

OPTIMAL CONSUMPTION WITH STOCHASTIC INCOME: DEVIATIONS FROM CERTAINTY EQUIVALENCE*

STEPHEN P. ZELDES

No one has derived closed-form solutions for consumption with stochastic labor income and constant relative risk aversion utility. A numerical technique is used here to give an accurate approximation to the solution. The resulting consumption function is often dramatically different than the certainty equivalence solution typically used, in which consumption is proportional to the sum of financial wealth and the present value of expected future income. The results help explain three important empirical consumption puzzles: excess sensitivity of consumption to transitory income, high growth of consumption in the presence of a low risk-free interest rate, and underspending of the elderly.

I. INTRODUCTION

A great deal of recent research derives and tests implications of the life cycle/permanent income hypothesis under uncertainty. One of the most important sources of uncertainty facing individuals is that of labor income. Yet, closed-form decision rules for optimal consumption in the presence of uncertain labor income have not, in general, been derived. It seems strange that so much theoretical and empirical work has been done studying consumption, and yet we do not even know what the optimal level of consumption or sensitivity of consumption to income should be in most very simple settings. In this paper I use numerical methods to closely approximate the optimal consumption function and the corresponding value function for some simple multiperiod problems. I then examine how consumption behavior differs from that implied by the certainty or certainty-equivalence models used by most authors who write down closed-form decision rules for consumption with random labor income. Thus, this paper returns to an older tradition of looking at the function for the optimal level of consumption, rather than at the Euler equation which relates consumption at two different points in time.

The technique I use enables me to address some important

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questions about the optimal consumption function that could previously not be addressed because of the lack of closed-form solutions. First, I can calculate the optimal amount of precautionary saving—saving done to hedge against the uncertainty in future labor income. Second, I can calculate the optimal sensitivity of consumption to both permanent and transitory changes in income. Finally, I can calculate the expected growth in consumption over time.

The results of this paper potentially shed light on three empirical puzzles that have arisen in the literature: the excess sensitivity of consumption to transitory innovations in income, the high growth of consumption in the presence of a low or negative real interest rate, and the underspending of the elderly. I provide examples in which individuals have constant relative risk aversion utility functions, face uncertain labor income streams and a riskless technology for borrowing and lending, and are fully optimizing. Relative to a certainty model, the individual optimally chooses to have the sensitivity of consumption to transitory income “too” high, the expected growth of consumption “too” high, and the level of consumption “too” low. All of these results arise, even though I assume throughout the paper that there are *no* constraints on riskless borrowing or lending.¹ In other words, the results suggest that we may not need to rely on borrowing constraints or nonrational behavior to explain these empirical puzzles.

The paper is structured as follows. Section II briefly discusses the empirical puzzles. Section III defines the optimization problem faced by individuals in the economy and discusses the assumptions necessary to derive the certainty equivalence (or simple permanent income hypothesis) solution. In Section IV, I briefly review the existing literature and discuss some of the theoretical issues involved. The numerical results are presented and discussed in Section V, and Section VI concludes the paper.

II. EMPIRICAL PUZZLES

At least three empirical puzzles have arisen in the literature recently.² The first of these relates to the sensitivity of consumption

1. An earlier version of this paper contained results about the effects of borrowing constraints on consumption in a model with income uncertainty. At the suggestion of the editors, these results have been excluded from the paper and can now be found in Zeldes [1987]. In the current paper the only constraint imposed is the standard terminal condition on debt.

2. See, for example, Dornbusch and Fischer [1984].

to current income. Hall and Mishkin [1982] consider the response of current consumption to an *innovation* in income. They define excess sensitivity as the difference between the response in consumption and the annuity value of the increase in human and nonhuman wealth that occurs as a result of this innovation. Using panel data from the Panel Study of Income Dynamics (PSID), they find that (food) consumption is excessively sensitive to innovations in the transitory component of income. (Note that this definition of excess sensitivity differs from Flavin's [1981], who tests for excess sensitivity to *anticipated* changes in income. She tests the implication that the response of consumption to a change in income that was previously anticipated to occur should be equal to zero.³ The focus in this paper is on the excess sensitivity of consumption to innovations in income.)

The second puzzle concerns the expected growth of consumption over time. Under any certainty model of consumption with time-separable utility, if the interest rate is less than the rate of time preference, the growth rate of consumption must be negative. For example, with constant relative risk aversion utility, the growth rate of consumption under certainty is equal to $[(1+r)/(1+\delta)]^{1/A} - 1$, where A is the coefficient of relative risk aversion, r is the real interest rate, and δ is the rate of time preference. Deaton [1986] and Singleton [1985], among others, have pointed out that there have been long periods of time in which average U. S. aggregate consumption growth has been positive despite real interest rates that were very low (close to zero) and rates of time preference that were assumed to be positive.

The third puzzle relates to the savings behavior of the elderly. There has been controversy recently over whether the elderly dissave at all during retirement. Mirer [1979] and Danziger et al. [1983], using cross-section data, argue that the elderly do not dissave during retirement. Hurd [1987] uses panel data to challenge that view, arguing that the wealth of elderly families does decline over time. Bernheim [1987] argues that when private pension and Social Security annuities are appropriately valued and included in wealth there is little or no tendency for retirees to draw down their resources. In their recent surveys of this literature, both Modigliani

3. Using U. S. aggregate time series data, Flavin [1981] finds that consumption responds positively to changes in income that were previously anticipated. Note that under an alternative hypothesis which says that consumption is equal to total income and thus does not distinguish between anticipated and unanticipated income, and assuming that the income process is not "too" persistent (see Campbell and Deaton [1987]), both types of excess sensitivity will be present. However, under other alternative hypotheses this need not be the case.

[1988] and Kotlikoff [1988] conclude that retirees do not draw down their wealth sufficiently fast to be consistent with the benchmark life cycle model with certainty and no bequest motive.

Each of these puzzles is based on a certainty or certainty-equivalence benchmark.⁴ The implications drawn in these studies are often that consumers are not rational or forward looking, or that borrowing restrictions are important. In this paper I examine a different benchmark: a model with random labor income and constant relative risk aversion utility. The results may help explain each of these puzzles without having to resort to a borrowing constraint or irrationality explanation.

III. THE CONSUMER'S PROBLEM

Consider the standard problem of a consumer who lives for many periods and chooses optimal current consumption and contingency plans for future consumption to maximize the expected value of a lifetime time-separable utility function. The only source of uncertainty considered is in exogenous future labor income, and I assume that no markets exist in which individuals can hedge against this uncertainty by trading contingent claims.⁵ In other words, the focus here is on uncertainty in nontraded labor income, as opposed to uncertainty about the rate of return on traded assets.⁶ The apparent absence in modern economies of markets for human capital is most likely due to problems of moral hazard and adverse selection, but these are not incorporated directly in the model below.

The formal problem can be summarized by an objective function, a transformation equation, and an initial and terminal condition. In each period t ($t = 1, \dots, T$, where $T < \infty$) the consumer chooses C_t in order to maximize

$$(1) \quad E_t \sum_{j=0}^{T-t} \left(\frac{1}{1 + \delta} \right)^j U(C_{t+j}),$$

4. Singleton [1985] examines an uncertainty model with constant relative risk aversion preferences. However, he does not consider the effects of individual uncertainty discussed below.

5. Labor supply is taken as exogenous in this paper. If, instead, workers faced a given wage and were allowed to choose their labor supply, this would provide another channel for hedging wage uncertainty. The effects of wage uncertainty on consumption would be similar to, but somewhat less dramatic than, those presented here.

6. We can think of labor income as the dividend on a share of human capital, but the individual is forced to hold a fixed number of these shares and cannot trade them. When all assets are traded, closed-form solutions for consumption can be derived. See, for example, Merton [1971].

subject to

$$(2) \quad W_{t+1} = (W_t - C_t)(1 + r_t) + Y_{t+1}$$

$$(3) \quad C_t \geq 0$$

$$(4) \quad W_T - C_T \geq 0,$$

where W_t equals financial wealth in period t (after receiving income and before consuming), r_t equals the real interest rate between t and $t + 1$, Y_t equals labor income in period t , C_t equals consumption in period t , E_t is the expectation operator conditional on information available at time t , U is the one-period utility function, and T is the nonstochastic date of death.⁷ Define J_{T-t+1} as the maximized value of (1), i.e., the value function for a $T - t + 1$ period problem in terms of period t utility. Income is received at the beginning of the period, and then consumption is chosen. The remaining financial wealth earns a rate of return r_t between the current and subsequent period. Equations (2) and (4) imply that the future value at time T of initial wealth and subsequent realizations of income must be at least as great as the future value of the chosen consumption path; i.e., the individual must pay back all loans with probability one before the end of life.⁸ Note that other than this terminal constraint, there are no constraints on borrowing—individuals may borrow and lend freely at the riskless rate of interest.

I assume throughout that r is constant over time and equal to δ . This problem might appear at first glance to be an easy one to solve. For any concave utility function in the case of variable but *nonstochastic* future income, the following is the solution:⁹

$$(5) \quad C_{CEQ,t} = k_{T-t+1} [W_t + HW_t],$$

7. In this paper I investigate arbitrarily long, but finite and known, horizons. The techniques used could be modified, however, to incorporate infinite or random horizons.

8. This budget constraint is the appropriate one to use, and is stronger than an expected value budget constraint.

In a model with many assets with uncertain rates of return, the problem would look quite similar except that the consumer would also choose portfolio shares. In that case, r_t would be defined as the arithmetic weighted average of the realized return on the financial (nonhuman capital) portfolio, and the future value calculations in the budget constraint would be based on ex post (realized) rates of return on the individual's portfolio.

9. The proportionality hypothesis which underlies the simple life cycle or permanent income model says that in any period consumption is proportional to the present value of lifetime resources, with the constant of proportionality independent of lifetime resources. Yaari [1964] showed for a certainty model that if the proportionality hypothesis holds (for all r and δ), then the utility function must be a constant relative risk aversion utility function. When $r = \delta$, the proportionality hypothesis holds under certainty for any concave utility function. For this reason, I focus on the $r = \delta$ case throughout this paper.

where

$$k_{T-t+1} = \left(\frac{r}{1+r} \right) \left[\frac{1}{1 - (1/(1+r))^{T-t+1}} \right]$$

and

$$HW_t = E_t \sum_{j=1}^{T-t} (1+r)^j \cdot Y_{t+j}.$$

This smoothing solution is what is commonly referred to as the life cycle or permanent income hypothesis and is the consumption function that is routinely used in the literature [Flavin, 1981; Hall and Mishkin, 1982]. Consumption is proportional to the expected present value of lifetime resources, which consist of human wealth (HW_t) plus nonhuman wealth (W_t). Human wealth is the present discounted value as of time t of expected future labor income. The constant of proportionality k_{T-t+1} is equal to the annual payment on a $T - t + 1$ period \$1 annuity; i.e., it is the equal amount that one could receive in each period left of life by giving up one dollar today. When $r = 0$, k_{T-t+1} is simply equal to $1/(T - t + 1)$, the inverse of the number of periods left in life.¹⁰ This solution implies that $C_t = E_t C_{t+j}$ for all $j \geq 0$; i.e., expected consumption is constant over one's lifetime. For any concave utility function the above solution is correct in a certainty model. However, this is *not* in general the correct solution to a maximizing model when income is stochastic. The necessary conditions for the above to be the general solution under stochastic income are

- (1) the period utility function $U(C_t)$ is quadratic;
- (2) C_t is allowed to range from $-\infty$ to $+\infty$.

If these conditions hold, then consumption is identical to what it would be with no uncertainty. This is known as the certainty-equivalence (CEQ) solution. I define C_{CEQ} as the optimal consumption (function) under certainty equivalence, the solution represented by (5).

Unless otherwise stated, I shall use the term consumption "function" to refer to the optimal level of consumption at a given point in time as a function of existing financial assets. The time t consumption function under certainty or certainty equivalence is an upward sloping straight line with slope $(\partial C_t / \partial W_t)$ equal to k_{T-t+1} ; if we give this individual one extra dollar of financial wealth, k_{T-t+1} of it would be spent today.

10. This can be seen by applying L'Hôpital's rule to the formula for k .

When riskless borrowing is freely allowed at rate r , the present value of any future income that will be received with certainty (i.e., a floor of future income) can, without loss of generality, be redefined as part of financial assets instead of income. Therefore, W_t should be thought of as the "certain" component of lifetime resources, including both current financial assets and the present value of the floor of future income, and Y_t should be thought of as the deviation of income from that floor.

Under certainty equivalence the sensitivity of consumption to current income is

$$(6) \quad \frac{dC_{CEQ,t}}{dY_t} = k_{T-t+1} \left[\sum_{j=0}^{T-t} \left(\frac{\partial E_t Y_{t+j}}{\partial Y_t} \right) \left(\frac{1}{1+r} \right)^j \right].$$

As shown by Flavin [1981], under certainty equivalence the marginal propensity to consume (MPC) out of current income depends crucially on the time series properties of income and the extent to which current income signals changes in expected future income. For the case of i.i.d. income, $\partial E_t Y_{t+j} / \partial Y_t = 0$ for $j > 0$, and therefore the MPC out of current income is simply equal to k_{T-t+1} . In this case, consumers respond in the same way to an extra dollar of income as to an extra dollar of wealth.

In this paper I examine what happens to optimal consumption when one simple change is made from the CEQ assumptions: the consumer is assumed to have a standard constant relative risk aversion utility function rather than a quadratic utility function. Thus, $U(C) = C^{1-A}/(1-A)$, where A is the coefficient of relative risk aversion. We shall see that in many circumstances the consumption functions look very different from those implied by certainty equivalence. Note that the property of constant relative risk aversion utility that $U'(0) = \infty$ endogenously bounds optimal consumption away from negative or zero consumption. There is therefore no need in this paper to impose any exogenous constraints on consumption or borrowing.¹¹

IV. THE THEORETICAL LITERATURE

The certainty-equivalence solution above requires a quadratic utility function; i.e., one in which the third derivative is equal to

11. Elsewhere [Zeldes, 1987] I maintain the assumption of quadratic utility and examine the effects of imposing the realistic condition that consumption be non-negative. This is done by imposing a form of borrowing constraint, and it also leads to substantial deviations from certainty equivalence.

zero. The first papers on the subject of the effects of a nonzero third derivative on optimal consumption in a model with labor income uncertainty were by Leland [1968], Sandmo [1970], and Drèze and Modigliani [1972]. Leland defined "precautionary" saving as the difference between consumption when (a) income is certain and (b) income is uncertain but with the same mean as in (a). Each of these papers used a two-period model, and their results imply that with time-separable utility precautionary saving is positive if and only if the third derivative of the utility is positive; i.e., marginal utility is convex. Put another way, with $U''' > 0$, C is less than C_{CEQ} at all levels of financial assets.

Sibley [1975] and Miller [1976] independently showed that the above result holds for a multiperiod model (with i.i.d. income) and that precautionary saving increases with a Rothschild-Stiglitz [1970] mean-preserving spread on income, beginning at *any* initial level of uncertainty.¹² Note that a positive third derivative of the utility function is implied by the commonly accepted assumption of decreasing absolute risk aversion.¹³

The above results do not imply anything about the sensitivity of consumption to wealth or transitory income; i.e., they deal with the level of the consumption function (versus wealth) but not with the slope of the consumption function. Elsewhere [Zeldes, 1984] I use a two-period model and a second-order Taylor expansion of marginal utility to show that with constant relative risk aversion, adding uncertainty (beginning with none) raises the slope of the consumption function, implying a greater sensitivity of consumption to transitory income than under certainty equivalence. However, this result of "excess sensitivity" depends on higher derivatives of the utility function than the third.¹⁴ The effects on the level

12. Grossman, Levhari, and Mirman [1979] showed that the value function inherits the positive third derivative property of the period utility function.

Yaari [1976] examines a model in which $r = \delta = 0$, income is i.i.d., and the utility function must be such that utility is well defined over all consumption values between $-\infty$ and $+\infty$. In his model the fact that income is independently distributed over time enables people to diversify away, across time, the risk inherent in income. By using the law of large numbers, he shows that, as the horizon is extended, optimal consumption approaches the mean of income—the same solution as in the certainty case with income equal to the mean (CEQ). Yaari's result seems to depend crucially on all of the above assumptions. Note, in particular, that utility is not defined over negative consumption for constant relative risk aversion utility and therefore Yaari's proof does not apply to that case.

13. Recall that decreasing absolute risk aversion means that the dollar amount that an individual would be willing to pay to avoid a fair gamble of a given size decreases as the level of consumption rises.

14. Since this paper was originally written, work has been done that generalizes this result and shows that excess sensitivity will occur for a class of utility functions that includes constant relative risk aversion and excludes constant absolute risk

and the slope of the utility function are generated by two different properties of the utility function.¹⁵

A closed-form solution for consumption with random, non-traded, labor income has not been derived, except for some specific examples such as constant absolute risk aversion [Merton, 1971; Schechtman and Escudero, 1977; Cantor, 1985; Roell, 1984; Caballero, 1987; Kimball and Mankiw, 1987]. No exact closed-form solutions have been derived for constant relative risk aversion utility.¹⁶ Therefore, I shall use numerical techniques in order to examine and quantify the effects of labor income uncertainty in a multiperiod model.

V. IMPLEMENTING THE MODEL

A. Calibrating the Model

The model is calibrated based on panel data estimates of the amount of income uncertainty facing individuals in the economy. Hall and Mishkin [1982] assume that income can be decomposed into the sum of two separate components, one of which follows a random walk (the “permanent” component) and the other of which follows an MA(2) (the “transitory” component), and that the shocks to these components are separately observable. MaCurdy [1982] assumes that the log of earnings follows an IMA(1,2), and thus that there is only one type of shock to income. In both cases the conditional variance of the j step ahead forecast of income grows with j . Hall and Mishkin’s formulation is convenient for the purposes of this paper because it allows us to distinguish between the effects on consumption of transitory and permanent disturbances to income.

I use two sets of examples. The first is a multiplicative version of Hall and Mishkin’s process. YL_t is the “lifetime” component of income. It follows a geometric random walk and is hit each period by the i.i.d. shock EL_t . EL_t is meant to capture the effects of raises, job changes, health changes, and other persistent factors. Total

aversion. See Roell [1986] and Kimball [1988]. These authors analytically verify (but cannot quantify) the numerical results in Section VI on the presence of excess sensitivity. Other recent related work includes Caballero [1987].

15. For example, with constant absolute risk aversion utility, the consumption function would be shifted down in a parallel way when uncertainty is added, leaving the slope unchanged.

16. Skinner [1988] examines the magnitude of precautionary savings with constant relative risk aversion utility by using a second-order Taylor expansion of the Euler equation.

labor income Y_t is equal to this component times an i.i.d. shock ES_t . ES_t is meant to capture the effects of one-time bonuses, unemployment spells, and other transitory factors. This gives

$$(7) \quad \begin{aligned} Y_t &= YL_t \cdot ES_t \\ YL_t &= YL_{t-1} \cdot EL_t. \end{aligned}$$

ES and EL are i.i.d., have mean equal to one, and are assumed to be *separately observable*. Expected income in period $t + j$ conditional on time t information is equal to YL_t . I calibrate the income process so that the j step ahead coefficient of variation is roughly in line with that implied by both Hall and Mishkin and MaCurdy. To keep things simple, I use the following three point distributions for ES and EL (I call this distribution #2):

| EL | outcome | probability | ES | outcome | probability |
|------|---------|-------------|------|---------|-------------|
| | 0.9 | 0.25 | | 0 | 0.05 |
| | 1.0 | 0.50 | | 1.0 | 0.45 |
| | 1.1 | 0.25 | | 1.1 | 0.50 |

In any period the permanent component of income can rise or fall by 10 percent (this corresponds to raises or permanent pay cuts). Also there is a 5 percent chance that income will equal zero in a given period (temporary unemployment), a 45 percent chance that income will equal the permanent component, and a 50 percent chance that income will be 10 percent higher than the permanent component (a transitory bonus).¹⁷ Figure I plots the coefficient of variations for the j year ahead forecasts of the level of earnings for this process against those implied by the estimates in Hall and Mishkin and MaCurdy. The coefficients of variation lie everywhere below MaCurdy's and generally below Hall and Mishkin's and thus provide a conservative estimate of the income uncertainty facing households.

In the second set of examples I assume income is i.i.d. ($EL \equiv 1$)

17. As discussed previously, the present value of any positive floor of future income should be included as part of financial assets, leaving future income with a floor of zero. As discussed in Barsky, Mankiw, and Zeldes [1986], however, it may also be reasonable to assume a minimum survival level of consumption (C_s) and to define utility as a function of consumption in excess of that level (e.g., $U = (C - C_s)^{1-A}/(1 - A)$). If the minimum survival level of consumption is equal to the floor of income, then the results are the same as in the text, with W interpreted as including only tangible nonhuman wealth and *not* including the present value of the floor of future income.

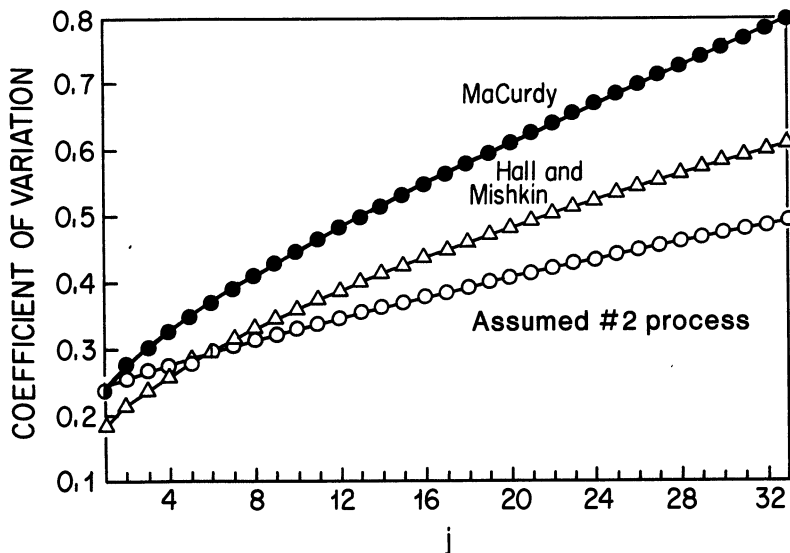


FIGURE I
Coefficient of Variation: j Step Ahead Forecast

and choose the income process so that the coefficient of variation (= 0.55) is roughly that of an average forecast horizon with either the Hall and Mishkin or MaCurdy process.¹⁸ The results for the two income processes in fact turn out to be quite similar.

B. The Numerical Technique

The method used to calculate the optimal consumption is stochastic dynamic programming.¹⁹ The technique for the case of i.i.d. income is described in detail in Barsky, Mankiw, and Zeldes [1986].²⁰ The problem is formulated as a one-state (wealth), one-control (consumption), one-disturbance (income) stochastic control

18. For this distribution (#1A), ES takes on the values 0, 1.0, and 2.0, with probabilities 0.15, 0.70, and 0.15, respectively. The coefficient of variation is equal to that of 16- and 27-year forecasts of income for the MaCurdy and Hall and Mishkin processes, respectively. Note that there are two important differences between the income processes that I use. First, a change in current income changes expectations of future income for the non-i.i.d. process, but not for the i.i.d. process. Second, the variance of the j step ahead forecast of income grows with j in the non-i.i.d. case, but is constant in the i.i.d. case.

19. I thank Andy Abel for helpful suggestions about the technique used here.

20. Barsky, Mankiw, and Zeldes [1986] focus on the issue of Ricardian equivalence, whereas the focus here is on the optimal consumption function. A more complicated technique is used here, in order to examine the effects when the income process contains both permanent and transitory components.

problem. The state space is discretized into an S element grid using a technique suggested in Bertsekas [1976]. Beginning in the last period, backwards induction is used to solve for the value function and corresponding optimal consumption. The computer was programmed to search, for each state and time, over all feasible levels of consumption and choose the one that maximizes the sum of current utility and the discounted expected value of next period's value function. The results are two $S \times T$ matrices which give optimal consumption and the value function at all possible levels of wealth and times left to live.

The procedure is similar for the more complicated income process except that another state variable—the lagged value of the random walk component of income—has to be added, yielding two three-dimensional matrices. This is conceptually straightforward, but unfortunately, due to the “curse of dimensionality,” it requires dramatically more computer memory and CPU time, so that only relatively short horizons could be examined.

While the numbers calculated with this technique are an approximation to the actual solution, the approximation error can be made arbitrarily small by narrowing the width of the grid used for discretization [Bertsekas, 1976].²¹ In part because of the high cost of this computer search,²² I restrict my attention to relatively simple problems.

VI. RESULTS AND IMPLICATIONS

A. Results

In what follows, I compare the consumption function with constant relative risk aversion (CRRA) utility to the consumption function under certainty or certainty equivalence (CEQ). For simplicity, I assume that $r = \delta = 0$. I examine the level of consumption, the sensitivity of consumption to permanent and to transitory changes in income, and the expected path of consumption. Recall that the marginal propensity to consume out of wealth is identical to the marginal propensity to consume out of transitory changes.

21. The grid solutions are very close to the analytical solutions for some simple three-period problems that I checked.

22. The problems with i.i.d. income were calculated at relatively low cost on a VAX 8600. However, the fifteen-period example with non-i.i.d. income involves considerable memory requirements and CPU time. This example requires creating two matrices with about 625,000 elements each. The optimal consumption (and value function) then had to be determined for each of the 625,000 possible nodes, for each of the fifteen periods.

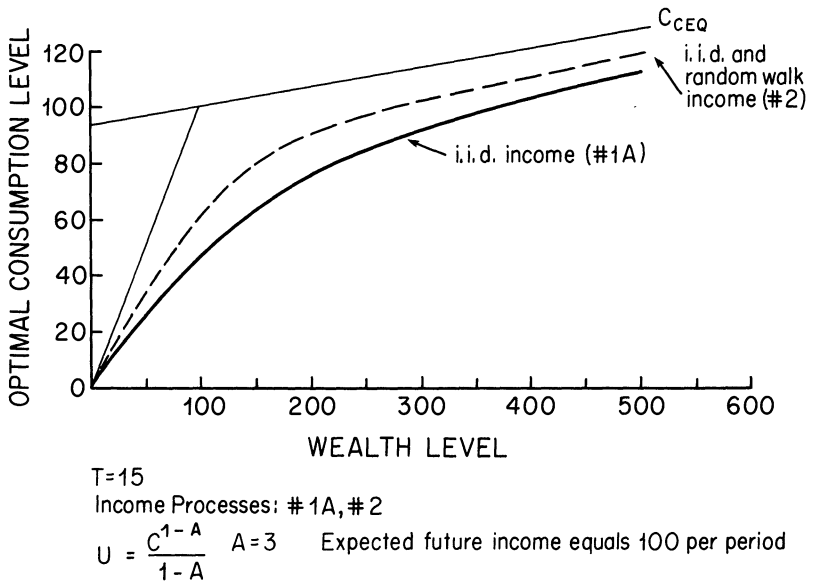


FIGURE II
 Optimal Consumption for Two Different Income Processes

The first example uses distribution #2, the combination random walk/i.i.d. process. The time horizon is 15 periods, and the coefficient of relative risk aversion is set equal to three.²³ Consumption as a function of initial financial assets is plotted as the middle line in Figure II. Both initial income and expected income in all future periods are equal to 100. The results, especially at low levels of assets, are quite striking. First, notice that the slope of the curve, which is equal to the marginal propensity to consume out of a transitory change in income, is considerably larger than that predicted by certainty equivalence. For example, a household with two years worth of expected income in the form of assets would have an MPC of over twice that which would be predicted by a certainty or certainty-equivalence model. A family with one year's worth of expected income as assets would have an MPC seven times as great as under CEQ. This fully optimizing unconstrained household exhibits dramatic "excess sensitivity" relative to the certainty-

23. Empirical estimates of A cover a wide range. They include Friend and Blume [1975] (2), Mankiw [1981, 1985] (4, 3, respectively), Hansen and Singleton [1983] (1), Mankiw, Rotemberg, and Summers [1985] (0.5), and Zeldes [1989] (2.3).

equivalence benchmark, suggesting that CEQ is not a terribly good benchmark.

For a household with very high assets relative to expected future income (e.g., four to five years worth) the MPC is almost the same as that implied by the simple smoothing model.²⁴ This means that we should expect MPCs out of transitory changes in income to be much larger for households with low current (certain) assets relative to expected future (uncertain) income than for the rest of the population. Note that this does not necessarily correspond to “poor” versus “wealthy” households, because what is relevant is not the absolute amount of current wealth, but current assets *relative* to expected future (uncertain) labor income.

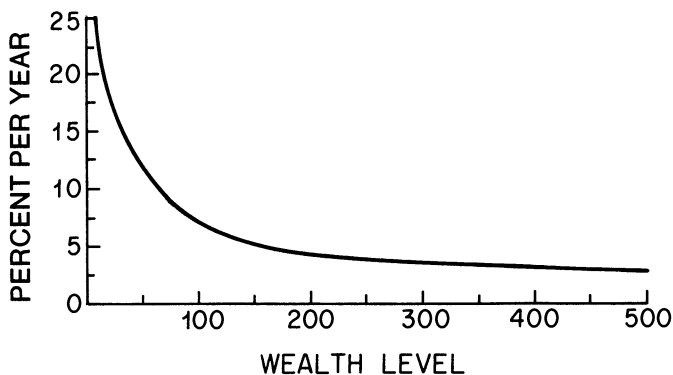
Next, I examine the sensitivity of consumption to changes in expected future income. Consider the experiment (not shown in figures) of varying YL_t , the random walk component of income, holding current assets constant at 200. A one-unit increase in YL_t increases expected income in every future period by one unit. The slope of the consumption line under CEQ is $(T - 1)/T$.²⁵ In fact, however, the sensitivity of consumption to expected future income is only about 60 percent as large as the CEQ benchmark. If initial wealth is instead equal to 400, the sensitivity of consumption to expected future income is much closer to the CEQ benchmark.²⁶

Overall, what we are seeing is quite interesting. Relative to the certainty-equivalence benchmark, individuals optimally “overrespond” to changes in current income or wealth and “underrespond” to changes in expected future income. This is especially true when current assets are relatively low. In these cases, current assets play a much more important role in the consumption function relative to risky future labor income than would be predicted under CEQ.

24. Recall that what is referred to as assets here is equal to the sum of financial wealth and the present discounted value of the *floor* of future income. See footnote 17.

25. The change in expected lifetime income equals $T - 1$ (one for each period left to receive income), and this is divided by T , the number of periods left to consume. Note that because the process for the lifetime component of income is a geometric random walk, an increase in YL_t also raises the conditional variance of future income, which contributes to the undersensitivity result below.

26. One might also wish to examine the following experiment. Assuming that the current realization of the transitory component was equal to its mean, what would be the MPC out of change in the current random walk component that raised current (and expected future) income by one dollar. Note that the MPC out of such a change would be equal to the sum to the MPCs out of current wealth (which was too large relative to CEQ) and expected future income (which was too small relative to CEQ) discussed in the text. (Under CEQ it would equal $(T - 1)/T + 1/T = 1$). As long as wealth before income is received is positive (i.e., wealth after income is at least one year's worth of expected income), the result of this experiment is still an underresponsiveness relative to CEQ.



$T=15$

Income Process: # 2

$$U = \frac{C^{1-A}}{1-A} \quad A = 3 \quad \text{Expected future income equals 100 per period}$$

FIGURE III
Expected Lifetime Consumption Growth

Finally, I examine the level of precautionary saving defined as $C - C_{CEQ}$. At a wealth of 200, precautionary saving is 20 percent of optimal consumption. That is, if there were no income uncertainty, consumption would be 20 percent higher than it is with the uncertainty. At low levels of assets, the certainty-equivalent benchmark dramatically overstates the optimal level of consumption. At a wealth of 500, precautionary saving is still about 7 percent of the optimal consumption level. This suggests that a significant fraction of the capital accumulation that occurs in the United States may be due to precautionary savings.

Next consider the expected growth of consumption between this period and next, which, since $r = \delta$, is equal to zero under *CEQ*. The results (not shown) indicate that at a wealth of 100, consumption is expected to grow by 25 percent between the first and second period.²⁷ At a wealth of 200, this number drops down considerably to 2.5 percent.

Figure III is a plot of the annualized expected growth of consumption between the current period and the last period of life.

27. The intuition behind this result is that the extra precautionary saving done today will lower today's consumption and will raise expected consumption later in life when the extra accumulated assets are spent. Also, one can see that the expected growth of consumption will be higher than under *CEQ* by looking at a Taylor expansion of the standard Euler equation (see, for example, Zeldes [1989]).

At an initial wealth of 200, consumption is expected to grow at an average rate of 4 percent per year between the current period and the end of life. That is, $E_1(C_T/C_1) = (1.04)^{T-1}$.²⁸ Even at an initial wealth of 500, this number only drops to 2.5 percent per year.

In order to be able to experiment with longer horizons without hitting memory constraints on the computer, the i.i.d. income process #1A is used next. Figure II shows that the optimal consumption function for this distribution is similar to that of the random walk/i.i.d. process used earlier. The results from extending the time horizon to 30 and 60 years (not shown) indicate that, holding financial assets constant, the percentage difference from CEQ is approximately the same regardless of age (i.e., horizon). Thus, the fact that the young have more uncertainty about lifetime resources is approximately offset by the fact that they also have more periods left to spread out any unexpected changes in current income. Increasing the time horizon does not diminish the effects of income uncertainty on the level of consumption.²⁹

Finally, a mean preserving spread is performed on the income distribution which raises the coefficient of variation from 0.55 to 0.71 (see Figure IV).³⁰ A more risky income stream leads to lower consumption levels [Miller, 1976; Sibley, 1975]. At a wealth of 300, precautionary saving is 25 percent of C_{CEQ} , as opposed to 17 percent for the less risky income stream. Greater uncertainty does not necessarily imply a higher MPC out of transitory income, however. The rankings for the MPCs depend on the level of initial wealth.³¹

B. Relationship to Consumption Puzzles

Overall, when we examine consumption under constant relative risk aversion utility, we see fairly dramatic deviations of optimal consumption from what is implied by a simple smoothing or certainty model. This is especially noticeable when current assets are low relative to expected future income.

28. Note that this involves calculating the true expected value of period T consumption, which takes into account all of the possible outcomes for the disturbances and the corresponding consumption responses along the entire path. It is not equal to the value of consumption if income is simply set equal to its mean along the path.

29. The results also indicate that the MPC out of wealth always falls as the horizon is lengthened, holding assets constant, a result which is also true in a certainty model.

30. For the higher variance income process (#1B), income is equal to 0, 100, and 200, with probabilities 0.25, 0.50, and 0.25, respectively.

31. I also experimented with different coefficients of relative risk aversion ($A = 1$ and $A = 6$). An increase in risk aversion (from 3 to 6) had an effect that was qualitatively similar to (and slightly stronger than) the increase in the uncertainty in income reported in the text.

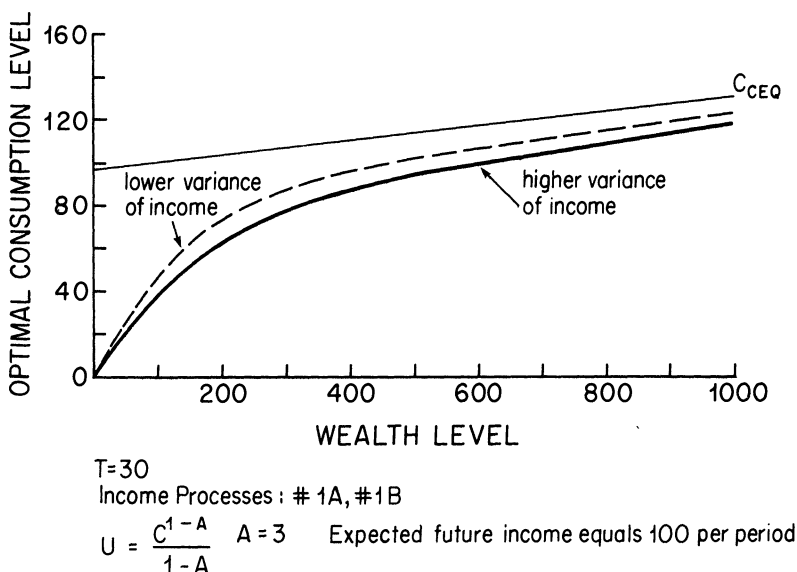


FIGURE IV
Optimal Consumption with Different Income Risk

Consider how this relates to the consumption puzzles described in Section II. Using an income process similar to that assumed by Hall and Mishkin, an optimizing unconstrained model with constant relative risk aversion utility yields a consumption function that exhibits “excess” sensitivity as defined by Hall and Mishkin. They find that the ratio of the MPCs out of transitory income and permanent income was 0.29 instead of the CEQ benchmark of 0.10 (about 2.9 times as great). In the fifteen-period model examined here, the CEQ benchmark is $1/15 = 0.067$, and the optimizing consumer with wealth of two years’ expected income in fact has a ratio of 0.21, about 3.1 times as great as under CEQ. While this is by no means conclusive evidence, it suggests that Hall and Mishkin’s results may have been consistent with optimizing behavior.³²

The second puzzle is related to the expected growth in consumption. In the examples presented here the rate of interest and the rate of time preference were both equal to zero. In a certainty model this would imply that the growth of consumption

32. It should be noted that as long as taxes are lump sum, Ricardian equivalence will hold in this model, despite the “excess” sensitivity. With taxes positively related to income, however, Ricardian equivalence will fail. See Barsky, Mankiw, and Zeldes [1986].

must be equal to zero in all periods. When uncertainty is explicitly modeled, however, the expected growth in consumption for an individual with two years worth of income in the form of wealth and a fifteen-year horizon was found to be 2.5 percent over the first year and 4 percent per year over the entire horizon. Thus, a negative rate of time preference is not required in order to explain positive expected growth rates of individual consumption with low or negative real risk-free interest rates. There exist models in which this result will show up in aggregate consumption as well. This helps resolve the puzzle of how a low risk-free interest rate can be compatible in equilibrium with a high growth in aggregate consumption. Note that the extra growth in aggregate consumption will be a function of the uncertainty in *individual* income, which is significantly larger than that in average aggregate income.³³

Finally, consider the puzzle relating to the saving behavior of the elderly. I have not tried to calibrate the model for this case explicitly, but one can model uninsured medical expenses or other emergency expenses that the elderly must sometimes incur as equivalent to negative draws of income. The possibility of having to incur severe costs for catastrophic illness is likely to be an important source of uncertainty for the elderly.³⁴ The results here show that the level of precautionary saving in the face of plausible amounts of income uncertainty is quite high. This suggests that the same may be true for the elderly facing uncertainty about expenses. Optimal consumption and average wealth decumulation may therefore be substantially less than what would be predicted on the basis of *expected* medical outlays alone. More work clearly needs to be done calibrating such a model, but the results here suggest that the low dissaving by the elderly may be consistent with optimizing behavior under uncertainty.

C. Implications for Empirical Consumption Functions

The standard consumption function posits a linear relationship between consumption and “permanent income,” defined as the

33. Consider a simple economy with N individuals, each of whom starts out with the same initial wealth and income, and let each of the shocks to income be i.i.d. across individuals. For large N , aggregate per capita consumption in each period will be equal to the expected consumption of each individual. In this case, aggregate consumption will grow by an average of 4 percent per year over the fifteen-year period. Here there is virtually no aggregate uncertainty, but the individual uncertainty causes the growth in aggregate consumption to be significantly higher. Note, however, that Deaton [1986] points out that in a steady state with finite horizon individuals the growth of aggregate consumption need bear no relation to the growth rate of each individual's consumption.

34. See Kotlikoff [1989] for an attempt to calibrate these effects explicitly.

annuity value of the sum of nonhuman wealth and the present discounted value of expected future labor income. The results here indicate that such a consumption function is likely to be severely misspecified, especially at low levels of wealth.

One possible remedy to this problem would be to put a weight of less than one on human wealth before adding it to nonhuman wealth, or to discount expected future income at a higher discount rate. This would be an improvement because it would shift the certainty equivalence line downward in a parallel way.³⁵ However, it would still miss important aspects of the consumption function because the weight or the discount rate that should be placed on human wealth is not constant, but depends on the amount of human wealth relative to nonhuman wealth. This suggests the following as a better approximation:

$$C = k \cdot [W + x(W/HW) \cdot HW],$$

where x is the weight ($0 \leq x \leq 1$) placed on human wealth, and is an increasing function of the fraction of lifetime resources in nonhuman wealth. This will help capture the curvature in the consumption function seen in the preceding figures.³⁶ While the resulting function is still not correct, it could give a significantly better approximation to the true optimal consumption function.³⁷

D. Unresolved Issues

While the above model of precautionary savings can potentially help explain a number of empirical puzzles, there are some additional implications of the model that either are not consistent with the empirical literature or have not been carefully tested to date.

First, most of the Euler equation tests of consumption need not assume quadratic utility, and in fact are frequently based on

35. Nagatani [1972] examined a model with constant relative risk aversion utility, and *assumed* that the solution involved discounting the expected future income by a higher risk-adjusted discount rate and using this value of human wealth in the certainty solution. Under his assumed solution he demonstrates that the expected growth of consumption depends positively on the ratio of human wealth to total resources, and that the time profile of average consumption may approximately match the time profile of expected income, even though neither of these would be true under certainty.

36. In a similar vein Friedman [1957, pp. 16–17] conjectured that the introduction of income uncertainty should cause consumption to depend positively on the ratio of nonhuman wealth to permanent income. However, his motivation was based on the difficulty of borrowing against future labor income. In the present paper consumption is a function of this ratio even in the absence of borrowing constraints.

37. One might also want to model x as a function of the amount of individual income uncertainty and utility function parameters.

constant relative risk aversion utility functions. Nevertheless, tests using aggregate data generally reject the Euler equation (e.g., Hansen and Singleton [1983]; Mankiw, Rotemberg, and Summers [1985]). This means that the model of precautionary savings presented here cannot explain these Euler equation violations.

Second, if precautionary savings are important, one would expect individuals on average to accumulate substantial wealth. However, the evidence seems to suggest that many households accumulate relatively small amounts of wealth. I have not attempted to calculate a steady-state distribution of wealth under this model, and therefore cannot directly address this question. This issue clearly deserves further attention.

Third, the theory implies that for a given mean of future income and for a given level of financial assets, individuals with greater uncertainty about future income should save more. The empirical evidence on this issue is mixed. Friedman [1957] found that average saving rates were higher for farmers and for the self-employed than for the rest of the population and attributed the difference to an assumed greater uncertainty in income for these groups.³⁸ Malcolm Fisher [1956] found a similar result for the self-employed using British data. In more recent work, however, Skinner [1988] finds that the self-employed and farmers save less than do other occupations. As Mayer [1972] points out, however, none of these studies presents evidence showing that income uncertainty over the lifetime is actually greater for these groups than for the general population. Also, the empirical tests are not able to hold constant variables such as the level of financial assets, the tilt of the age income path, and the degree of risk aversion, making interpretation very difficult.³⁹ It seems clear that additional work is needed to test this implication of the theory of precautionary saving.

38. Friedman also found a lower MPC for farmers and self-employed, and argued that this was due to a higher variance of the transitory component of income relative to the permanent component. Note that this argument need not be related to precautionary saving. Even under CEQ, the MPC out of income depends on the change in human wealth signaled by the change in income, which in turn depends on the importance of transitory versus permanent changes in income.

39. For example, if high uncertainty individuals save more on average when young, thereby accumulating more assets, they must save *less* on average (or dissave more) later in life. Thus, if financial wealth is not held constant, savings of high-risk individuals will be lower in one stage of life and higher in another stage of life, relative to low-risk individuals. Skinner attempts to adjust for this by excluding individuals over age 50.

VII. CONCLUSIONS

The simple life cycle/permanent income model described in most macroeconomic texts says that individuals base their current consumption on the sum of their financial assets and the expected discounted value of their future labor income, in such a way that consumption is expected to be constant over their lifetimes. This result is generally true in a certainty model, but when future labor income is random, it is necessary to assume that the utility function is quadratic and that consumption can be negative. When the utility function is instead assumed to be of the standard constant relative risk aversion form, closed-form solutions can no longer be derived. For this reason, many researchers have confined their attention to the testable implications implied by the intertemporal Euler equation. The numerical technique used here gives a very accurate approximation to the optimal consumption function and thus allows us to examine the current level of consumption as a function of financial wealth and the level of income, and the contingent path of consumption over time.

The resulting consumption function looks quite different from the common certainty equivalence benchmark described above. This is especially true when the "certain" component of lifetime resources is small relative to the risky components of lifetime resources; i.e., when financial assets are small relative to human capital. In such circumstances, the level of precautionary saving calculated for individuals is large, suggesting that precautionary saving may represent a significant fraction of the total saving of U. S. households. Thus, the growth of unemployment and other forms of insurance may help explain the secular decline in the U. S. savings rate.⁴⁰ In addition, the results show that current assets (which include the income just received) and nonstochastic future receipts are optimally given much more weight than future random labor income in making the current consumption decision. This cannot be compensated for merely by discounting future labor income at a higher rate, because the appropriate rate would vary with the level of initial assets.

The results indicate that rational individuals with constant relative risk aversion utility will optimally exhibit "excess" sensitivity to transitory income, save "too" much, and have expected growths of consumption that are "too" high, relative to the simple

40. For evidence of this decline see Summers and Carroll [1987].

permanent income hypothesis benchmark, even in the absence of borrowing constraints. This suggests that we should rethink our presumption that the certainty equivalent model is the appropriate benchmark, especially at low levels of financial wealth. These results have the potential to explain at least partially three important empirical puzzles in the consumption literature: the excess sensitivity of consumption to transitory changes in income, the high growth in consumption with a low risk-free interest rate, and the high savings rate of the elderly.

THE WHARTON SCHOOL, UNIVERSITY OF PENNSYLVANIA, AND
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