Risk Neutrality Regions

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Abstract

An Expected Utility maximizer can be risk neutral over a set of non-degenerate multivariate distributions even though her NM (von Neumann Morgenstern) index is not linear. We provide necessary and sufficient conditions for an individual with a concave NM utility to exhibit risk neutral behavior and characterize the regions of the choice space over which risk neutrality is exhibited. The least concave decomposition of the NM index introduced by Debreu [3] plays an important role in our analysis as do the notions of minimum concavity points and minimum concavity directions. For the special case where one choice variable is certain, the analysis of risk neutrality requires modification of the Debreu decomposition. The existence of risk neutrality regions is shown to have important implications for the classic consumption-savings and representative agent equilibrium asset pricing models.

KEYWORDS. Risk Neutrality, Expected Utility, Least Concave Utility and Minimum Concavity Points.

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1 Introduction

Standard textbook treatments of the economics of risk typically show that an Expected Utility maximizer will be risk neutral in the univariate case if and only if her NM (von Neumann and Morgenstern [23]) index is linear (e.g., Mas-Colell, Whinston and Green [16], p. 186). However for bivariate lotteries or distributions, the assumption that the NM utility takes the following linear form

$$U(c_1, c_2) = \alpha c_1 + \beta c_2 + \gamma, \tag{1}$$

where c_1 and c_2 denote units of two different commodities, is only sufficient and not necessary for risk neutrality.[1] An individual with a concave NM index not taking the form (1) can be risk neutral toward subsets of multivariate lotteries. We refer to these subsets as risk neutrality regions of the full choice space of lotteries or distributions.[2]

In this paper, individuals are assumed to possess Expected Utility preferences. We derive necessary and sufficient conditions for when a consumer with an NM index not taking the linear form (1) is risk neutral toward a subset of lotteries and characterize the regions of the choice space in which risk neutrality is exhibited. Since the results for the bivariate case naturally generalize to multivariate distributions, we focus on just the simpler bivariate case.[3] We also consider the important special case where one attribute is certain and the second is random.

The existence of risk neutrality regions can have important implications for popular application problems where the NM index is concave and does not take the form (1). First, suppose a consumer faces the classic two period consumption-savings, capital risk problem with a single risky asset. In response to a pure increase in the risk of the asset's return (i.e., a mean preserving spread), optimal savings can remain unchanged which is consistent with risk neutral behavior. Second assume a two period representative agent exchange economy in which there exists one risky and one risk free asset. Then for a particular endowment of period one consumption, the equilibrium risk premium can go to zero which is consistent with the representative agent being risk neutral.

[1] Whereas the discussion of risk neutrality is commonplace for the case of univariate distributions, the multivariate case is much less thoroughly investigated. One interesting exception is Safra and Segal [21], who consider multivariate risk neutrality for non-Expected Utility preferences.

[2] In a number of papers that have sought to extend the notion of risk aversion to multivariate Expected Utility preferences, the authors have tended to define risk attitudes in terms of utility indices. Reference to risk neutrality arises as the extreme of an individual being both risk averse and risk prone. See, for instance, Duncan [4], Karni [11], Kihlstrom and Mirman [12] and Hellwig [8].

[3] In the online Appendix L, we consider a specific example which illustrates how several of the key concepts investigated in this paper extend to more than two choice variables.

A key element in our analysis is Debreu's [3] classic decomposition of a concave, multicommodity NM index into a least concave utility representing certainty preferences over commodity vectors and a univariate concave transformation reflecting risk preferences. Debreu's focus was on proving the existence of a least concave utility given that a concave NM utility is known to exist. However in order to distinguish the specific set of lotteries toward which the consumer is risk neutral, one must go beyond existence and actually derive from a given concave U the specific form of the least concave utility. One must also identify the set of minimum concavity points and minimum concavity directions (characterizing where and in which directions the Hessian of the least concave utility vanishes). The very popular homothetic and quasihomothetic NM utilities[4] permit particularly clear characterizations of the subset of risk neutral lotteries. This follows from the very special properties of the minimum concavity points and minimum concavity directions of these utilities (see Kannai and Selden [10]). However we emphasize that risk neutrality regions also occur for non-homothetic and non-quasihomothetic preferences.

To extend our analysis of risk neutrality to the case where one of the commodities is certain, it is necessary to modify Debreu's decomposition result. For a given NM index, the set of minimum concavity points, minimum concavity directions and least concave utilities can differ when one commodity is certain versus when no commodity is certain. As a result, the risk neutrality regions will typically change. As we discuss in Section 5, the fact that the least concave utilities can diverge when one of the commodities become certain seems to have been missed in the literature. Not recognizing this point can result in decompositions of a given NM index into certainty preferences and risk preferences which are erroneous and ultimately lead to incorrect behavioral predictions such as how an individual will react to increasing risk.

The rest of the paper is organized as follows. In the next section, we give two motivating examples. The first illustrates the existence of risk neutrality regions in a standard lottery choice setting. The second demonstrates that risk neutrality regions can have interesting implications for optimal savings behavior. Section 3 reviews the definitions of risk neutrality toward univariate and bivariate probability distributions. In Section 4, we first discuss the notions of least concave utility, minimum concavity points and minimum concavity directions and then use them to characterize the subsets of distributions where an individual will be risk neutral. Section 5 considers the special case where one preference attribute is certain. Section

[4]The terms homothetic and quasihomothetic are defined as customary (see Deaton and Muellbauer [2], pp. 143–145). It should be noted that in the Expected Utility setting, if the NM index is a member of the HARA (hyperbolic absolute risk aversion) family of utilities, then preferences are homothetic or quasihomothetic.

6 provides two additional economic applications. Selected proofs are provided in the Appendix of this paper and the remaining proofs and supplemental materials are available in an online Appendix.

2 Motivating Examples

Before formally defining risk neutrality, we consider two motivating examples which illustrate that an individual or an economy can exhibit risk neutral behavior even though the assumed bivariate NM index does not take the linear form of eqn. (1).[5]

The following example shows that an individual can be risk neutral toward some (but not all) lotteries if her NM index does not take the linear form (1).

Example 1 *Assume the individual's NM index is given by*

$$U(c_1, c_2) = 600c_1 + 600c_2 - (c_2 - c_1)^2 - (c_2 - c_1)^4, \tag{2}$$

where $(c_1, c_2) \in (0, 5)^2$*. It can be verified that this utility function is strictly increasing and concave, implying that the indifference curves are well behaved (convex) in* $(0, 5)^2$*. Consider the lottery* $L_1 = \left((1, 1.5), \frac{1}{2}; (2, 2.5), \frac{1}{2}\right)$*, where* $(1, 1.5)$ *and* $(2, 2.5)$ *are the vector payoffs and* $\frac{1}{2}$ *is the probability of each payoff.* $L_2 = ((1.5, 2), 1)$ *is a degenerate lottery with its payoff equal to the means of the payoffs of* L_1*. Following Safra and Segal [21], an individual is said to be risk neutral toward the risky lottery* L_1 *if she is indifferent between it and the special degenerate lottery* L_2 *(see Definition 2 below). Lotteries* L_1 *and* L_2 *are plotted in Figures 1 and lie on the common ray*

$$c_2 = c_1 + 0.5.$$

Using the NM index (2), computation of the Expected Utility for L_1 *and* L_2 *yields*

$$EU(L_1) = \frac{1}{2}(600 + 900 - 0.25 - 0.0625) + \frac{1}{2}(1200 + 1500 - 0.25 - 0.0625) = 2099.6875$$

and

$$EU(L_2) = 900 + 1200 - 0.25 - 0.0625 = 2099.6875.$$

Since $EU(L_1) = EU(L_2)$*, the individual is risk neutral toward* L_1*. We next show that although the individual is risk neutral toward* L_1*, she is not risk neutral toward all lotteries since the NM index (2) does not take the linear form (1). Consider*

[5] As noted in footnote 10, the case where the NM index is linear over a portion of its domain and the payoffs corresponding to a distribution or lottery are defined on this subdomain will not be distinguished from the case where the index is linear over its entire domain.

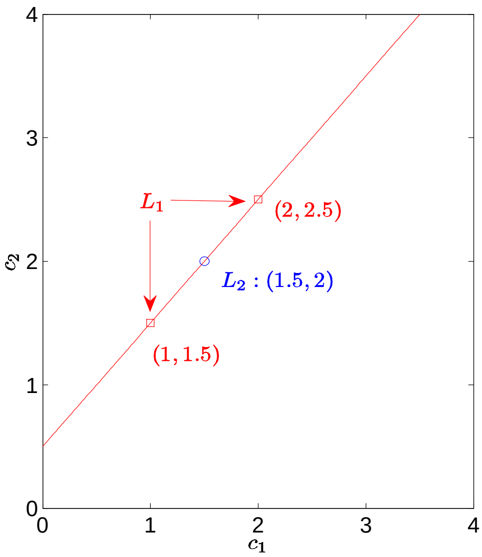

Figure 1: Risk Neutral Lottery

lotteries $L_3 = \left((1, 1.5), \frac{1}{2}; (2, 3.5), \frac{1}{2}\right)$ *and* $L_4 = ((1.5, 2.5), 1)$, *where* L_4 *is the degenerate lottery with payoff corresponding to the means of the payoffs of* L_3. *Lotteries* L_3 *and* L_4 *also lie along a common ray in* $c_1 - c_2$ *payoff space which is defined by*

$$c_2 = 2c_1 - 0.5.$$

However in this case since

$$EU(L_3) = \frac{1}{2}(600 + 900 - 0.25 - 0.0625) + \frac{1}{2}(1200 + 2100 - 2.25 - 5.0625) = 2396.1875$$

and

$$EU(L_4) = 900 + 1500 - 1 - 1 = 2398,$$

the individual is not risk neutral but rather is risk averse toward L_3 *since her Expected Utility is lower for* L_3 *than for* L_4.

The next example is based on the classic two period consumption-savings, capital risk problem. Certain first period and random second period consumption are denoted, respectively, by c_1 and $\widetilde{c}_2$. In period one, the consumer is endowed with income I and chooses how much to consume c_1 and save $I - c_1$. Saving takes place via a risky asset paying a risky (gross) rate of return $\widetilde{R}$. Random period two consumption is given by

$$\widetilde{c}_2 = \widetilde{R}\,(I - c_1)\,. \tag{3}$$

The consumer chooses optimal period one consumption so as to maximize the Expected Utility $EU\,(c_1, \widetilde{c}_2)$, where the NM index U does not take linear form (1).

The following demonstrates that the resulting optimal savings behavior need not change in response to a MPS (mean preserving spread) in the return $\widetilde{R}$.[6]

Example 2 *Assume the consumer's NM index U is given by*

$$U(c_1, c_2) = 5.5c_1 + 5c_2 - (c_1 - 0.5)^2 \left(c_2^2 + 4\right), \tag{4}$$

where $(c_1, c_2) \in (0,1)^2$. Although optimal consumption and savings behavior can be obtained by maximizing the Expected Utility corresponding to the NM index (4) subject to the constraint (3), one can more clearly see the impact of a MPS in $\widetilde{R}$ on optimal savings behavior by considering a dual formulation. The first step is to eliminate the risk associated with the random return by using the certainty equivalent period two consumption $\widehat{c}_2$ defined by[7]

$$U(c_1, \widehat{c}_2) = EU(c_1, \widetilde{c}_2). \tag{5}$$

Substituting (3) and (4) into (5) yields

$$5.5c_1 + 5\widehat{c}_2 - (c_1 - 0.5)^2 \left(\widehat{c}_2^2 + 4\right) = 5.5c_1 + 5(I - c_1)E\widetilde{R} - (c_1 - 0.5)^2 \left((I - c_1)^2 E\widetilde{R}^2 + 4\right). \tag{6}$$

Solving (6) for the certainty equivalent as a function of c_1, the consumer can be viewed as solving the certainty problem

$$\max_{c_1, \widehat{c}_2} U(c_1, \widehat{c}_2) \qquad S.T.\ \widehat{c}_2 = \widehat{c}_2(c_1).$$

The solution to this problem is illustrated in Figure 2(a), where $I = 1$ and the risky return $\widetilde{R}$ pays off 1.3 and 0.9 with equal probability. The concave curve corresponds to the certainty equivalent constraint $\widehat{c}_2(c_1)$. It is straightforward to show that the consumption-savings optimum, corresponding to the tangency of this constraint and the certainty indifference curve in the $c_1 - \widehat{c}_2$ plane, is given by $c_1 = 0.5$ which does not depend on the value of I and $E\widetilde{R}^2$. Thus, optimal consumption and savings are independent of a MPS in $\widetilde{R}$. To demonstrate this, consider a MPS of the equiprobable risky investment's return where the payoffs go from $(1.3, 0.9)$ to $(2.1, 0.1)$ resulting in the same mean $E\widetilde{R} = 1.1$ but a larger variance. This results in the new, lower certainty equivalent constraint in Figure 2(b), where corresponding to each value of period one consumption except for $c_1 = 0.5$, $\widehat{c}_2$ declines. Since the tangency point is unchanged, optimal consumption and savings are unaffected by the MPS and the consumer exhibits risk neutrality despite the fact that her NM index does not take the linear form (1).[8]

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[6] As is standard for any cumulative distribution functions F and G, G is a mean preserving spread of F if and only if $\widetilde{y} = \widetilde{x} + \widetilde{\epsilon}$, where $\widetilde{x}$ and $\widetilde{y}$ are respectively the random variables corresponding to F and G, and $E[\widetilde{\epsilon}|\,\widetilde{x}] = 0$.

[7] This process has been used more generally in Selden [22].

[8] As is clear from the form of the NM index (4), the consumer will be risk neutral toward all c_2-lotteries when her first period consumption satisfies $c_1 = 0.5$.

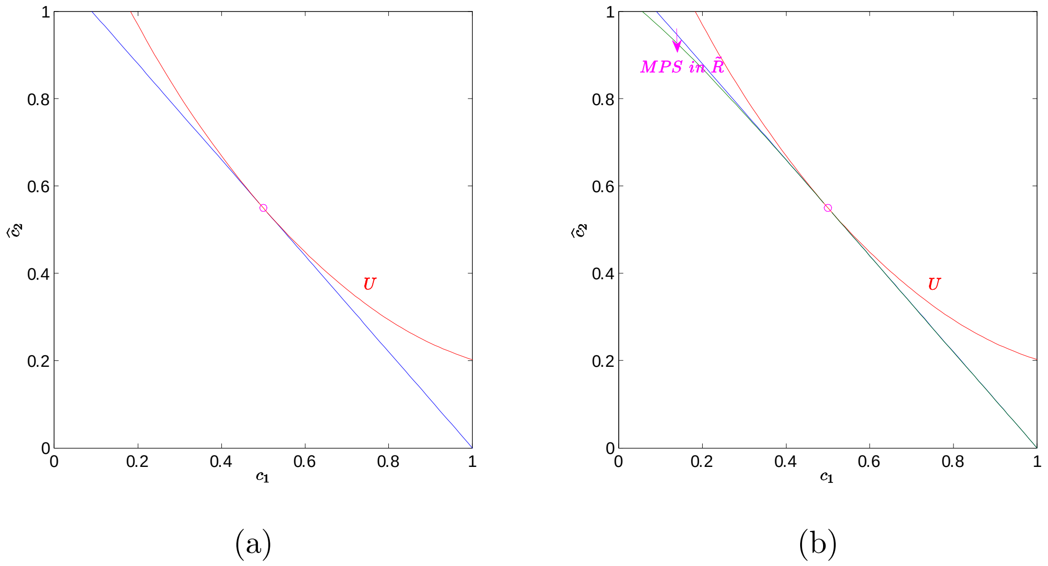


Figure 2: Optimal Period One Consumption with a MPS in $\tilde{R}$

3 Classic Definitions of Risk Neutrality

In this section, we review the definitions of risk neutrality for both univariate and bivariate risks. Let $c_1 \in C_1$ and $c_2 \in C_2$ denote the quantities of two commodities, where unless stated otherwise $C_1 = C_2 = (0, \infty)$. Define $C = C_1 \times C_2$. Let $\mathcal{F}_i$ $(i = 1, 2)$ denote the set of c.d.f.s (cumulative distribution functions) defined on C_i and F_i be an element in $\mathcal{F}_i$. $\mathcal{J}$ denotes the set of joint cumulative distribution functions defined on $C_1 \times C_2$ and J is an element in $\mathcal{J}$. The c.d.f. $J(c_1, c_2)$ corresponds to the random variable $(\widetilde{c}_1, \widetilde{c}_2)$ which maps states of nature into specific consumption vectors (c_1, c_2). As is standard, to simplify subsequent discussions lotteries and c.d.f.s will be used interchangeably. It will be understood that when referring to a bivariate lottery L as being an element in $\mathcal{J}$, we are referring to the uniquely determined c.d.f. J defined by the payoffs and probabilities of the lottery L (see Machina [15]). Similarly, the degenerate lotteries $\delta_{c_i} \in \mathcal{F}_i$ $(i = 1, 2)$ and $\delta_{\mathbf{c}} \in \mathcal{J}$ with certain payoffs $c_i \in C_i$ and $\mathbf{c} = (c_1, c_2) \in C$, respectively, will be referred to as one point c.d.f.s. The set of degenerate lotteries δ_{c_i} is denoted as $\mathcal{F}_i^*$.

Throughout this section and the next, preferences over the choice space $\mathcal{J}$ are denoted by $\succeq^{\mathcal{J}}$ and are assumed to be representable according to the Expected Utility principle where $U : C_1 \times C_2 \to \mathbb{R}$ is a twice continuously differentiable, strictly increasing and (weakly) concave NM index such that, for all $J_1, J_2 \in \mathcal{J}$,

$J_1 \succeq^{\mathcal{J}} J_2$ iff

$$\int_{C_1}\int_{C_2} U(c_1,c_2)\, dJ_1(c_1,c_2) \geq \int_{C_1}\int_{C_2} U(c_1,c_2)\, dJ_2(c_1,c_2).$$

First we define risk neutrality in the single attribute case. Assume preferences are defined over the space of univariate distributions $\mathcal{F}_2$ and are denoted by $\succeq^{\mathcal{F}_2}$.[9] Following Mas-Colell, Whinston and Green [16], p. 185, one can define risk neutrality toward a given nondegenerate lottery as follows.

Definition 1 *An individual is risk neutral toward a given nondegenerate lottery $F_2(c_2) \in \mathcal{F}_2$ if and only if the degenerate lottery $\delta_{\overline{c}_2}$ that pays off $\overline{c}_2 = \int c_2 F_2(c_2)$ with certainty satisfies $\delta_{\overline{c}_2} \sim^{\mathcal{F}_2} F_2$.*

If the single attribute preferences admit an Expected Utility representation, it is well-known that the NM index must be linear for an individual to be risk neutral toward all nondegenerate lotteries $F_2(c_2) \in \mathcal{F}_2$ in the sense of Definition 1.[10]

In order to define risk neutrality in the case where lotteries pay off vectors $\mathbf{c} = (c_1, c_2) \in C$, we adopt the following natural extension of Definition 1.[11]

Definition 2 *(Safra and Segal [21]) An individual is risk neutral toward a given nondegenerate lottery $J(c_1,c_2) \in \mathcal{J}$ if and only if the degenerate lottery $\delta_{(\overline{c}_1,\overline{c}_2)}$ that pays off $(\overline{c}_1,\overline{c}_2)$ with certainty satisfies $\delta_{(\overline{c}_1,\overline{c}_2)} \sim^{\mathcal{J}} J$, where*

$$\overline{c}_1 = \int c_1 dF_1(c_1) \quad \text{and} \quad \overline{c}_2 = \int c_2 dF_2(c_2),$$

$$F_1(x) = \int_0^x \int_{C_2} dJ(c_1,c_2)$$

and

$$F_2(x) = \int_0^x \int_{C_1} dJ(c_1,c_2).$$

[9] We have chosen to use the notation c_2 and F_2 for the univariate case, since it can be directly applied to the special bivariate case considered in Section 5 where the first commodity is certain and the second is risky.

[10] It should be noted that if we do not require an individual to be risk neutral toward all lotteries, then risk neutral behavior can occur if the NM index is linear over an interval of its domain. Since risk neutral behavior for this case is very similar to the case where the NM index is linear over its entire domain, we will simply include it as a special case of the linear form. A similar assumption will be made for the analogous multivariate case.

[11] In the following definition and elsewhere when a lottery $J(c_1,c_2) \in \mathcal{J}$ is referred to as being nondegenerate, this will be understood to exclude lotteries of the form $\delta_{(c_1,c_2)}$ but not lotteries associated with the pairs $(\delta_{c_1}, F_2(c_2))$ or $(F_1(c_1), \delta_{c_2})$.

Remark 1 *It should be noted that when defining risk neutrality along the lines of Definition 2 above, Safra and Segal [21] (i) assume the individual is risk neutral toward all lotteries* $J(c_1, c_2) \in \mathcal{J}$ *and (ii) do not assume the preferences are Expected Utility representable. But they prove that risk neutrality together with the Axiom of Degenerate Independence (ADI), i.e.,* $\forall J \in \mathcal{J}$, $\mathbf{c}, \mathbf{c}' \in C_1 \times C_2$ *and* $\alpha \in [0, 1]$

$$\mathbf{c} \succeq^C \mathbf{c}' \Leftrightarrow \alpha J + (1 - \alpha)\, \delta_{\mathbf{c}} \succeq^{\mathcal{J}} \alpha J + (1 - \alpha)\, \delta_{\mathbf{c}'},$$

implies the Strong Independence Axiom and hence the preferences are Expected Utility representable. Since the individual is assumed to be risk neutral to all lotteries, the NM index will take the linear form (1). Grant, Kajii and Polak [7] introduce Bifurcation Risk Neutrality (BRN), where preferences $\succeq^{\mathcal{J}}$ *over bivariate lotteries are said to exhibit BRN if and only if* $\forall J \in \mathcal{J}$, $\mathbf{c}, \mathbf{c}' \in C_1 \times C_2$ *and* $\alpha, \beta \in [0, 1]$

$$\alpha J + (1 - \alpha)\, \delta_{\beta \mathbf{c} + (1-\beta)\mathbf{c}'} \sim^{\mathcal{J}} \alpha J + (1 - \alpha) \left(\beta \delta_{\mathbf{c}} + (1 - \beta)\, \delta_{\mathbf{c}'}\right).$$

Grant, Kajii and Polak argue that BRN is stronger than the Safra and Segal risk neutrality definition and hence also implies the Strong Independence Axiom together with ADI. Therefore BRN is also consistent with our Definition 2 and assumption that preferences are Expected Utility representable although it differs in defining risk neutrality as holding for all lotteries rather than for just a subset of nondegenerate lotteries. Richard [18] proposes another definition for multivariate risk neutrality, where a consumer's Expected Utility preferences are said to be risk neutral if and only if her NM index $U(c_1, c_2)$ *satisfies*

$$\frac{\partial^2 U(c_1, c_2)}{\partial c_1 \partial c_2} = 0.$$

This definition is based on the correlation between the random variables $\widetilde{c}_1$ *and* $\widetilde{c}_2$*, which is totally different from the notion of risk neutrality discussed in this paper.*

If the Definition 2 characterization of risk neutrality holds for all nondegenerate lotteries $J(c_1, c_2) \in \mathcal{J}$, then we obtain the following natural analogue of the restriction on the NM index for the univariate case (the proof is provided in the online Appendix F).[12]

Corollary 1 *An individual is risk neutral toward all nondegenerate lotteries* $J \in \mathcal{J}$ *in the sense of Definition 2 if and only if her NM index takes the linear form in eqn. (1).*

[12] Corollary 1 can be viewed as a special case of Theorem 7 in Safra and Segal [21]. However, it should be emphasized that Safra and Segal [21] only consider the case where the consumer is risk neutral toward all distributions.

The notion of a risk neutrality region follows naturally from Definition 2.

Definition 3 *A risk neutrality region is a set of nondegenerate distributions* $\widehat{\mathcal{J}} \subset \mathcal{J}$ *where each* $J \in \widehat{\mathcal{J}}$ *satisfies* $J \sim^{\mathcal{J}} \delta_{(\bar{c}_1,\bar{c}_2)}$ *and* $\delta_{(\bar{c}_1,\bar{c}_2)}$ *is defined as in Definition 2.*[13]

As illustrated by Example 1, an Expected Utility maximizer can exhibit risk neutral behavior toward a subset of distributions even if her NM index does not take the linear form (1). As a result, Corollary 1 is too strong in that it gives conditions when an individual is or is not risk neutral toward all $J \in \mathcal{J}$. In the next section we consider when, as in Example 1, an individual with an NM index U can exhibit risk neutral behavior toward a subset of distributions in $\mathcal{J}$.

4 Risk Neutrality Regions

4.1 Minimum Concavity Points, Minimum Concavity Directions and Least Concave Utility

To understand why the consumer in Example 1 (i) is risk neutral toward lottery L_1 even though the NM index does not take the linear form (1) and (ii) is not risk neutral toward L_3, it will prove useful to review de Finetti's [5] notion of least concave utility (also see Debreu [3]). Having assumed the existence of Expected Utility preferences where the NM index U is twice continuously differentiable, strictly increasing and concave, we can introduce the following definition.

Definition 4 *Let* $\mathcal{U}$ *denote the set of all increasing monotone transforms of the twice continuously differentiable, concave, strictly increasing real-valued function* U*. Then* $u \in \mathcal{U}$ *is said to be least concave if and only if all other concave members of* $\mathcal{U}$ *are concave transforms of* u*, where* u *will always be assumed to be defined up to a positive affine transform.*

Then we have the following existence result.

Proposition 1 *(Debreu [3]) Assume that preferences over* $\mathcal{J}$ *are representable by an Expected Utility function where the concave NM index* $U \in \mathcal{U}$*. Then there exists a least concave representation* u *defined on* C *such that*

$$U = f \circ u, \tag{7}$$

where f *is concave.*

[13] Since risk neutrality toward degenerate lotteries is automatically satisfied, in the rest of paper when discussing the risk neutrality region $\widehat{\mathcal{J}}$, we will exclude the degenerate lotteries.

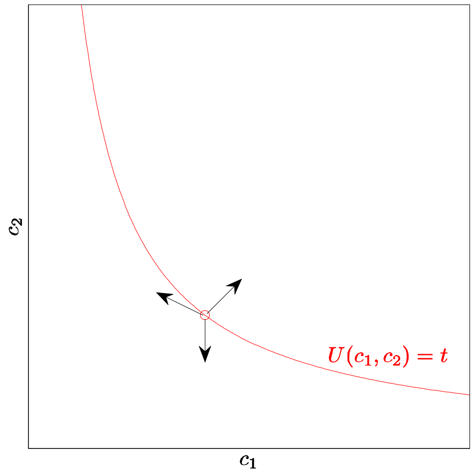


Figure 3: Minimum Concavity Directions

Given a concave NM index U, the process of finding its least concave utility u can be broken into two steps. First, at each point $\mathbf{c}$, identify the one or more directions of “maximum” convexity (or "minimum" concavity) as represented by the arrows in Figure 3. Following Fenchel [6], this search process corresponds to finding the set of directions (ξ_1, ξ_2) as functions of (c_1, c_2) (if they exist) that yield the following infimum[14]

$$a(\mathbf{c}) = \inf_{\{\boldsymbol{\xi}: U_1(\mathbf{c})\xi_1 + U_2(\mathbf{c})\xi_2 \neq 0\}} -\frac{U_{11}(\mathbf{c})\xi_1^2 + 2U_{12}(\mathbf{c})\xi_1\xi_2 + U_{22}(\mathbf{c})\xi_2^2}{\left(U_1(\mathbf{c})\xi_1 + U_2(\mathbf{c})\xi_2\right)^2}, \tag{8}$$

where $\mathbf{c} = (c_1, c_2)$ and the partial derivatives $U_i(c_1, c_2) = \partial U(c_1, c_2)/\partial c_i$ and $U_{ij}(c_1, c_2) = \partial^2 U(c_1, c_2)/\partial c_i \partial c_j \quad (i, j = 1, 2)$. The vector (ξ_1, ξ_2) where the infimum is attained (if it exists) will be referred to as a minimum concavity direction (MCD). For the special cases, where the MCD corresponds to the slope of vertical or horizontal rays through (c_1, c_2), we follow the standard convention of referring to the directions as $(0, 1)$ and $(1, 0)$, respectively.[15]

Having found the directions, the second step is to pick an indifference curve $U = t$ and then search along this curve for the specific point(s) $\mathbf{c} = (c_1, c_2)$ where

[14] As additional clarification, defining

$$q\left(c_1, c_2, \xi_1, \xi_2\right) = -\frac{U_{11}(\mathbf{c})\xi_1^2 + 2U_{12}(\mathbf{c})\xi_1\xi_2 + U_{22}(\mathbf{c})\xi_2^2}{\left(U_1(\mathbf{c})\xi_1 + U_2(\mathbf{c})\xi_2\right)^2},$$

the process of selecting directions in eqn. (8) generates the functions $\xi_1(c_1, c_2)$ and $\xi_2(c_1, c_2)$. Substituting these functions back into q yields $a(\mathbf{c})$.

[15] Since in the two dimensional case, the MCD can be characterized equivalently by a vector or a slope, we use these two terms interchangeably.

concavity is minimized according to

$$G(t) = \inf_{\{\mathbf{c}:U(\mathbf{c})=t\}} a(\mathbf{c}) > -\infty. \tag{9}$$

Such a point will be referred to as a minimum concavity point (MCP). The least concave utility u can be derived from the following integration of $G(t)$[16]

$$u = F(U) = \int^U \exp\left(\int^t G(s)ds\right) dt. \tag{10}$$

This formula defines u as a function of U (for each constant U). If U itself is a function of $\mathbf{c}$, then the repeated integral (10) yields a composite function $u(\mathbf{c}) = u(U(\mathbf{c}))$. Given that U is assumed to be twice continuously differentiable, if we further assume that the Gaussian curvature of the indifference curves is positive and the indifference curves are compact on their compactified domain, then $a(\mathbf{c})$ and $G(t)$ are continuous (see Kannai [9] for a detailed discussion). (See the discussion of compactifying the set C prior to Definition 5 below.) It follows that u is also twice continuously differentiable. One can interpret eqn. (10) as using the transform F to adjust the concavity of U at the point(s) determined by (9) in the (ξ_1, ξ_2) direction obtained from (8) to the linear level, resulting in the least concave utility u. It should be noted that for a general U, although $a(\mathbf{c})$ is minimized at the MCP in the corresponding MCD, it may not reach zero. However if one uses the least concave utility, then at each MCP, denoted $\mathbf{c}^* = (c_1^*, c_2^*)$, $a(\mathbf{c}^*)$ becomes zero, i.e.,

$$\begin{bmatrix} \xi_1(c_1^*, c_2^*) & \xi_2(c_1^*, c_2^*) \end{bmatrix} H_{u(c_1^*,c_2^*)} \begin{bmatrix} \xi_1(c_1^*, c_2^*) \\ \xi_2(c_1^*, c_2^*) \end{bmatrix} = 0. \tag{11}$$

Since (ξ_1, ξ_2) cannot be a zero vector and the Hessian matrix is negative semidefinite, eqn. (11) implies that $\det H_u = 0$. For a strictly concave utility U, the Hessian matrix is negative definite and hence $\det H_U$ can be zero only at limit (boundary) points. Therefore given a candidate concave utility u, if $\det H_u = 0$ at some interior points, then u is least concave (for further discussion, see Kannai and Selden [10], footnote 8).

The MCD at the MCP (c_1^*, c_2^*) based on the least concave utility u is given in the following proposition.[17]

Proposition 2 *At the MCP (c_1^*, c_2^*), if the Hessian matrix based on u is not the zero matrix and satisfies*

$$u_1u_{22} - u_2u_{12}|_{(c_1^*,c_2^*)} \neq 0 \ \ or \ \ u_2u_{11} - u_1u_{12}|_{(c_1^*,c_2^*)} \neq 0,$$

[16]It should be noted that throughout this paper, all such double integrations give rise to constants which can be ignored given that the resulting representations are defined only up to positive affine transformations.

[17]Unless indicated otherwise, proofs are provided in the Appendix to this article.

then the MCD is proportional (or parallel) to the vector[18]

$$\left(u_1 u_{22} - u_2 u_{12}, u_2 u_{11} - u_1 u_{12}\right)\big|_{\left(c_1^*, c_2^*\right)}.$$

If the Hessian matrix based on u is the zero matrix, then every direction is a MCD.

We next discuss the connection between the set of MCPs and the MCDs. It will be convenient to compactify the commodity space. In fact given that a MCP may not be in the commodity space C, it is advantageous for certain forms of U to compactify C so as to include its boundary points (which may be infinite). Then one can define minimum concavity at a boundary point $\mathbf{c}$ by the asymptotic vanishing of the ratio of the Hessian and the Bordered Hessian determinants as points in C approach $\mathbf{c}$. This compactified domain is referred to as the extended domain $\overline{C}$. The notion of extending the domain to include limit points is commonplace. See, for instance, Rockafellar ([19], pp. 24-25). We may now state the following definitions.

Definition 5 *C^* is the set of all MCPs in the extended domain $\overline{C}$ associated with u.*

Definition 6 *$C[ac_1 + bc_2 + d] = \{(c_1, c_2) \in \overline{C} \mid ac_1 + bc_2 + d = 0\}$. $S[ac_1 + bc_2 + d]$ is a connected subset of $C[ac_1 + bc_2 + d]$.*

Remark 2 *Since $C[ac_1 + bc_2 + d]$ represents a ray in $\overline{C}$, its connected subset $S[ac_1 + bc_2 + d]$ can be understood as a line segment (or ray). The reason to introduce the line segment is that for some utility functions, the set of MCPs may consist of a part or several parts of a ray but not the whole ray.*[19] *This is illustrated explicitly in the example considered in the online Appendix J.*

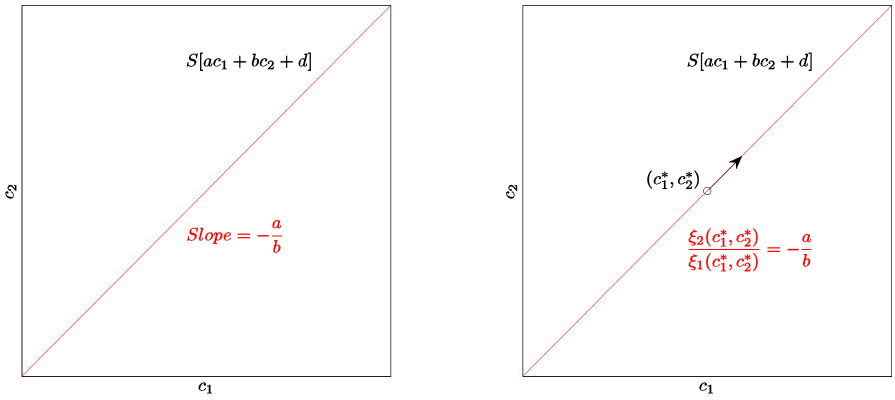
Assume that $S[ac_1 + bc_2 + d] \subset C^*$, implying that each point along the line segment is a MCP. Then as shown in Figure 4(a), the slope of this MCP line segment is given by $-\frac{a}{b}$. If for each MCP

$$\left(c_1^*, c_2^*\right) \in S[ac_1 + bc_2 + d],$$

[18] For example, assuming the Cobb-Douglas utility, we have $u = \sqrt{c_1 c_2}$. Since the set of MCPs is the whole space, it can be verified that

$$\frac{\xi_2\left(c_1^*, c_2^*\right)}{\xi_1\left(c_1^*, c_2^*\right)} = \left.\frac{u_2 u_{11} - u_1 u_{12}}{u_1 u_{22} - u_2 u_{12}}\right|_{\left(c_1^*, c_2^*\right)} = \frac{c_2^*}{c_1^*}.$$

[19] Given that consumption is not allowed to be negative, the set $C[ac_1 + bc_2 + d]$ corresponds to a ray or line segment rather than a line since one end of the ray is always in the non-negative orthant.



(a) (b)

Figure 4: Slope of the MCP line segment and MCD

the MCD always equals $-\frac{a}{b}$, i.e.,

$$\frac{\xi_2(c_1^*, c_2^*)}{\xi_1(c_1^*, c_2^*)} = -\frac{a}{b},$$

then the slope of the MCP line segment and the MCD are the same. This is shown in Figure 4(b).

In Example 1, it can be verified that the assumed NM index (2) is affinely equivalent to the least concave utility u and the set of MCPs is the whole space $(0, 5)^2$.[20] It should be emphasized that Kannai and Selden ([10], Proposition 4) have proved that if the utility function is (ordinally) additively separable (i.e., there exists a utility function U representing the preference such that $U(\mathbf{c}) = \sum_{i=1}^n U_i(c_i)$), then the set of MCPs is the whole space if and only if preferences are homothetic or quasihomothetic. Example 1 demonstrates that if U is not additively separable, which is the case for the utility (2), then the set of MCPs can still be the whole space even though preferences are neither homothetic nor quasihomothetic. For this U in Example 1 at any MCP (c_1^*, c_2^*), the MCD is given by $(\xi_1(c_1^*, c_2^*), \xi_2(c_1^*, c_2^*)) = (1, 1)$. Therefore for any line segment parallel to the 45° degree ray, the MCD is the same as the slope of the ray. Thus an individual with the NM index (2) was seen to be risk neutral toward lottery L_1 which has its payoffs on a line segment parallel to the 45° ray. For other lotteries such as L_3 with payoffs lying on line segments not parallel to the 45° degree ray, the individual does not exhibit risk neutrality. Clearly for this utility there exist risk neutrality regions, and the least concave utility, the set

[20]It should be noted that utilities being affinely equivalent will be understood to be short-hand for the utilities being equivalent up to a positive affine transformation.

of MCPs and the MCDs play important roles in determining risk neutral behavior.

4.2 Generalized Conditions for Risk Neutrality

We next (i) provide necessary and sufficient conditions for when an individual will exhibit risk neutral behavior and (ii) characterize the subset of lotteries or equivalently distributions $\widehat{\mathcal{J}} \subset \mathcal{J}$ toward which the individual is risk neutral.

Proposition 3 *Assume that the NM index $U = f \circ u$ where f is a strictly increasing and concave transformation, u is the least concave utility as derived from eqn. (10) and U does not take the linear form (1). The consumer will be risk neutral toward a given nondegenerate lottery $J \in \mathcal{J}$ if and only if*

(i) *f is a positive affine transformation;*

(ii) *$\exists a, b, d \in \mathbb{R}$, such that the payoffs of J are in the set $S[ac_1 + bc_2 + d]$, where $S[ac_1 + bc_2 + d] \subseteq C^*$ and*

(iii) *$\forall (c_1, c_2) \in S[ac_1 + bc_2 + d]$, $(\xi_1(c_1, c_2), \xi_2(c_1, c_2)) \propto (b, -a)$.*

Remark 3 *If the Hessian matrix becomes the zero matrix at all of the MCPs, then for every vector (ξ_1, ξ_2), one always have*

$$\begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix} H_{u(c_1^*, c_2^*)} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0,$$

implying that every direction is a MCD and hence condition (iii) in Proposition 3 automatically holds.

Remark 4 *In Proposition 3, an individual will be risk neutral toward a given nondegenerate lottery J only if the payoffs of the lottery are collinear in the MCD. Thus, the only set of distributions that can satisfy Proposition 3 and comprise $\widehat{\mathcal{J}}$ are those with perfectly correlated payoffs.*

It should be emphasized that Proposition 3 gives necessary and sufficient conditions for when an individual can simultaneously exhibit multivariate risk neutrality and possess an NM index not taking the linear form (1). Moreover, the risk neutral region $\widehat{\mathcal{J}}$ of the choice space is comprised only of lotteries with collinear payoffs lying on one or more rays $C[ac_1 + bc_2 + d]$ or line segments $S[ac_1 + bc_2 + d]$. We want to emphasize that an NM utility taking the least concave form is not sufficient for the existence of risk neutrality regions. The restrictions on the set of MCPs and the MCDs play a crucial role. First although the set of MCPs can be a ray, its slope can diverge from the MCD. This is illustrated by least concave utility corresponding to the NM index

$$U(c_1, c_2) = 2\sqrt{c_1} - c_2^{-1},$$

where $(c_1, c_2) \in (0, 3] \times (0, \infty)$. Second, a least concave utility need not become linear at the set of MCPs. Third, the set of MCPs can correspond to a curve as is the case for the following least concave utility

$$u(c_1, c_2) = 8\left(c_2 - c_1^2\right) - \left(c_2 - c_1^2\right)^4 + 24c_1 + 10c_2,$$

where $(c_1, c_2) \in (0, 1)^2$. In each of these cases, the set $\widehat{\mathcal{J}}$ is empty. (See the online Appendix H for additional discussion.)

Remark 5 *In Kihlstrom and Mirman [13] and Kihlstrom [14], the authors use Debreu's [3] decomposition $U = f \circ u$ to analyze multiperiod savings behavior and asset pricing. They consider the special case of homothetic preferences. Kihlstrom [14], p. 638, argues that "in the Kihlstrom-Mirman approach, u is the risk neutral representation of preferences". However it should be emphasized that in the multivariate case, the least concave utility u for general preferences results in risk neutral behavior only for the special subset of lotteries characterized in Proposition 3.*

We next illustrate the application of Proposition 3 assuming an individual's NM index takes the classic CES (constant elasticity of substitution) form. As is well-known, the corresponding least concave utility is given by

$$u(c_1, c_2) = (c_1^{-\delta} + c_2^{-\delta})^{-1/\delta} \qquad (\delta \neq 0 \text{ and } \delta > -1)\,, \tag{12}$$

where $C = (0, \infty)^2$. Since CES preferences are homothetic, every point in $\overline{C}$ is a MCP.[21] To find the MCD, it is straightforward to show that

$$(1, k)^T H_{u(c_1, kc_1)}(1, k) = 0 \;\forall c_1 \in C_1,\; k > 0.$$

It follows that each MCP lies on a ray through the origin. Corresponding to the MCP (c_1, kc_1), both the MCD and the slope of the ray through this point are equal to k. Hence following Proposition 3, each ray through the origin corresponds to a risk neutral ray. See Figure 5.[22]

Given the ability of CES preferences to meet the requirements for the existence of risk neutrality regions, it is natural to wonder if homotheticity is perhaps a necessary

[21] See Kannai and Selden [10], Proposition 3.

[22] It should be noted that although the risk neutral rays in Figure 5 look like expansion paths corresponding to different price ratios, any geometric similarity is purely coincidental. This is confirmed by Example 1, where we continue to have risk neutral line segments but since preferences are not homothetic or quasihomothetic, the expansion paths are not linear.

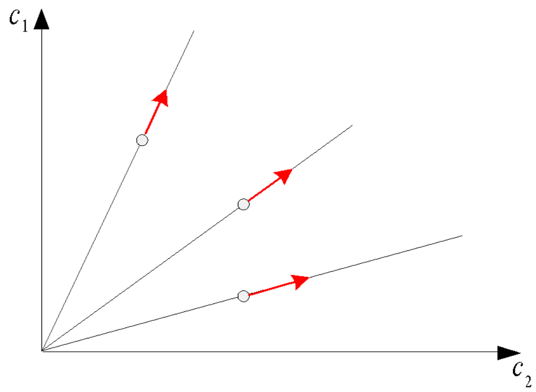


Figure 5: CES Risk Neutral Rays

condition. However, this is not the case since the NM index (2) in Example 1 is clearly neither homothetic nor quasihomothetic.[23]

4.3 Summary: Special Cases

Eight classes of utility functions are considered in Table 1. The first column gives the number of the utility class, the second column gives the form of the utility and the third column gives parametric restrictions that can result in different entries in other columns. Columns 4-6, respectively, provide the corresponding MCD, least concave utility and MCPs. Columns 7 and 8 will be discussed in the next section. All eight classes are ordinally additively separable. The special cases of CES utilities, classes 2-4, are homothetic. Class 1 and the negative exponential class 5 are quasihomothetic.[24] Classes 6-8 are neither homothetic nor quasihomothetic. The class 8 utility function was introduced by Wold and Jureen [25] to illustrate Giffen good behavior.

[23] Although the CES and Example 1 utilities, (12) and (2) respectively, are characterized by multiple risk neutral rays and line segments, this need not always be the case. Consider the following utility

$$u(c_1, c_2) = 10c_1 + 20c_2 - (c_2 - c_1)^2 \left(c_2^2 + 4\right),$$

where $(c_1, c_2) \in (0,1)^2$. It can be verified that u is strictly increasing, concave and satisfies

$$u_{11}u_{22} - u_{12}^2 = (c_1 - c_2)^2 \left(4 - 3c_2^2\right) \geq 0,$$

where the equal sign holds if and only if $c_1 = c_2$. Thus, the set of minimum concavity points corresponds to the single line segment $c_2 = c_1$ $(0 < c_1 < 1)$. Since this utility function becomes linear along this line segment, it results in risk neutral behavior.

[24] Pollak [17] observes that the negative exponential utility is homothetic to the translated origin $(-\infty, -\infty)$.

	U	Restriction	$(\xi_1(c_1^*,c_2^*),\xi_2(c_1^*,c_2^*))$	u	(c_1^*,c_2^*)	$\overline{u}$	x^*
1	$\frac{(a+bc_1)^{-\delta}}{-\delta}+\frac{(d+ec_2)^{-\delta}}{-\delta}$ $C_1=(0,\infty),C_2=(0,\infty)$	$\delta>0$ $\delta<0$	$\begin{pmatrix}\frac{a}{b}+c_1^*,\\ \frac{d}{e}+c_2^*\end{pmatrix}$	$\begin{pmatrix}(a+bc_1)^{-\delta}+\\(d+ec_2)^{-\delta}\end{pmatrix}^{-\frac{1}{\delta}}$	$C_1\times C_2$	$\begin{pmatrix}(a+bc_1)^{-\delta}+\\(d+ec_2)^{-\delta}\end{pmatrix}^{-\frac{1}{\delta}}$ $\begin{pmatrix}(a+bc_1)^{-\delta}+\\(d+ec_2)^{-\delta}-a^{-\delta}\end{pmatrix}^{-\frac{1}{\delta}}$	∞ 0
2	$-\frac{c_1^{-\delta}}{\delta}-\frac{c_2^{-\delta}}{\delta}$ $C_1=(0,\infty),C_2=(0,\infty)$	$\delta>0$ $\delta<0$	(c_1^*,c_2^*)	$\left(c_1^{-\delta}+c_2^{-\delta}\right)^{-1/\delta}$	$C_1\times C_2$	$\left(c_1^{-\delta}+c_2^{-\delta}\right)^{-1/\delta}$	∞ 0
3	$-\frac{c_1^{-\delta}}{\delta}-\frac{c_2^{-\delta}}{\delta}$ $C_1=[a,b],C_2=(0,\infty)$	$\delta>0$ $\delta<0$	(c_1^*,c_2^*)	$\left(c_1^{-\delta}+c_2^{-\delta}\right)^{-1/\delta}$	$C_1\times C_2$	$\left(c_1^{-\delta}+c_2^{-\delta}-b^{-\delta}\right)^{-1/\delta}$ $\left(c_1^{-\delta}+c_2^{-\delta}-a^{-\delta}\right)^{-1/\delta}$	b a
4	$c_1^a c_2^b$ $C_1=(0,\infty),C_2=(0,\infty)$		(c_1^*,c_2^*)	$c_1^{\frac{a}{a+b}}c_2^{\frac{b}{a+b}}$	$C_1\times C_2$	$c_1^{\frac{a}{b}}c_2$	$(0,\infty)$
5	$-\frac{e^{-\alpha_1c_1}}{\alpha_1}-\frac{e^{-\alpha_2c_2}}{\alpha_2}$ $C_1=(0,\infty),C_2=(0,\infty)$		(α_2,α_1)	$-\ln\left(\frac{e^{-\alpha_1c_1}}{\alpha_1}+\frac{e^{-\alpha_2c_2}}{\alpha_2}\right)$	$C_1\times C_2$	$-\ln\left(\frac{e^{-\alpha_1c_1}}{\alpha_1}+\frac{e^{-\alpha_2c_2}}{\alpha_2}\right)$	∞
6	$-\frac{c_1^{-\delta_1}}{\delta_1}-\frac{c_2^{-\delta_2}}{\delta_2}$ $C_1=(0,\infty),C_2=(0,\infty)$	$\delta_1>\delta_2>0$ $\delta_2>\delta_1>0$ $0>\delta_1>\delta_2$ $0>\delta_2>\delta_1$	$\left(\frac{c_1^*}{\delta_1+1},\frac{c_2^*}{\delta_2+1}\right)$	$\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_1}$ $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_2}$ $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_2}$ $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_1}$	$c_2^*=\infty$ $c_1^*=\infty$ $c_1^*=0$ $c_2^*=0$	$\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_2}$ $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_2}$ $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_2}$ $\left(\frac{\delta_2}{\delta_1}c_1^{-\delta_1}+c_2^{-\delta_2}\right)^{-1/\delta_2}$	∞ ∞ 0 0
7	$-\frac{c_1^{-\delta_1}}{\delta_1}-\frac{c_2^{-\delta_2}}{\delta_2}$ $C_1=[a,b],C_2=(0,\infty)$	$\delta_1>0>\delta_2$ $\delta_2>0>\delta_1$	$\left(\frac{c_1^*}{\delta_1+1},\frac{c_2^*}{\delta_2+1}\right)$	$\begin{pmatrix}\frac{\delta_2}{\delta_1}\left(c_1^{-\delta_1}-a^{-\delta_1}\right)\\+c_2^{-\delta_2}+\frac{\delta_2+1}{\delta_1+1}a^{-\delta_1}\end{pmatrix}^{\frac{-1}{\delta_2}}$ $\begin{pmatrix}\frac{\delta_2}{\delta_1}\left(c_1^{-\delta_1}-b^{-\delta_1}\right)\\+c_2^{-\delta_2}+\frac{\delta_2+1}{\delta_1+1}b^{-\delta_1}\end{pmatrix}^{\frac{-1}{\delta_2}}$	$c_1^*=a$ $c_1^*=b$	$\begin{pmatrix}\frac{\delta_2}{\delta_1}\left(c_1^{-\delta_1}-a^{-\delta_1}\right)\\+c_2^{-\delta_2}\end{pmatrix}^{\frac{-1}{\delta_2}}$ $\begin{pmatrix}\frac{\delta_2}{\delta_1}\left(c_1^{-\delta_1}-b^{-\delta_1}\right)\\+c_2^{-\delta_2}\end{pmatrix}^{\frac{-1}{\delta_2}}$	a b
8	$U=\frac{(c_1-1)}{(c_2-2)^2}$ $C_1=(1,\infty),C_2=(0,2)$		(c_1^*-1,c_2^*-2)	$-\frac{(c_2-2)^2}{(c_1-1)}$	$C_1\times C_2$	$\frac{c_2-2}{\sqrt{c_1-1}}$	$(1,\infty)$

Table 1: Least concave utility, MCPs and MCD for classic forms of U

* $\delta,\delta_1,\delta_2>-1$, $a,b,d,e>0$ and $\alpha_1,\alpha_2>0$ for all the examples on the table.

5 The Certain $\times$ Risky Special Case

So far, we have focused on the case where both choice variables are risky. However in a number of important applications, such as the classic consumption-savings, consumption-portfolio and multiperiod asset pricing problems, it is common to assume that consumption in the first period is certain and risky in subsequent periods. For this certain $\times$ risky case, the necessary and sufficient conditions for risk neutrality require modification.

Moreover except in special cases, the least concave utility derived in the prior section does not extend to the certain $\times$ risky setting. This fact does not seem to have been recognized in the literature (see, for example, the analysis in Kihlstrom and Mirman [13] and Kihlstrom [14]). Failure to make this distinction can result in an improper characterization of risk neutrality regions in the certain $\times$ risky choice space.

5.1 Definition of Risk Neutrality

In seeking to extend the necessary and sufficient conditions for risk neutrality summarized in Proposition 3 to the certain $\times$ risky setting, one essential difference is that the certainty of c_1 forces the MCD to be $(\xi_1, \xi_2) = (0, 1)$ in $C_1 \times C_2$ and requires that the least concave utility be linear in this direction. This results in a generally different least concave utility from the bivariate risk case. Consistent with this restriction on the MCD, in this section we will denote the certain $\times$ risky choice space as $C_1 \times \mathcal{F}_2$ and assume that the preferences $\succeq^{C_1 \times \mathcal{F}_2}$ are representable by an Expected Utility function $\int_{C_2} U(c_1, c_2)\, dF_2(c_2)$, where the NM index U is twice continuously differentiable, strictly increasing and (weakly) concave in c_2. This concavity assumption differs from of the bivariate risky case in Sections 3 and 4 where U was assumed to be (weakly) concave in both c_1 and c_2. (The significance of this difference is illustrated by the case of Cobb-Douglas utility discussed in Subsection 6.1 below.) It will prove convenient to introduce the following definition of risk neutrality for the certain $\times$ risky case.[25]

Definition 7 *For a given pair $(c_1, F_2(c_2)) \in C_1 \times \mathcal{F}_2$, an individual is said to be risk neutral toward the nondegenerate lottery $F_2(c_2)$ if and only if $(c_1, F_2(c_2)) \sim^{C_1 \times \mathcal{F}_2}$*

[25] Given our assumption that preferences in the bivariate risky setting, $\succeq^{\mathcal{J}}$, are representable by an Expected Utility function, it follows that preferences in the certain $\times$ risky setting will also be representable by an Expected Utility function if the latter preferences agree with the former when restricted to $C_1 \times \mathcal{F}_2$. In this case, clearly Definition 7 is a special case of Definition 2. However as noted by Rossman and Selden [20], p. 75, there may be good reasons for not making this embedding argument, such as the consumer simply not possessing preferences outside the economically meaningful world of $C_1 \times \mathcal{F}_2$.

$(c_1, \delta_{\overline{c}_2})$*, where* $\delta_{\overline{c}_2}$ *denotes the degenerate distribution paying off* $\overline{c}_2 = \int c_2 dF_2\left(c_2\right)$ *with certainty.*

It is clear that for all pairs $(c_1, F_2(c_2)) \in C_1 \times \mathcal{F}_2$, an investor will be risk neutral toward each of the corresponding nondegenerate lotteries $F_2(c_2)$ in the sense of Definition 7 if and only if her NM index $U\left(c_1, c_2\right)$ takes the following form which is linear in c_2 but not necessarily linear in c_1

$$U\left(c_1, c_2\right) = h\left(c_1\right) c_2 + g\left(c_1\right), \tag{13}$$

where $h\left(c_1\right) > 0$ and $g\left(c_1\right)$ are strictly increasing functions. The fact that the NM index U in (13) can be nonlinear in c_1, implies that it differs from the restriction (1) associated with risk neutrality in the bivariate risky case. However if U takes the form (1), then it also satisfies (13) as can be seen by defining $\beta = h(c_1)$ and $\alpha c_1 + \gamma = g(c_1)$.

5.2 Minimum Concavity Points, Minimum Concavity Direction and Least Concave Utility

A key element in the derivation of the least concave utility u is the determination of the minimum concavity direction (ξ_1, ξ_2). However given the change in the choice space from $\mathcal{J}$ to $C_1 \times \mathcal{F}_2$, the MCD can no longer be any direction as in Figure 6(a). Rather, the MCD can only be the direction $(\xi_1, \xi_2) = (0, 1)$ as indicated in Figure 6(b) since movement between points not on vertical rays is precluded by the first variable being certain. The least concave utility u, derived from eqn. (10), despite satisfying $\det H_u = 0$ at minimum concavity points, will in general fail to (i) be "linear" in the required direction $(\xi_1, \xi_2) = (0, 1)$ and (ii) satisfy $u_{22} = 0$. Hence one cannot use u as the basis for defining risk neutrality regions in $C_1 \times \mathcal{F}_2$.

We next derive the least concave utility which is appropriate for the choice space $C_1 \times \mathcal{F}_2$. Substitution of the required direction $(\xi_1, \xi_2) = (0, 1)$ into eqn. (8) and following the same process as was used to derive u yields

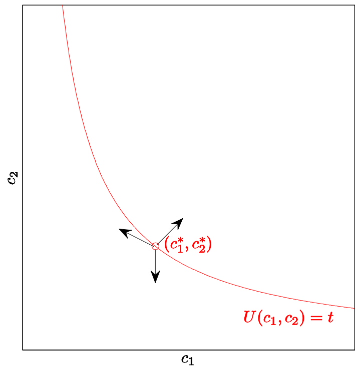
$$\overline{a}(\mathbf{c}) = -\frac{U_{22}(\mathbf{c})}{(U_2(\mathbf{c}))^2},$$

$$\overline{G}(t) = \inf_{\{\mathbf{c}: U(\mathbf{c}) = t\}} \overline{a}(\mathbf{c}) > -\infty$$

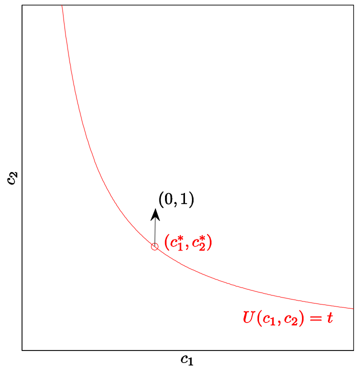
and

$$\overline{u} = \overline{F}(U) = \int^U \exp\left(\int^t \overline{G}(s) ds\right) dt, \tag{14}$$

where the "bars" over a, G, F and u indicate that the functions correspond to the certain $\times$ risky choice space. Given that U is assumed to be twice continuously



(a)



(b)

Figure 6: MCD Comparison

differentiable, it is clear that $\overline{a}(\mathbf{c})$ and $\overline{G}(t)$ are continuous. It follows that $\overline{u}$ is also twice continuously differentiable. It should be noted that $\overline{u}$ may not be concave, although it satisfies $\overline{u}_{22} \leq 0$.

The following defines a minimum concavity point for $\overline{u}$.

Definition 8 *Along any given indifference curve $U = t$, a MCP (c_1^*, c_2^*) for $\overline{u}$ is defined by*[26]

$$\overline{u}_{22}(c_1^*, c_2^*) = 0.$$

(See the online Appendix I for a more complete analysis of $\overline{u}$.)

We next characterize the relationship between u and $\overline{u}$.

Proposition 4 *The utility functions $\overline{u}$ and u are equivalent up to a positive affine transformation if and only if $\overline{u}$ is concave.*

It should be noted that if the NM index U is not concave, then u may not exist. But $\overline{u}$ can still exist.[27] This is illustrated in the following example.

[26] There can be one or many c_1-values associated with MCPs in the $(0, 1)$ direction. Consider the Cobb-Douglas case referenced as class 4 in Table 1. For the least concave utility $\overline{u}(c_1, c_2)$ every point in $C_1 \times C_2$ is a MCP and the utility is linear in the $(0, 1)$ direction for any vertical ray. For the other members of the CES class of utility functions (classes 1-3 in Table 1), $\overline{u}_{22} = 0$ only at a single c_1-value.

[27] When each choice variable is risky, Debreu's [3] sufficient condition for the existence of u is the concavity of U. Analogously in the certain $\times$ risky case, $U_{22} \leq 0$ is sufficient for the existence of $\overline{u}$.

Example 3 *Assume the NM index takes the form*

$$U(c_1, c_2) = \frac{c_2}{2 - c_1},$$

where $c_1 \in (0, 2)$ *and* $c_2 \in C_2$. *Clearly* U *is not concave. A simple computation shows that the right hand side of eqn. (8) blows up if* $\frac{\xi_1}{\xi_2} = \frac{c_1 - 2}{c_2}$ *and* u *fails to exist. (A non-computational argument on this point due to Aumann [1] is reproduced in Kannai [9].) On the other hand, there exists a* $\overline{u}$ *which is equivalent up to a positive affine transform to* U *since*

$$\overline{a}(\mathbf{c})|_{(\xi_1,\xi_2)=(0,1)} = -\frac{U_{22}}{U_2^2} = 0 < \infty.$$

When u and $\overline{u}$ fail to be affinely equivalent, one obtains the following decomposition of a given NM index U, which differs from the decomposition in eqn. (7),

$$U = \overline{f} \circ \overline{u},$$

where $\overline{f}$ is the modified Debreu representation of an individual's risk preferences.

To connect the set of MCPs to risk neutrality in the certain $\times$ risky setting, we next introduce the following definitions analogous to Definitions 5 and 6.

Definition 9 $\overline{C}^*$ *is the set of all the minimum concavity points in the extended domain* $\overline{C}$ *associated with* $\overline{u}$.

Reflecting the requirement that the MCD must correspond to $(\xi_1, \xi_2) = (0, 1)$, the definitions of a ray and line segments in Definition 6 are next modified to ensure that $c_1 = d$.

Definition 10 $C[c_1 - d] = \{(c_1, c_2) \in \overline{C} \mid c_1 - d = 0\}$. $S[c_1 - d]$ *is a connected subset of* $C[c_1 - d]$.

We next give necessary and sufficient conditions for risk neutrality regions in $C_1 \times \mathcal{F}_2$, where the NM index U can take forms other than just the linear utility (13) which ensures risk neutrality toward all lotteries $F_2(c_2) \in \mathcal{F}_2$ in the choice space.

Proposition 5 *Assume that* $U = \overline{f} \circ \overline{u}$ *where* $\overline{f}$ *is a strictly increasing and concave transformation,* $\overline{u}$ *is the least concave form as derived from eqn. (14). For a given pair* $(c_1, F_2(c_2)) \in C_1 \times \mathcal{F}_2$, *the consumer will be risk neutral toward the nondegenerate lottery* $F_2(c_2)$ *if and only if*

(i) $\overline{f}$ *is a positive affine transformation and*

(ii) *$\exists\, d \in \mathbb{R}_+$, such that the lottery payoffs are in the set $S[c_1 - d]$, where $S[c_1 - d] \subseteq \overline{C}^*$.*

In the certain $\times$ risky setting, if the conditions in Proposition 5 are satisfied, then an individual will be risk neutral toward any lottery with the payoffs along the vertical line segment (or ray) $S\,[c_1 - d]$.[28]

Table 1 permits a direct comparison between the cases where both c_1 and c_2 are risky and only c_2 is risky for the eight classes of utilities discussed earlier. For u the MCDs $(\xi_1\,(c_1^*, c_2^*)\,, \xi_2\,(c_1^*, c_2^*))$ are given in column 4, whereas for $\overline{u}$ it is understood that $(\xi_1\,(c_1^*, c_2^*)\,, \xi_2\,(c_1^*, c_2^*)) = (0, 1)$ holds for each of the 8 classes. Column 7 gives $\overline{u}$ and column 8 provides the special c_1 values, denoted x^*, which characterizes the vertical ray containing the MCPs where $\overline{u}_{22} = 0$. From inspecting the table, it will be noted that in a number of cases, u and $\overline{u}$ are not affinely equivalent. For instance the class 1 utility, corresponding to the HARA (hyperbolic absolute risk aversion) family, provides an interesting example of how the least concave utilities can diverge.

It can be seen from the last column in Table 1 that for each of the eight classes of utility, x^* either lies at a boundary of C_1 or corresponds to any point in C_1. However in general, this is not the case as can be seen by considering the NM index (4) used in Examples 2 and 4. It can be verified that this U is concave. Noticing that

$$U_{22}(c_1, c_2) = -2\,(c_1 - 0.5)^2 \leq 0,$$

where the equal sign is reached at $c_1 = 0.5$, it follows that U and $\overline{u}$ are affinely equivalent. The set of MCPs corresponds to (and only to) the vertical line segment defined by $c_1 = 0.5$ $(c_2 \in (0, 1))$ and the consumer is risk neutral toward all lotteries along this line segment.[29] This explains why in the examples, when $c_1 = 0.5$ the

[28] It should be noted that a least concave $\overline{u}$ may exist, but condition (ii) in Proposition 5 may be violated and no risk neutrality region will exist. Consider Examples H.2 and H.3 in the online Appendix H, where that $\overline{u}$ and u are affinely equivalent. For $\overline{u}$ corresponding to eqn. (H.3) in Example H.2, the set of MCPs lies along a set of interior horizontal rays. An individual with this utility is not risk neutral toward any points along a vertical ray as required by the MCD equalling $(0, 1)$. For $\overline{u}$ corresponding to eqn. (H.8) in Example H.3, the set of MCPs corresponds to an interior curve $c_2 = c_1^2$ and not along a vertical ray.

[29] There also exist NM indices such that the set of MCPs corresponds to a set of discrete vertical line segments. Suppose the NM index is affinely equivalent to

$$\overline{u}\,(c_1, c_2) = \alpha_1 c_1 + \beta_1 c_2 - \left(\prod_{i=1}^{n} (c_1 - i)\right)^2 \left(\alpha_2 c_2^2 + \beta_2 c_2 + \gamma_2\right),$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_2$ are appropriately chosen to ensure that $\overline{u}$ is strictly increasing. Then $\overline{u}_{22} = 0$ for each $i \in (1, ..., n)$. However, in general, $\overline{u}$ will be neither concave nor affinely equivalent to u.

consumer exhibits an indifference to a mean preserving spread in risk and why at an aggregate economy endowment of $\bar{c}_1$ the equilibrium risk premium equals 0.

It should be noted that for each of the utilities considered in Table 1, $\overline{u}$ is linear in c_2 along one or all vertical rays where every point on each ray is a MCP. In each case, an individual will be risk neutral toward all lotteries with payoffs on these vertical rays. However, this is not true in general as illustrated by the discussion in footnote 28 and online Appendix J. Corresponding to two different c_1-values, one portion of the MCPs lies along a line segment and a second portion lies along a ray. In terms of condition (ii) in Proposition 5, there are two different vertical sets $S[c_1 - d']$ and $S[c_1 - d'']$ corresponding to the two portions of $\overline{C}^*$. As a result the choice space is comprised of two discrete risk neutrality regions.

6 Applications

In this section, we investigate the implications of our risk neutrality analysis for the classic consumption-savings problem and a representative agent equilibrium asset pricing model, which assumes the two period certain $\times$ risky setting. In the online Appendix K, a single period consumption-bequest optimization model is considered where both variables are risky. We show that Proposition 3 can be directly employed in this setting to characterize when the equilibrium expected return on a risky asset equals the risk free rate.

6.1 Income Risk: Confounding u and $\overline{u}$

As referenced in Remark 5, the Debreu decomposition $U = f \circ u$ has been employed in consumption-savings applications. Unfortunately this decomposition has been used in cases where the first period is a certain and the choice space corresponds to $C_1 \times \mathcal{F}_2$. To illustrate the type of incorrect conclusions that can be reached by not using the appropriate $\overline{u}$ introduced in the prior section, assume the consumer's preferences are represented by the familiar Cobb-Douglas form referred to as class 4 in Table 1

$$U(c_1, c_2) = c_1^a c_2^b \quad (a, b > 0).$$

From columns 5 and 7, respectively, in the table, the least concave utilities are given by[30]

$$u(c_1, c_2) = c_1^{\frac{a}{a+b}} c_2^{\frac{b}{a+b}} \qquad \text{and} \qquad \overline{u}(c_1, c_2) = c_1^{\frac{a}{b}} c_2.$$

[30] Kihlstrom [14], p. 637, discusses the application of Cobb-Douglas utility in a two period consumption-savings problem, where the first period is certain. However, he incorrectly uses u instead of $\overline{u}$ to characterize risk neutral behavior.

Assume two time periods, where certain first period and random second period consumption are denoted, respectively, by c_1 and $\widetilde{c}_2$. The consumer is endowed with certain period one income I_1 and random period two income $\widetilde{I}_2$. Let s_1 denote period one savings. The consumer maximizes $EU(c_1, \widetilde{c}_2)$ subject to

$$c_1 = I_1 - s_1 \text{ and } \widetilde{c}_2 = \widetilde{I}_2 + s_1.$$

The optimization problem can be expressed as

$$\max_{s_1} EU\left(I_1 - s_1, \widetilde{I}_2 + s_1\right). \tag{15}$$

We next consider the effect of a mean preserving spread (see footnote 6) of $\widetilde{I}_2$ on optimal s_1 where the NM index takes the Cobb-Douglas form.

Result 1 *Consider the consumption-savings problem (15). If $U = f \circ u$, where f is a positive affine transformation and*

$$u(c_1, c_2) = c_1^a c_2^b \quad (a, b > 0 \textit{ and } a + b = 1),$$

then the optimal saving s_1^* strictly increases with a MPS of $\widetilde{I}_2$. If $U = \overline{f} \circ \overline{u}$, where $\overline{f}$ is a positive affine transformation and

$$\overline{u}(c_1, c_2) = c_1^{\frac{a}{b}} c_2 \quad (a, b > 0 \textit{ and } a + b = 1),$$

then the optimal saving s_1^* is unchanged with a MPS of $\widetilde{I}_2$.

From Result 1, it can be seen that s_1^* is unchanged with a mean preserving spread of $\widetilde{I}_2$, or the consumer is risk neutral, if the NM index is affinely equivalent to $\overline{u}$. However when period one consumption is certain, erroneously using u as the least concave utility leads to the incorrect conclusion that the consumer changes her savings behavior in response to increased period two income risk. The reason for this can be seen from observing that

$$u = \overline{u}^b.$$

Thus, incorrectly using u results in the consumer being risk averse since u is more concave than $\overline{u}$. It should be also noted that the widely used log additive two period NM index U can as well be viewed as a concave transformation of $\overline{u}$, i.e., $\ln \overline{u}$. Therefore, the consumer with the log NM index is also risk averse. This is consistent with the well known result that if utility takes the additively separable form $U(c_1, c_2) = U_1(c_1) + U_2(c_2)$ and the third derivative of the component utility $U_2(c_2)$ is positive, then savings strictly increase with a MPS of second period income.

6.2 Two Period Consumption-Portfolio Equilibrium

Consider a two period representative agent exchange equilibrium based on a consumption-portfolio optimization. In period one, the agent chooses c_1 as well as asset holdings in a risky asset and risk free asset. Let n and n_f denote the number of units of the risky asset and risk free asset, respectively. The period two payoffs on the risky and risk free assets are given respectively by $\widetilde{\zeta}$ and $\zeta_f > 0$. Random period two consumption is given by

$$\widetilde{c}_2 = \widetilde{\zeta} n + \zeta_f n_f.$$

The agent maximizes $EU(c_1, \widetilde{c}_2)$ subject to the budget constraint

$$c_1 + pn + p_f n_f \leq \overline{c}_1 + p\overline{n} + p_f \overline{n}_f, \tag{16}$$

where p and p_f are the prices of the risky and risk free assets and $\overline{c}_1$, $\overline{n}$ and $\overline{n}_f$ are respectively the endowments of period one consumption, the risky asset and the risk free asset. The following demonstrates that the equilibrium risk premium can be zero for a specific endowment of period one consumption and is positive for other values of $\overline{c}_1$ even though the NM index does not take the linear form (13).

Example 4 *Assume a representative agent pure exchange economy, where the representative agent's NM index is given by eqn. (4) as in Example 2. Maximizing (4) subject to the budget constraint (16), it follows from the first order conditions that equilibrium asset prices are given by*

$$p = \frac{E\left[5\widetilde{\zeta} - 2(c_1 - 0.5)^2 \widetilde{\zeta}\left(\widetilde{\zeta} n + \zeta_f n_f\right)\right]}{5.5 - 2(c_1 - 0.5) E\left[\left(\widetilde{\zeta} n + \zeta_f n_f\right)^2 + 4\right]} \tag{17}$$

and

$$p_f = \frac{E\left[5\zeta_f - 2(c_1 - 0.5)^2 \zeta_f\left(\widetilde{\zeta} n + \zeta_f n_f\right)\right]}{5.5 - 2(c_1 - 0.5) E\left[\left(\widetilde{\zeta} n + \zeta_f n_f\right)^2 + 4\right]}. \tag{18}$$

As is standard, define the equilibrium risk premium by

$$E\widetilde{R} - R_f = \frac{E\widetilde{\zeta}}{p} - \frac{\zeta_f}{p_f}.$$

Substituting from (17) and (18), yields

$$\begin{aligned} E\widetilde{R} - R_f \quad = \quad & \frac{E\widetilde{\zeta}\left(5.5 - 2(\overline{c}_1 - 0.5) E\left[\left(\widetilde{\zeta}\overline{n} + \zeta_f \overline{n}_f\right)^2 + 4\right]\right)}{E\left[5\widetilde{\zeta} - 2(\overline{c}_1 - 0.5)^2 \widetilde{\zeta}\left(\widetilde{\zeta}\overline{n} + \zeta_f \overline{n}_f\right)\right]} \\ & - \frac{5.5 - 2(\overline{c}_1 - 0.5) E\left[\left(\widetilde{\zeta}\overline{n} + \zeta_f \overline{n}_f\right)^2 + 4\right]}{E\left[5 - 2(\overline{c}_1 - 0.5)^2 \left(\widetilde{\zeta}\overline{n} + \zeta_f \overline{n}_f\right)\right]}. \end{aligned} \tag{19}$$

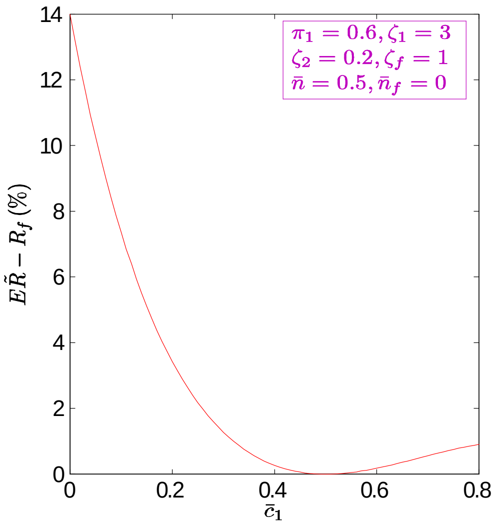


Figure 7: Risk Premium versus Period One Consumption Endowment

Assuming the following parameters

$$\zeta_1 = 3, \quad \zeta_2 = 0.2, \quad \zeta_f = 1, \quad \pi_1 = 0.6, \quad \bar{n} = 0.5 \quad \textit{and} \quad \bar{n}_f = 0,$$

where ζ_i *is the payoff of* $\widetilde{\zeta}$ *with the probability* π_i $(i = 1, 2)$*, we plot in Figure 7 the risk premium* $E\widetilde{R} - R_f$ *for different values of the endowment* $\bar{c}_1$*. It will be noted that the risk premium is strictly positive for all values of* $\bar{c}_1 \neq 0.5$*, implying the representative agent and the economy are risk averse. The fact that the risk premium equals* 0 *at* $\bar{c}_1 = 0.5$*, can be confirmed easily by substituting* $\bar{c}_1 = 0.5$ *into eqn. (19), which yields*

$$E\widetilde{R} - R_f = \frac{5.5E\widetilde{\zeta}}{5E\widetilde{\zeta}} - \frac{5.5}{5} = 0.$$

Thus for this particular endowment, the economy is risk neutral even though the representative agent's NM index (4) does not take the linear form (13). This is because the representative agent's utility (4) assumed here is a least concave utility $\bar{u}$ *as discussed in Subsection 5.2. As a result, it follows from Proposition 5 that the risk neutrality region* $S[c_1 - d]$ *corresponds to the vertical* $c_1 = 0.5$*.*[31]

[31] It will be observed that when $\bar{c}_1 = 0.5$, no matter what values $\bar{n}$ and $\bar{n}_f$ take, we always have $E\widetilde{R} - R_f = 0$. The reason is that when $c_1 = 0.5$, the utility function (4) becomes $U = 2.75 + 5c_2$, which is linear. Therefore, the indifference curves corresponding to

$$EU = 2.75 + 5\left(E\widetilde{\xi}n + \xi_f n_f\right)$$

are parallel lines in $n - n_f$ space with slope equal to $-E\widetilde{\xi}/\xi_f$. Therefore the representative agent's indifference curves will coincide with the budget line with the same slope, implying that

7 Conclusion

In this paper, we extend the univariate characterization of risk neutrality to the multivariate case for Expected Utility preferences where for the latter the NM index is assumed not to be linear in each of its arguments. Building on Debreu's [3] decomposition of the NM index into a risk preference function and a least concave utility and utilizing the notions of minimum concavity points and minimum concavity directions, we characterize risk neutral behavior for the case where each of the choice variables is risky and for the case where one variable is certain. As applications we show that, even though the bivariate NM index of an Expected Utility maximizer is not linear in both arguments, (i) an individual consumer's optimal savings behavior can be unaffected by a mean preserving spread in income risk and (ii) a representative agent two period equilibrium risk premium can be zero for certain specific consumption endowments.

In this paper we focus only on the Expected Utility case. However since considerable laboratory evidence suggests that the observed behavior of individuals can be inconsistent with the Expected Utility axioms, an area of potentially interesting future research would seem to be to consider the extension of our analysis to non-Expected Utility models such as Cumulative Prospect Theory, Rank Dependent Utility and Ambiguity preferences (see, for example, Wakker [24] for an excellent summary of these models).

Appendix

A Proof of Proposition 2

Denoting[32]

$$\xi = \frac{\xi_2}{\xi_1},$$

for the least concave form u, we have

$$a(\mathbf{c}) = \inf_{\{\boldsymbol{\xi}: u_1(\mathbf{c})\xi_1 + u_2(\mathbf{c})\xi_2 \neq 0\}} -\frac{u_{11}(\mathbf{c}) + 2u_{12}(\mathbf{c})\xi + u_{22}(\mathbf{c})\xi^2}{(u_1(\mathbf{c}) + u_2(\mathbf{c})\xi)^2}.$$

If at $\mathbf{c}^* = (c_1^*, c_2^*)$ the Hessian matrix based on u is the zero matrix, then for any ξ, the following must always hold

$$u_{11}(\mathbf{c}^*) + 2u_{12}(\mathbf{c}^*)\xi + u_{22}(\mathbf{c}^*)\xi^2 = 0,$$

if $p/p_f = E\widetilde{\xi}/\xi_f$, there are infinite number of optimal asset allocations. If $p/p_f \neq E\widetilde{\xi}/\xi_f$, there is no interior optimum. Thus in equilibrium when $\overline{c}_1 = 0.5$, $p/p_f = E\widetilde{\xi}/\xi_f$ holds for any $(\overline{n}, \overline{n}_f)$ values implying that $E\widetilde{R} - R_f = 0$.

[32] As noted below prior to Remark 6, this discussion is valid even if $\xi_1 = 0$.

implying that $a(\mathbf{c}^*)$ reaches an infimum and hence every direction is a MCD. Otherwise, taking the derivative of

$$\frac{u_{11}(\mathbf{c}^*) + 2u_{12}(\mathbf{c}^*)\xi + u_{22}(\mathbf{c}^*)\xi^2}{\left(u_1(\mathbf{c}^*) + u_2(\mathbf{c}^*)\xi\right)^2}$$

with respect to ξ and setting it to be zero yields

$$\begin{aligned}
& \frac{\left(2u_{12} + 2u_{22}\xi\right)\left(u_1 + u_2\xi\right)^2 - 2\left(u_1 + u_2\xi\right)u_2\left(u_{11} + 2u_{12}\xi + u_{22}\xi^2\right)}{\left(u_1 + u_2\xi\right)^4} \\
= & \frac{\left(2u_{12} + 2u_{22}\xi\right)\left(u_1 + u_2\xi\right) - 2u_2\left(u_{11} + 2u_{12}\xi + u_{22}\xi^2\right)}{\left(u_1 + u_2\xi\right)^3} \\
= & \frac{2\left(u_1u_{12} - u_2u_{11}\right) + 2\xi\left(u_1u_{22} - u_2u_{12}\right)}{\left(u_1 + u_2\xi\right)^3} = 0,
\end{aligned}$$

implying that

$$\xi\left(c_1^*, c_2^*\right) = \frac{\xi_2\left(c_1^*, c_2^*\right)}{\xi_1\left(c_1^*, c_2^*\right)} = \left.\frac{u_2u_{11} - u_1u_{12}}{u_1u_{22} - u_2u_{12}}\right|_{\left(c_1^*, c_2^*\right)}.$$

When $\left.u_1u_{22} - u_2u_{12}\right|_{\left(c_1^*, c_2^*\right)} \neq 0$, the MCD is proportional (or parallel) to the vector

$$\left.\left(u_1u_{22} - u_2u_{12}, u_2u_{11} - u_1u_{12}\right)\right|_{\left(c_1^*, c_2^*\right)}.$$

When $\left.u_1u_{22} - u_2u_{12}\right|_{\left(c_1^*, c_2^*\right)} = 0$ (corresponding to $\xi_1 = 0$) and $\left.u_2u_{11} - u_1u_{12}\right|_{\left(c_1^*, c_2^*\right)} \neq 0$, it is clear that the MCD continues to be proportional (or parallel) to the vector

$$\left.\left(u_1u_{22} - u_2u_{12}, u_2u_{11} - u_1u_{12}\right)\right|_{\left(c_1^*, c_2^*\right)}.$$

Remark 6 *If one uses the coordinate system where c_1 corresponds to the direction tangent to the indifference curve and c_2 corresponds to the direction normal to the indifference curve (the price direction), then the MCD can be obtained from*

$$\frac{\xi_2\left(c_1^*, c_2^*\right)}{\xi_1\left(c_1^*, c_2^*\right)} = -\frac{u_{12}}{u_{11}},$$

where u_{11} measures the strict concavity of u. (See Kannai [9], p. 301.)

B Proof of Proposition 3

To prove this proposition, we first state and prove the following lemma which can be used to directly verify whether u becomes linear on $S[ac_1 + bc_2 + d]$ without deriving the MCPs and MCDs.

Lemma 1 *Assume that the NM index $U = f \circ u$ where f is a strictly increasing and concave transformation, u is the least concave utility derived from eqn. (10) and*

U does not take the linear form defined in (1). The consumer will be risk neutral toward a given nondegenerate lottery $J \in \mathcal{J}$ if and only if the lottery's payoffs lie on $S[ac_1 + bc_2 + d]$, f is a positive affine transformation and u becomes linear along this line segment.

First prove sufficiency. Suppose the lottery's payoffs lie on the line segment $S[ac_1+bc_2+d]$. Without loss of generality, assume that $b \neq 0$. Since f is a positive affine transformation and the consumer's NM index becomes linear along this line segment, we can conclude that

$$u(c_1, c_2) = u\left(c_1, -\frac{d + ac_1}{b}\right) = \alpha c_1 + \beta.$$

For any S-state lottery with the payoffs along the line segment $S[ac_1 + bc_2 + d]$

$$\sum_{s=1}^{S} \pi_s u(c_{1s}, c_{2s}) = \alpha \sum_{s=1}^{S} \pi_s c_{1s} + \beta = u\left(\sum_{s=1}^{S} \pi_s c_{1s}, \sum_{s=1}^{S} \pi_s c_{2s}\right),$$

implying that the consumer is risk neutral. Next prove necessity. Without loss of generality, consider a lottery with two states.[33] Assume the lottery pays (c_1, c_2) with probability π and (c_1', c_2') with probability $1 - \pi$, where $c_1 \neq c_1'$. Denote the line segment going through (c_1, c_2) and (c_1', c_2') as $ac_1 + bc_2 + d = 0$. Since $c_1 \neq c_1'$, it follows that $b \neq 0$. Then

$$U(c_1, c_2) = U\left(c_1, -\frac{d + ac_1}{b}\right) \text{ and } U(c_1', c_2') = U\left(c_1', -\frac{d + ac_1'}{b}\right).$$

Therefore, along the line segment $ac_1 + bc_2 + d = 0$, $U(c_1, c_2)$ can be viewed as a function of c_1. It can be verified that

$$\frac{\partial^2 U(c_1, c_2(c_1))}{\partial c_1^2} = U_{11} + 2U_{12}\frac{\partial c_2}{\partial c_1} + U_{22}\left(\frac{\partial c_2}{\partial c_1}\right)^2 + U_2\frac{\partial^2 c_2}{\partial c_1^2}.$$

On the line segment $ac_1 + bc_2 + d = 0$, $\frac{\partial^2 c_2}{\partial c_1^2} = 0$ and $\frac{\partial c_2}{\partial c_1} = -\frac{a}{b}$ and hence

$$\frac{\partial^2 U(c_1, c_2(c_1))}{\partial c_1^2} = U_{11} - \frac{2a}{b}U_{12} + \frac{a^2}{b^2}U_{22} = \begin{bmatrix} 1 & -\frac{a}{b} \end{bmatrix} H_U \begin{bmatrix} 1 \\ -\frac{a}{b} \end{bmatrix},$$

where H_U is the Hessian Matrix. Since U is concave, H_U is negative semidefinite, implying that $\frac{\partial^2 U(c_1, c_2(c_1))}{\partial c_1^2} \leq 0$. Therefore along the line segment $ac_1 + bc_2 + d = 0$,

[33] For the lotteries with more than two states, one can use the argument in this proof to consider the comparison between the payoffs in any two arbitrary states and their mean. Then one can treat their mean as a new state and apply the argument in the proof again to this new state and another state. An example illustrating this process is given in the online Appendix G.

$U(c_1, c_2(c_1))$ is a concave function in c_1. It follows from Jensen's inequality that

$$\begin{aligned} & \pi U(c_1, c_2) + (1-\pi) U(c_1', c_2') \\ = & \pi U\left(c_1, -\frac{d + ac_1}{b}\right) + (1-\pi) U\left(c_1', -\frac{d + ac_1'}{b}\right) \\ \leq & U\left(\pi c_1 + (1-\pi) c_1', -\frac{d + a(\pi c_1 + (1-\pi) c_1')}{b}\right) \\ = & U(\pi c_1 + (1-\pi) c_1', \pi c_2 + (1-\pi) c_2'), \end{aligned}$$

where the equal sign holds if and only if $c_1 = c_1'$ or $U(c_1, c_2(c_1))$ is linear in c_1. Since it is assumed that $c_1 \neq c_1'$, the consumer will be risk neutral only if her NM index becomes linear on the line segment $ac_1 + bc_2 + d = 0$. If f is not an affine transformation, then U is strictly concave. Therefore H_U is negative definite, implying that $\frac{\partial^2 U(c_1, c_2(c_1))}{\partial c_1^2} < 0$ always holds and hence the consumer's NM index cannot be linear in c_1, which contradicts the conclusion above. Therefore, we require that f is an affine transformation and u becomes linear along $ac_1 + bc_2 + d = 0$. This completes the proof.

Next we prove Proposition 3. Following Lemma 1, we only need to show that the consumer's NM index will become linear along the line segment $S[ac_1 + bc_2 + d]$ if and only if all points on this line segment are MCPs and the MCD is the same as the slope of this line segment. The consumer's NM index $U = u$ (we ignore the affine transformation for simplicity) is linear along the line segment $S[ac_1 + bc_2 + d]$ if and only if $\forall (c_1, c_2) \in S[ac_1 + bc_2 + d]$,

$$\frac{\partial^2 u(c_1, c_2(c_1))}{\partial c_1^2} = 0.$$

This implies that

$$u_{11} + 2u_{12}\frac{\partial c_2}{\partial c_1} + u_{22}\left(\frac{\partial c_2}{\partial c_1}\right)^2 + u_2\frac{\partial^2 c_2}{\partial c_1^2} = 0.$$

Noticing that $\frac{\partial^2 c_2}{\partial c_1^2} = 0$ and $\frac{\partial c_2}{\partial c_1} = -\frac{a}{b}$, we can obtain

$$u_{11} - \frac{2a}{b}u_{12} + \frac{a^2}{b^2}u_{22} = 0,$$

or equivalently

$$\begin{bmatrix} 1 & -\frac{a}{b} \end{bmatrix} H_u \begin{bmatrix} 1 \\ -\frac{a}{b} \end{bmatrix} = 0.$$

Following the derivation of the set of MCPs, this implies that (c_1, c_2) is a minimum concavity point, or equivalently $S[ac_1 + bc_2 + d] \subseteq C^*$ and

$$\frac{\xi_2(c_1, c_2)}{\xi_1(c_1, c_2)} = -\frac{a}{b}$$

is a MCD.

C Proof of Proposition 4

First prove necessity. If $\overline{u}(c) = au(c) + b$ for some $a > 0$, then clearly $\overline{u}$ is concave. Next prove sufficiency. If $\overline{u}$ is concave, then $\overline{u} \in \mathcal{U}$, and there exists a strictly monotone real-valued concave function g defined on the range of u such that $\overline{u} = g(u)$. If $g''(t) < 0$ with $t = u(c_1, c_2)$, then straightforward computation shows that $\frac{\overline{u}_{22}}{\overline{u}_2^2} \leq \frac{g''(t)}{g'(t)^2}$, so that $\frac{\overline{u}_{22}}{\overline{u}_2^2}$ cannot vanish (or tend to zero) along the indifference curve $u(c_1, c_2) = t$. Therefore u and $\overline{u}$ are equivalent up to a positive affine transformation.

D Proof of Proposition 5

For the certain $\times$ risky setting, since c_1 is fixed, the lottery's payoffs must stay on the vertical line segment $S[c_1 - d]$. Since $U(d, c_2)$ is concave in c_2, following Jensen's inequality, the consumer will become risk neutral toward a nondegenerate lottery with payoffs along the vertical line segment $S[c_1 - d]$ if and only if $U(d, c_2)$ is linear in c_2, which is equivalent to the condition that f is a positive affine transformation and $S[c_1 - d] \subseteq \overline{C}^*$.

E Proof of Result 1

First we prove that using u, s_1^* increases with a MPS of $\widetilde{I}_2$. Ignoring the affine transformation, we can assume $U = u$. Then the optimization problem is given by

$$\max_{s_1} \quad E\left[(I_1 - s_1)^a \left(\widetilde{I}_2 + s_1\right)^b\right],$$

where $a + b = 1$. It follows from the first order condition that

$$-a(I_1 - s_1)^{a-1} E\left[\left(\widetilde{I}_2 + s_1\right)^b\right] + b(I_1 - s_1)^a E\left[\left(\widetilde{I}_2 + s_1\right)^{b-1}\right] = 0,$$

or equivalently,

$$\frac{E\left[\left(\widetilde{I}_2 + s_1\right)^{b-1}\right]}{E\left[\left(\widetilde{I}_2 + s_1\right)^b\right]} = \frac{a}{b(I_1 - s_1)}. \tag{20}$$

Since $0 < b < 1$, $\left(\widetilde{I}_2 + s_1\right)^{b-1}$ is a convex and decreasing function of $\widetilde{I}_2$ and $\left(\widetilde{I}_2 + s_1\right)^b$ is a concave and increasing function of $\widetilde{I}_2$. Therefore, assuming that

$$\widetilde{I}_2(t) = \widetilde{I}_2 + t\left(\widetilde{I}_2 - E\widetilde{I}_2\right),$$

we have

$$A = \frac{\partial}{\partial t} E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b-1}\right] > 0$$

and

$$B = \frac{\partial}{\partial t} E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b}\right] < 0.$$

Since $\left(\widetilde{I}_2(t) + s_1\right)^{b-1}$ is a decreasing function of s_1 and $\left(\widetilde{I}_2(t) + s_1\right)^{b}$ is an increasing function of s_1,

$$C = \frac{\partial}{\partial s_1} E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b-1}\right] < 0$$

and

$$D = \frac{\partial}{\partial s_1} E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b}\right] > 0.$$

Taking the natural logarithm on both sides of equation (20), yields

$$\ln E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b-1}\right] - \ln E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b}\right] = \ln \frac{a}{b} - \ln\left(I_1 - s_1\right). \tag{21}$$

Then implicitly differentiating (21) with respect to t, one obtains

$$\begin{aligned}
&\frac{\frac{\partial}{\partial t} E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b-1}\right] + \frac{ds_1}{dt}\frac{\partial}{\partial s_1} E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b-1}\right]}{E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b-1}\right]} \\
&- \frac{\frac{\partial}{\partial t} E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b}\right] + \frac{ds_1}{dt}\frac{\partial}{\partial s_1} E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b}\right]}{E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b}\right]} \\
= \quad & \frac{1}{I_1 - s_1}\frac{ds_1}{dt},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{A}{E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b-1}\right]} - \frac{B}{E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b}\right]} \\
= \quad & \frac{ds_1}{dt}\left[-\frac{C}{E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b-1}\right]} + \frac{D}{E\left[\left(\widetilde{I}_2(t) + s_1\right)^{b}\right]} + \frac{1}{I_1 - s_1}\right].
\end{aligned}$$

Since $A > 0$, $B < 0$, $C < 0$ and $D > 0$,

$$\frac{ds_1}{dt} > 0.$$

Next we show that using $\overline{u}$, s_1^* is unaffected by a MPS of $\widetilde{I}_2$. When $U = \overline{f} \circ \overline{u}$, ignoring the positive affine transformation, the optimization problem is

$$\max_{s_1} \quad E\left[(I_1 - s_1)^{\frac{a}{b}} \left(\widetilde{I}_2(t) + s_1 \right) \right].$$

It follows from the first order condition that

$$-\frac{a}{b}(I_1 - s_1)^{\frac{a}{b}-1} \left(E\widetilde{I}_2 + s_1 \right) + (I_1 - s_1)^{\frac{a}{b}} = 0,$$

or equivalently,

$$(I_1 - s_1)^{\frac{a}{b}} = \frac{a}{b}(I_1 - s_1)^{\frac{a}{b}-1} \left(E\widetilde{I}_2 + s_1 \right). \tag{22}$$

Taking the natural logarithm on both sides of equation (22), yields

$$\frac{a}{b}\ln(I_1 - s_1) = \ln\frac{a}{b} + \left(\frac{a}{b} - 1\right)\ln(I_1 - s_1) + \ln\left(E\widetilde{I}_2 + s_1 \right). \tag{23}$$

Implicitly differentiating (23) with respect to t, one obtains

$$-\frac{a}{b(I_1 - s_1)}\frac{ds_1}{dt} = -\frac{a-b}{b(I_1 - s_1)}\frac{ds_1}{dt} + \frac{1}{E\widetilde{I}_2 + s_1}\frac{ds_1}{dt},$$

implying that

$$\frac{ds_1}{dt} = 0.$$

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