Variance Reduction Techniques for Estimating Value-at-Risk

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This paper describes, analyzes and evaluates an algorithm for estimating portfolio loss probabilities using Monte Carlo simulation. Obtaining accurate estimates of such loss probabilities is essential to calculating value-at-risk, which is a quantile of the loss distribution. The method employs a quadratic (''delta-gamma'') approximation to the change in portfolio value to guide the selection of effective variance reduction techniques; specifically importance sampling and stratified sampling. If the approximation is exact, then the importance sampling is shown to be asymptotically optimal. Numerical results indicate that an appropriate combination of importance sampling and stratified sampling can result in large variance reductions when estimating the probability of large portfolio losses.
(*Value-At-Risk*; *Monte Carlo*; *Simulation*; *Variance Reduction Technique*; *Importance Sampling*; *Stratified Sampling*; *Rare event*)

1. Introduction

An important concept for quantifying and managing portfolio risk is value-at-risk (VAR) (Jorion 1997, Wilson 1999). VAR is defined as a quantile of the loss in portfolio value during a holding period of specified duration. If the value of the portfolio at time t is $V(t)$, the holding period is Δt, and the value of the portfolio at time $t+\Delta t$ is $V(t+\Delta t)$, then the loss in portfolio value during the holding period is $L = V(t) - V(t+\Delta t)$. For a given probability p, the VAR, x_p, is defined to be the $(1-p)$th quantile of the loss distribution:

$$P\{L > x_p\} = p. \tag{1}$$

Typically, the interval Δt is one day or two weeks and p is close to zero, often $p \approx 0.01$. Monte Carlo simulation is frequently used to estimate the VAR. In such a simulation, changes in the portfolio's ''risk factors'' (e.g., interest rates, currency exchange rates, stock prices, etc.) during the holding period are generated and the portfolio is reevaluated using these new values for the risk factors. This is repeated many times so that the loss distribution may be estimated. We should note that this type of Monte Carlo VAR analysis is often augmented with ''stress'' tests using predetermined stress scenarios. This is done primarily because the underlying distributional assumptions (especially correlations) implicit in the analysis may tend to break down in extreme circumstances.

The computational cost required to obtain accurate Monte Carlo VAR estimates is often enormous. This is because of two factors. First, the portfolio may consist of a very large number of financial instruments. Furthermore, computing the value of an individual instrument may itself require substantial computational effort. Thus each portfolio evaluation may be costly. Second, a large number of runs (portfolio evaluations) are required to obtain accurate estimates of the loss distribution in the region of interest. We focus on this second issue: the development of variance reduction techniques designed to dramatically reduce the number of runs required to achieve accurate estimates of low probabilities. A general discussion

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on variance reduction techniques may be found in Hammersley and Handscomb (1964). The technique described in this paper builds on the methods of Glasserman et al. (1999a, c), which were developed to reduce the variance when pricing a single instrument. Those methods combine specific implementations of two general purpose variance reduction techniques: importance sampling and stratified sampling. In this paper we also combine these two techniques, but in a way that is tailored to the VAR setting so the method is quite different from that of Glasserman et al. (1999a, c). Preliminary numerical results for the technique described in this paper, and for related techniques, were reported in Glasserman et al. (1999b). We now focus on the most promising approach tried in Glasserman et al. (1999b), provide a rigorous analysis of this approach, and perform more extensive experiments on it.

Our approach is to approximate the portfolio loss by a quadratic function of the underlying risk factors and to use this approximation to design variance reduction techniques. Quadratic approximations are widely used without simulation; indeed the second-order Taylor series approximation is commonly called the ''delta-gamma approximation'' (Britten-Jones and Schaefer 1999, Jorion 1997, Rouvinez 1997, Wilson 1999). While our approach could be combined with other quadratic approximations, many of the first and second derivatives needed for the delta-gamma approximation are routinely computed for other purposes quite apart from the calculation of VAR. One premise of this paper is that these derivatives are thus readily available as inputs to be used in a VAR simulation and do not represent an additional computational burden.

When the change in risk factors has a multivariate normal distribution, as is commonly assumed (and as we will assume), then the distribution of the delta-gamma approximation can be computed numerically (Imhof 1961, Rouvinez 1997). While this approximation is not always accurate enough to provide precise VAR estimates, we describe how it may be used to guide selection of an importance sampling (IS) change of measure for sampling the changes in risk factors. IS is a particularly appropriate technique for ''rare event'' simulations, which corresponds to the VAR problem with a small value of p. See Bucklew (1990), Chen et al. (1993), Glasserman et al. (1999a, c), and Heidelberger (1995) and the references therein for detailed discussions of IS. As the distribution of the quadratic approximation can be computed numerically, it can also be used as either a control variable or for stratified sampling. Numerical results in Glasserman et al. (1999b) showed that while the effectiveness of the control variable decreases as p decreases, the effectiveness of a combination of IS and stratified sampling increases as p decreases. This is the method we focus on in this paper. Independent of our work, Cárdenas et al. (1999) have studied using the delta-gamma approximation as a control variable and in a simple form of stratified sampling (with two strata) without IS.

The rest of the paper is organized as follows. In §2 we develop the quadratic approximation and describe the proposed IS change of measure based on this approximation. When this approximation is exact, we show that the IS is ''asymptotically optimal'' for estimating $P\{L>x\}$ for large x, meaning that the second moment of the estimator decreases at the fastest possible exponential rate as x increases. We also consider asymptotics as the number of risk factors becomes large and establish effectiveness of the method in this limit as well. Stratified sampling and its combination with IS are described in §3. In §4, we show how accurate estimates for $P\{L>x\}$ can translate to accurate estimates of the VAR x_p. First, we show that when the IS is selected so as to optimize estimation of $P\{L>x\}$, it is simultaneously asymptotically optimal for estimating $P\{L>y\}$ for a wide range of ys about x. In addition, we establish a central limit theorem (CLT) for the quantile estimate of the VAR x_p under IS and stratification. The form of the asymptotic variance in this CLT is typical of quantile estimates in other settings and implies that any variance reduction obtained for estimating $P\{L>x\}$ in a neighborhood of x_p carries over to a VAR estimate. The complete algorithm is stated in §5 and numerical results are given for a variety of sample portfolios. In most cases, the variance is reduced by at least one order of magnitude and often more than two orders of magnitude improvement are obtained. For one rather extreme case there is no improvement and further investigation reveals