Capacitated Location Problems on a Line[1]

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This paper examines capacitated facility location problems on a straight line. To serve a customer, a facility must be located within a corresponding customer neighborhood. The fixed costs of locating facilities and the unit production costs of serving a customer from a facility can depend upon their locations on the line. We discuss the computational complexity of several capacitated location models. For capacitated problems on a line with non-nested customer intervals, and for general capacitated problems that satisfy a certain "monotonicity" property, we develop polynomial-time dynamic programming algorithms for (i) locating minimum cost facilities to serve all customers, and (ii) maximizing the profit by locating up to q facilities that serve some or all customers with idiosyncratic returns, penalties and demands.

Capacitated location problems arise frequently in practice, but are notoriously difficult to solve. In this paper, we describe, model, and develop solution approaches for capacitated location problems in which customers lie on a straight line. To serve a customer, a facility must be located in a designated customer neighborhood. Problems of this variety arise, either directly or as subproblems, in a number of contexts:

- *Scheduling deliveries.* A retailer has to deliver merchandise to various households in the same neighborhood. Each household prefers delivery during a certain time period on a certain day, and the time required for delivering the merchandise is negligible. What is the smallest number of delivery trips required and when should the deliveries be scheduled?
- *Rest area location.* Cars enter a highway at different points. What is the smallest number of rest areas that are needed along the highway to ensure that each car can access a rest area within a given distance from its point of entry?
- *Transformer location.* A high-voltage power line runs through rural townships. To limit power losses, step-down transformers must be installed within certain distances of the townships. What is the smallest number of transformers required to service all communities?

In the above examples, households, entry ramps, etc., correspond to the "customers," and deliveries, rest areas, etc., correspond to the "facilities." The preferred delivery time periods, distance, etc., define the "customer neighborhoods (intervals)." In these examples, the objective is to select the smallest number of "facilities" to serve all "customers." In related examples, the objective may be to maximize total profit from serving some or all "customers" using q "facilities."

The common aspects of each of the above applications are that (i) a single attribute characterizes preferences of all the customers, and (ii) each customer has a range of indifference for this attribute.

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These two characteristics allow us to model the above situations as location problems on a straight line that represents the critical attribute. Customer intervals on the line represent indifference ranges for the customers.

This paper examines a dynamic-programming approach for solving capacitated location problems on the straight line. Customers, represented by intervals on the straight line, have known (integral) demands. Capacitated facilities, can be set up at a fixed cost, and incur a production (or a service) cost for satisfying each unit of demand. We first examine a covering problem, which requires the selection of facilities that together satisfy the demands of all customers at minimum total cost. Next, we examine a profit-maximization problem in which a return (penalty) is obtained by serving (not serving) a customer and the objective is to maximize profit by selecting at most q facilities that supply some or all customers. A transformation from the knapsack problem shows that both problems are NP-Hard. We then assume that no customer interval is nested in any other customer interval. Even though general versions of the problems continue to be NP-Hard, the assumption of non-nested customer intervals provides sufficient structure to permit polynomial-time dynamic programs in special cases. We extend our analysis to a more general setting of problems for which the optimal solution satisfies a certain "monotonicity" property.

1. LITERATURE REVIEW

GIVEN A SPATIAL distribution of customers, the objective of a location problem is to identify facility locations serving some or all customers so that a cost (profit) function is minimized (maximized). *Continuous location problems* permit facilities to be located anywhere in a (one or higher dimensional) space. *Discrete location problems* restrict facility locations to a finite set of points, typically by restricting feasible locations in a continuous space, or by defining the problem over a network and reducing the search for optimal facility locations to its nodes.

Perhaps the best known discrete location problem is the uncapacitated (simple) plant location problem that requires the selection of facilities from a finite feasible set so that the total (fixed and variable) cost of serving a finite number of customers is minimized. CORNUEJOLS, NEMHAUSER and WOLSEY (1990) review exact and approximate solution procedures that have been proposed for this problem. Variations in which no fixed costs are incurred and the objective is to maximize a profit function from serving a subset of the customers have also been considered (see MIRCHANDANI, 1990 for a review), as have capacitated versions of the problem (e.g., GUIGNARD and SPIELBERG, 1979, CHRISTOFIDES and BEASLEY, 1983, LEUNG and MAGNANTI, 1989). Uncapacitated location problems on trees in which variable costs depend on distances between nodes have been examined by, among others, BARANY, EDMONDS, and WOLSEY (1986), KOLEN (1983), MEGIDDO, ZEMEL and HAKIMI (1983), and KOLEN and TAMIR (1990). For a more detailed discussion of the literature on general location models, we refer the reader to MIRCHANDANI and FRANCIS (1990), and to the review of location problems by BRANDEAU and CHIU (1989).

Several researchers have developed efficient algorithms for solving a variety of discrete location problems on the real line. We mention only the most recent and relevant ones. The reader is referred to the references cited in these recent papers for earlier results.

HASSIN and TAMIR (1991) develop linear time algorithms for uncapacitated (simple) location problems, where the variable costs (transportation costs) are linear in the distances between the customers and their respective serving facilities. Observe that the variable cost functions we consider in our paper can be viewed as stepwise functions with 0, $+\infty$ values. AGGARWAL and PARK (1993) present linear time algorithms for the related classical uncapacitated lot-sizing model.

Efficient algorithms for capacitated location problems on the real line and related problems are known for only a few cases. FLORIAN and KLEIN (1971) describe a polynomial $O(n^4)$ algorithm for the n-period lot-sizing problem with constant production capacities. This bound has been recently reduced to $O(n^3)$ by WAGELMANS and VAN HOESEL (1994). JAEGER and GOLDGERG (1991) discuss the equal capacity p-center problem and present a polynomial time algorithm. In our context, this model represents the case of constant facility set up costs and stepwise transportation cost functions. Approximation algorithms for this capacitated p-center problem on general networks are given in BAR-ILAN, KORTSARZ and PELEG (1993).

2. CAPACITATED COVER WITH NON-NESTED CUSTOMER INTERVALS

LET $M = u_1, u_2, \ldots, u_m$, $u_1 < u_2 \ldots < u_m$, denote a set of potential sites (points on the real line), for establishing facilities, and let $N = 1, 2, \ldots, n$ denote a set of customers. Each customer j in N can be served only by a nonempty contiguous subset of sites, M^j, called customer interval j. (Observe that

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we have defined the customer intervals using facility locations, even though our motivating examples implicitly assumed that the customer intervals are continuous. Note, however, that the two representations are equivalent for the problems studied in this paper since we can transform a problem in which the customer intervals are continuous into one in which they are defined using facility locations by first identifying the potential facility locations.)

Let $u'_{i(j)}$ and $u''_{i(j)}$ denote respectively the smallest and the largest facility sites in M^j. Customer j has an integral demand d_j, which can be served (in integral units) by one or more facilities in M^j. For each $i = 1, 2, \ldots, m$, we assume that at most one facility can be established at u_i and call it facility i. Facility i can be set up at a fixed cost f_i, has a capacity constraint of C_i units, and incurs a cost p_i for producing or serving each unit of demand.

Consider the following minimum cost capacitated cover problem on the straight line, which we denote by Problem P1:

> Determine which facilities should be set up, and how much should each opened facility supply to each customer, so that the total cost of setting up the facilities and serving all customers is minimized.

It is easily observed that Problem P1 is (weakly) NP-hard even for the simple case where there is a single customer and all unit production costs are zero. (In this case, the problem is equivalent to the knapsack problem.) We therefore consider the case when all customer demands are unity. The complexity of this case is still open. However, we focus on the case where the family of customer intervals is non-nested and develop a polynomial-time algorithm for this case.

NON-NESTEDNESS ASSUMPTION. *For any pair of customers j and k, $u'_{i(j)} < u'_{i(k)}$ implies $u''_{i(j)} \le u''_{i(k)}$.*

In particular, the non-nestedness assumption implies that there is no pair of distinct customers j, k in N such that M^j is a proper subset of M^k. The non-nestedness assumption was motivated by two applications. The first is a problem of tool selection in flexible manufacturing systems for the sheet metal industry. DASKIN, JONES and LOWE (1990) develop and analyze (uncapacitated) models for this application. The second motivating application is the location problem in which customers are represented by points on the real line, and each customer interval j is defined by all points in M whose distance from the point representing j is at most r, where r is a "proximity" parameter which has the same value for all customers.

To facilitate the subsequent discussion, we reindex the customers in N, and suppose that if $j < k$, then either $u'_{i(j)} < u'_{i(k)}$, or $u'_{i(j)} = u'_{i(k)}$ and $u''_{i(j)} \le u''_{i(k)}$. Thus, identical customer intervals are indexed in consecutive, arbitrary order. If two customer intervals are not identical, they are indexed in ascending order of their lower endpoints, and in the case of ties in the lower endpoints, in ascending order of their upper endpoints. With the above assumption and reindexing, we have the following property.

CONTIGUITY PROPERTY. *Each facility can only serve a contiguous set of customers and each customer can be served only by a contiguous set of facilities.*

Let a_i and b_i denote the lowest- and highest-indexed customers, respectively, that can be served by facility i. Note that feasibility implies $a_1 = 1$ and $b_m = n$. The following Lemma is an immediate consequence of our indexing procedure.

LEMMA 1. *If i_1 and i_2 are two facilities such at $i_1 < i_2$, then $a_{i_1} \le a_{i_2}$* and $b_{i_1} \le b_{i_2}$.

Let Problem $P1(i, j)$ denote the subproblem of selecting the minimum cost facilities from $1, 2, \ldots, i$ to serve all customers $1, 2, \ldots, j$. Lemma 2 characterizes two properties of the optimal solution to Problem $P1\,(i, j)$. The first property is similar to the subtree-cover property described by Kolen (1983) for an uncapacitated cover problem on trees. These properties are then used to construct and prove the validity of a dynamic program for solving Problem $P1$.

LEMMA 2. *There exists an optimal solution to $P1(i, j)$ that satisfies the following two properties:*

Property 1. *If customers j_1 and j_2 are served by different facilities $i(j_1)$ and $i(j_2)$, respectively, then $j_1 < j_2$ if and only if $i(j_1) < i(j_2)$.*

Property 2. *Suppose the unit production costs of all facilities are equal. If i_l denotes the highest-indexed facility selected, then facility i_l serves customers s, $s + 1$, $\ldots, j$* where $s = \max\{j - C_{i_l} + 1, a_{i_l}\}$.

Proof. Consider an optimal solution to Problem $P1(i, j)$ that selects facilities $i_1 < i_2 < \cdots < i_l$, where $i_l \le i$. Assume that this solution does not satisfy Property 1. The following procedure constructs an optimal solution satisfying Property 1. Start with facility i_1. Let j_1 denote the lowest-indexed customer that is currently being served by facility i_1 and that violates Property 1; i.e., customer $j_1 - 1$ is served by some facility $i^* > i_1$. Then $a_{i^*} \le j_1 - 1$ because facility i^* can serve customer $j_1 - 1$, and $j_1 \le b_{i_1}$ because

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customer j_1 can be served by facility i_1. As $i_1 < i^*$, Lemma 1 implies that $a_{i_1} \le a_{i^*}$ and $b_{i_1} \le b_{i^*}$. Thus

$$a_{i_1} \le a_{i^*} \le j_1 - 1 < j_1 \le b_{i_1} \le b_{i^*},$$

which implies that it is possible to reassign customer $j_1 - 1$ to facility i_1 and customer j_1 to facility i^*. Note that this reassignment does not violate the capacity constraint because each customer has unit demand. Repeating this procedure for all customers served by facility i_1 in the current optimal solution constructs an optimal solution for which Property 1 is satisfied for all pairs of customers, one of which is served by facility i_1. Sequentially repeating this procedure for facilities $i_2, i_3, \ldots, i_{l-1}$ constructs an optimal solution satisfying Property 1.

To satisfy Property 2 when all unit production costs are equal, start with an optimal solution that satisfies Property 1; i.e., each facility serves a contiguous set of customers. Because of the contiguity property, customers $a_{i_l}, a_{i_l} + 1, \ldots, j$ can be served by facility i_l. As $s = \max\{j - C_{i_l} + 1, a_{i_l}\} \ge a_{i_l}$, reassign, if necessary, customers $s, s + 1, \ldots, j$ to facility i_l. This reassignment satisfies Property 2, preserves Property 1, and does not violate the capacity constraints or change the solution value. ■

We now describe algorithm $DP1$ for solving Problem $P1$. As mentioned earlier, we first assume that the customers have unit demand. We denote the optimal solution value to Problem $P1(i,j)$ by $v(i,j)$.

DP1

Initialization:

$$v(i, 0) = 0 \quad \text{for all} \quad i = 0, 1, 2, \ldots, m.$$

$$v(0, j) = \infty \quad \text{for all} \quad j = 1, 2, \ldots, n.$$

Recursion: For all $1 \le i \le m$, for all $1 \le j \le n$,

$$v(i,j) = \begin{cases} v(i-1,j), & \text{if } j < a_i; \\ \min\left\{ v(i-1,j), \; f_i + \min_{k:\max\{a_i - 1, j - C_i\} \le k < j} \{v(i-1,k) + p_i \cdot (j-k)\} \right\}, & \text{if } a_i \le j \le b_i \\ \infty, & \text{if } j > b_i. \end{cases}$$

Termination: Stop when $j = n$, $i = m$.

THEOREM 3. *$DP1$ finds the optimal solution to Problem $P1$, with unit demand (and non-nested customer intervals), in $O(mn \min(C_{\max}, n))$ time where $C_{\max}$ denotes $\max_i C_i$.*

We omit the proof which uses Lemma 2.

$DP1$ can be improved if the unit production cost for all facilities equals p. First, notice that since all customers must be served, we can solve the problem without the unit production costs and then add pn to the optimal solution value. Thus, we can modify the recursion step for $a_i \le j \le b_i$ as follows:

$$v(i,j) = \min\{v(i-1,j), f_i + v(i-1, \max\{a_i - 1, j - C_i\})\}.$$

This change in the recursion decreases the complexity of the $DP1$ to $O(mn)$ since the inner minimization over k is no longer required.

Finally, consider the case when the customers are allowed to have arbitrary integral demand d_j. Since this version of the Problem $P1$ is NP-hard, we do not expect to develop a polynomial time algorithm. If we represent a customer with demand d_j by d_j dummy customers, the complexity of the above algorithm for arbitrary demand is $O(mD \min(C_{\max}, D))$ where $D = \sum_{j \in N} d_j$. Note that the dummy customer representation is not required if the facilities are uncapacitated (i.e., $\min_i C_i \ge D$). In this case, the inner minimization over k is also not required, and the complexity of $DP1$ (with $C = \infty$) is $O(mn)$ even for arbitrary demand.

3. MAXIMIZING PROFIT FROM q CAPACITATED FACILITIES

WE NOW CONSIDER Problem $P2'$ in which, as earlier, a fixed cost of f_i is incurred in setting up facility i, which has capacity C_i and unit production cost p_i. Each customer j is assumed to have integral demand which provides a unit return of r_j if it is satisfied but imposes a unit penalty of h_j if it is not satisfied. Problem $P2'$ is the following profit maximization problem:

> Select up to q facilities that maximize the total profit from serving some or all (non-nested) customers on the straight line.

Problem $P1$ can be represented as a special case of Problem $P2'$ by permitting all customers to have equal but "large" returns, zero penalties, and allowing $q = m$ possible facilities. Thus, Problem $P2'$ is also NP-hard. However, as in Section 2, we will first consider a polynomial time algorithm for the unit demand case, and then generalize this algorithm for the arbitrary demand model for which the complexity of the dynamic program becomes pseudo-polynomial. As we will see, our algorithm applies whenever the "monotonicity" property holds and can be used for problems that are more general than Problem $P2'$.

Let $P2'(i, j, k, t)$ denote the subproblem of selecting up to t facilities from $1, 2, \ldots, i$ to maximize the profit

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from serving some or all customers 1, 2, ..., j such that exactly k customers are served by facility i. Lemma 4 describes a property of the optimal solution to $P2'(i, j, k, t)$ that is analogous to Property 1 described in Lemma 2. If the returns/penalties of all customers and the unit production costs of all facilities are equal, then a property analogous to Property 2 of Lemma 2 also holds. We omit the proof of Lemma 4 which is based on an "interchange" argument similar to the one used in the proof of Lemma 2.

LEMMA 4. *There exists an optimal solution to* $P2'(i, j, k, t)$ *such that if customers* j_1 *and* j_2 *are served by different facilities* $i(j_1)$ *and* $i(j_2)$, *respectively, then* $j_1 < j_2$ *if and only if* $i(j_1) < i(j_2)$.

To consider Problem $P2'$ in a more general context, let c_{ij} denote the unit contribution received by serving customer j from facility i. As earlier, f_i denotes the fixed cost of establishing facility i with capacity C_i, and h_j denotes the penalty incurred if we do not serve customer j. The objective is find the optimal locations of at most q facilities that will maximize total profit. We denote this problem by $P2$ and its specialization to the quadruplet (i, j, k, t) by $P2\ (i, j, k, t)$. Observe that in Problem $P2$, we do not restrict the customers to lie on a straight line.

The recursive algorithm $DP2$ solves Problem $P2$ in polynomial time as long as the following property holds.

MONOTONICITY PROPERTY. *There is an optimal solution to Problem* $P2\ (i, j, k, t)$ *such that for every pair of customers* j_1 *and* j_2, $1 \le j_1 < j_2 \le n$, *which are both served, but not by the same facility, the respective facility, say* $i(j_1)$ *and* $i(j_2)$, *satisfy* $i(j_1) < i(j_2)$.

Let us consider the following specialization of Problem $P2$: the contribution c_{ij} that accrues from serving customer j from facility i equals $r_j - p_i$, customers lie on a straight line, and satisfy the non-nestedness assumption. This specialization of Problem $P2$ is Problem $P2'$ and Lemma 4 implies that the monotonicity property holds. Similarly, Lemma 2 implies that the monotonicity property holds when Problem $P2$ is specialized to Problem $P1$.

We solve Problem $P2$ recursively. For each quadruplet (i, j, k, t), $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 0, 1, \ldots, \min(C_i, j)$; and $t = 0, 1, \ldots \min(q, i, j)$; let $v(i, j, k, t)$ be the optimal value of Problem $P2(i, j, k, t)$, i.e., the subproblem restricted to the customers $1, 2, \ldots, j$, the locations in $1, 2, \ldots, i$, and subject to the constraint that a total of t facilities are established, and exactly k customers are served by location i. (Note that if $k = 0$, no facility is established at location i.) For convenience we set $v(i, j, k, t) = -\infty$ if the quadruplet is not in the above ranges.

DP2

Initialization:

$v(1, 1, 0, 0) = -h_j$

$v(1, 1, 0, 1) = -\infty$

$v(1, 1, 1, 1) = -f_i + c_{ij}$

Recursion: For $j = 1$ and for all $i = 2, \ldots, m$,

$v(i, 1, 1, 1) = -f_i + c_{i1}$, and

$v(i, 1, 0, 1) = \max[v(i-1, 1, 1, 1); v(i-1, 1, 0, 1)]$.

For $i = 1$ and for all $j = 2, \ldots, n$,

$v(1, j, 0, 0) = -h_j + v(1, j-1, 0, 0)$,

$v(1, j, 1, 1) = \max[-h_j + v(1, j-1, 1, 1), -f_1 + c_{1j} + v(1, j-1, 0, 0)]$,

and for $k \ge 2$,

$v(1, j, k, 1) = \max[-h_j + v(1, j-1, k, 1); c_{1j} + v(1, j-1, k-1, 1)]$.

For all $i = 2, \ldots, m$,

for all $j = 2, \ldots, n$,

for all $t = 1, \ldots, \min(q, i, j)$,

$v(i, j, 0, t) = \max[v(i-1, j, k, t): k = 0, 1 \ldots, \min(C_{i-1}, j)]$.

$v(i, j, 1, t) = \max[-h_j + v(i, j-1, 1, t); -f_i + c_{ij} + v(i, j-1, 0, t-1)]$,

and for $k \ge 2$,

$v(i, j, k, t) = \max[-h_j + v(i, j-1, k, t); c_{ij} + v(i, j-1, k-1, t)]$.

Termination:

Stop when $j = n$, $i = m$, $k = \min(C_m, n)$, $t = q$.

The optimal solution value for the original problem is given by

$\max[v(m, n, k, t): k = 0, 1, \ldots, \min(C_m, n); t = 0, 1, \ldots, \min(m, n, q)]$.

The validity of the above dynamic programming algorithm is clear from the recursive equations. For example, consider the equation defining $v(i, j, k, t)$, where $i \ge 2$, $j \ge 2$, $k \ge 2$, and $t \ge 1$. In this case exactly k customers are served by facility i. If customer j is not one of them, then, by the monotonicity assumption, it is not served at all, (since we are considering only the problem restricted to the sites $1, 2, \ldots, i$). Thus, the maximum profit is $-h_j + v(i, j-1, k, t)$. Otherwise, customer j is served by facility i, and the maximum profit equals $c_{ij} + v(i, j-1, k-1, t)$.

To evaluate the total complexity of this dynamic programming algorithm, note that for a given triplet (i, j, t) the total effort to compute $v(i, j, k, t)$ for all values of k, $k = 0, 1, \ldots, \min(C_i, j)$ is $O(\min(C_{\max}, n))$, where $C_{\max}$, as earlier, equals $\max(C_i: i = 1, 2, \ldots, m)$. Therefore, the complexity bound of the algorithm is $O(qmn \min(C_{\max}, n))$.

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We have thus proved

THEOREM 5. *Problem P2, with unit customer demand (and with the monotonicity property assumption), can be solved in* $O(qmn \min(C_{\max}, n))$ *time.*

COROLLARY 6. *P2′, with unit customer demand (and non-nested customer intervals), can be solved in* $O(qmn \min(C_{\max}, n))$ *time.*

As with *DP*1, by defining "dummy" customers, we can implement *DP*2 in $O(qmD \min(C_{\max}, D))$ time if customers have arbitrary demands. Observe that if all the capacity bounds $C_i \geq D$ for all i, the dynamic programming equations are simplified considerably since the paramenter k is not required anymore. In particular, the total complexity bound reduces to $O(qmn)$. Similarly, if, in addition, q, the upper bound on the number of facilities satisfies $q \geq m$, the complexity bound becomes $O(mn)$. This uncapacitated model is discussed by JONES et al. (1992), where it is solved as a linear flow problem in $O(mn)$ time.

4. CONCLUSIONS

IN THIS PAPER we consider capacitated location problems on a straight line. The problems we consider arise in a variety of situations in which customers are indifferent about the location of a "facility," so long as it is located within a specified interval. We permit facilities to have capacity constraints, allow fixed costs to vary by the location of a facility, and incorporate both customer returns (which accrue when a demand is satisfied) and penalties (which arise when a demand is not satisfied). We focus on two problem classes: (i) locating minimum cost facilities to serve all customers, and (ii) maximizing the profit by locating up to q facilities that serve some or all customers with idiosyncratic returns and penalties. If customers have general demands, then these problems are NP-hard for both nested and non-nested cases (since they both include the knapsack problem as a special case). For problems involving non-nested customer intervals, or for those satisfying the monotonicity property, we have developed pseudo-polynomial dynamic programming algorithms. If each customer has unit demand, then our algorithms run in polynomial time for the non-nested case. However, the computational complexity of the unit demand case with nested customer intervals remains an open question.

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