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# THE TERM STRUCTURE OF SIMPLE FORWARD RATES WITH JUMP RISK

Paul Glasserman

*Graduate School of Business, Columbia University, New York*

S. G. Kou

*IEOR Department, Columbia University, New York*

This paper characterizes the arbitrage-free dynamics of interest rates, in the presence of both jumps and diffusion, when the term structure is modeled through simple forward rates (i.e., through discretely compounded forward rates evolving continuously in time) or forward swap rates. Whereas instantaneous continuously compounded rates form the basis of most traditional interest rate models, simply compounded rates and their parameters are more directly observable in practice and are the basis of recent research on "market models." We consider very general types of jump processes, modeled through marked point processes, allowing randomness in jump sizes and dependence between jump sizes, jump times, and interest rates. We make explicit how jump and diffusion risk premia enter into the dynamics of simple forward rates. We also formulate reasonably tractable subclasses of models and provide pricing formulas for some derivative securities, including interest rate caps and options on swaps. Through these formulas, we illustrate the effect of jumps on implied volatilities in interest rate derivatives.

Key Words: interest rate models, interest rate derivative securities, jump-diffusion models

## 1. INTRODUCTION

This paper characterizes the arbitrage-free dynamics of interest rates, in the presence of both jumps and diffusion, when the term structure is modeled through simple forward rates—that is, through discretely compounded forward rates evolving continuously in time—or through forward swap rates. We consider very general types of jump processes (allowing randomness in jump sizes and dependence between jump sizes, jump times, and interest rates) and identify how jump and diffusion risk premia enter into the dynamics of simple forward rates. We also formulate a reasonably tractable subclass of models and provide pricing formulas for some term structure derivatives.

Our investigation builds on several strands of research, in particular on the dynamics of instantaneous continuously compounded rates (as in Heath, Jarrow, and Morton 1992), option pricing with jumps (as in Merton 1976), LIBOR and swap rate market models (including Brace, Gatarek, and Musiela 1997; Jamshidian 1997; Miltersen,

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Address correspondence to Paul Glasserman, 403 Uris Hall, Graduate School of Business, Columbia University, New York, NY 10027; e-mail: pg20@columbia.edu or S.G. Kou, IEOR Dept., Columbia University, New York, NY 10027; e-mail: sk75@columbia.edu.

Sandmann, and Sondermann 1997; Musiela and Rutkowski 1997a), and especially the marked point process framework of Björk, Kabanov, and Runggaldier (1997). The motivation for models based on simple forwards (in contrast to the instantaneous rates traditionally treated in continuous-time models) lies in building a model based on observable quantities. Most market rates are indeed based on simple compounding, so instantaneous continuously compounded rates often represent an idealized approximation to market data. This point is relevant whether one tries to infer model parameters from time-series data or from prices of derivative securities because most derivatives contracts are tied to simple rates.

Motivation for including jumps comes from both time-series properties and derivative prices. Specific sources of jumps in interest rates, including economic news and moves by central banks, are put forward in Babbs and Webber (1997), Das (1999b), El-Jahel, Lindberg, and Perraudin (1997), and Johannes (2003). These studies find compelling empirical evidence for jumps. Das (1999b) and Johannes argue that the kurtosis in short-term interest rates is incompatible with a pure-diffusion model. Jumps in interest rates can also be used to try to reproduce the patterns in implied volatilities derived from market prices of interest rate derivatives. The pricing of interest rate derivatives in the presence of jumps is considered in Björk et al. (1997), Burnetas and Ritchken (1997), Das (1999a), Das and Foresi (1996), Duffie and Kan (1996), Duffie, Pan, and Singleton (2000), Jarrow and Madan (1995, 1999) and Shirakawa (1991). The possibility of default (as modeled in Duffie and Singleton 1999 and Jarrow and Turnbull 1995) provides further motivation for including jumps, though we do not consider credit risk here.

Implied volatilities extracted from interest rate caps are putative parameters of simple forward rates, which again motivates adopting simple forwards as the building blocks of a model. (Similarly, implied volatilities extracted from options on interest rate swaps are putative parameters of forward swap rates.) In special cases of the general framework we develop, interest rate caps or swaptions can be priced explicitly, making it possible to investigate what types of patterns in implied volatility can be produced through jumps. The general framework is necessary for the formulation of tractable special cases: it turns out that for caps to be priced using a Poisson-based formula, the actual process of jumps must be substantially more complex than a Poisson process. The additional complexity needed follows from general considerations on precluding arbitrage; in particular, the Poisson property is not in general preserved by the necessary changes of measure.

The rest of this paper is organized as follows. Section 2 develops further motivation and background on modeling simple forward rates and on representing jump processes. Section 3 presents our main results: a general formulation of the arbitrage-free dynamics of simple forwards subject to jumps, and reduction to a tractable subclass. Section 4 presents some pricing formulas and numerical results on implied volatilities. Section 5 undertakes a similar analysis based on swap rates rather than forward rates: we present the arbitrage-free dynamics of the term structure of swap rates with both jumps and diffusion and then provide pricing formulas for options on swaps. All proofs are collected in the Appendixes.

## 2. MOTIVATION AND BACKGROUND

### 2.1. Simple Forwards

As in Brace et al. (1997), Jamshidian (1997), and Miltersen et al. (1997), we consider models of the term structure based on simple forward rates with a fixed accrual period $\delta$, expressed as a fraction of a year (e.g., to model 3-month rates we would take $\delta = 1/4$).

With $\delta$ fixed, we denote by $L(t, T)$ the forward rate for the interval from $T$ to $T + \delta$ as of time $t \leq T$. Thus, a party entering into a contract at time $t$ to borrow \$1 over the interval $[T, T + \delta]$ will receive \$1 at time $T$ and will return to the lender $\$(1 + \delta L(t, T))$ at time $T + \delta$. Denoting by $B(t, \tau)$ the time-$t$ price of a zero-coupon bond maturing at $\tau$, the forward rate satisfies

$$L(t, T) = \frac{1}{\delta}\left(\frac{B(t, T)}{B(t, T + \delta)} - 1\right). \tag{2.1}$$

Conversely, for any $k = 1, 2, \ldots,$

$$B(t, t + k\delta) = \prod_{i=0}^{k-1} \frac{1}{1 + \delta L(t, t + i\delta)}. \tag{2.2}$$

Simple compounding of this type is characteristic of 3-month or 6-month LIBOR. We will, however, treat the forward rates and associated bonds as default-free, though in practice LIBOR reflect some credit risk.[1]

These simple forward rates should be contrasted with the instantaneous, continuously compounded short rate of classical models and also with the instantaneous forward rates modeled in the framework of Heath et al. (1992). The instantaneous forwards $f(t, T)$ of the Heath-Jarrow-Morton framework satisfy

$$L(t, T) = \frac{1}{\delta}\left(\exp\left\{\int_T^{T+\delta} f(t, s)\, ds\right\} - 1\right), \tag{2.3}$$

but this relation cannot in general be inverted, so the distinction is not simply one of choice of variables. Arbitrage-free models based on simple forwards have been advanced by Brace et al. (1997), Jamshidian (1997), and Miltersen et al. (1997), and this work has given rise to a rapidly expanding related literature. Among other attractive features, these models are based on quantities that are more directly observable in the market than are the instantaneous rates of much of the earlier literature.

Working with simple forward rates often facilitates calibration to derivatives prices, in particular caps and floors. The information about the underlying forward rates in the market prices of caps and floors is commonly summarized through an implied volatility derived from the (so-called) Black (1976) formula (see our equation (4.1) in Section 4). These implied volatilities are frequently used as inputs to models for pricing other derivatives. In more detail, a caplet for the period $[T, T + \delta]$ struck at $K$ pays $\delta(L(T, T) - K)^+$ at $T + \delta$. The Black formula may be viewed as evaluating the discounted expected payoff

$$B(0, T + \delta)E[\delta(L(T, T) - K)^+], \tag{2.4}$$

under the assumption that $L(T, T)$ is lognormally distributed with mean $L(0, T)$ and $\log L(T, T)$ having variance $\sigma_T^2 T$. The implied volatility is the value of $\sigma_T$ that equates (2.4) to the market price.

A simple way to introduce dynamics that yield a lognormal distribution for $L(t, T)$ specifies

$$\frac{dL(t, T)}{L(t, T)} = \sigma_T\, dW_t, \tag{2.5}$$

[1] Miltersen et al. note that, through results of Duffie and Singleton (1999), their model can be used to represent defaultable interest rates under appropriate assumptions and with some redefinition of terms. A similar reinterpretation should be possible in our setting as well. See also Duffie et al. (2000).

with $W_t$ a standard Brownian motion. The absence of a drift in this specification further implies that the conditional expectation of $L(T, T)$ at time $t$ is $L(t, T)$, as is implicit in the Black formula. It turns out, however, that a model specifying (2.5) for all $T$ fails to be arbitrage free. More precisely, there is no probability measure under which forward rates for all maturities simultaneously evolve according to (2.5) in an arbitrage-free model.

Despite this apparent inconsistency, Brace et al. (1997), Jamshidian (1997), Miltersen et al. (1997), and Musiela, and Rutkowski (1997a) were nevertheless able to construct arbitrage-free models of the term structure in which cap prices indeed conform to the Black formula. The models are, in effect, kept arbitrage free through inclusion of an appropriate stochastic drift in (2.5) for each maturity $T$. The forward rates are thus not simultaneously lognormal, but each becomes lognormal under a maturity-specific change of measure. Each such change of measure is associated with a change of numeraire which further serves to justify discounting by a zero-coupon bond. These ideas are discussed in greater detail in Section 3 and Appendix B.

On one hand, these models provide a theoretical basis for the market convention of quoting or interpreting cap prices through the Black formula; on the other hand, they also make evident an incompatibility between market prices and the models intended to explain them. For in these models the same implied volatility should apply to all caps and floors of a given maturity, regardless of strike price, whereas volatilities implied by market prices vary systematically with strike. This *volatility skew* is particularly pronounced in the Japanese market, but is also present in the US dollar market.

There are various means by which one might try to incorporate an implied volatility skew. These include adding a stochastic volatility, changing from a lognormal to constant elasticity of variance (CEV) form of volatility (as in Andersen and Andreasen 2000), or allowing for jumps. Empirical evidence in equity markets (Bakshi, Cao, and Chen 1997; Bates 2000; Das and Sundaram 1999) suggests that both jumps and stochastic volatility play an important role in the implied volatility skew observed there. It is therefore natural to investigate how jumps can be incorporated in a model of simple forwards.

A naive extension of the naive "Black model" in (2.5) specifies

$$dL(t, T) = -\lambda m L(t, T)\,dt + \sigma_T L(t, T)\,dW_t + L(t-, T)\,d\left(\sum_{i=1}^{N_t}(Y_i - 1)\right), \tag{2.6}$$

where $N_t$ is a Poisson process with arrival rate $\lambda$ and the $Y_i$ are i.i.d. lognormal random variables with mean $1 + m$. (By writing $L(t-, T)$ we specify the value of $L(\cdot, T)$ just before a possible jump at $t$.) This is a jump-diffusion of the type considered by Merton (1976) as a model of a stock price, with the drift modified to make $L(t, T)$ a martingale. The marginal distributions of $L(t, T)$ under (2.6) are Poisson mixtures of lognormal distributions. "Pricing" a caplet according to (2.4) therefore results in a "Merton-Black formula,"

$$\sum_{k=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^k}{k!}\mathrm{BC}_k, \tag{2.7}$$

where each $\mathrm{BC}_k$ is an evaluation of the Black formula but with arguments depending on $k$; this will be made explicit in Corollary 4.1. This pricing formula is nearly as tractable as the Black formula. Moreover, if caplets are priced according to (2.7), their Black-implied volatilities will vary with strike. Indeed, by varying the parameters of (2.6) it is possible to reproduce a variety of patterns in implied volatilities as functions of strike; see Section 4.1.

This simple example serves to motivate the questions we investigate: Is (2.6) consistent with an arbitrage-free model of the term structure? Can the naive pricing formula (2.7) be reconciled with a genuine pricing model? More generally, when is a jump-diffusion model of the term structure of simple forward rates arbitrage-free? We provide answers to these questions (and their analogs for forward swap rates) in subsequent sections.

## 2.2. Modeling Jumps

Addressing these questions requires an investigation of the dynamics of the term structure with respect to different choices of numeraire asset and under the associated probability measures. This in turn requires consideration of jump processes more general than the compound Poisson process appearing in (2.6). The marked point process framework developed by Björk et al. (1997) provides a convenient framework.

A marked point process (MPP) is characterized by a sequence $\{(\tau_n, X_n), n = 1, 2, \ldots\}$. The $\tau_n$ take values in $(0, \infty)$ and satisfy $\tau_1 < \tau_2 < \cdots < \tau_n < \tau_{n+1} < \cdots$, $\sup_n \tau_n = \infty$; interpret these as the times of *potential* jumps. The *marks* $X_n$ may in general take values in an abstract space; we will use them to determine the sizes of the jumps at the points $\tau_n$, though they are not themselves the jump sizes. Forward rates of different maturities may respond to the marks with jumps of different magnitudes. For our purposes, it will suffice to consider marks taking values in $[0, \infty)$.

To construct a jump process, first let $N_t$ be the number of points in $[0, t]$: $N_t = \sup\{n \geq 0 : \tau_n \leq t\}$. Let $h$ be a real-valued function of the marks (and possibly also of the points) and consider the jump process $J(t) = \sum_{n=1}^{N_t} h(X_n, \tau_n)$. The function $h$ transforms the abstract mark $X_n$ into a jump magnitude. In (2.6), it takes the form $h(x, \tau) \equiv h(x) = x - 1$.

We construct our models on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathsf{P})$ on which are defined a multidimensional Brownian motion $W$ and $r$ marked point process $\{(\tau_n^{(i)}, X_n^{(i)}), n = 1, 2, \ldots\}, i = 1, 2, \ldots, r$, not necessarily independent of each other or the Brownian motion. With each forward rate we associate jump-size functions $H_i, i = 1, \ldots, r$, and define

$$J(t) = \sum_{i=1}^{r} \sum_{n=1}^{N_t^{(i)}} H_i\big(X_n^{(i)}, \tau_n^{(i)}\big), \tag{2.8}$$

with $N_t^{(i)}$ the counting process associated with the $i$th marked point process. The dynamics of a forward rate $L(t, T)$ take the form

$$dL(t, T) = \alpha(t)L(t, T)\,dt + \gamma(t)L(t, T)\,dW(t) + L(t-, T)\,dJ(t), \tag{2.9}$$

for adapted processes $\alpha$ and $\gamma$ satisfying regularity conditions. The $r$ marked point processes in (2.8) can be dependent on each other but we require that the jump times $\tau_n^{(i)}, n \geq 1, 1 \leq i \leq r$, be distinct (this is needed for a generalization of the Girsanov theorem).

We assume that each marked point process $\{(\tau_n, X_n)\}$ has an *intensity* $\lambda(dx, t)$. Intuitively, $\lambda(dx, t)$ is the arrival rate of points with marks in $dx$. More precisely, the intensity has the property that, for all suitably integrable $h$,

$$\sum_{n=1}^{N_t} h(X_n, \tau_n) - \int_0^t \int_0^\infty h(x, s)\lambda(dx, s)\,dt_s \tag{2.10}$$

is a martingale in $t$, the inner integral being over the mark space. (The key assumption here is that the arrival rate is absolutely continuous in time; otherwise, in addition to the $dt$ term in (2.9) we would need a $\lambda(dx, dt)$ term.) For a marked point process in which the points follow a Poisson process and the marks are i.i.d. random variables (as would be the case in the Merton model (2.6)) the intensity takes the form

$$\lambda(dx, t) = \lambda \cdot f(x)\, dx,$$

where $\lambda$ is the Poisson arrival rate and $f$ is the common density of the marks.

A valuable feature of modeling jumps through marked point processes and their intensities arises in considering term structure dynamics under different probability measures. In a pure-diffusion setting, changing probability measures typically corresponds to adding a drift to a Brownian motion. In a model with jumps, changing probability measures can involve changing the jump intensity as well. We will see that even if we want the jumps in a forward rate to follow a Poisson process under one measure, we have no choice but to suppose that they follow a more general marked point process under other measures.

## 3. MODEL CONSTRUCTION

We now proceed to investigate conditions under which a term structure model of the general form (2.9) is consistent with the absence of arbitrage. The main task lies in identifying the appropriate form of the risk premium determining the drift in (2.9), once the other parameters have been specified.

A prerequisite to this investigation is a precise notion of what it means for a model to be arbitrage free in the presence of jumps, which further entails defining a class of admissible trading strategies. Our objective is the construction of models formulated purely in terms of simple forwards and their parameters; one could in principle specify a class of admissible trading strategies and develop the associated theory in this setting. Rather than make such a digression here, we choose a more efficient and only slightly less general route: We define a model of the term structure of simple forwards $L(t, T)$ to be arbitrage free if it can be embedded in an arbitrage-free model of instantaneous forwards $f(t, T)$ via (2.3). The necessary theory for instantaneous forwards has been developed by Björk et al. (1997), so we may invoke their results. We stress, however, that the models we construct are purely models of simple forwards and make no reference to hypothetical instantaneous forwards. Indeed, the instantaneous forwards appear nowhere in the rest of this section. Jamshidian (1999) has recently developed a model of simple forwards driven by very general discontinuous semimartingales; his framework does not use underlying instantaneous rates but rather works with simple rates throughout.

To simplify both the analysis and notation, we formulate our results in a discrete-tenor setting in which the maturity $T$ is restricted to a finite set of dates $0 = T_0 < T_1 < \cdots < T_M < T_{M+1}$. (In Appendix A we prove an intermediate result that does hold simultaneously for all maturities $T$ and from which we prove Theorem 3.1.) We will further assume that the intervals $T_{i+1} - T_i$ are equally spaced with a common spacing of $\delta$ (e.g., a quarter year or a half year). Let $L_n(t) = L(t, T_n)$ so that $L_n$ is the forward rate for the accrual period $[T_n, T_{n+1}]$. Similarly, let $B_n(t) = B(t, T_n)$ denote the price of a zero coupon bond maturing at $T_n$. Let $\eta(t) = \inf\{k \geq 0 : T_k \geq t\}$ so that $\eta(t)$ is the index of the next maturity as of time $t$.

The results of this section are proved in Appendixes A and B under regularity conditions. Ideally, all conditions would be made explicit in the statements of the results. As it