

Joint performance of greedy heuristics for the integer knapsack problem

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Abstract

This paper analyzes the worst-case performance of combinations of greedy heuristics for the integer knapsack problem. If the knapsack is large enough to accommodate at least m units of any item, then the joint performance of the total-value and density-ordered greedy heuristics is no smaller than $(m + 1)/(m + 2)$. For combinations of greedy heuristics that do not involve both the density-ordered and total-value greedy heuristics, the worst-case performance of the combination is no better than the worst-case performance of the single best heuristic in the combination.

1. Introduction

This paper examines the worst-case performance of a combination of greedy heuristics for the integer knapsack problem. A solution is obtained using each heuristic in the combination and a bound placed on the performance ratio of the best solution. “Composite” heuristics of this kind have been analyzed by Frederickson et al. [2] for a variant of the travelling salesman problem, by Yao [10] and Friesen and Langston [3] for the bin-packing problem, and by Langston [8] for a job transportation problem in a flow shop.

The time complexity of the composite is the highest time complexity among the constituent heuristics. However, if the heuristics complement each other, the composite solution value can be closer to the optimal than the solution values of the individual heuristics. An analysis of composite heuristics therefore provides insight into why one heuristic performs well when another does poorly. Also, partial-enumeration methods (e.g., branch and bound) and approximation schemes sometimes use a heuristic to bound the value of the optimal solution [4]. A composite that has a better

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performance than a single heuristic can therefore be used to potentially improve the performance of the partial enumeration or approximation scheme. For example, Lawler's modification [9] of Ibarra and Kim's approximation scheme [6] for the integer knapsack problem employs the density-ordered greedy heuristic to bound the optimal solution. Using the higher bound for a composite heuristic (which has the same time complexity as the density-ordered greedy heuristic) can be shown to improve the constant, if not the order, of the time complexity of the approximation scheme.

The objective of this paper is to analyze the performance of a composite algorithm comprising of the greedy heuristics that have been suggested for the integer knapsack problem. The knapsack problem considers n items which can differ in their weights and values. Without loss of generality, let $W = 1$ denote the knapsack capacity. The problem is to select a collection of items that have the largest total value and no more total weight than the knapsack capacity. If w_i and v_i denote the unit weight and unit value, respectively, of item $i = 1, 2, \dots, n$, the integer knapsack problem is described by the following integer program with optimal value Z :

$$Z = \max \sum_{i=1}^n v_i x_i,$$

$$\sum_{i=1}^n w_i x_i \leq 1,$$

$$x_i \geq 0 \text{ and integral, for all } i, i = 1, \dots, n.$$

Given an ordering of the items, a greedy heuristic for the integer knapsack problem identifies the lowest-indexed item that has not yet been considered and selects as many units of the item as will fit in the remaining knapsack capacity. Four criteria have been proposed for ordering the items: (i) in increasing order of weight w_i , (ii) in decreasing order of value v_i , (iii) in decreasing order of density v_i/w_i and (iv) in decreasing order of the total value $\lfloor W/w_i \rfloor v_i = \lfloor 1/w_i \rfloor v_i$.

The weight-ordered and value-ordered greedy heuristics have arbitrarily bad worst-case bounds [5]. The density-ordered greedy heuristic has a worst-case bound of $m/(m+1)$, where $m = \min_i \lfloor 1/w_i \rfloor \lceil 1 \rceil$. For the first five integer values of $m \geq 1$, the worst-case bound for the total-value greedy heuristic is 0.59, 0.70, 0.76, 0.81 and 0.83 [7]. Note that the value of m can be obtained in $O(n)$ time for any problem. Also, the computation of the worst-case performance ratio considers the choice of only the first greedy item. Hence the preceding performance bounds are guaranteed in $O(n)$ time.

We begin by showing that a composite of the density-ordered and total-value greedy heuristics guarantees a worst-case performance bound of $(m+1)/(m+2)$. This bound is tight and strictly dominates the independent worst-case performance of either heuristic. However, using the weight-ordered and value-ordered greedy heuristics with each other or with either or both of the density-ordered and total-value greedy heuristics yields no further improvement in the worst-case performance of the best greedy heuristic solution. Thus, even if all four greedy heuristics are used to

solve the integer knapsack problem, the worst-case value of the best heuristic solution is no higher than if only the total-value and density-ordered greedy heuristics are used.

2. Joint performance of greedy heuristics

Let Z_v , Z_w , Z_d and Z_t denote the solution values for the value-ordered, weight-ordered, density-ordered, and total-value greedy heuristics, respectively. Let Z denote the value of the optimal solution. Let $Z_{td} = \max\{Z_t, Z_d\}$ denote the higher of the solution values for the total-value and density-ordered greedy heuristics. We begin by examining the lower bound for $r_{td} = Z_{td}/Z$.

Without loss of generality, let the highest total value across the n available items be 1; i.e., let

$$\max_{1 \leq i \leq n} \left\lfloor \frac{1}{w_i} \right\rfloor v_i = 1.$$

At the first step, the total-value greedy heuristic selects as many units as possible of the item with the highest total value. Hence $Z_t \geq 1$ and

$$r_{td} = \frac{Z_{td}}{Z} \geq \frac{Z_t}{Z} \geq \frac{1}{Z}.$$

Let d_i , v_i , and w_i , $i = 1, 2, \dots, k$, $k \leq n$, denote in non-increasing density order the density, unit value, and unit weight, respectively, of the k items selected by the density-ordered greedy heuristic. Similarly, let D_i , V_i and W_i , $i = 1, 2, \dots, K$, $K \leq n$, denote in non-increasing density order the density, unit value, and unit weight, respectively, of the K items in the optimal solution. If the optimal solution consists of $K = 1$ item, then $Z_{td} = Z_t = Z$ and $r_{td} = 1$. Thus, we consider $K \geq 2$ in the following analysis. Also, if the densest item available occupies the entire knapsack capacity (i.e., $w_1 = 1$), then $Z_{td} = Z_d = Z$ and $r_{td} = 1$. Thus, we consider $w_1 < 1$ in the following analysis. Further, if the density-ordered greedy heuristic selects only one item, we define a "dummy" item with unit value $v_2 = 0$ and unit weight $w_2 = 1 - w_1$. The dummy item has zero density and contributes no value to any solution. However, it permits us to assume without loss of generality that the density-ordered greedy heuristic selects at least two items, one of which may be the dummy item. We begin by proving the following lemma.

Lemma 1. *If an item has weight w and unit value v , and if $\lfloor 1/w \rfloor \geq t$, then it has density $v/w < (t + 1)/t$, where $t \geq 1$ is an integer.*

Proof. Let $\lfloor 1/w \rfloor = t + j$, where $j \geq 0$ is an integer. Then $1/(t + j) \geq w > 1/(t + j + 1)$. As no item has total value greater than 1,

$$v \left\lfloor \frac{1}{w} \right\rfloor = v(t + j) \leq 1.$$

Thus, $v \leq 1/(t+j)$ and

$$\frac{v}{w} < \frac{1/(t+j)}{1/(t+j+1)} = \frac{t+j+1}{t+j} \leq \frac{t+1}{t}. \quad \square$$

Theorem 2. $r_{td} \geq (m+1)/(m+2)$.

Proof. Let $\lfloor 1/w_1 \rfloor \geq m+j$ and let $\lfloor 1/W_1 \rfloor \geq m+j'$, where $j, j' \geq 0$ are integers. If $j \geq 1$, then $d_1 < (m+2)/(m+1)$ from Lemma 1. Hence

$$Z \leq 1 \times D_1 \leq 1 \times d_1 < \frac{m+2}{m+1}$$

and

$$r_{td} \geq \frac{1}{Z} > \frac{m+1}{m+2}.$$

Similarly, if $j' \geq 1$, then $D_1 < (m+2)/(m+1)$ from Lemma 1. Hence

$$Z \leq 1 \times D_1 < \frac{m+2}{m+1}$$

and

$$r_{td} \geq \frac{1}{Z} > \frac{m+1}{m+2}.$$

We therefore consider $j = j' = 0$. Note that if $0 < W_1 \leq 1/(m+1)$, then $1/W_1 \geq m+1$ and hence $j' \geq 1$. Consequently, we consider below the following two cases for W_1 : Case 1: $1/(m+1) < W_1 \leq (m+1)/[m(m+2)]$, and Case 2: $(m+1)/[m(m+2)] < W_1 \leq 1/m$.

Case 1: $1/(m+1) < W_1 \leq (m+1)/[m(m+2)]$. If $\lfloor 1/W_i \rfloor = m$ for all K items in the optimal, then $V_i \leq 1/m$ for all items $i, i = 1, \dots, K$. Also, at most m units comprise the optimal since for all items $i, i = 1, \dots, K$, $W_i \geq (1+\varepsilon)/(m+1)$ and consequently $\lfloor 1/[(1+\varepsilon)/(m+1)] \rfloor \leq m$. Thus, $Z \leq m(1/m) = 1$ and $r_{td} \geq 1/Z = 1$. We therefore need to show that $r_{td} \geq (m+1)/(m+2)$ if $\lfloor 1/W_i \rfloor \geq m+1$ for *some* item l in the optimal.

Let $i = 1, \dots, l-1$ denote items for which $\lfloor 1/W_i \rfloor = m$. Without loss of generality, let item l have the highest density D_l among items for which $\lfloor 1/W_i \rfloor \geq m+1, l \leq K$. Let $\lfloor 1/W_l \rfloor = m+s$, where $s \geq 1$ is integer. Let n_i denote the number of units of item $i, i = 1, \dots, l-1$ in the optimal. Let $W'_{l-1} = \sum_{i=1}^{l-1} n_i W_i$. We examine two subcases: (i) $D_l \geq d_2$ and (ii) $D_l < d_2$.

(i) $D_l \geq d_2$. As $D_l \geq d_2$, the weight of item l is no smaller than $1 - mw_1$, the remaining capacity after m units of the densest item are selected by the density-ordered greedy heuristic, i.e., $W_l \geq 1 - mw_1$. Also, as item l appears in the optimal, $W_l \leq 1 - W'_{l-1}$. Thus,

$$1 - mw_1 \leq W_l \leq 1 - W'_{l-1},$$

which implies

$$mw_1 \geq W'_{l-1}.$$

An upper bound on the optimal solution value is

$$Z \leq W'_{l-1}D_1 + (1 - W'_{l-1})D_l.$$

As $W'_{l-1} \leq mw_1$ and $d_1 \geq D_1 \geq D_l$,

$$Z \leq mw_1D_1 + (1 - mw_1)D_l \leq mw_1d_1 + (1 - mw_1)D_l.$$

Now $\lfloor 1/W_l \rfloor = m + s \geq m + 1$. As each item has total value no larger than 1, $(m + s)V_l \leq 1$. Hence

$$V_l \leq \frac{1}{m + s} \leq \frac{1}{m + 1}.$$

An upper bound on the density of item l is

$$D_l = \frac{V_l}{W_l} \leq \frac{1}{(m + 1)W_l} \leq \frac{1}{(m + 1)(1 - mw_1)}.$$

Thus,

$$Z \leq mw_1d_1 + (1 - mw_1) \frac{1}{(m + 1)(1 - mw_1)} \leq 1 + \frac{1}{m + 1} = \frac{m + 2}{m + 1},$$

where $mw_1d_1 \leq 1$ because no item has total value greater than 1. It follows that

$$r_{ld} \geq \frac{1}{Z} \geq \frac{m + 1}{m + 2}.$$

(ii) $D_l < d_2$. We separately examine (a) $D_l < d_2 \leq D_2$ and (b) $D_l \leq D_2 < d_2$ or $D_l < D_2 \leq d_2$.

(a) $D_l < d_2 \leq D_2$. We begin by showing that $m = 1$ is not feasible if $D_l < D_2$.

Assume $m = 1$. Then $W_1 > \frac{1}{2}$ (as per the assumption in Case 1), or equivalently $1 - W_1 < \frac{1}{2}$. Now $W_2 \leq 1 - W_1$. Thus, $W_2 < \frac{1}{2}$, or equivalently $1/W_2 > 2$. Hence $\lfloor 1/W_2 \rfloor \geq 2$. As item l is the densest optimal item for which $\lfloor 1/W_l \rfloor \geq m + 1 = 2$, it follows that $l = 2$. But $D_l < D_2$ implies $l > 2$, which is a contradiction.

We thus show that $r_{ld} \geq (m + 1)/(m + 2)$ for $m \geq 2$. For all items $i = 1, \dots, l - 1$, $\lfloor 1/W_i \rfloor = m$ implies $W_i > 1/(m + 1)$. Also, $V_i \lfloor 1/W_i \rfloor \leq 1$ implies $V_i \leq 1/\lfloor 1/W_i \rfloor = 1/m$. As $n_i \leq m$ (where n_i denotes the number of units of item i , $i = 1, \dots, l - 1$ in the optimal), $t = \sum_{i=1}^{l-1} n_i \leq m$. Thus $W'_{l-1} = \sum_{i=1}^{l-1} n_i W_i > t/(m + 1)$ and $V = \sum_{i=1}^{l-1} n_i V_i \leq t/m$. Thus an upper bound on the optimal solution value is

$$Z \leq V + (1 - W'_{l-1})D_l \leq \frac{t}{m} + \left(1 - \frac{t}{m + 1}\right)D_l = t \left(\frac{1}{m} - \frac{D_l}{m + 1}\right) + D_l.$$

As $\lfloor 1/W_l \rfloor = m + s \geq m + 1$, Lemma 1 implies $D_l < (m + s + 1)/(m + s) \leq (m + 2)/(m + 1)$. Now, $1/m > \lfloor 1/(m + 1) \rfloor (m + 2)/(m + 1) > \lfloor 1/(m + 1) \rfloor D_l$.

Therefore, the right-hand side of the above inequality for Z is maximized when t is maximum. As $t \leq m$,

$$Z \leq 1 + \left(1 - \frac{m}{m+1}\right) D_1 \leq 1 + \frac{D_1}{m+1}.$$

Also, a lower bound on the density-ordered greedy heuristic solution is

$$Z_d \geq mw_1 + \left\lfloor \frac{1 - mw_1}{w_2} \right\rfloor v_2 = mw_1 d_1 + \left\lfloor \frac{1 - mw_1}{w_2} \right\rfloor w_2 d_2,$$

where $\lfloor (1 - mw_1)/w_2 \rfloor$ is the number of units of the second densest item in the density-ordered greedy heuristic solution. Now

$$\left\lfloor \frac{1 - mw_1}{w_2} \right\rfloor w_2 > \frac{1 - mw_1}{2},$$

where $1 - mw_1$ is the remaining capacity after m units of the densest item are selected by the density-ordered greedy heuristic. Hence,

$$Z_d > mw_1 d_1 + \frac{1 - mw_1}{2} d_2.$$

As $w_1 > 1/(m+1)$ and $d_1 \geq d_2$,

$$Z_d > \frac{m}{m+1} d_1 + \frac{1}{2(m+1)} d_2.$$

Since $d_2 > D_1$,

$$Z_d > \frac{m}{m+1} d_1 + \frac{1}{2(m+1)} D_1.$$

Therefore,

$$r_{td} \geq \frac{Z_d}{Z} > \frac{[m/(m+1)]d_1 + D_1/[2(m+1)]}{1 + D_1/(m+1)}.$$

The right-hand side of the above expression is minimized when $D_1 \leq (m+2)/(m+1)$ attains its maximum value. Hence

$$r_{td} > \frac{(m/(m+1))d_1 + (m+2)/[2(m+1)^2]}{1 + (m+2)/(m+1)^2}.$$

If $d_1 \leq (m+2)/(m+1)$, then

$$Z \leq 1 \times D_1 \leq 1 \times d_1 \leq 1 \times \frac{m+2}{m+1}$$

and

$$r_{td} \geq \frac{1}{Z} \geq \frac{m+1}{m+2}.$$

We therefore consider $d_1 > (m + 2)/(m + 1)$. As the right-hand side of the above expression for r_{id} is minimum when d_1 has its smallest value, it follows that

$$r_{id} > \frac{m(m + 2)/(m + 1)^2 + (m + 2)/[2(m + 1)^2]}{1 + (m + 2)/(m + 1)^2}.$$

Simplifying the above expression yields

$$r_{id} > \frac{(m + 2)(2m + 1)}{2((m + 1)^2 + (m + 2))} > \frac{m + 1}{m + 2} \quad \text{for } m \geq 2.$$

(b) $D_1 \leq D_2 < d_2$ or $D_1 < D_2 \leq d_2$. We use the condition $D_2 \leq d_2$ (which subsumes the condition $D_2 < d_2$) to prove that $r_{id} \geq (m + 1)/(m + 2)$.

Let W'_1 be the total weight capacity occupied by the densest item in the optimal solution. An upper bound on the optimal solution value is

$$Z \leq W'_1 D_1 + (1 - W'_1) D_2.$$

Also, a lower bound on the density-ordered greedy heuristic solution value is

$$Z_d > mw_1 d_1 + \frac{1 - mw_1}{2} d_2,$$

where mw_1 is the total knapsack capacity occupied by the densest item and $(1 - mw_1)/2$ is the minimum capacity occupied by the second-densest item in the density-ordered greedy heuristic solution. Thus,

$$r_{id} > \frac{mw_1 d_1 + [(1 - mw_1)/2] d_2}{W'_1 D_1 + (1 - W'_1) D_2}.$$

As $D_1 \geq D_2$, the largest value for the denominator, and hence the smallest value for the expression on the right-hand side of the above inequality, is attained when W'_1 is maximum. As $W'_1 \leq mW_1$ and $W_1 \leq (m + 1)/[m(m + 2)]$ (as per the assumption in Case 1),

$$r_{id} > \frac{mw_1 d_1 + [(1 - mw_1)/2] d_2}{[(m + 1)/(m + 2)] D_1 + (1 - (m + 1)/(m + 2)) D_2}.$$

Since $d_1 \geq d_2$, the right-hand side of the above expression is minimized when w_1 is minimum. As m units of the densest item are selected by the density-ordered greedy heuristic, $w_1 > 1/(m + 1)$. Thus,

$$r_{id} > \frac{[m/(m + 1)] d_1 + \{[1 - m/(m + 1)]/2\} d_2}{[(m + 1)/(m + 2)] D_1 + (1 - (m + 1)/(m + 2)) D_2}$$

which on simplification yields

$$r_{id} > \frac{m + 2}{2(m + 1)} \frac{2md_1 + d_2}{(m + 1)D_1 + D_2}.$$

As $d_1 \geq D_1$ and $d_2 \geq D_2$ (as per the assumption in this subcase),

$$r_{id} > \frac{m+2}{2(m+1)} \frac{2mD_1 + D_2}{(m+1)D_1 + D_2} = \frac{m+2}{2(m+1)} \left(1 + \frac{(m-1)D_1}{(m+1)D_1 + D_2} \right).$$

Also, $D_1 \geq D_2$, which implies

$$r_{id} > \frac{m+2}{2(m+1)} \left(1 + \frac{(m-1)D_1}{(m+1)D_1 + D_1} \right) = \frac{2m+1}{2(m+1)} > \frac{m+1}{m+2}.$$

Case 2: $(m+1)/[m(m+2)] < W_1 \leq 1/m$. As $W_1 \leq 1/m$, $1/W_1 \geq m$ and hence $\lfloor 1/W_1 \rfloor \geq m$. Also, $V_1 \lfloor 1/W_1 \rfloor \leq 1$, which implies $V_1 \leq 1/\lfloor 1/W_1 \rfloor \leq 1/m$. Since $W_1 > (m+1)/[m(m+2)]$,

$$D_1 = \frac{V_1}{W_1} < \frac{1/m}{(m+1)/[m(m+2)]} = \frac{m+2}{m+1}.$$

Therefore,

$$Z \leq 1 \times D_1 < \frac{m+2}{m+1},$$

which implies

$$r_{id} \geq \frac{1}{Z} > \frac{m+1}{m+2}. \quad \square$$

To illustrate that the bound obtained in Theorem 2 is tight, consider the example in Fig. 1. Both the total-value and density-ordered greedy heuristics select m units of item 1 and one unit of item 2, so that $Z_t = Z_d = 1$. The optimal solution consists of m units of item 3 and one unit of item 4 and hence $Z = (m+2)/(m+1) - \epsilon$. The joint performance ratio for the two greedy heuristics is $r_{id} = (m+1)/[(m+2) - (m+1)\epsilon]$, which for arbitrarily small ϵ approaches $(m+1)/(m+2)$.

Item	Weight	Value	Density	Total value
1	$\frac{1+2\epsilon}{m+1}$	$\frac{1}{m}$	$\frac{m+1}{m(1+2\epsilon)}$	1
2	$\frac{1-2\epsilon m}{m+1}$	0	0	0
3	$\frac{1+\epsilon}{m+1}$	$\frac{1-\epsilon}{m}$	$\frac{(m+1)(1-\epsilon)}{m(1+\epsilon)}$	$1-\epsilon$
4	$\frac{1-\epsilon m}{m+1}$	$\frac{1}{m+1}$	$\frac{1}{1-\epsilon m}$	1

Fig. 1. Worst-case example for joint performance of density-ordered and total-value greedy heuristics.

Now consider the joint performance of all four greedy heuristics. Observe that for the example in Fig. 1, the value-ordered greedy heuristic selects m units of item 1 and one unit of item 2 so that $Z_v = 1$. The weight-ordered greedy heuristic selects $m + 1$ units of item 2 and hence $Z_w = 0$. It follows that using either or both of the weight-ordered and value-ordered greedy heuristics with the density-ordered and the total-value greedy heuristics does not improve the worst-case performance ratio beyond $(m + 1)/(m + 2)$.

If the total-value greedy heuristic is used with either or both of the value-ordered and weight-ordered greedy heuristics, the joint worst-case performance ratio does not improve beyond that for the total-value greedy heuristic. Similarly, if the density-ordered greedy heuristic is used with either or both of the value-ordered and weight-ordered greedy heuristics, the joint worst-case performance ratio does not improve beyond that for the density-ordered greedy heuristic. Finally, if the weight-ordered greedy heuristic is used with the value-ordered greedy heuristic, the joint worst-case performance ratio continues to be arbitrarily bad. The examples in Figs. 2–4 suffice to prove these results.

Consider the example in Fig. 2 for the joint performance of the total-value greedy heuristic with the weight-ordered and/or value-ordered greedy heuristic. The value-ordered greedy heuristic selects m units of item $(n - 1)$ and has solution value $Z_v = 1$. The weight-ordered and total-value greedy heuristics select p units of item n (where p is an arbitrarily large integer) and have solution value $Z_w = Z_t = 1$. The optimal

Item	Weight	Value	Density	Total value
1	$\frac{1 + \epsilon}{m + 1}$	$\frac{1}{m}$	$\frac{m + 1}{m(1 + \epsilon)}$	1
2	$\frac{1 + \epsilon}{h(0) + 1} = \frac{1 + \epsilon}{m + 2}$	$\frac{1}{h(0)} = \frac{1}{m + 1}$	$\frac{m + 2}{(m + 1)(1 + \epsilon)}$	1
...
i	$\frac{1 + \epsilon}{h(i - 2) + 1}$	$\frac{1}{h(i - 2)}$	$\frac{h(i - 2) + 1}{h(i - 2)(1 + \epsilon)}$	1
...
$n - 2$	$\frac{1 + \epsilon}{h(n - 4) + 1}$	$\frac{1}{h(n - 4)}$	$\frac{h(n - 4) + 1}{h(n - 4)(1 + \epsilon)}$	1
$n - 1$	$\frac{1}{m}$	$\frac{1}{m}$	1	1
n	$\frac{1}{p} \left(< \frac{1}{m} \right)$	$\frac{1}{p}$	1	1

Fig. 2. Worst-case example for joint performance of value-ordered, weight-ordered, and total-value greedy heuristics.

solution can be verified to consist of m units of item 1, and one unit each of items 2, ..., $n - 2$, and has solution value $Z = 1 + \sum_{i=0}^{n-4} h(i)$, where $h(0) = m + 1$, and $h(i) = h(i - 1) \times (h(i - 1) + 1)$. Hence the performance ratio for a combination of the three greedy heuristics, for a combination of the total-value and value-ordered greedy heuristics, and for a combination of the total-value and weight-ordered greedy heuristics, is $1/(1 + \sum_{i=0}^{n-4} h(i))$, which is the strict worst-case bound for the total-value greedy heuristic [7].

Now consider the example in Fig. 3 for the joint performance of the density-ordered greedy heuristic with the weight-ordered and/or value-ordered greedy heuristic. Both the density-ordered and value-ordered greedy heuristics select m units of item 1, and hence $Z_d = Z_v = m/(m + 1)$. The weight-ordered greedy heuristic selects $m + 1$ units of item 2 and hence $Z_w = 0$. The optimal solution consists of $m + 1$ units of item 3 and has value $Z = 1 - \epsilon$. Hence the joint performance ratio for a combination of the three greedy heuristics, for a combination of the density-ordered and value-ordered greedy heuristic, and for a combination of the density-ordered and weight-ordered greedy heuristic, is $m/[(m + 1)(1 - \epsilon)]$, which for arbitrarily small ϵ approaches $m/(m + 1)$, the worst-case bound for the density-ordered greedy heuristic.

Finally, consider the example in Fig. 4 for the joint performance of the value-ordered and weight-ordered greedy heuristics. The value-ordered greedy heuristic

Item	Weight	Value	Density
1	$\frac{1 + \epsilon}{m + 1}$	$\frac{1}{m + 1}$	$\frac{1}{1 + \epsilon}$
2	$\frac{1 - \epsilon m}{m + 1}$	0	0
3	$\frac{1}{m + 1}$	$\frac{1 - \epsilon}{m + 1}$	$1 - \epsilon$

Fig. 3. Worst-case example for joint performance of value-ordered, weight-ordered and density-ordered greedy heuristics.

Item	Weight	Value
1	$\frac{1}{m}$	$\frac{1 + \epsilon}{pm}$
2	$\frac{1}{pm} - \epsilon$	0
3	$\frac{1}{pm}$	$\frac{1}{pm}$

Fig. 4. Worst-case example for joint performance of value-ordered and weight-ordered greedy heuristics.

selects m units of item 1 and has solution value $Z_v = (1 + \varepsilon)/p$. The weight-ordered greedy heuristic selects pm units of item 2, and has solution value $Z_w = 0$. The optimal solution consists of pm units of item 3 and has solution value $Z = 1$. The joint performance ratio of the value-ordered and weight-ordered greedy heuristics is $(1 + \varepsilon)/p$, which approaches zero as p tends to infinity. Thus, the joint worst-case performance of the two heuristics is arbitrarily bad, and hence no better than the worst-case performance of either heuristic alone.

Thus, only the density-ordered and the total-value greedy heuristics complement each other, their joint performance ratio exceeding the independent performance ratio for either of the two heuristics. For combinations of greedy heuristics that do not involve both the density-ordered and total-value greedy, the joint worst-case performance is no better than the worst-case performance of the single best heuristic in the combination.

3. Conclusion

There is little work analyzing the joint performance of algorithms, partly because such analysis can be complicated. In this paper, we have analyzed the performance of a composite algorithm comprising the greedy heuristics that have been suggested for the knapsack problem. The principal benefit of using multiple greedy heuristics to solve the knapsack problem is that when one greedy heuristic performs poorly, another may do well. By itself, the density-ordered greedy heuristic appears to perform most poorly when the densest item leaves a significant capacity of the knapsack unused, yet leaves insufficient weight capacity to fit any other item. For instance, the density-ordered greedy heuristic leaves nearly $1/(m + 1)$ of the knapsack capacity unused in the worst case (Fig. 3). The total-value greedy heuristic appears to compensate for this limitation by preferring items with lower density that fill more of the knapsack and hence contribute more to the total solution value. On the other hand, the total-value greedy heuristic has the limitation that it cannot discriminate between items that have the same total-value contribution but differing densities. The density-ordered greedy heuristic appears to compensate for this limitation of the total-value greedy heuristic by preferring items that fill the knapsack at a higher rate. The density-ordered and total-value greedy heuristics appear to complement each other in this sense. However, the weight-ordered and value-ordered greedy heuristics use so little information regarding the problem that they seem to neither complement each other, nor complement the density-ordered and total-value greedy heuristics.

Finally, a combination of the density-ordered and the total-value greedy heuristics can be used to provide better lower bounds for the optimal solution in approximation schemes. The most efficient approximation scheme is due to Lawler [9], which uses the density-ordered greedy heuristic to improve the lower bound on the optimal solution. A combination of the density-ordered and the total-value greedy heuristics

provides a better lower bound on the optimal solution, improving the constant, if not the order, of the time complexity of the approximation scheme.

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