over the set of outcomes. This suggests that the result can be further simplified by aggregating outcomes, and considering the probabilities of each team advancing to the final. Indeed, we may rewrite the arbitrage conditions in terms of $z^i = \sum_{s\in\mathcal{T}:i\in s} z_s$, and $y^i = \sum_{s\in\mathcal{T}:i\in s} y_s$. It is clear that every distribution over outcomes induces a distribution over teams, but the converse does not necessarily hold. Lemmas 3 and 4 identify the set of attainable distributions over teams for different tournament structures. For example in the case of a single elimination tournament we require that the teams in both branches sum up to the same value, that is, $\sum_{i\in\mathcal{T}_1} y^i = \sum_{i\in\mathcal{T}_2} y^i$ and $\sum_{i\in\mathcal{T}_1} z^i = \sum_{i\in\mathcal{T}_2} z^i$. Hence, after eliminating z^i from the system, we obtain that there is no arbitrage if there exists that a strictly positive $y \in \mathbb{R}^N$ such that

$$p_a = \sum_{i \in \mathcal{T}_1} p_o^i + y^i p_e^i = \sum_{i \in \mathcal{T}_2} p_o^i + y^i p_e^i,$$
$$\sum_{i \in \mathcal{T}_1} y^i = \sum_{i \in \mathcal{T}_2} y^i.$$

Lemma 3. *Consider a general tournament, and let the cone* $\mathcal{C} = \{\alpha \in \mathbb{R}^N \mid \alpha = \Lambda y, y \geq 0\}$. *Then,* $\alpha \in \mathcal{C}$ *if and only if* $2\alpha_i \leq \sum_{j=1}^N \alpha^j$ *for all* $i = 1, \ldots, N$, *and* $\alpha \geq 0$.

Lemma 4. *Consider a single elimination tournament, and let the cone* $\mathcal{C} = \{\alpha \in \mathbb{R}^N \mid \alpha = \Lambda y, y \geq 0\}$. *Then,* $\alpha \in \mathcal{C}$ *if and only if* $\sum_{i \in \mathcal{T}_1} \alpha^i = \sum_{i \in \mathcal{T}_2} \alpha^i$, *and* $\alpha \geq 0$.

Proof. For the only if, take any $\alpha \in \mathcal{C}$ and observe that

$$\sum_{i \in \mathcal{T}_1} \alpha^i = \sum_{i \in \mathcal{T}_1} \sum_{s \in \mathcal{T}} (\Lambda)_{is} y_s = \sum_{i \in \mathcal{T}_1} \sum_{j \in \mathcal{T}_2} y_{(i,j)} = \sum_{j \in \mathcal{T}_2} \sum_{i \in \mathcal{T}_1} y_{(i,j)} = \sum_{j \in \mathcal{T}_2} \sum_{s \in \mathcal{T}} (\Lambda)_{js} y_s = \sum_{j \in \mathcal{T}_2} \alpha^j.$$

For the if part we proceed by contradiction. If $\alpha = 0$ the result is trivial, so we assume that $\alpha \neq 0$. Since the cone $\mathcal{C}$ is closed and convex and $\alpha \notin \mathcal{C}$, by the Strictly Separating Hyperplane Theorem there exists an hyperplane that strictly separates them Boyd and Vandenberghe (2004). Alternatively, there is a vector $q \in \mathbb{R}^N$ such that $q^T \alpha < q^T \Lambda y$ for all $y \geq 0$.

Pick any $i' \in \mathcal{T}_1$, and set y such that $y_{(i,j)} = \alpha_j$ if $i = i'$ and 0 otherwise. Evaluating the right hand side at y we get

$$q^T \Lambda y = \sum_{i \in \mathcal{T}_1} \sum_{j \in \mathcal{T}_2} (q_i + q_j) y_{(i,j)} = \sum_{j \in \mathcal{T}_2} (q_i' + q_j) \alpha_j = q_i' \sum_{j \in \mathcal{T}_2} \alpha_j + \sum_{j \in \mathcal{T}_2} q_j \alpha_j = \sum_{i \in \mathcal{T}_1} q_i' \alpha_i + \sum_{j \in \mathcal{T}_2} q_j \alpha_j,$$

where the last equality follows from the hypothesis. Hence, $\sum_{i \in \mathcal{T}_1 \setminus i'} (q^i - q^{i'}) \alpha_i < 0$ from the separating hyperplane theorem, and we conclude that $q^{i''} < q^{i'}$ for some $i'' \in \mathcal{T}_1 \setminus i'$ since $\alpha \geq 0$. Repeating the argument with i'' we obtain that $q^{i'''} < q^{i''} < q^{i'}$ for some $i''' \in \mathcal{T}_1 \setminus i''$. Repeatedly applying the same argument we reach a contradiction. □

7 Numerical Examples

In this section we describe numerical experiments conducted to evaluate the revenue improvements from offering options to consumers. We check the sensitivity of revenue improvements for various distributions of consumer valuation, love of the game, and different levels of arrival intensity.

Our numerical example is based on Superbowl XLVI which will take place on February 12th, 2012. For the sake of computational simplicity we assume that pricing decisions are made at the Conference Championship level where only four teams are left and we also assume that the teams that will play in the finals will be the winners of New Orleans Saints vs. Minnesota

Vikings, and Indianapolis Colts vs. New York Jets games. Let us note that those teams were the divisional round winners in the 2009 season. We used the betting odds from Vegas.com to calculate the probabilities of each team playing in the Superbowl which gave $q = (.6, .4, .65, .35)$ for (Saints, Vikings, Colts, Jets). We estimated arrival rates arrival rates proportional to the population of each team's hometown, $\lambda = (0.1271, 0.0477, 0.0675, 0.7576)$.

We tried two different distributions for fans' valuation V, uniform and truncated normal, with equal means and variances. Also, we checked the sensitivity of our results against different love of the game parameters and *load factors*, where the load factor l_f is defined by $l_f = (T * \lambda_{tot})/C$. We tried five different values of $\ell(0.001, 0.1, 0.2, 0.5$ and $0.9)$ and two different values of $l_f(1$ and $3)$. Each set of parameters is represented by $NFL_{a,b,c}$ where a equals u if the distribution is uniform and tn if it is truncated normal, b is equal to the load factor and c is equal to ℓ.

The policy that we used in the simulation was to construct the offer sets and times from the solution of the MBLP, and offer the products according to it. Since MBLP is a deterministic approximation to our problem, it does not give any information about the ordering of the offer sets, so we offered the sets randomly. The simulation results were compared to the results of the deterministic advance selling problem, MBLP and simulated advance selling problem. The simulation outputs for different sets of parameters are given in Table 3. OPT SIM corresponds to the average sample path revenue when the policy obtained from MBLP is used and NAIVE SIM corresponds to the average sample path revenue obtained when all products, advance tickets and options, are offered throughout the whole sales horizon with prices obtained from the MBLP.

Param. Set	ADV DET	OPT DET	ADV SIM	OPT SIM	NAIVE SIM
$NFL_{u,3,0.01}$	\$71.92	\$80.73	\$71.80	\$80.62	\$80.62
$NFL_{u,3,0.1}$	\$83.84	\$87.88	\$83.70	\$87.74	\$87.74
$NFL_{u,3,0.2}$	\$95.68	\$98.30	\$95.53	\$98.13	\$98.13
$NFL_{u,3,0.5}$	\$130.41	\$130.87	\$130.22	\$130.67	\$94.52
$NFL_{u,3,0.9}$	\$175.53	\$175.53	\$175.25	\$175.25	\$142.25
$NFL_{u,1,0.01}$	\$26.97	\$28.31	\$26.97	\$28.31	\$26.43
$NFL_{u,1,0.1}$	\$31.44	\$31.44	\$31.44	\$31.44	\$26.18
$NFL_{tn,3,0.01}$	\$72.01	\$81.95	\$71.89	\$81.82	\$81.82
$NFL_{tn,3,0.1}$	\$83.70	\$88.91	\$83.55	\$88.71	\$83.70
$NFL_{tn,3,0.2}$	\$95.29	\$99.03	\$95.14	\$98.86	\$98.86
$NFL_{tn,3,0.5}$	\$130.06	\$130.06	\$129.10	\$129.10	\$105.90
$NFL_{tn,3,0.9}$	\$173.69	\$173.69	\$173.42	\$173.42	\$132.79
$NFL_{tn,1,0.01}$	\$28.88	\$30.50	\$28.89	\$30.51	\$29.65
$NFL_{tn,1,0.1}$	\$33.74	\$33.74	\$33.74	\$33.74	\$30.59

Table 3: Revenues for different parameters and policies(in millions)

Let us first analyze how our approximation performs. Table 4 gives the gaps between the deterministic approximation and simulation results for different load factors. We can see that all gaps are below 1%. So, our approximation performs very well for both distributions and is robust for different values of ℓ and l_f.

	$l_f = 3$		$l_f = 1$	
ℓ	Uniform	Trunc. Norm	Uniform	Trunc. Norm
0	0.13%	0.15%	0.00%	0.01%
0.1	0.16%	0.22%	0.00%	0.00%
0.2	0.17%	0.18%	0.00%	0.00%
0.5	0.15%	0.74%		
0.9	0.16%	0.15%		

Table 4: Optimality gaps for different parameters and distributions

Now, let us examine how offering options affects the organizer's revenue. Table 5 lists revenue improvements for different sets of parameters. From the table above we see that offering options is most beneficial when ℓ is low. As it is increased the potential benefit the organizer can get from offering options decreases. This result is intuitive since options target fans who care about the teams playing in the finals. As ℓ is increased the utility that the fans get from watching other teams increases, so fans will care less about the finalists and be more willing to buy advance tickets. Consequently, options become less attractive for the fans and the organizer does not benefit as much from offering them. Table 5 also confirms that options are most beneficial when capacity is scarce which coincides with the results obtained about symmetric tournaments in §4. It is seen from the table above that once the load factor is decreased, which is equivalent to making capacity abundant, options do not result in any revenue improvement. In order to keep options attractive the organizer has to set its sales price lower than the advance ticket price. Thus, the organizer has to sell multiple options to get the same revenue it does from selling a single advance ticket. When the capacity is abundant, the total number of advance tickets or options that the organizer can sell is less than the stadium's capacity. So, the organizer will prefer selling advance tickets over options. Lastly, we see that our results do not appear to be very sensitive to the shape of the distribution.

8 Conclusion

In this paper, we analyzed consumer options that are contingent on a specific team reaching the tournament final. Offering options, in addition to advance tickets, allows an organizer to segment fans. The organizer targets fans with a higher willingness to pay, who are less sensitive to the outcome, with advance tickets, whereas options aim fans who are mostly willing to attend

	$l_f = 3$		$l_f = 1$	
ℓ	Uniform	Trunc. Norm	Uniform	Trunc. Norm
0	12.24%	13.79%	0.00%	0.01%
0.1	4.83%	6.23%	0.00%	0.00%
0.2	2.74%	3.93%	0.00%	0.00%
0.5	0.35%	0.00%		
0.9	0.00%	0.00%		

Table 5: Revenue improvements from options for different parameters and distributions

a game with their preferred team playing. Our results show that the organizer can potentially increase its profits by taking advantage of this segmentation, and offering options is beneficial for the fans as well.

In this work we specifically addressed the problem of pricing and capacity control of such options and advance tickets, under a stochastic and price-sensitive demand. The organizer faces the problem of pricing the tickets and options, and determining the number of tickets to offer so as to maximize her expected revenue. We propose solving the organizer's problem using a two-stage optimization problem. The first stage optimizes over the prices, while the second the optimizes the expected revenue by controlling the subset of products that is offered at each point in time using a discrete choice revenue management model. The second-stage problem is a stochastic control problem, and in most cases is very difficult to solve. Hence, we propose an efficient deterministic approximation, which is shown to be asymptotically optimal and numerical results confirm that this approximation turns out to be accurate in our setting.

To develop some insight, we provide a theoretical characterization of the problem in the symmetric case, i.e., when all teams are equal in terms of arrival rate and other characteristics. Under some mild assumptions, we show that when the seats are scarce and fans strictly prefer their own team over any other, introducing options increases both the revenue of the organizer, and the surplus of the consumers. Surprisingly, we show that the benefits of options subside as the number of teams grow.

As for future research directions, approaching this problem from a dynamic pricing point of view should be considered. Dynamic pricing may provide higher revenues at the cost of substantially increased complexity. Two natural extensions are relaxing the no-resale restriction and allowing secondary markets, and selling tickets after the tournament starts. Relaxing these two assumptions can affect the fans' decisions substantially, and deserve special attention. However, this may result in an intricate model since the fans may now delay their decisions of buying tickets and options. Lastly, the single quality seat restriction may be relaxed by dividing the stadium according to seat quality. This has the possibility of complicating the consumer choice part of the model considerably. Instead of having to choose from at most two products, fans now may face a wide array of choices.

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