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Robert Phillips
Columbia University, Nomis Solutions

A. Serdar Simsek
Columbia University

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Robert Phillips*　　A. Serdar Simsek †

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Abstract

We consider a network in which products consist of combinations of connecting edges and each edge corresponds to a perishable resource. In our model, different revenue-maximizing "controllers" determine the prices associated with different resources and the price of the product is the sum of the prices of the constituent resources. At one extreme, a single controller might set all resource prices – at the other extreme there would be a different controller associated with each resource. We show that decentralized pricing always leads to lower total revenue relative to centralized pricing. For the uncapacitated networks, we develop bounds on the "price of anarchy" – the loss from totally decentralized control versus centralized control – as the number of controllers increases. We present provably convergent algorithms for calculating Nash equilibrium prices for both the uncapacitated and capacitated cases. We present numerical analyses to illustrate the effect on producer and consumer surplus of decentralization. While we develop our model in the context of airline pricing, it is applicable to any service network such as freight transportation, pipelines, and toll roads as well as to the more general case of supply chain networks.

*Columbia Business School and Nomis Solutions, e-mail: rp2051@columbia.edu

†Columbia Business School, e-mail: as3497@columbia.edu

1 Introduction

We consider the problem of pricing on a network. The classic example is an airline that operates a flight network with connections. In this case, it is standard to refer to the individual flights as *resources* and the itineraries that passengers can fly using one or more combinations of flights as *products*. The problem of how an airline should price and manage its products given an underlying flight network has been widely studied – see Chapter 5 of Talluri and van Ryzin (2004b) for a survey.

In this paper, we consider the case in which the the prices and availability of resources in a network may be set by different firms, which we call controllers, each of whom is seeking to maximize expected revenue or profit. The price that is quoted to a customer for a product is the sum of the prices quoted by the controllers. This can be illustrated by the passenger airline example in Figure 1. In this example, there are two flights, one from New York to Chicago and one from Chicago to Los Angeles. We assume that the two flights connect so that three separate products can be sold: a New York to Chicago product, a Chicago to Los Angles product, and a New York to Los Angeles product connecting in Chicago. In the *decentralized case*, there are two separate controllers, one for each flight. Each controller can set the price for his direct product and a (possibly different) price for the connecting New York to Los Angeles product. The market price for the connecting product is the sum of the prices set by the two controllers. In airline terms, the New York to Los Angeles product is an *interline connection* and, assuming that the two airlines involved are not partners, the price charged to the consumer would be the sum of the prices set by the individual carriers. In contrast, a centralized controller would set the prices for all three products in order to maximize total revenue.[1]

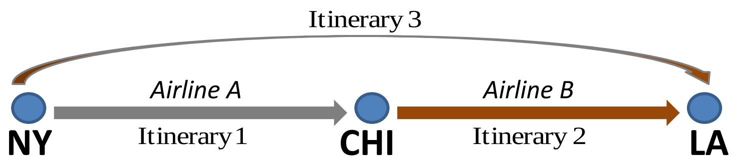

Figure 1: A simple network example from passenger airline industry

It is well known that two controllers setting the price for a single product in a network such as that in Figure 1 will set a higher price than a single controller resulting in both lower total

[1] The calculation of airline interline fares is actually considerably more complex, however the basic concept is the same; see Barnes (2012) for more details.

producer surplus and consumer surplus. This is the classic case of *double marginalization* first identified by Spengler (1950). Our work extends the concept of double marginalization in two ways. First, we consider general networks of resources with arbitrary control structures. Second, we consider the case in which individual resources may have constrained capacity as in an airline or other service network. Our findings are quite consistent with the classic double marginalization result – in the absence of capacity constraints, increasing the number of controllers tends to increase price and decrease the total amount of both consumer and producer surplus. For this case, we also provide bounds for the loss of consumer and producer surplus as a function of the number of controllers. These bounds approach zero as the number of controllers approaches infinity, which implies that the decentralized pricing case can be arbitrarily inefficient in terms of revenue and consumer surplus loss relative to the centralized pricing case. The classic summary of double marginalization is that "*The only thing worse than a monopolist is two monopolists.*" We would extend that result to say that "*The only thing worse than two monopolists is a network of monopolists.*"

The analysis of capacitated networks is more complex and we cannot make strong statements about the relationship of consumer and producer surplus with the number of controllers. However, we present a provably convergent algorithm for calculating Nash equilibrium prices on capacitated networks. The proposed algorithm is based on a successive under-relaxation method (Phillips (1984)) and efficiently computes the equilibrium prices for general networks. This type of fixed point algorithm has been used to compute equilibrium of supply and demand in multicommodity markets (Khilnani and Tse (1985)). To the best of our knowledge, this paper presents the first algorithm offered to compute equilibrium prices for a general network with perishable resources. Additionally, we present numerical examples illustrating different situations and show that, contrary to the unconstrained cases, there are situations in which consumer surplus can increase with more controllers.

We note that our results are often described in terms of airlines, however, the results are quite general and apply to any situation in which the mechanics of supply and demand can be described by a network. This would include service networks such as telecommunications and transportation as well as supply chain networks.

2 Literature Review

Our work is related to several major areas of research. Our model is similar to the classical revenue management models described in papers such as Gallego and van Ryzin (1994, 1997). It is well-known that Nash equilibria in decentralized networks are not generally the most efficient structure, in the sense that both the total revenue and consumer surplus can be improved (Dubey (1986)). There has been considerable research about quantifying the efficiency gap between centralized and decentralized networks –the so-called "price of anarchy". Much of this research is focused on competition in networks with congestion effects (Johari and Tsitsiklis (2004), Acemoglu and Ozdaglar (2007)). Farahat and Perakis (2009, 2011) analyze the efficiency of price competition among multi-product firms in differentiated oligopolies which offers gross substitute products to consumers. Granot and Yin (2008) studies the pricing inefficiencies and stability of coalitions under push and pull assembly systems. Yin (2010) investigates the conditions that can lead to stable coalitions among perfectly complementary suppliers.

Our work is also related to the literature regarding supply chain coordination and double marginalization. Basic references for supply chain coordination include the survey papers Cachon (2003) and Chen (2003). The double marginalization concept dates back to Spengler (1950) and has been widely applied to supply chains (Lariviere and Porteus (2001), Perakis and Roels (2007)). A good survey of pricing efficiencies and inefficiencies in supply chain networks can be found in Kaya and Őzer (2012).

One application of our work is airline networks where alliances and mergers have led to increasing consolidation of the industry over the past 20 years. Park (1997), Park and Zhang (2000) and Brueckner and Whalen (2000) use structural econometric models to measure the effects of alliances on airfares and conclude that fares decrease significantly under alliance of flight legs with vertical competition. Park (1997) and Park and Zhang (2000) also show (again empirically) that economic welfare increases under alliances when the size of the markets is sufficiently large.

In addition to incorporating competition introduced to markets by decentralization to the classical network model of Gallego and van Ryzin (1997), our work also generalizes Yin (2010)'s work by considering the capacity constraints for the network resources. Introducing capacity constraints for perishable resources makes our model better suited to pricing problems in the revenue management context. Another important contribution of our paper is using the successive under-relaxation method to efficiently calculate the Nash equilibrium prices in such capacitated network structures. Using this algorithm, we are able to make counterintuitive observations for capacitated networks, such as the fact that there are situations in which consumer surplus increases after decentralization.

3 The Model

The key elements of our model are:

1. *Products* require one or more resources.
2. *Resources* can be combined according to a network structure to produce different products.
3. *Controllers* control one or more resources and set prices for the resources that they control.
4. Resources may be constrained or unconstrained. If a resource is constrained, the total production of products using that resource cannot exceed its capacity.
5. All resources are perishable and have no residual value if they are not consumed.
6. Controllers seek to maximize revenue.
7. The price of each product is the sum of its resource prices.
8. The controllers can set different prices for the same resource based on the product using that resource.
9. Prices are set once and do not change.
10. Demand for a product is a deterministic function of its price alone. There is no competition among products in the network.

Elements 1 through 6 are standard in the network revenue management literature (Talluri and van Ryzin (2004a), Phillips (2005)). The assumption of revenue maximization can be replaced by profit-maximization with fixed unit costs without changing the nature of our results: more complex unit cost structures would require additional analysis. Number 7 is non-restrictive: if there is an additional party – say a distributor – that requires additional compensation to distribute a network product, this can be represented by adding an additional edge to the network with the distributor as the controller of that edge. Number 8 assumes that each controller can distinguish among products using her resources and charge different prices based on product. The assumptions of static prices and deterministic demand go together and are clearly simplifications. The assumption that products in a network do not compete with each other is also a simplification. We note that it is consistent with the many of the models originally used in revenue management, however, there has recently been considerable research into incorporating competition among products via so-called *consumer choice models* into network revenue management – the reader is referred to Talluri and van Ryzin (2004b), Gallego et al. (2004) and Liu and van Ryzin (2008) for details.

Notation:

We reserve the letters i, j and k to index resources, products, and controllers, respectively. Other notation is described as follows.

- M = number of resources (edges). Resources will also be called "legs".
- N = number of products in the market. Each product is a combination of one or more resources.
- K = number of controllers in the market. We must have $1 \leq K \leq M$.
- $C_i > 0$ = capacity of resource i.
- S_j = set of resources used by product j.
- T_k = set of resources controlled by controller k.
- U_k = set of products that use at least one resource controlled by controller k.
- y_k = number of products that use at least one resource controlled by controller k, i.e., $|U_k| = y_k$
- a_{ij} = resource to product incidence factor. In particular, $a_{ij} = 1$ if resource i is used in product j and $a_{ij} = 0$ otherwise.
- b_{ik} = resource to controller incidence factor. In particular, $b_{ik} = 1$ if resource i is controlled by controller k and $b_{ik} = 0$ otherwise.

- e_{jk} = product to controller mapping. In particular, $e_{jk} = 1$ if $a_{ij}b_{ik} = 1$ for some $i = 1, 2, \ldots, M$ and $e_{jk} = 0$ otherwise. That is, $e_{jk} = 1$ if controller k controls at least one resource used in product j.

- K_j = number of controllers whose resources are used by product j. $K_j = \sum_k e_{jk}$

- p_{jk} = price that controller k charges for resources used in product j. We note that p_{jk} is only defined when $e_{jk} = 1$. The number of prices is $\sum_j \sum_k e_{jk}$.

- $p_j = \sum_k e_{jk} p_{jk}$ is the price of product j.

- $\lambda_j = D_j \bar{F}_j(p_j)$ is the demand for product j as a function of price p_j. We assume that demand for each product is a continuous, downward sloping function of price. Hence demand for each product j can be represented as the product of a constant $D_j > 0$ and a function $\bar{F}_j$ that is the c.c.d.f. of some probability distribution with density f_j.

- $r_j = p_j \lambda_j(p_j)$ is the revenue from product j.

We assume that, for all products j, the demand functions are "regular" as defined in Gallego and van Ryzin (1994). In particular, $r_j(\lambda_j)$ is continuous, bounded, and concave; and has a finite maximizer λ_j^*. Additionally, for each product j, there exists a null price $p_j^\infty > 0$ (possibly ∞) such that $lim_{p \to p_j^\infty} \lambda(p) = lim_{p \to p_j^\infty} p\lambda(p) = 0$. We also assume that all of the demand functions display the Increasing Failure Rate (IFR) property (Lariviere (2006)), that is $h_j(p) = f_j(p)/\bar{F}_j(p)$ is an increasing function for all j and all $0 \le p \le p_j^\infty$.

We define $\mathbf{T} = \{T_1, T_2, \ldots, T_K\}$ as a *network control structure.* We call the case of a single controller ($K = 1$) *centralized pricing.* The case where there is more than one controller, i.e., $K \ge 2$, is called *decentralized pricing*, and $K = M$ is the *fully decentralized pricing* case. We are concerned with the set of prices associated with different network control structures.

In decentralized networks, each controller maximizes her own revenue. Controller k's optimization problem can be formulated as:

$$\begin{aligned} \max_{p_{jk}} \quad & \sum_{j \in U_k} p_{jk} D_j \bar{F}_j(p_{jk} + p_{jk}^-) \qquad (1) \\ s.t. \quad & \sum_j a_{ij} D_j \bar{F}_j(p_{jk} + p_{jk}^-) \le C_i \quad \text{for} \quad i \in T_k \\ & p_{jk} \ge 0 \end{aligned}$$

where

$$p_{jk}^{-} = \sum_{\ell: j \in U_\ell, \ell \neq k} p_{j\ell}$$

and we maintain the convention that the sum over an empty set equals 0.

For centralized networks, i.e., $K = 1$, this problem simplifies as follows:

$$\begin{aligned} \max_{p_j} \quad & \sum_j p_j D_j \bar{F}_j(p_j) \qquad (2) \\ s.t. \quad & \sum_j a_{ij} D_j \bar{F}_j(p_j) \leq C_i \quad \text{for} \quad i = 1, 2, \ldots, M \\ & p_j \geq 0 \end{aligned}$$

This is the standard problem of pricing on a constrained network (see Talluri and van Ryzin (2004b)).

4 The Unconstrained Case

We first consider the case in which resources are unlimited – that is $C_i = \infty$ for $i = 1, 2, \ldots, M$. In this case, the network pricing problem (1) is separable, hence we can optimize the revenue of each product independently. Also, it is well-known that if $h_j(p)$ is increasing for $0 \leq p \leq p_j^\infty$ (IFR), then $r_j(p)$ is quasi-concave on $0 \leq p \leq p_j^\infty$, and a unique maximizer p_j^* is guaranteed to exist (Ziya et al. (2004)).

For centralized networks, let p_j^c be the maximizer of product j's revenue function ('c' stands for "centralized"). Then p_j^c must satisfy the first order condition:

$$\begin{aligned} \bar{F}_j(p_j^c) - p_j^c f_j(p_j^c) &= 0 \\ \Rightarrow p_j^c &= \frac{\bar{F}_j(p_j^c)}{f_j(p_j^c)} = \frac{1}{h_j(p_j^c)} \qquad (3) \end{aligned}$$

For decentralized networks, let p_{jk}^{eq} and p_j^{eq} denote the equilibrium price that controller k charges for resources used in product j and total equilibrium price of product j, respectively ('e' stands for "equilibrium"). The equilibrium prices of products are characterized by the following proposition.

Proposition 1: When capacity is unconstrained, i.e. $C_i = \infty$ for $i = 1, 2, \ldots, M$, the decentralized problem (1) has a Nash equilibrium in which the equilibrium prices p_j^{eq} all satisfy the condition:

$$p_j^{eq} \;=\; K_j/h_j(p_j^{eq}) \tag{4}$$

Furthermore, this Nash equilibrium is unique among those that satisfy the condition $p_j < p_j^{\infty}$.

Proof For the case of a single controller, the result is immediate and corresponds to the well-known revenue-maximizing monopoly price. In the case of multiple controllers ($j : K_j \geq 2$) the best response of player k is:

$$\begin{aligned} \bar{F}_j(p_{jk} + p_{jk}^-) - p_{jk} f_j(p_{jk} + p_{jk}^-) &= 0 \\ \Rightarrow p_{jk} &= 1/h_j(p_{jk} + p_{jk}^-) \end{aligned} \tag{5}$$

Since the right hand sides of (5) are equal for all controllers k, the equilibrium is symmetric. By virtue of the IFR property, this equilibrium is unique.

In this symmetric equilibrium, we have $p_j^e = K_j p_{jk}^{eq}$. Therefore (5) becomes:

$$p_{jk}^{eq} \;=\; 1/h_j(K_j p_{jk}^{eq}) \tag{6}$$

$$\Rightarrow p_j^{eq} \;=\; K_j/h_j(p_j^{eq}) \tag{7}$$

Q.E.D.

We note, that in the case of a finite null price, p_j^{∞} for a product, there exist Nash Equilibria at which $p_{jk} > p_j^{\infty}$ for some k. However since no revenue is generated for any controller at these equilibria, they are of no practical interest and we will henceforth ignore them.

4.1 The Price of Anarchy

We consider two components of the price of anarchy – producer surplus and consumer surplus. We note that under our assumption of zero unit costs, producer surplus is equal to revenue.

4.1.1 The Effect of Decentralization on Revenue

We utilize equation (6) to derive our primary result on the price of anarchy in an unconstrained network. To do so, we make the number of controllers involved in a product explicit by letting p_j^K be the equilibrium price of product j in a decentralized network where K controllers manage the resources used by product j.

Proposition 2: When capacity is unconstrained for all resources:

- The price of a product in a decentralized network is strictly increasing in the number of controllers involved in the product. Additionally, $p_j^K \leq \frac{K}{L} p_j^L$, $\forall K \geq L$.
- p_{jk}^{eq} is strictly decreasing in the number of controllers involved in the product
- The total revenue of a product and the revenue of each controller is strictly decreasing in the number of controllers involved in the product
- As the number of controllers increases, the equilibrium product price approaches the cutoff price and revenue approaches 0, that is: $\lim_{K\to\infty} p_j^K = p_j^\infty$ and $\lim_{K\to\infty} p_j^K \bar{F}_j(p_j^K) = 0$.

Proof The first part of the Proposition is immediate from equation (7) and the fact that, among the demand functions with IFR property, the maximum price increase occurs for the one with constant hazard rate (exponential demand function), and $\frac{K}{L}$ is the increase rate in this case. The second part is immediate from equation (6) noting that $h_j(K_j p_{jk}^{eq})$ is an increasing function of K_j. The third part follows since, as a consequence of the IFR condition, the revenue function is quasi-concave and $r_j(p_j^K) := p_j^K \lambda_j(p_j^K)$ is strictly decreasing in K considering the fact that r_j is maximized when $K = 1$.

Finally, the last part follows since for any $K \geq 1$, p_j^K is the unique solution of $\Phi_j(p_j^K) = K$ where $\Phi_j(p) = p h_j(p)$. We show that $\lim_{p_j \to p_j^\infty} \Phi_j(p_j) = \infty$. This is clearly true if $p_j^\infty = \infty$. If p_j^∞ is finite, then, $\bar{F}_j(p_j^\infty) = 0$ implying that $\lim_{p_j \to p_j^\infty} h_j(p_j) = \infty$ and hence $\lim_{p_j \to p_j^\infty} \Phi_j(p_j) = \infty$. Since $\Phi_j(p)$ is strictly increasing, we can write $p^K = \Phi^{-1}(K)$ and from the limit results, $\lim_{K\to\infty} \Phi_j^{-1}(K) = p_j^\infty$ as required. $\lim_{K\to\infty} p_j^K \bar{F}_j(p_j^K) = 0$ follows from the assumption that the demand function is regular. **Q.E.D.**

In the infinite-capacity case, the optimization problem is separable and the equilibrium prices of products are independent. Therefore, we can express the prices of individual products as a function of the number of controllers for some commonly used demand functions. Recall that p_j^K is the equilibrium price for product j with K controllers and we define $r_j^K = p_j^K \lambda_j(K p_j^K)$ as the corresponding total product revenue. Table 1 shows the equilibrium price and demand values along with the ratios of the fully decentralized to centralized revenues for the products with K controllers, for exponential and linear demand functions in an infinite capacity network.

Remark 1: The revenue ratio, (r_j^K / r_j^1), goes to zero as K goes to infinity by the last part

Demand Function	$\lambda_j(p_j)$	p_j^K	$\lambda_j(p_j^K)$	r_j^K/r_j^1
Exponential	$e^{a_j-b_jp_j}$	$\frac{K}{b_j}$	e^{a_j-K}	Ke^{1-K}
Linear	$(a_j-b_jp_j)^+$	$\frac{Ka_j}{(K+1)b_j}$	$\frac{a_j}{K+1}$	$\frac{4K}{(K+1)^2}$

Table 1: Effects of decentralization on revenues for exponential and linear demands

of Proposition 2. Hence the decentralized equilibrium solution can be arbitrarily inefficient in terms of revenue loss relative to the centralized case.

Remark 2: Figure 2 shows that the revenue loss bound goes to zero much faster for the case of exponential demand than linear demand.

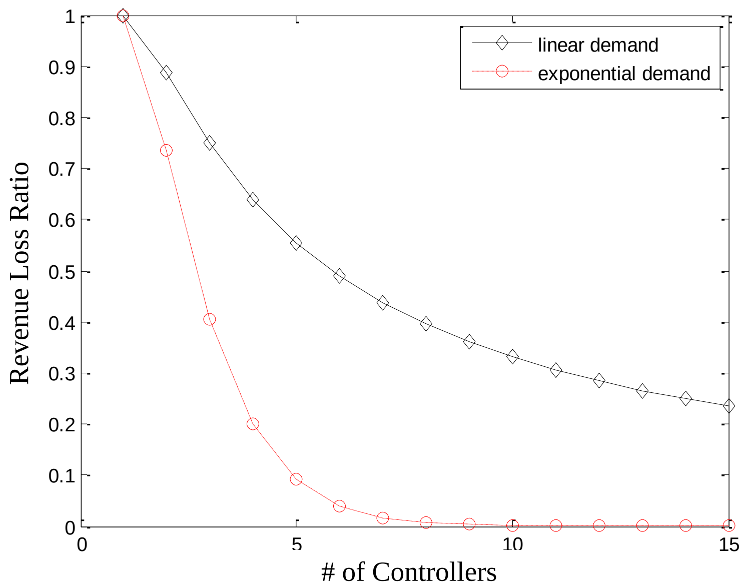

Figure 2: Revenue loss ratios by the number of controllers

Remark 3: Even though Proposition 2 suggests that centralization of an unconstrained network is beneficial to controllers, such coalitions may not be stable. Proposition 6 of Yin (2010) shows that coalitions with more than two perfectly complementary suppliers are not stable[2] when resource capacities are infinite for both exponential and linear-power ($\lambda(p) = (a - bp)^\gamma$ for $a, b, \gamma > 0$) demand functions (for linear-power demand functions, coalitions with two suppliers may not be stable either depending on the value of γ). If we consider coalitions for only one product, our model leads to the same result. However, coalitions with more than two controllers

[2] A stable coalition is defined as a coalition structure in which no controller has a strictly profitable and feasible deviation (Yin (2010)).

Demand Function	$\lambda_j(p_j)$	CS_j^K	CS_j^K/CS_j^1
Exponential	$e^{a_j-b_jp_j}$	$\frac{1}{b_j}e^{a_j-K}$	e^{1-K}
Linear	$(a_j-b_jp_j)^+$	$\frac{1}{2(K+1)^2}\frac{a_j^2}{b_j}$	$\frac{4}{(K+1)^2}$

Table 2: Effects of decentralization on consumer surplus for exponential and linear demands

can be stable in our model with both exponential and linear demand functions, due to the fact that our model takes the network structure of the resources into account. To see this, assume that each controller controls a single resource. In this case, for stability of a coalition with more than two controllers, coalition members need to jointly control both products with two resources and products with more than two resources (instead of controlling only one product using more than two resources). For example, in a network with three serially connected resources, a stable coalition can be formed among all three resource controllers, if the products with two resources have sufficiently large demand relative to the product with three resources.

4.1.2 The Effect of Decentralization on Consumer Surplus

Decentralization of the networks affects consumers as well as producers. We quantify this effect by calculating the ratio of consumer surplus under decentralized control to that under centralized control. Let CS_j^K be the total consumer surplus generated for product j with K controllers. Note that consumer surplus from product j is $CS_j = \int_{p_j}^{\infty} \lambda_j(p)dp$, which is decreasing in p_j. Table 2 shows the equilibrium consumer surplus values along with the ratios of the fully decentralized to centralized consumer surpluses for the products with K controllers for exponential and linear demand functions in an infinite capacity network.

Note that these ratios are smaller than their counterparts for producer surplus (see Figure 3). We conjecture that, as the number of controllers increases, "the chain of monopolies" reduces total consumer surplus more rapidly than total producer surplus.

5 The Constrained Case

Capacitated networks do not lend themselves to the straightforward analysis that we applied to unconstrained networks. The solution of the centralized problem (2) can be characterized

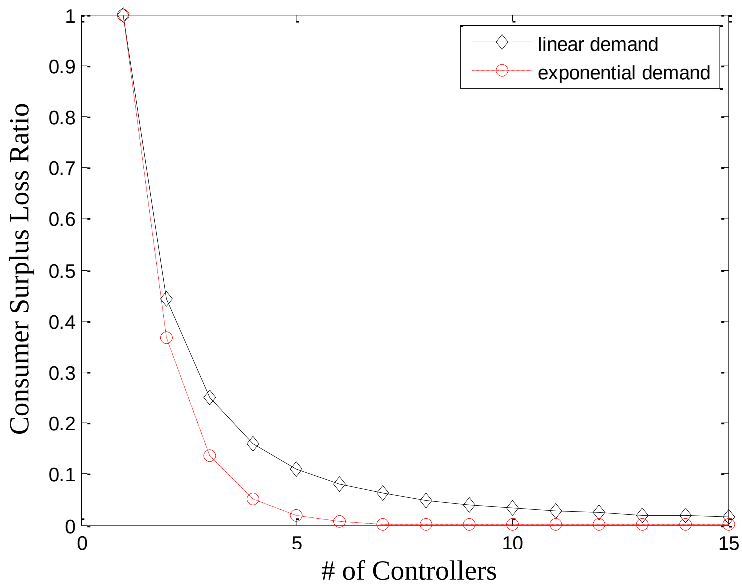

Figure 3: Consumer surplus loss ratios by the number of controllers

by its KKT conditions. This problem can also be formulated with the demand rate being the decision variable. By virtue of the assumption that demand functions are *regular*, this alternative formulation becomes a strictly convex optimization problem with a unique solution (see Talluri and van Ryzin (2004b)). Also, since there is a one-to-one relation between the demand and pricing functions, we can conclude that the following KKT conditions of (2) are both necessary and sufficient conditions for optimality:

$$p_j = \frac{1}{h_j(p_j)} + \sum_i a_{ij}\mu_i \quad j = 1...N \tag{8}$$

$$\mu_i(\sum_j a_{ij} D_j \bar{F}_j(p_j) - C_i) = 0 \quad i = 1...M \tag{9}$$

$$\mu_i \geq 0 \quad i = 1...M \tag{10}$$

where $\mu_i \geq 0$, $i = 1...M$ are the Lagrange multipliers with the interpretation as the marginal opportunity costs for resources $i = 1...M$, respectively. Condition (10) assures that these marginal opportunity costs are nonnegative. Condition (8) is the usual Lagrangian optimality condition.

We can use the KKT conditions of Problem (1) to derive a set of simultaneous equations for the

Nash Equilibrium prices in the decentralized problem.

$$p_{jk} = \frac{1}{h_j(p_{jk} + p_{\bar{jk}})} + \sum_i a_{ij} b_{ik} \mu_i \quad \forall\ (j,k) \text{ pair} \tag{11}$$

$$\mu_i(\sum_j a_{ij} D_j \bar{F}_j(p_{jk} + p_{\bar{jk}}) - C_i) = 0 \quad i = 1...M \tag{12}$$

$$\mu_i \geq 0 \quad i = 1...M \tag{13}$$

So, the NE of the capacitated decentralized problem (1) can be derived as a fixed point solution of (11), given that we calculate the optimal dual prices using the complementary slackness conditions given in equation (12).

5.1 The Price of Anarchy

For capacitated networks, finding analytical bounds for the price of anarchy is more complicated than the uncapacitated case and closed form solutions are generally not computable. However, we can still obtain some useful insights. Recall that the optimality and equilibrium conditions for the centralized and decentralized prices are:

$$p_j^c = \frac{1}{h_j(p_j^c)} + \sum_i a_{ij} \mu_i^c \quad j = 1...N \tag{14}$$

$$p_j^{eq} = \frac{K_j}{h_j(p_j^{eq})} + \sum_i a_{ij} \mu_i^{eq} \quad j = 1...N \tag{15}$$

respectively, where the superscripts of μ_i's are used to distinguish the centralized and decentralized Lagrange multipliers, and we calculate μ_i^c and μ_i^{eq}, $i = 1...M$ using the corresponding complementary slackness conditions. Note that (15) is derived by summing (11) over all k.

We know by Proposition 2 that the centralized optimal prices will always be smaller than the decentralized prices for infinite capacity networks. Therefore, product demands are always larger in decentralized systems and if a resource's capacity constraint is binding in a decentralized system, it will be binding for the centralized system as well. However, the reverse case may not be true, i.e., there might be network structures whose centralized problem is capacity constrained but decentralized problem is not. For such cases, we can derive equilibrium prices using equation (4). We also know that, in the capacitated case, centrally optimal prices will be larger than their uncapacitated counterparts in order to satisfy the capacity constraints. Hence, constrained-centralized to unconstrained-decentralized revenue and consumer surplus ratios are always larger than the networks with infinite capacities. Therefore, revenue ratios shown in Table 1 and consumer surplus ratios shown in Table 2 are lower bounds for this case as well.

We next study a simple network with two resources and three products (as in Figure 1).

Proposition 3: In the network structure of Figure 1, if all the capacity constraints are binding when pricing is fully decentralized (K=2), then

$$p_j^c > p_j^{eq} \quad j = 1, 2 \quad \text{(single-resource products)}$$
$$p_j^c < p_j^{eq} \quad j = 3 \quad \text{(multiple-resource product)}$$

Proof If the capacity constraints are binding in the decentralized system, then they are also binding in the centralized system as explained above. Hence we have

$$\lambda_1(p_1^c) + \lambda_3(p_3^c) = \lambda_1(p_1^{eq}) + \lambda_3(p_3^{eq}) = C_1$$
$$\lambda_2(p_2^c) + \lambda_3(p_3^c) = \lambda_2(p_2^{eq}) + \lambda_3(p_3^{eq}) = C_2$$

Clearly, if the price of one of the products using a single resource ($i = 1, 2$) increases after decentralization, the price of the other single-resource product will also increase, and vice versa. Now, assume to the contrary that

$$p_j^c < p_j^{eq} \quad j = 1, 2 \quad \text{(single-resource products)} \tag{16}$$
$$p_j^c > p_j^{eq} \quad j = 3 \quad \text{(multiple-resource product)}$$

Then:

$$p_1^c < p_1^{eq} \Rightarrow h_1(p_1^c) < h_1(p_1^{eq}) \quad \text{by the IFR property}$$
$$\Rightarrow \frac{1}{p_1^c - \mu_1^c} < \frac{1}{p_1^{eq} - \mu_1^{eq}} \quad \text{by (14) and (15)} \tag{17}$$
$$\Rightarrow \mu_1^c < \mu_1^{eq} \tag{18}$$

and similarly

$$p_3^c > p_3^{eq} \Rightarrow h_3(p_3^c) > h_3(p_3^{eq}) \quad \text{by the IFR property}$$
$$\Rightarrow \frac{1}{p_3^c - \mu_1^c - \mu_2^c} > \frac{2}{p_3^{eq} - \mu_1^{eq} - \mu_2^{eq}} > \frac{1}{p_3^{eq} - \mu_1^{eq} - \mu_2^{eq}} \quad \text{by (14) and (15)} \tag{19}$$
$$\Rightarrow p_3^c - \mu_1^c - \mu_2^c < p_3^{eq} - \mu_1^{eq} - \mu_2^{eq} \tag{20}$$
$$\Rightarrow \mu_2^c > \mu_2^{eq} \quad \text{by (18)} \tag{21}$$
$$\Rightarrow p_2^c > p_2^{eq}$$

which contradicts (16). **Q.E.D.**

We also consider an arbitrary network structure where only a single resource's capacity constraint is binding in the decentralized problem (hence in the centralized problem as well). In such a case, it is easy to conclude that the price of the product that uses only that specific resource (say product k) will decrease after fully decentralization of the network. We can show this as follows: Assume that resource i has a binding capacity constraint, then:

$$\sum_j a_{ij}\lambda_j(p_j^c) = \sum_j a_{ij}\lambda_j(p_j^{eq}) \tag{22}$$

If we assume to the contrary that $p_k^c \leq p_k^{eq}$, we have

$$\begin{aligned} h_k(p_k^c) \leq h_j(p_k^{eq}) &\Rightarrow \mu_i^c \leq \mu_i^{eq} \\ &\Rightarrow \frac{K_j}{p-\mu_i^{eq}} > \frac{1}{p-\mu_i^c} \quad \forall p \text{ and } \forall j \text{ s.t. } a_{ij}=1\ ,\ j \neq k \\ &\Rightarrow p_j^{eq} > p_j^c \quad \text{by the IFR property} \end{aligned}$$

which contradicts (22).

Even for these simple cases, we cannot derive conclusions about how much total revenue or total consumer surplus changes as pricing is decentralized. We know that the total revenue always (weakly) decreases after decentralization by the structure of the controllers' optimization problem. However, there are cases where total consumer surplus increases with more controllers when the decentralized problem is capacity constrained. In fact, we derived numerical examples for all possible cases, i.e., examples in which the price of a product with single/multiple controller(s) inreases/decreases after decentralization of the network. We present these results in Section 7.

6 Computing Equilibrium Prices

We next present provably convergent algorithms to calculate the Nash equilibrium prices, first for the unconstrained and then for more general (constrained) networks.

6.1 Computing Equilibrium Prices in an Unconstrained Network

As noted in Section 4.1.1, the unconstrained problem is entirely separable. It is thus possible to calculate the optimal price for each product separately by finding the unique value of p_j

such that $p_j = K_j/h_j(p_j)$ for each product. The prices charged by each controller can then be calculated as $p_{jk} = p_j/K_j$. Since the revenue function is continuous and quasi-concave, this can be done using line search. However, we have found that a Successive Under-Relaxation algorithm is faster.

Proposition 4: Assume that $f_j(x)$ is IFR with corresponding hazard rate $h_j(x)$. Furthermore, assume that $h_j(0) > 0$ and $(h'_j(p)/h_j^2(p)) < E$ for all $p \in (0, p_j^\infty)$. Then, there exists an $\epsilon > 0$ such that the iterative process $p_j^k(m) = \alpha K/h_j(p_j^k(m)) + (1-\alpha)p_j^k(m)$ will converge to the unique equilibrium product price p_j when initiated from any $0 < p_j^0 < p_j^\infty$.

Proof Define $g(p) = \alpha K/h_j(p) + (1-\alpha)p$. Then the desired equilibrium price is the unique fixed point of $g(p)$. Furthermore, it is well known that if there exists a $0 \le \delta < 1$ such that $||g'(p)|| < \delta$ for all $p \in (0, p_j^\infty)$, the iterative process defined in the proposition is contractive and will converge to the fixed point (Isaacson and Keller (1966)). For the proposed process, this condition is equivalent to:

$$0 < \alpha < \frac{2h^2(p)}{Kh'(p) + h^2(p)}$$

Set $\delta = \frac{2}{KE+1}$. Then, $\delta < \frac{2h^2(p)}{Kh'(p)+h^2(p)}$ and any $\alpha < \delta$ will generate a convergent sequence. **Q.E.D.**

6.2 Computing Equilibrium Prices in a General Network

Recall that y_k denotes the number of products that use at least one resource controlled by controller k. Let $Y = y_1 + \ldots + y_K$ and $P \in \mathbb{R}^Y$ denote the vector of controller actions (all controller-price combinations), i.e., it consists of prices charged by each controller k for each product j that she controls. We can always restructure P so that first n_1 of its elements are prices of the products with single controllers. We also group the prices of a specific product charged by different controllers together, i.e., $P = [p_1\ p_2\ ...\ p_{n_1}\ p_{n_1+1,1}\ ...\ p_{n_1+1,K_{n_1+1}}\ ...\ p_{N,K_N}]^T$ (note that $n_1 + (n_1+1) * K_{n_1+1} + ... + N * K_N = Y$). Let ℓ be the index that enumerates the entries of P. Also let $\mu = [\mu_1\ ...\ \mu_M]$ be the vector of Lagrange multipliers.

Considering the optimality conditions derived in equations (11) - (13), define the transformation $G(P)$ such that

$$G_\ell(P) = \frac{1}{h_j(p_{jk} + p_{jk}^-)} + \sum_i a_{ij} b_{ik} \mu_i \quad \ell = 1...Y$$

and the transformation $Z(\mu)$ such that

$$Z_i(\mu) = \left\{ \mu_i + \gamma \left(\sum_j a_{ij}\lambda_j(p_{jk} + p_{jk}^-) - C_i \right) \right\}^+ \qquad i = 1...M$$

for some $\gamma > 0$. Then, we can apply the following Successive Under-Relaxation (SUR) algorithm to compute the Nash equilibrium prices for Problem (1).

SUR Algorithm for Capacitated Networks:

Step 0: Initialize $P^0 = [...\ p_{jk}(0)\ ...]$ with $0 < p_{jk}(0) < p_j^\infty$ for all (j,k) and $\mu = 0$. Set t=0;

Step 1:

while $|p_{jk}(t) - p_{jk}(t-1)| > \epsilon$ for any (j,k) pair and for some $\epsilon > 0$

$\mu^{t+1} = Z(\mu^t)$

$P^{t+1} = \alpha G(P^t) + (1-\alpha)P^t$ where $0 < \alpha \leq 1$ is the relaxation coefficient

$t = t + 1$

end

Our convergence proof for this algorithm uses the following theorem.

Theorem 1: Phillips (1984). Let S be a closed, convex subset of $\mathbb{R}^n$. Let Q be the set of transformations on $\mathbb{R}^n$, $H : S \to S$, with the following properties:

1. H is continuous and differentiable on S.
2. The spectral radius of H', $s(H')$, is a norm everywhere in S.
3. The eigenvalues of $H'(\tilde{P})$ are real for all $\tilde{P} \in S$, and there exists a and b, $a \leq b < 1$ such that all the eigenvalues of $H'(\tilde{P})$ lie between a and b for all $\tilde{P} \in S$.

Let $T \in Q$. Then T has a unique fixed point $\tilde{P}^* \in S$ such that $T(\tilde{P}^*) = \tilde{P}^*$. Furthermore, there exists an $0 < \epsilon \leq 1$ such that the SUR algorithm will converge to this unique fixed point for any $0 < \alpha < \epsilon$.

Proposition 5: Assume that $f_j(x)$ is IFR with corresponding hazard rate $h_j(x)$ and $h_j(0) > 0$, $\forall j$. Then, there exists an $0 < \alpha \leq 1$ such that the **SUR Algorithm for Capacitated Networks** converges to the unique Nash equilibrium (NE).

Proof Let $S = [0, p_1^\infty] \times ... \times [0, p_N^\infty]$. The first condition of Theorem 1 holds by assumption. We show the third condition holds by proving that $G^{'}(P)$ is negative semi-definite for $\forall P \in S$.

In every iteration t, after calculating the new μ_i's, i.e. $\mu_i(t+1)$'s, we perform the following update for each p_{jk}:

$$p_{jk}(t+1) = \frac{1}{h_j(p_{jk}(t) + p_{\bar{jk}}(t))} + \sum_i a_{ij} b_{ik} \left[\mu_i(t) + \gamma \left(\sum_j a_{ij} \lambda_j (p_{jk}(t) + p_{\bar{jk}}(t)) - C_i \right) \right]^+$$

Therefore, $G^{'}(P)$ is a block diagonal matrix, i.e.,

$$G^{'}(P) = \begin{bmatrix} A_1 & 0 & 0 & ... & & 0 \\ 0 & A_2 & 0 & ... & & 0 \\ 0 & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & ... & & 0 & A_n \end{bmatrix}$$

where $A_j = \beta_j U_{K_j}$, where

$$\beta_j = \begin{cases} (\frac{1}{h_j(p_{jk} + p_{\bar{jk}})})^{'} + \gamma \lambda_j^{'}(p_{jk} + p_{\bar{jk}}) & \text{if } -\gamma \left(\sum_j a_{ij} \lambda_j (p_{jk} + p_{\bar{jk}}) - C_i \right) < \mu_i \\ (\frac{1}{h_j(p_{jk} + p_{\bar{jk}})})^{'} & \text{otherwise} \end{cases}$$

and U_{K_j} is a $K_j \times K_j$ matrix of ones. The first term of β_j is negative since the demand distributions have the IFR property. The second term is also negative since $\gamma > 0$ and demand is a decreasing function of price. Hence $\beta_j < 0, \forall j$. Therefore each A_j is negative semi-definite, so $G^{'}(P)$ is also negative semi-definite, which implies that all eigenvalues of $G^{'}(P)$ are non-positive $\forall P \in S$.

The smallest eigenvalue of $G^{'}(P)$ is $K^{max} \beta_{\bar{j}}$, where $K^{max} = \max_j K_j$ and $\bar{j}$ is the index where that maximum occurs. Since we assume $h_j(0) > 0, \forall j$ and IFR, we have $h_j(p_{jk}) > 0, \forall P \in S$, and hence $\frac{1}{h_j(p_{jk})} < \infty, \forall P \in S$. The second term of $\beta_{\bar{j}}$ is bounded by the assumption of *regular* demand functions. So, all the eigenvalues of $G^{'}(P)$ are lower bounded.

The second condition of Theorem 1 follows from the fact that $G^{'}(P)$ is symmetric (therefore, all its eigenvalues are real). **Q.E.D.**

Remark 4: Since the exponential demand function has a constant hazard rate, the first term of β_j vanishes. Hence we can make $\|G^{'}(P)\| < 1$ by choosing γ sufficiently small, which means that we can make G a contraction mapping for exponential demand functions. Therefore, we can use a simple line search algorithm by taking $\alpha = 1$ in the **SUR Algorithm for Capacitated Networks**.

7 Numerical Analysis

We calculated the prices and consumer and producer surpluses under different regimes for the networks shown in Figure 4. Network 1 is a serial network. The other networks have "hub and spoke" structures such as those commonly found in airlines and supply chains. In Networks 1,

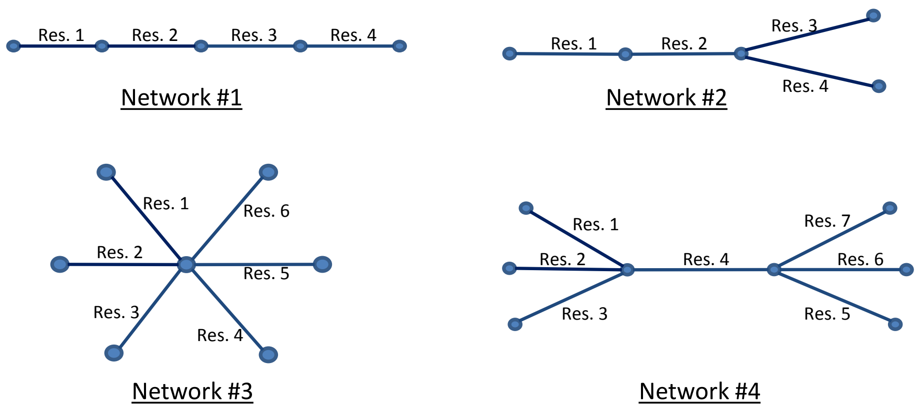

Figure 4: Common network structures

3, and 4, we calculated optimal prices under the two extreme control structures: centralized pricing ($K = 1$) and fully decentralized pricing ($\mathbf{T} = \{\{1\}, \{2\}, \ldots, \{M\}\}$). For Network 2, we compared centralized pricing with the case in which resources 1 and 2 had different controllers, but resources 3 and 4 had the same controller ($\mathbf{T} = \{\{1\}, \{2\}, \{3, 4\}\}$). In every case, we assume that there is a product for all possible combinations of adjacent resources. We calculated optimal prices in each network using both linear ($\lambda(p) = (a - bp)^+$) and exponential ($\lambda(p) = e^{a-bp}$) demand functions. Parameter values for the demand functions can be found in the Appendix.

We compared the centralized and decentralized network revenue and consumer surplus under three different scenarios: (1) infinite capacity for every resource, (2) capacity levels that only constrain the centralized problem, and (3) capacity levels that constrain both centralized and decentralized problems. By Proposition 2, the prices for the multi-controller scenarios will be higher than the centralized problem, so the capacity levels in scenario 2 are higher than those in scenario 3. The capacity levels that we used in our examples can also be found in the Appendix. Table 3 presents the changes in the total generated revenue and consumer surplus from centralized to decentralized systems under these three scenarios.

Demand Function	Network	Change in Total Revenue			Change in Total Consumer Surplus		
		Scenario 1	Scenario 2	Scenario 3	Scenario 1	Scenario 2	Scenario 3
Linear	1	-12.49%	-10.75%	-2.87%	-41.37%	-25.50%	-1.02%
	2	-9.00%	-5.65%	-3.22%	-33.06%	-12.12%	-3.76%
	3	-8.19%	-8.00%	-0.90%	-40.96%	-37.85%	5.73%
	4	-12.66%	-8.92%	-4.87%	-48.05%	-30.84%	-17.46%
Exponential	1	-30.86%	-26.60%	-13.70%	-48.56%	-31.00%	-5.76%
	2	-24.74%	-22.32%	-13.55%	-41.39%	-30.00%	-8.41%
	3	-18.51%	-13.35%	-2.09%	-44.29%	-20.74%	7.86%
	4	-33.48%	-30.04%	-12.54%	-57.20%	-41.98%	-5.35%

Table 3: Changes in total revenue and consumer surplus from centralized to decentralized networks

We used the algorithms described in Section 6 to solve for the prices in each scenario. As algorithm parameters, we chose $\epsilon = 10^{-6}$ and $\gamma = 10^{-5}$. We used $\alpha = 0.1$ for the linear demand case and $\alpha = 1$ for the exponential demand case and initialized all the prices to one. The SUR algorithm converged quite quickly for these small examples –the longest convergence time was 0.23 seconds for 780 iterations (for Network 4) using a laptop with 8GB RAM.

In all cases, total revenue was reduced by decentralization. Total consumer surplus was usually reduced under decentralization, however, for Network 3 under Scenario 3, total consumer surplus actually increased under decentralization for both the linear and exponential demand functions. Under Scenario 3, the network is highly constrained so that all of the capacity constraints are binding under both the centralized and decentralized cases. When pricing is centralized, the prices of the single-resource products are higher than their optimal levels in order to satisfy the capacity constraints. Under decentralization, double marginalization results in an increase in the price of the multi-resource products resulting in a corresponding decrease in their demand. This increases the residual capacity available for the single-product resources and their prices drop towards their centralized values. For these cases, on the balance, the decreased prices of the single-resource products outweighed the increased prices of the multi-resource products and both consumer surplus and total surplus increased.

For the unconstrained problems, revenue and consumer surplus loss ratios are ordered differently for the four networks: even though Network 2's revenue loss ratio is larger than Network 3, the case is reversed for the consumer surplus loss ratio. The reason is that Network 2 has three products with two distinct controllers and Network 3 has 15 products with two distinct controllers and the revenue loss ratio of products with two controllers is smaller (9/8) than the consumer surplus loss ratio (9/4).

We also explored the effects of different type of control structures on the network revenue. For Network 1 with unconstrained resources, we compared two control structures. In one case there are two controllers, one managing resources 1 and 2 and the other managing 3 and 4. In the other case there are again two controllers, but one managing resources 1 and 3 and the other managing resources 2 and 4. When we use the same parameters as the previous examples, the revenue and consumer surplus loss ratios are 4.93% and 24.63% for the first control structure, and 6.79% and 33.97% for the second control structure. This is in line with expectations since there are more products with multiple controllers under the second control structure.

To answer the question of whether the capacity constraints of all the resources have similar effects on the revenue losses, we decreased the capacity levels of resources one at a time for the above network examples under both centralized and decentralized systems and observed the effects on the total revenues. Table 4 presents the total revenue changes from uncapacitated to one-resource-capacitated networks for Network 4: the decrease in the total network revenue is significantly larger for a 50% cut in resource 4's capacity (with respect to the optimal/equilibrium demand in uncapacitated case) compared to similar cuts in other resources' capacities (we obtained similar results for the other network examples). This distinction arises from the number of products that use each resource: cuts to the capacities of the resources used by more products are more effective in revenue losses.

8 Summary and Conclusions

In this study, we showed that decentralized pricing in a network always leads to a reduction in total product revenue relative to the centralized case and that more decentralization leads to greater loss. In the case in which resources are unconstrained, decentralization also leads to a loss of consumer surplus. We derived closed-form solutions for the revenue and price as a

Resource #	Capacity/UD*	Total Revenue Change Centralized Linear	 Exponential	 Decentralized Linear	 Exponential
1	0.5	-5.25%	-3.02%	-7.57%	-5.70%
2	0.5	-5.72%	-3.12%	-8.11%	-5.69%
3	0.5	-6.01%	-3.51%	-8.59%	-6.81%
4	0.5	-12.77%	-7.37%	-18.51%	-13.22%
5	0.5	-5.65%	-2.98%	-8.10%	-5.66%
6	0.5	-5.90%	-3.14%	-8.49%	-5.94%
7	0.5	-5.85%	-3.40%	-8.38%	-6.45%

Table 4: Effects of capacity cuts on total network revenue for Network 4. *UD: Uncapacitated Demand

function of the number of controllers for the cases in which product demand curves are linear or exponential.

When resources have constrained capacity, the situation is more complex. Total revenue always decreases as the number of controllers increases. However, in certain cases, consumer surplus - and total social surplus - can actually increase. When the capacity constraint is binding on one or more resources, closed form solutions are no longer available for price or revenue. However, we show that a Successive Under-Relaxation algorithm with suitable choice of relaxation parameter is guaranteed to converge given mild assumptions on the forms of the demand functions.

Our study can be viewed as an extension of the well-known concept of double marginalization (or horizontal externality) to networks: as the number of controllers in an unconstrained network increases, both producer and consumer surpluses decrease relative to the single controller case. This indicates that there can be an increase in consumer and producer surplus from the consolidation of network industries such as airlines and pipelines in which there are many shared products prior to the merger.

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Appendix

		Linear		Exponential	
Network	Product	a	b	a	b
1	1	100	2	4.61	2
	2	95	2	4.59	2
	3	110	2	4.62	2
	4	105	2	4.6	2
	5	140	4	5.01	4
	6	150	4	5	4
	7	130	4	4.98	4
	8	80	1	4.41	1
	9	85	1	4.4	1
	10	120	3	4.8	3
2	1	100	2	4.61	2
	2	95	2	4.59	2
	3	110	2	4.62	2
	4	105	2	4.6	2
	5	140	4	5.01	4
	6	150	4	5	4
	7	130	4	4.98	4
	8	145	4	4.99	4
	9	85	1	4.4	1
	10	75	1	4.41	1
3	1	100	3	4.61	3
	2	110	3	4.59	3
	3	105	2	4.62	2
	4	95	2	4.6	2
	5	90	3	4.61	3
	6	100	3	4.58	3
	7	140	4	4.98	2
	8	150	5	4.99	5
	9	130	4	5	4
	10	145	5	5.04	5
	11	154	5	5.02	5
	12	148	4	5	4
	13	135	4	4.97	5
	14	140	4	4.98	4
	15	126	5	4.99	5
3	16	128	4	5.02	4
	17	138	5	5	5
	18	150	5	4.97	5
	19	130	4	4.98	4
	20	130	4	4.99	3
	21	145	5	5.05	5
4	1	100	3	4.61	3
	2	110	3	4.59	3
	3	105	2	4.62	2
	4	95	2	4.6	2
	5	90	3	4.61	3
	6	100	3	4.58	3
	7	105	2	4.6	2
	8	150	5	4.99	5
	9	130	4	5	4
	10	145	5	5.04	5
	11	154	5	5.02	5
	12	148	4	5	4
	13	140	5	4.97	5
	14	125	4	4.98	4
	15	150	5	4.99	5
	16	130	4	5.02	4
	17	145	5	5	5
	18	154	5	4.97	5
	19	148	4	4.98	4
	20	72	1	4.38	1
	21	82	2	4.4	2
	22	76	3	4.37	3
	23	75	2	4.38	2
	24	80	2	4.4	2
	25	84	1	4.37	1
	26	77	1	4.42	1
	27	88	2	4.41	2
	28	80	2	4.4	2

Table 5: Demand function parameters used in Section 2.5: Numerical Analysis

		Capacity			
		Constrained Centralized Problem		Constrained Decentralized Problem	
Network	Resource	Linear Demand	Exponential Demand	Linear Demand	Exponential Demand
1	1	280	150	180	70
	2	280	150	180	70
	3	280	150	180	70
	4	280	150	180	70
2	1	220	170	180	70
	2	250	170	200	70
	3	220	170	180	70
	4	220	170	180	70
3	1	390	200	278	100
	2	390	200	278	100
	3	390	200	278	100
	4	390	200	278	100
	5	390	200	278	100
	6	390	200	278	100
4	1	380	200	255	100
	2	380	200	255	100
	3	380	200	255	100
	4	600	400	350	100
	5	380	200	255	100
	6	380	200	255	100
	7	380	200	255	100

Table 6: Capacity levels used in Section 2.5: Numerical Analysis