On Integrability and Changing Tastes*

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Abstract

The classic integrability problem asks (i) what conditions guarantee that demand functions can be rationalized by a well-behaved utility function and (ii) if such a utility exists, how can it be recovered. Hurwicz and Uzawa (1971) provided answers to both questions. However for the popular case of changing tastes, as represented by a sequence of non-nested utilities, the Hurwicz and Uzawa conditions fail to hold in general. Following Strotz (1956), an individual can determine her dynamic demands via a naïve or sophisticated solution technique. For given dynamic demands, we provide necessary and sufficient conditions such that the demands are rationalized by a set of utilities using the sophisticated solution process. Moreover we provide a means for recovering the generating sequence of utilities, although this sequence of utilities is not unique. We also give sufficient conditions for demands to be rationalized by a sequence of utilities using the naive solution process and for recovering the complete set of generating utilities for two special cases.

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1 Introduction

The classic integrability problem of finding conditions guaranteeing that a given set of individual demand functions could have been generated by a budget constrained, utility maximizing consumer was formulated in its modern form in the mathematical notes section of Samuelson (1950). Hurwicz and Uzawa (1971) proved that multicommodity demand functions can be rationalized by a strictly quasiconcave utility if the corresponding Slutsky matrix is symmetric and negative semidefinite and also provided a process for recovering the utility function.[1] These results were further refined in Hurwicz and Richter (1979). One important area of economic analysis where the Slutsky symmetry and negative semidefiniteness conditions are known to possibly fail is in dynamic choice problems. Following the classic papers of Strotz (1956), Pollak (1968) and Phelps and Pollak (1968), it is now popular to allow for the possibility that decision makers might not have a single well-behaved intertemporal utility, but rather may exhibit changing tastes which can reflect a bias for the present and self control issues.[2] Whereas the integrability problem is largely solved for the classic static setting, the question of finding conditions for rationalizing demands corresponding to changing tastes seems not to have been investigated. This paper seeks to contribute to filling this void.

The changing tastes optimization problem can most simply be framed in a three period certainty setting with a single consumption good c_t $(t = 1, 2, 3)$ in each period t. Assume preferences in period one are defined over (c_1, c_2, c_3) triples and represented by $U^{(1)}$. Preferences in period two defined over (c_2, c_3) pairs are represented by $U^{(2)}$. For any fixed $c_1 = \bar{c}_1$, $U^{(1)}(\bar{c}_1, c_2, c_3)$ and $U^{(2)}(c_2, c_3 \,|\bar{c}_1)$ differ by more than a strictly increasing transformation. To determine an optimal plan, an individual can follow naive choice by using $U^{(1)}$ to make the period one consumption decision and then in period two given remaining resources, use $U^{(2)}$ to make the allocation between c_2 and c_3. Since in general the period one demands for c_2 and c_3 will be revised in period two, the consumer is said to exhibit time inconsistency. Alternatively, she could follow sophisticated choice and solve the problem recursively using $U^{(2)}$ to make the allocation between c_2 and c_3 conditional on c_1 and then use $U^{(1)}$ to select c_1.[3]

For the case of sophisticated choice, we derive necessary and sufficient con-

[1] Several other standard properties are assumed – see Theorem 1 in Section 2.

[2] See, for example, Laibson (1997), O'Donoghue and Rabin (1999, 2015) and Loewenstein, O'Donoghue and Rabin (2003) and the relevant references cited in these papers.

[3] An alternative game theoretic solution is proposed in Phelps and Pollak (1968), which will not be considered in this paper.

ditions for demands to be rationalizable by a $(U^{(1)}, U^{(2)})$-pair. To establish the existence of $U^{(2)}$, conditions are given for deriving the period two and period three conditional demands (corresponding to the first stage of the sophisticated optimization process) from the given unconditional demands. Based on these conditional demands, it is possible to directly apply the Hurwicz and Uzawa (1971) theorem to test whether there exists a unique $U^{(2)}$. For the existence of $U^{(1)}$ in place of the Slutsky symmetry and negative semidefiniteness properties assumed by Hurwicz and Uzawa (1971), we require the invertibility of the (unconditional) given demands and the existence of the conditional demands (as well as a few other standard properties).[4] While our result ensures the existence of a utility pair, it does not suggest that the period one utility is unique. $U^{(1)}$ satisfies a partial differential equation, and surprisingly the solution to this equation can yield a set $\{U^{(1)}\}$, where each utility in the set differs from other members of the set by more than an increasing transformation.[5] In this recovery process which is quite different from that of Hurwicz and Uzawa (1971), the multiplicity of $U^{(1)}$ functions arises because the sophisticated solution does not correspond to the optimal demand functions maximizing $U^{(1)}$. The latter is the only set of demands that can uniquely determine $U^{(1)}$ up to an increasing transformation. It follows from our integrability result that when the classic Hurwicz and Uzawa conditions (see Theorem 1 below) hold, it not only implies the existence of a non-changing tastes utility rationalizing sophisticated choice but if the resulting utility is twice continuously differentiable also implies the existence of a set of changing tastes utilities rationalizing the same demands. However without the partial differential equations we derive, it is unclear how one would recover the changing tastes set of utilities.

Selden and Wei (2015) have shown that a given set of dynamic demand functions can sometimes be rationalized both as sophisticated choice corresponding to a changing tastes pair $(U^{(1)}, U^{(2)})$ and as a standard utility maximization based on a non-changing tastes U.[6] In this case, the dynamic demands satisfy both our

[4] Shafer (1974, 1975) also considers a representation for the intransitive preferences. However his discussion is based on the classical static setting instead of an intertemporal setting. Therefore in contrast to our changing tastes assumption, no specific argument is given for why the consumer would have intransitive preferences. Moreover, no sufficient condition is given such that demands can be rationalized by his representation.

[5] Each ordinally equivalent collection of utilities is treated as one member in $\{U^{(1)}\}$.

[6] Since the observed consumption demands can come from both a non-changing tastes U and a changing tastes $(U^{(1)}, U^{(2)})$-pair, one can never distinguish which are the true generating utilities. It follows that if a given data set of (demand, price) observations satisfies the classic Afriat (1967) - Varian (1983) Generalized Axiom of Revealed Preference (GARP), implying that the data is consistent with the maximization of a concave utility, there is no guarantee that

necessary and sufficient conditions for the existence of $(\{U^{(1)}\}, U^{(2)})$-pairs and the Hurwicz and Uzawa (1971) conditions. However using the Hurwicz and Uzawa recovery process one can only recover the one member of $\{U^{(1)}\}$ corresponding to the non-changing tastes utility U and the period two continuation of this utility corresponding to the unique $U^{(2)}$. Our recovery process allows one to recover the full set $\{U^{(1)}\}$ including U. Motivating Example 1 in Section 3 illustrates a case in which a given set of demands is rationalized by both a $(U^{(1)}, U^{(2)})$ pair and a different non-changing tastes U where all three utilities are strictly increasing and quasiconcave on the full choice space.

Motivating Example 2 demonstrates that a given dynamic demand function can be rationalized as naive choice based on one $(U^{(1)}, U^{(2)})$-pair and as sophisticated choice based on another $(U^{(1)}, U^{(2)})$-pair where the $U^{(1)}$ functions differ but the $U^{(2)}$ functions are the same. We also extend our analysis to the case where there are two time periods with multiple goods in each period. In this case, one must include in the necessary and sufficient conditions a type of symmetry restriction on the derivatives of the inverse demand functions.

The reader may well note a historical irony in the role of invertibility conditions in our integrability theorems for sophisticated choice and the disavowal of invertibility in favor of Slutsky symmetry in the classic paper of Samuelson (1950, pp. 377-385). In our analysis since Slutsky symmetry is not satisfied in general for the changing tastes case, it cannot be assumed. Interestingly we show that for sophisticated choice, after assuming one can solve for conditional demands to guarantee the existence of $U^{(2)}$, assuming the invertibility of demands (EI) can guarantee the existence of $\{U^{(1)}\}$. Therefore, although the Property (EI) has nothing essential to do with the traditional integrability problem as argued by Samuelson (1950), it plays an important role in the integrability problem with changing tastes.

For the case of naive demands, to find a rationalizing $(U^{(1)}, U^{(2)})$-pair exactly the same conditional demand properties are required for the existence of $U^{(2)}$ as in the sophisticated case. However to find $U^{(1)}$, it is only possible to identify a quite weak set of sufficient conditions since one knows only the c_1 demand function optimizing $U^{(1)}$ and has no information about the optimal c_2 and c_3 demands

the individual doesn't exhibit changing tastes. Kubler (2004) notes that for dynamic demand tests of revealed preference where there is only a single extended observation, one can test time-separable and time-invariant utility but not more general forms such as Kreps-Porteus utility. In fact given that the demands corresponding to time-separable and time-invariant preferences could also have also been generated by a changing tastes $(U^{(1)}, U^{(2)})$-pair, the ability to reach strong conclusions based on revealed preference tests of these demand functions seems quite limited.

(given that the period two optimization is based on $U^{(2)}$). This is in contrast to the sophisticated case where one has information relating to the c_1-dependence of the c_2 and c_3 conditional demands when maximizing $U^{(1)}$ with respect to c_1. Indeed the naive problem of finding $U^{(1)}$ is formally analogous to the static integrability problem of incomplete demand systems often observed and analyzed in empirical multicommodity demand applications. For the static problem, Epstein (1982) provides some weak sufficient conditions which can be directly applied to the naive choice problem of finding $U^{(1)}$. However these conditions do not result in the recovery of all possible $U^{(1)}$ functions. In this paper we identify two special cases where the complete set of period one utilities $\{U^{(1)}\}$ together with $U^{(2)}$, which rationalize the given demands as naive choice, can be recovered. Interestingly, these two special cases are related to the class of effectively consistent preferences discussed in Selden and Wei (2015).

The rest of the paper is organized as follows. Section 2 gives the basic setup. Two motivating examples are provided in Section 3. In Section 4, we first derive necessary and sufficient conditions for the existence of $(\{U^{(1)}\}, U^{(2)})$ assuming the given demands correspond to sophisticated choice in the three period case, where there is only one commodity in each period. Then we generalize our results to the two period case with multiple commodities in each period. In Section 5, we give sufficient conditions for the existence of $(\{U^{(1)}\}, U^{(2)})$ assuming the given demands correspond to naive choice and discuss two special cases where the complete set of utilities $\{U^{(1)}\}$ rationalizing the given demands as naive choice can be identified. Concluding comments are provided in Section 6. Proofs of the results and supplemental materials are provided in the Appendix.

2 Preliminaries

Assume that there are two periods and M ($M \geq 3$) commodities. The first K commodities are consumed in period one and the remaining $M-K \geq 2$ commodities are consumed in period two.[7] Let $\mathbf{c} = (c_1, ..., c_M) \in \mathbb{R}_+^M$ denote the quantities of the M commodities and $\mathbf{p} = (p_1, ..., p_M)$ denote the corresponding prices. A consumer is endowed with income or initial wealth of y_1 which she seeks to allocate over time periods $t = 1, 2$. Preferences for periods one and two are represented respectively by

$$U^{(1)}(c_1, ..., c_M) : C_1 \times ... \times C_M \to \mathbb{R} \tag{1}$$

[7]Although in this paper we focus on the case of a single taste change, we demonstrate in Appendix E that our integrability result for sophisticated choice, Theorem 2, does indeed extend to a setting with two taste changes. The other results can be extended similarly.

and $\forall\,(c_1,...,c_K)\in C_1\times...\times C_K$,

$$U^{(2)}(c_{K+1},...,c_M|\ c_1,...,c_K):C_{K+1}\times...\times C_M\to\mathbb{R}, \tag{2}$$

where C_i denotes the set of possible consumption values for commodity i, which is (a subset of) $\mathbb{R}_+$. We frequently will refer to utilities as satisfying the following property.

Property 1 *Utility U is (i) a real-valued function defined on (a subset of) the positive orthant of a Euclidean space, (ii) continuous, (iii) strictly increasing in each of its arguments and (iv) strictly quasiconcave.*

In this paper, we focus on the case of changing tastes which will be said to occur if and only if there exists a $(\bar{c}_1,...,\bar{c}_K)\in C_1\times...\times C_K$ such that for every strictly increasing transformation T

$$U^{(2)}(c_{K+1},...,c_M|\ \bar{c}_1,...,\bar{c}_K)\neq T(U^{(1)}(\bar{c}_1,...,\bar{c}_K,c_{K+1},...,c_M)). \tag{3}$$

Thus, preferences will change if and only if $U^{(2)}$ is not nested in $U^{(1)}$.

The consumer faces the optimization problems

$$P_1:\ \max_{c_1,...,c_M} U^{(1)}(c_1,...,c_M) \qquad S.T.\ y_1\geq\sum_{i=1}^{M}p_ic_i \tag{4}$$

and

$$P_2:\ \max_{c_{K+1},...,c_M} U^{(2)}(c_{K+1},...,c_M|\ c_1,...,c_K) \qquad S.T.\ \ y_2=y_1-\sum_{i=1}^{K}p_ic_i\geq\sum_{j=K+1}^{M}p_jc_j. \tag{5}$$

Let $\mathbf{c}^\circ=(c_1^\circ,..,c_M^\circ)$ denote the optimal two period consumption plan for P_1. Applying terminology from Machina (1989) and McClennen (1990), the $\mathbf{c}^\circ$ plan is said to be resolute if and only if the consumer does not modify her $(c_{K+1}^\circ,...,c_M^\circ)$ plan even if her tastes change.

Following Strotz (1956) and Pollak (1968), it is standard to consider the following solution techniques for problems P_1 and P_2.

Definition 1 *P_1 and P_2 are said to be solved by* ***naive choice*** *if first P_1 is solved for the optimal $(c_1^*,...,c_K^*,c_{K+1},...,c_M)=(c_1^\circ,..,c_K^\circ,c_{K+1}^\circ,...,c_M^\circ)$ and then the optimal $(c_{K+1}^*,...,c_M^*)$ is determined from solving P_2 conditional on $(c_1^*,...,c_K^*)$.*

Definition 2 *P_1 and P_2 are said to be solved by* ***sophisticated choice*** *if first P_2 is solved for the conditionally optimal $(c_{K+1}^{**}(c_1,...,c_K)\,,\ ...,\ c_M^{**}(c_1,...,c_K))$ and then the optimal $(c_1^{**},...,c_K^{**})$ is determined from solving P_1 conditional on $(c_{K+1}^{**}(c_1,...,c_K)\,,\ ...,\ c_M^{**}(c_1,...,c_K))$.*

The vectors $\mathbf{c}^* = (c_1^*, .., c_M^*)$ and $\mathbf{c}^{**} = (c_1^{**}, .., c_M^{**})$ denote respectively the solutions resulting from the naive and sophisticated optimizations. A consumption plan will be said to be consistent if and only if $\left(c_{K+1}^*, ..., c_M^*\right) = \left(c_{K+1}^\circ, ..., c_M^\circ\right)$ for any $(p_1, ..., p_M, y_1)$. Although the consistency of the plan is sufficient to ensure that $\mathbf{c}^* = \mathbf{c}^{**}$, it is not necessary. As introduced in Selden and Wei (2015), one can also have the following case of effectively consistent demands and preferences.

Definition 3 *Given* $\left(U^{(1)}, U^{(2)}\right)$*, if there exists a unique naive and sophisticated pair* $(\mathbf{c}^*, \mathbf{c}^{**})$ *as characterized in Definitions 1 and 2 which for every* $(\mathbf{p}, y_1)$ *satisfies* $\mathbf{c}^* = \mathbf{c}^{**}$ *and is rationalizable by a non-changing tastes* U *satisfying Property 1, i.e.,*

$$\mathbf{c}^* = \mathbf{c}^{**} = \underset{c_1,...,c_M}{\arg\max}\ U(c_1, ..., c_M) \qquad S.T.\ y_1 \geq \sum_{i=1}^{M} p_i c_i, \tag{6}$$

then this common plan and associated preferences are said to be ***effectively consistent****. Otherwise, they are effectively inconsistent.*[8]

In Subsection 4.1.3, we show how to recover both the changing tastes utilities $(U^{(1)}, U^{(2)})$ and a non-changing tastes U from a given set of effectively consistent demands.

Let $c_i(\mathbf{p}, y_1) = c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ denote a given set of dynamic demand functions and $\mathbf{c}(\mathbf{p}, y_1)$ denote the corresponding demand vector. If $c_i(\mathbf{p}, y_1)$ $(i = 1, ..., M)$ are generated based on naive or sophisticated choice using a $(U^{(1)}, U^{(2)})$-pair, they will be referred to as the **unconditional** demands resulting from solving the period one and period two decision problems P_1 and P_2. The demand functions

$$c_i = c_i\left(p_{K+1}, ..., p_M, y_2 \middle|\ c_1, ..., c_K\right) \quad (i = K+1, ..., M) \tag{7}$$

resulting from solving just the period two optimization P_2 will be referred to as the **conditional** demands. These functions can depend on $p_1, ..., p_K$ but only through $(c_1, ..., c_K, y_2)$. The dependence on period one goods arises from $U^{(2)}(c_{K+1}, ..., c_M|\ c_1, ..., c_K)$ where $(c_1, ..., c_K)$ can enter as preference parameters.

Throughout this paper, we will refer to the following properties for all $(\mathbf{p}, y_1) \in \mathbb{R}_{++}^M \times \mathbb{R}_+$ (unless otherwise stated) of a given set of unconditional demand functions:[9]

[8] For effective consistency, we always assume that there is a unique sophisticated choice solution. It should be noted that this is not true in general, since as pointed out by Peleg and Yaari (1973), the sophisticated choice process need not always generate an optimal plan. This problem arises when substitution of the P_2 solution into the P_1 optimization results in $U^{(1)}$ not being concave in c_1.

[9] Although these properties are defined for the complete set of demand functions $c_i(\mathbf{p}, y_1)$ $(i = 1, ..., M)$, one can also apply them to the partial demand system by specifying the demand

(P) $c_i(\mathbf{p}, y_1)$ $(i = 1, ..., M)$ are real-valued functions defined on (a subset of) the positive orthant of a Euclidean space;

(TD) $c_i(\mathbf{p}, y_1)$ $(i = 1, ..., M)$ are twice continuously differentiable;

(H) $c_i(\mathbf{p}, y_1)$ $(i = 1, ..., M)$ are homogeneous of degree zero with respect to prices and income;

(B) Budget Balancedness:[10]

$$\sum_{i=1}^{M} p_i c_i = y_1; \tag{8}$$

(EC) Existence of Conditional Demands: based on the unconditional demands $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ and $y_2 = y_1 - \Sigma_{i=1}^{K} p_i c_i$, one can solve for the continuously differentiable functions

$$c_i = c_i(p_{K+1}, ..., p_M, y_2 | \, c_1, ..., c_K) \quad (i = K+1, ..., M); \tag{9}$$

(EI) Existence of Inverse (Unconditional) Demands: based on the unconditional demands $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$, one can solve for p_i $(i = 1, ..., M)$ as continuously differentiable functions of $(c_1, ..., c_M)$;

(S) Slutsky Symmetry: the corresponding Slutsky matrix $(\sigma_{ij})_{M \times M}$ is symmetric, where

$$\sigma_{ij} = \frac{\partial c_i}{\partial p_j} + c_j \frac{\partial c_i}{\partial y_1}; \tag{10}$$

(N) Slutsky Negative Semidefiniteness: the corresponding Slutsky matrix $(\sigma_{ij})_{M \times M}$ is negative semidefinite;

(ND) Slutsky Negative Definiteness: the corresponding Slutsky matrix $(\sigma_{ij})_{M \times M}$ is negative definite.

Although Properties (EC) and (EI) are stated globally, we next provide two simple tests for local versions of these properties. If either test fails, then one can conclude that Property (EC) or (EI) is not satisfied. If both tests are satisfied, then one can seek to directly solve the appropriate systems of equations for the conditional and inverse demand functions. First to test for (EC), it will prove

functions one considers. For example, if we say $c_i(\mathbf{p}, y_1)$ $(i = 1, ..., K)$ satisfy Property (S), we mean $(\sigma_{ij})_{K \times K}$ is symmetric.

[10] We use the term "Budget Balancedness" following Jehle and Reny (2011). This condition is also referred to as Walras' law in Mas-Colell, Whinston and Green (1995).

useful to denote the Jacobian matrix of derivatives of the vector $(c_1, ..., c_K, y_2)$ with respect to $(p_1, ..., p_K, y_1)$ evaluated at $(p_1^0, ..., p_K^0, y_1^0)$ as

$$J_c = \left. \frac{\partial (c_1, ..., c_K, y_2)}{\partial (p_1, ..., p_K, y_1)} \right|_{(p_1,...,p_K,y_1)=\left(p_1^0,...,p_K^0,y_1^0\right)}. \tag{11}$$

Lemma 1 *For the given unconditional demands* $c_i (p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ *and* $(p_1^0, ..., p_K^0, y_1^0) \in \mathbb{R}_{++}^K \times \mathbb{R}_+$*, there is an open neighborhood containing* $(p_1^0, ..., p_K^0, y_1^0)$ *such that there exist continuously differentiable conditional demands*

$$c_i (p_1, ..., p_M, y_1) = c_i (p_{K+1}, ..., p_M, y_2 |\; c_1, ..., c_K) \quad (i = K + 1, ..., M) \tag{12}$$

if and only if $\det J_c \neq 0$.

The actual process for solving for the conditional demands assuming the conditions in Lemma 1 are satisfied is discussed in Subsection 4.1.1 below. Next for the (EI) test, denote the Jacobian matrix of derivatives of the vector demand function $\mathbf{c}(\mathbf{p}, y_1)$ with respect to $(p_1, ..., p_M)$ evaluated at $(p_1^0, ..., p_M^0)$ as

$$J = \left. \frac{\partial (c_1, ..., c_M)}{\partial (p_1, ..., p_M)} \right|_{(p_1,...,p_M)=\left(p_1^0,...,p_M^0\right)}. \tag{13}$$

Lemma 2 *For the given unconditional demands* $c_i (p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ *and* $(p_1^0, ..., p_M^0) \in \mathbb{R}_{++}^M$*, there is an open neighborhood containing* $(p_1^0, ..., p_M^0)$ *such that* p_i $(i = 1, ..., M)$ *can be expressed as continuously differentiable functions* $(c_1, ..., c_M)$ *if and only if* $\det J \neq 0$.

Tests for properties (EC) and (EI), involving global assumptions, are given by Corollaries 5 and 6, respectively, in Appendix C. For each of the examples considered in this paper, if the conditions in Lemmas 1 (2) hold in the entire parameter space, it can be verified that the conditions in Corollary 5 (6) also are satisfied. For this reason, we apply the Lemmas 1 and 2 tests for (EC) and (EI), respectively, to each point in the parameter space.

Given a set of demand functions $c_i (\mathbf{p}, y_1)$ $(i = 1, ..., M)$, the classic integrability problem asks what properties of the demands guarantee that there exists a utility function $U(c_1, ..., c_M)$ such that

$$\mathbf{c}(\mathbf{p}, y_1) = \underset{c_1,...,c_M}{\arg\max}\, U(c_1, ..., c_M) \quad S.T. \;\; y_1 \geq \sum_{i=1}^{M} p_i c_i. \tag{14}$$

We can now state a version of the solution provided by Hurwicz and Uzawa (1971).[11]

[11] The statement of Theorem 1 is adapted from Mas-Colell, Whinston and Green (1995, pp. 75-76). Hurwicz and Uzawa (1971) assume $c_i (\mathbf{p}, y_1)$ $(i = 1, 2, ..., M)$ are differentiable rather than satisfying (TD) and assume a boundedness condition (also see Border 2014). A refined version of Theorem 1 is given in Hurwicz and Richter (1979).

Theorem 1 *Assume a given set of demands* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ *which have the Properties (P), (TD) and (B). Then there exists a unique (up to an increasing transformation)* $U(c_1, ..., c_M)$*, satisfying Property 1, which rationalizes the demands if and only if Properties (S) and (N) hold.*

Hurwicz and Uzawa (1971) also provide a recovery process based on the income compensation function for deriving the utility function U (up to a positive affine transformation) which rationalizes the given demands. Their existence result and recovery process generates in a single utility function which is well-behaved in the sense of Property 1. In the case of changing tastes, the integrability problem becomes one of finding conditions for the existence of potentially different $(U^{(1)}, U^{(2)})$-pairs corresponding to a given set naive or sophisticated demands and identifying a process for recovering the utility pairs.

3 Motivating Examples

In this section, we consider the special case where $M = 3$ and there are three periods with one commodity in each period. The consumer's changing tastes are represented by $U^{(1)}(c_1, c_2, c_3)$ and $U^{(2)}(c_2, c_3|\ c_1)$ and the optimization problems P_1 and P_2 are revised accordingly.

In each of the two examples discussed below, a set of dynamic demand functions $c_1(p_1, p_2, p_3, y_1)$, $c_2(p_1, p_2, p_3, y_1)$ and $c_3(p_1, p_2, p_3, y_1)$ is assumed. In the first example, a given set of demands is shown to satisfy Theorem 1 and hence is rationalizable by a non-changing tastes utility U. However the same demands are also shown to be rationalizable by a changing tastes $(U^{(1)}, U^{(2)})$-pair, satisfying Property 1, based on sophisticated choice. The second example demonstrates that a given set of demand functions can be rationalized both as naive choice based on one $(U^{(1)}, U^{(2)})$-pair and as sophisticated choice based on a different $(U^{(1)}, U^{(2)})$-pair.

Example 1 *Assume that*

$$c_1 = \frac{2p_1 + 3y_1 - 2\sqrt{p_1^2 + 3p_1 y_1}}{9p_1}, \tag{15}$$

$$c_2 = \frac{6y_1 + 2\sqrt{p_1^2 + 3p_1 y_1} - 2p_1}{18p_2} \quad \textit{and} \quad c_3 = \frac{6y_1 + 2\sqrt{p_1^2 + 3p_1 y_1} - 2p_1}{18p_3}. \tag{16}$$

These demands satisfy (P), (TD), (B), (S) and (N). Thus following Theorem 1, the demands can be rationalized by a non-changing tastes utility U*. Using the*

Hurwicz and Uzawa (1971) recovery process, one can obtain

$$U\left(c_1, c_2, c_3\right) = \left(\sqrt{c_1} + 1\right)\sqrt{c_2 c_3}. \tag{17}$$

It can be verified that the demands are also rationalizable as sophisticated choice based on the changing tastes $(U^{(1)}, U^{(2)})$*-pair*

$$U^{(1)}\left(c_1, c_2, c_3\right) = \sqrt{\left(\sqrt{c_1} + 1\right) c_2} + \left(\left(\sqrt{c_1} + 1\right) c_3\right)^{\frac{1}{4}}, \tag{18}$$

and

$$U^{(2)}(c_2, c_3 |\ c_1) = \ln c_2 + \ln c_3. \tag{19}$$

All three utilities (17) - (19) satisfy Property 1 over the full choice space. Clearly $U^{(1)}$ *differs from* U *by more than an increasing transformation.*[12]

Example 2 *Assume that*

$$c_1 = \frac{y_1}{p_1} - \frac{p_1}{p_2} - \frac{p_1}{p_3}, \qquad c_2 = \frac{\frac{p_1^2}{p_2} + \frac{p_1^2}{p_3}}{2p_2} \qquad \text{and} \qquad c_3 = \frac{\frac{p_1^2}{p_2} + \frac{p_1^2}{p_3}}{2p_3}, \tag{20}$$

where the following is assumed to ensure that Property (P) holds

$$y_1 \geq \frac{p_1^2}{p_2} + \frac{p_1^2}{p_3}. \tag{21}$$

These demands satisfy Properties (P), (TD), (B) and (H) but not (S) and hence cannot be rationalized by a non-changing tastes utility U. *However, it can be verified that the demand functions (20) correspond to naive choice utilizing*

$$U^{(1)}\left(c_1, c_2, c_3\right) = c_1 + 2\sqrt{c_2} + 2\sqrt{c_3} \qquad \text{and} \qquad U^{(2)}(c_2, c_3 |\ c_1) = \ln c_2 + \ln c_3 \tag{22}$$

and also sophisticated choice utilizing

$$U^{(1)}\left(c_1, c_2, c_3\right) = c_1 + 2\sqrt{2\left(c_2 + c_3\right)} \qquad \text{and} \qquad U^{(2)}(c_2, c_3 |\ c_1) = \ln c_2 + \ln c_3. \tag{23}$$

Each of the utilities in (22) and (23) satisfies Property 1 except for $U^{(1)}$ *in (23) which is quasiconcave but not strictly quasiconcave.*[13] *Clearly the two* $U^{(1)}$ *functions are not ordinally equivalent and thus from just knowing the demands (20) it is impossible to tell whether they have been generated by naive or sophisticated choice.*

[12] In the application of sophisticated choice, one can express the non-changing tastes utility (17) as a $(U^{(1)}, U^{(2)})$-pair, where $U^{(1)}$ corresponds to (17) and $U^{(2)}(c_2, c_3) = \sqrt{c_2 c_3}$ is the period two continuation of U which is ordinally equivalent to (19).

[13] For $U^{(1)}$ and $U^{(2)}$ in (23), if one were to consider naive choice, there would be an infinite number of optimal solutions for the optimization problem P_1 if $p_2 = p_3$ and no interior solution if $p_2 \neq p_3$. However, there is a unique sophisticated choice solution for any p_2 and p_3 satisfying (21).

These examples raise two questions we address in the next section. First, when will a $(U^{(1)}, U^{(2)})$-pair exist that rationalizes a given set of dynamic demands corresponding to sophisticated choice? Second, how can the generating utilities be found? The same questions are addressed for naive choice in Section 5.

4 Rationalizing Sophisticated Choice

In this section, we address the integrability question for sophisticated choice first assuming three goods and then more than three goods. In both settings, there is only one change in tastes. The final subsection explores a potential strengthening of our existence results.

4.1 Three Commodity Case

For the three commodity case, we first derive necessary and sufficient conditions for the existence of the utilities $(\{U^{(1)}\}, U^{(2)})$ which rationalize a given set of sophisticated demands and then describe a process for recovering the generating utilities. Our method for recovering the set of period one utilities is based on the solution to a partial differential equation and is quite different from the one used by Hurwicz and Uzawa (1971) to recover a non-changing tastes U. The application of our existence result and recovery process are illustrated in the second subsection with several examples. The third subsection considers a special case where the period two and three conditional demands are linear in period two income.

4.1.1 Existence

In general, a set of dynamic demand functions

$$c_1(p_1, p_2, p_3, y_1), \quad c_2(p_1, p_2, p_3, y_1) \quad \text{and} \quad c_3(p_1, p_2, p_3, y_1) \tag{24}$$

obtained via sophisticated choice will not satisfy (S) and (N), and a single U will fail to exist. If the unconditional demands (24) satisfy (EC),[14] then (p_1, y_1) can be solved for as functions of (c_1, p_2, p_3, y_2) from the pair of equations

$$c_1 = c_1(p_1, p_2, p_3, y_1) \quad \text{and} \quad y_2 = y_1 - p_1 c_1(p_1, p_2, p_3, y_1). \tag{25}$$

Substituting $p_1(c_1, p_2, p_3, y_2)$ and $y_1(c_1, p_2, p_3, y_2)$ into the period two and period three unconditional demands, one obtains the conditional demand functions $c_2 =$

[14] That is, they pass the tests in Lemma 1 or Corollary 5 in Appendix C.

$c_2(p_2, p_3, y_2|\ c_1)$ and $c_3 = c_3(p_2, p_3, y_2|\ c_1)$, respectively. (This process is illustrated in more detail in Example 4 below.)

In addition to Property (EC), to ensure the existence of $U^{(2)}$, it is also necessary to assume that $c_2(p_2, p_3, y_2|\ c_1)$ and $c_3(p_2, p_3, y_2|\ c_1)$ satisfy (S) and (N). However if the conditional demands satisfy (P), (TD), (B) and (H), it follows from Katzner (1970, Theorem 4.1-2 for the two commodity case) that Property (S) is always implied.[15] In order to ensure the existence of $U^{(1)}$, the unconditional demands have to satisfy (EI).

Theorem 2 *Assume a given set of demand functions* $c_i\,(p_1, p_2, p_3, y_1)$ $(i = 1, 2, 3)$ *which have the Properties (P), (TD), (H) and (B).*[16] *Then there exists a* $(U^{(1)}, U^{(2)})$*-pair which generates these demands as a result of sophisticated choice, where* $U^{(1)}$ *is twice continuously differentiable and* $U^{(2)}$ *satisfies Property 1,*[17] *if and only if the demand functions also have Properties (EC) and (EI) and the corresponding conditional demand functions* $c_2\,(p_2, p_3, y_2|\ \ c_1)$ *and* $c_3\,(p_2, p_3, y_2|\ \ c_1)$ *have Property (N). In this case,* $U^{(1)}$ *satisfies*

$$\frac{\partial U^{(1)}}{\partial c_1} + \frac{\partial c_2}{\partial c_1}\frac{\partial U^{(1)}}{\partial c_2} + \frac{\partial c_3}{\partial c_1}\frac{\partial U^{(1)}}{\partial c_3} = 0, \tag{26}$$

where c_2 *and* c_3 *are functions of* $(c_1, p_2, p_3, y_1 - p_1 c_1)$ *and* $\frac{\partial c_2}{\partial c_1}$ *and* $\frac{\partial c_3}{\partial c_1}$ *can be transformed into functions of* (c_1, c_2, c_3) *using the inverse demand functions.*

Remark 1 *The fact that Property (EI) is essential in Theorem 2 may at first blush seem at odds with the discussion of integrability in Samuelson (1950). Indeed in essence referring to (EI), he states emphatically that "reversibility as such has absolutely nothing to do with integrability" (Samuelson, 1950, p. 385). To see this, notice that once Properties (S) and (N) are assumed, (EI) is automatically*

[15] To see that Property (N) does not always hold for the two commodity case, assume that

$$c_1 = \frac{p_1 y_1}{p_1^2 + p_2^2} \qquad \text{and} \qquad c_2 = \frac{p_2 y_1}{p_1^2 + p_2^2}.$$

It can be verified that although the corresponding Slutsky matrix is symmetric, it is also positive semidefinite.

[16] In Theorem 1, it will be noted that (H) is not assumed. As shown by Jehle and Reny (2011, Theorem 2.5) if Properties (P) and (TD) hold, then (B) and (S) imply (H). But for Theorem 2, since (S) may fail to hold it is necessary to assume (H).

[17] Since the $U^{(1)}$ utility is recovered using eqn. (26), it must be twice continuously differentiable due to the Frobenius condition. It should be emphasized that when the sophisticated demands satisfy Properties (P), (TD), (H) and (B), it is possible that they are generated by a non-differentiable $(U^{(1)}, U^{(2)})$-pair. A simple example is $U^{(1)} = \min(c_1, c_2, c_3)$ and $U^{(2)} = \min(c_2, c_3)$. Since for Theorem 1, the generating utility is recovered via an integration process rather than solving a partial differential equation, this issue never arises.

satisfied and hence it adds nothing to solving the integrability problem, which is also the reason why Hurwicz and Uzawa (1971) do not include Property (EI) in their result (Theorem 1).

Although (EC) and (EI) together cannot ensure the existence of a single rationalizing non-changing tastes utility U, they are necessary and sufficient for the existence of a set $\{U^{(1)}\}$ associated with sophisticated choice. More specifically, (EC) ensures the existence of the conditional demands and (EI) ensures that the partial differential equation (26) for $U^{(1)}$ based on these conditional demands is solvable.

If the conditions in Theorem 2 are satisfied, then it follows from its proof (see Appendix D) that Theorem 1 can be employed to guarantee the existence of a unique (up to an increasing transformation) $U^{(2)}$. Moreover, one can use the Hurwicz and Uzawa (1971) recovery process to solve for $U^{(2)}$ based on the conditional demands. The partial differential equation (26) in Theorem 2 can be solved by finding the characteristic equations and deriving the two independent first integrals. Then the general solution $\{U^{(1)}\}$ is given by any combination of these two first integrals. This process is illustrated in several examples in the next subsection.

As Example 4 below shows, a $U^{(1)}$ obtained as a solution to the partial differential equation eqn. (26) may not be strictly increasing and quasiconcave. However if the conditional demands $c_2(p_2, p_3, y_2|\ c_1)$ and $c_3 = c_3(p_2, p_3, y_2|\ c_1)$ are independent of c_1 and at least one of the conditional demands is a normal good then it follows from Corollary 1 below that at least one utility in $\{U^{(1)}\}$ will be strictly increasing in a subspace of $C_1 \times C_2 \times C_3$. The independence of c_1 assumed in our motivating Examples 1 and 2, is commonplace in the changing tastes literature.[18] As will be seen from the corollary, this case corresponds to the assumption that the period two utility $U^{(2)}(c_2, c_3|\ c_1)$ is weakly separable in (c_2, c_3).[19] Finally, the conditional demands being independent of c_1 can simplify somewhat the process for recovering a $(U^{(1)}, U^{(2)})$-pair.

Corollary 1 *Suppose that the conditions in Theorem 2 hold and the conditional demands* $c_2\left(p_2, p_3, y_2|\ c_1\right)$ *and* $c_3\left(p_2, p_3, y_2|\ c_1\right)$ *are independent of* c_1*, then* $U^{(2)}$

[18] Indeed the popular additively separable quasihyperbolic discounting case exhibits this property. See, for example, Laibson (1997) and Diamond and Koszegi (2003).

[19] It follows that c_1 plays no role in the period two optimization and the simpler notation $U^{(2)}\left(c_2, c_3\right)$ will be used for this case throughout the rest of the paper. A similar notational convention will be employed in Subsection 4.2 for the general case of more than three commodities.

is weakly separable in (c_2, c_3) *and eqn. (26) becomes*

$$\frac{\partial U^{(1)}}{\partial c_1} - p_1 \frac{\partial c_2}{\partial y_2}\frac{\partial U^{(1)}}{\partial c_2} - p_1 \frac{\partial c_3}{\partial y_2}\frac{\partial U^{(1)}}{\partial c_3} = 0. \tag{27}$$

Moreover among the set of $U^{(1)}$ *functions that solve (27), there will exist at least one member that is strictly increasing at least in some subspace of* $C_1 \times C_2 \times C_3$*, if and only if* $\frac{\partial c_2}{\partial y_2} > 0$ *or* $\frac{\partial c_3}{\partial y_2} > 0$ *or equivalently,* c_2 *or* c_3 *is a normal good with respect to second period income.*

A necessary and a sufficient condition for the existence of at least one member in $\{U^{(1)}\}$ to be locally twice continuously differentiable, strictly increasing and quasiconcave is provided in Appendix F. Despite this condition being relatively strong, it is satisfied by most of the examples given in this paper. A conjecture relating to at least one $U^{(1)}$ satisfying these same properties over the full choice space is given in Subsection 4.3.

It should be emphasized that given a set of unconditional sophisticated demands, Theorem 2 and Corollary 1 ensure the existence of a unique $U^{(2)}$ (up to an increasing transform) and a set of period one utilities $\{U^{(1)}\}$ with infinitely many non-ordinally equivalent members. The latter set arises since there always exist two independent first integrals for the partial differential equations (26) and (27).

4.1.2 Recovery Process

In this subsection, we illustrate the use of the partial differential equations in Theorem 2 and Corollary 1 to derive the utilities $\left(\{U^{(1)}\}, U^{(2)}\right)$ that rationalize a given set of sophisticated demands. Before doing so, consider the following example in which a given set of demands fails to satisfy property (EI).

Example 3 *Assume that*[20]

$$c_1 = \frac{y_1}{p_1 + p_2 + p_3}, \qquad c_2 = \frac{(p_2 + p_3)\, y_1}{2p_2\,(p_1 + p_2 + p_3)}, \qquad c_3 = \frac{(p_2 + p_3)\, y_1}{2p_3\,(p_1 + p_2 + p_3)}. \tag{28}$$

It can be verified that these demands satisfy Properties (P), (TD), (H) and (B) but not Property (S), and thus a non-changing tastes U fails to exist. Also there does not exist a $U^{(1)}$ *which can rationalize the demands as sophisticated choice since* $\det J = 0$*, where the Jacobian matrix is given by*

$$J = \begin{bmatrix} -\frac{y_1}{(p_1+p_2+p_3)^2} & -\frac{y_1}{(p_1+p_2+p_3)^2} & -\frac{y_1}{(p_1+p_2+p_3)^2} \\ -\frac{(p_2+p_3)y_1}{2p_2(p_1+p_2+p_3)^2} & -\frac{\left(p_1p_3+(p_2+p_3)^2\right)y_1}{2p_2^2(p_1+p_2+p_3)^2} & \frac{p_1y_1}{2p_2(p_1+p_2+p_3)^2} \\ -\frac{(p_2+p_3)y_1}{2p_3(p_1+p_2+p_3)^2} & \frac{p_1y_1}{2p_3(p_1+p_2+p_3)^2} & -\frac{\left(p_1p_2+(p_2+p_3)^2\right)y_1}{2p_3^2(p_1+p_2+p_3)^2} \end{bmatrix}, \tag{29}$$

[20] This example is similar to Exercise 2.F.17 in Mas-Colell, Whinston and Green (1995).

and Property (EI) is not satisfied.[21]

Before considering an example illustrating the application of Theorem 2, we first analyze the simpler case of motivating Example 2 where the conditional demands are independent of c_1 and Corollary 1 applies. We discuss this example in some detail in order to illustrate the processes for (i) deriving the conditional demands from the given unconditional demands and (ii) solving the partial differential eqn. (27) for the set $\{U^{(1)}\}$.

Example 4 *Assume that the unconditional demands are given by eqn. (20) in Example 2. The demands satisfy Properties (P), (TD), (B) and (H). To verify that Property (EC) holds, first note*

$$c_1 = \frac{y_1}{p_1} - \frac{p_1}{p_2} - \frac{p_1}{p_3} \qquad \text{and} \qquad y_2 = y_1 - p_1 c_1 = \frac{p_1^2}{p_2} + \frac{p_1^2}{p_3}. \tag{30}$$

Since

$$\det \frac{\partial (c_1, y_2)}{\partial (p_1, y_1)} \neq 0 \qquad \forall (p_1, y_1) \in \mathbb{R}_{++} \times \mathbb{R}_+, \tag{31}$$

the unconditional demands satisfy (EC). Using (30) and solving for (p_1, y_1) as functions of (c_1, y_2), yields

$$p_1 = \sqrt{\frac{y_2}{\frac{1}{p_2} + \frac{1}{p_3}}} \qquad \text{and} \qquad y_1 = c_1 \sqrt{\frac{y_2}{\frac{1}{p_2} + \frac{1}{p_3}}} + y_2. \tag{32}$$

Substituting the above two expressions into the unconditional demands for c_2 and c_3 in (20), one obtains the conditional demand functions

$$c_2 (p_2, p_3, y_2 |\ c_1) = \frac{y_2}{2p_2} \qquad \text{and} \qquad c_3 (p_2, p_3, y_2 |\ c_1) = \frac{y_2}{2p_3}, \tag{33}$$

which are independent of c_1 and increasing in y_2. Moreover, it can be verified that the conditional demands (33) satisfy Property (N) and hence applying Corollary 1, $U^{(2)}$ exists and is independent of c_1. Following the Hurwicz and Uzawa (1971) recovery process yields

$$U^{(2)} (c_2, c_3) = \ln c_2 + \ln c_3, \tag{34}$$

[21] Interestingly, although the demands (28) do not correspond to sophisticated choice, following the analysis in Section 5 below they can be rationalized as naive choice by the changing tastes $(U^{(1)}, U^{(2)})$-pair

$$U^{(1)} (c_1, c_2, c_3) = \min (c_1, c_2, c_3) \qquad \text{and} \qquad U^{(2)} (c_2, c_3) = \ln c_2 + \ln c_3.$$

which satisfies Property 1. To establish the existence of a period one utility $U^{(1)}$, it can be verified that

$$\det \frac{\partial (c_1, c_2, c_3)}{\partial (p_1, p_2, p_3)} \neq 0 \qquad \forall (p_1, p_2, p_3) \in \mathbb{R}^3_{++} \tag{35}$$

and hence Property (EI) is satisfied. Given that the requisite demand properties hold, Corollary 1 applies and the partial differential equation (27) can be used to solve for $U^{(1)}$. First based on eqn. (33),

$$-p_1 \frac{\partial c_2}{\partial y_2} = -\frac{p_1}{2p_2} \qquad \textit{and} \qquad -p_1 \frac{\partial c_3}{\partial y_2} = -\frac{p_1}{2p_3}. \tag{36}$$

Solving for the inverse demands from (20), one obtains

$$\frac{p_1}{y_1} = \frac{1}{c_1 - \sqrt{2(c_2 + c_3)}}, \tag{37}$$

$$\frac{p_2}{y_1} = \frac{\sqrt{c_2 + c_3}}{\sqrt{2} c_2 \left(c_1 - \sqrt{2(c_2 + c_3)} \right)} \qquad \textit{and} \qquad \frac{p_3}{y_1} = \frac{\sqrt{c_2 + c_3}}{\sqrt{2} c_3 \left(c_1 - \sqrt{2(c_2 + c_3)} \right)}. \tag{38}$$

Substituting the above inverse demand functions into eqn. (36) yields

$$-\frac{p_1}{2p_2} = -\frac{\sqrt{2} c_2}{2\sqrt{c_2 + c_3}} \qquad \textit{and} \qquad -\frac{p_1}{2p_3} = -\frac{\sqrt{2} c_3}{2\sqrt{c_2 + c_3}}. \tag{39}$$

Thus, eqn. (27) can be rewritten as

$$2\sqrt{c_2 + c_3} \frac{\partial U^{(1)}}{\partial c_1} - \sqrt{2} c_2 \frac{\partial U^{(1)}}{\partial c_2} - \sqrt{2} c_3 \frac{\partial U^{(1)}}{\partial c_3} = 0. \tag{40}$$

The two characteristic equations are given by

$$\frac{dc_1}{2\sqrt{c_2 + c_3}} + \frac{dc_2}{\sqrt{2} c_2} = 0 \qquad \textit{and} \qquad \frac{dc_2}{\sqrt{2} c_2} - \frac{dc_3}{\sqrt{2} c_3} = 0, \tag{41}$$

and the two independent first integrals are

$$\psi_1 (c_1, c_2, c_3) = c_1 + 2\sqrt{2(c_2 + c_3)} \qquad \textit{and} \qquad \psi_2 (c_1, c_2, c_3) = \frac{c_2}{c_3}. \tag{42}$$

It follows that

$$U^{(1)} (c_1, c_2, c_3) = f \left(c_1 + 2\sqrt{2(c_2 + c_3)}, \frac{c_2}{c_3} \right). \tag{43}$$

Since f is an arbitrary function, in general the $U^{(1)}$ functions associated with different f functions differ by more than an increasing transformation. Therefore, eqn. (43) defines a set of $U^{(1)}$ function. One special form of the function $f(x, y) =$

x corresponds to the utility (23) referenced in Example 2. Another form of the period one utility corresponding to $f(x,y) = x + \ln y$ is given by

$$U^{(1)}(c_1, c_2, c_3) = c_1 + 2\sqrt{2(c_2 + c_3)} + \ln \frac{c_2}{c_3}. \tag{44}$$

This utility is strictly increasing and quasiconcave for all $(c_1, c_2, c_3) \in (0, \infty) \times (0, 1) \times (2, \infty)$.[22]

In the above example, multiple $U^{(1)}$'s are recovered based on Corollary 1 although as noted in the discussion following Theorem 2, not all of the utilities need be strictly increasing and quasiconcave everywhere. Since the conditions of Corollary 1 are satisfied, at least one member of the set of possible $U^{(1)}$ functions (defined by eqn. (43)) will be strictly increasing over some subset of the choice space. In the example, the $U^{(1)}$ corresponding to $f(x,y) = x$ is actually strictly increasing and quasiconcave in the whole space. But another member (44) is strictly increasing and quasiconcave when $(c_1, c_2, c_3) \in (0, \infty) \times (0, 1) \times (2, \infty)$ and decreasing in c_3 when $(c_1, c_2, c_3) \in (0, \infty) \times (2, \infty) \times (0, 1)$. As suggested in Section 1, this multiplicity of $U^{(1)}$ functions arises because the sophisticated solution does not result in a complete set of optimal demand functions corresponding to the period one optimization problem P_1.

The next example illustrates the application of Theorem 2, where the conditional demands depend on c_1 and Corollary 1 does not apply.

Example 5 *Assume that*

$$c_1 = \frac{y_1}{p_1 - p_2 - p_3} - (p_1 - p_2 - p_3)\left(\frac{1}{p_2} + \frac{1}{p_3}\right), \tag{45}$$

$$c_2 = (p_1 - p_2 - p_3)\left(\frac{p_1 + p_2 - p_3}{2p_2^2} + \frac{p_1 + p_2 - p_3}{2p_2 p_3}\right) - \frac{y_1}{p_1 - p_2 - p_3} \tag{46}$$

and

$$c_3 = (p_1 - p_2 - p_3)\left(\frac{p_1 + p_3 - p_2}{2p_3^2} + \frac{p_1 + p_3 - p_2}{2p_2 p_3}\right) - \frac{y_1}{p_1 - p_2 - p_3}, \tag{47}$$

where the following are assumed to ensure that Property (P) holds

$$p_1 > p_2 + p_3, \tag{48}$$

[22] If one assumes $f(x,y) = y$, then following sophisticated choice,

$$U^{(1)} = \frac{c_2}{c_3} = \frac{y_2/(2p_2)}{y_2/(2p_3)} = \frac{p_3}{p_2}$$

is a constant and hence the utility value is the same for all $c_1 < y_1/p_1$. Therefore, the given c_1 demand function can be viewed as an optimal solution.

$$y_1 \geq (p_1 - p_2 - p_3)^2 \left(\frac{1}{p_2} + \frac{1}{p_3}\right) \tag{49}$$

and

$$y_1 \leq \frac{1}{2}(p_1 - p_2 - p_3)^2 \left(\frac{1}{p_2} + \frac{1}{p_3} + \frac{p_1}{p_2 p_3}\right). \tag{50}$$

The demands satisfy Properties (P), (TD), (B) and (H) but not (S) and (N). Therefore, there is no U to rationalize these demands. It can be shown that the requisite demand properties for Theorem 2 apply and the partial differential equation (26) can be used to solve for $U^{(1)}$. Given the resulting two independent first integrals, it follows that

$$U^{(1)}(c_1, c_2, c_3) = f\left(c_1 + 2\sqrt{2(c_2 + c_3 + 2c_1)}, \frac{c_2 + c_1}{c_3 + c_1}\right). \tag{51}$$

If one assumes $f(x, y) = x$, then

$$U^{(1)}(c_1, c_2, c_3) = c_1 + 2\sqrt{2(c_2 + c_3 + 2c_1)}, \tag{52}$$

which satisfies Property 1 except that it is quasiconcave instead of strictly quasiconcave. (Supporting computations for this example are provided in Appendix H.)

If a given set of demands satisfy the conditions in Theorem 1, does this imply that the conditions in Theorem 2 hold as well? The following result establishes that this is indeed the case if the non-changing tastes utility U recovered based on the Hurwicz and Uzawa (1971) process is twice continuously differentiable.[23] However without the partial differential equation in Theorem 2, it is not clear how one would recover the changing tastes set of utilities.

Corollary 2 *If a given set of demands satisfy the conditions in Theorem 1, then there exists a non-changing tastes U satisfying Property 1 which rationalizes the demands. If U is twice continuously differentiable, then the given demands also satisfy the conditions in Theorem 2 and there exists a $(U^{(1)}, U^{(2)})$-pair which rationalizes the demands as sophisticated choice, where $U^{(1)}$ can be expressed as*

$$U^{(1)}(c_1, c_2, c_3) = f(g(c_1, c_2, c_3), U(c_1, c_2, c_3)), \tag{53}$$

$U(c_1, c_2, c_3)$ and $g(c_1, c_2, c_3)$ are the independent first integrals of the characteristic equations for the partial differential equation (26) and $U^{(2)}$ is the continuation of U. Moreover, there exists at least one $U^{(1)}$ in the set $\{U^{(1)}\}$ defined by (53) which satisfies Property 1.

[23] The reason that U is required to be twice continuously differentiable is to ensure that Property (EI) holds (refer to the proof of Corollary 2 in Appendix I).

Following Corollary 2, Theorem 2 can be viewed as a generalization of Theorem 1 since when Properties (S) and (N) are satisfied, one can also recover the non-changing tastes U using the partial differential equation in Theorem 2 (assuming U is twice continuously differentiable). In fact, one of the first integrals is always the non-changing tastes $U(c_1, c_2, c_3)$ which is a member of $\{U^{(1)}\}$.

The application of Corollary 2 is illustrated by the following example, where the conditional demands will be seen to exhibit c_1 dependence and one member of $\{U^{(1)}\}$ will be recognized to be the popular habit formation utility (see, for example, Constantinides 1990).

Example 6 *Assume that*

$$c_1 = \frac{y_1}{p_1 + \alpha(p_2 + p_3) + p_2\left(\frac{p_1}{p_2} + \alpha\left(1 + \frac{p_3}{p_2}\right)\right)^{\frac{1}{1+\delta}} + p_3\left(\frac{p_1}{p_3} + \alpha\left(1 + \frac{p_2}{p_3}\right)\right)^{\frac{1}{1+\delta}}}, \tag{54}$$

$$c_2 = \frac{\left(\alpha + \left(\frac{p_1}{p_2} + \alpha\left(1 + \frac{p_3}{p_2}\right)\right)^{\frac{1}{1+\delta}}\right) y_1}{p_1 + \alpha(p_2 + p_3) + p_2\left(\frac{p_1}{p_2} + \alpha\left(1 + \frac{p_3}{p_2}\right)\right)^{\frac{1}{1+\delta}} + p_3\left(\frac{p_1}{p_3} + \alpha\left(1 + \frac{p_2}{p_3}\right)\right)^{\frac{1}{1+\delta}}} \tag{55}$$

and

$$c_3 = \frac{\left(\alpha + \left(\frac{p_1}{p_3} + \alpha\left(1 + \frac{p_2}{p_3}\right)\right)^{\frac{1}{1+\delta}}\right) y_1}{p_1 + \alpha(p_2 + p_3) + p_2\left(\frac{p_1}{p_2} + \alpha\left(1 + \frac{p_3}{p_2}\right)\right)^{\frac{1}{1+\delta}} + p_3\left(\frac{p_1}{p_3} + \alpha\left(1 + \frac{p_2}{p_3}\right)\right)^{\frac{1}{1+\delta}}}, \tag{56}$$

where $\delta > -1$ and $0 < \alpha < 1$. These demands satisfy Properties (P), (TD), (B), (S) and (N) and using Theorem 1 one can recover the non-changing tastes utility

$$U(c_1, c_2, c_3) = -\frac{c_1^{-\delta}}{\delta} - \frac{(c_2 - \alpha c_1)^{-\delta}}{\delta} - \frac{(c_3 - \alpha c_1)^{-\delta}}{\delta}, \tag{57}$$

which satisfies Property 1 and is twice continuously differentiable. It can be verified that the requisite demand properties for Theorem 2 hold. It follows that $U^{(2)}$ is the period two continuation of (57)

$$U^{(2)}(c_2, c_3 |\ c_1) = -\frac{(c_2 - \alpha c_1)^{-\delta}}{\delta} - \frac{(c_3 - \alpha c_1)^{-\delta}}{\delta} \tag{58}$$

and one can use the partial differential equation (26) to solve for $U^{(1)}$, yielding

$$U^{(1)}(c_1, c_2, c_3) = f\left(-\frac{c_1^{-\delta}}{\delta} - \frac{(c_2 - \alpha c_1)^{-\delta}}{\delta} - \frac{(c_3 - \alpha c_1)^{-\delta}}{\delta}, \frac{c_2 - \alpha c_1}{c_3 - \alpha c_1}\right). \tag{59}$$

(Supporting computations for this example are provided in Appendix J.)

4.1.3 Conditional Demands Affine in Income

Based on Examples 4 - 6, it would seem that when the conditional demands are affine in y_2, the $U^{(1)}$ obtained from solving the partial differential equation in Theorem 2 always has $\frac{c_2-a(c_1)}{c_3-b(c_1)}$ as one of its independent first integrals, where $a(c_1)$ and $b(c_1)$ are arbitrary functions which can depend on c_1. We next verify that this is the case.

Corollary 3 *Assume that the conditions in Theorem 2 hold and the conditional demands $c_2(p_2, p_3, y_2|\ c_1)$ and $c_3(p_2, p_3, y_2|\ c_1)$ satisfy*

$$c_2(p_2, p_3, y_2|\ c_1) - a(c_1) = k_2(p_2, p_3)\, y_2 \tag{60}$$

and

$$c_3(p_2, p_3, y_2|\ c_1) - b(c_1) = k_3(p_2, p_3)\, y_2, \tag{61}$$

where $a(c_1)$ and $b(c_1)$ are arbitrary functions of c_1 and $k_2(p_2, p_3)$ and $k_3(p_2, p_3)$ are arbitrary functions of (p_2, p_3).[24] *Then there exists a $(U^{(1)}, U^{(2)})$-pair which generates these demands as a result of sophisticated choice and $U^{(1)}$ can be expressed as*

$$U^{(1)}(c_1, c_2, c_3) = f\left(g(c_1, c_2, c_3), \frac{c_2 - a(c_1)}{c_3 - b(c_1)}\right), \tag{62}$$

where $g(c_1, c_2, c_3)$ and $(c_2 - a(c_1))/(c_3 - b(c_1))$ are the two independent first integrals of the characteristic equations for the partial differential equation (26).

We next show that the assumption that conditional demands are affine in y_2 also plays a critical role in extending our integrability results to the case of demands associated with effectively consistent preferences (see Definition 3). Selden and Wei (2015) provide necessary and sufficient conditions in terms of the forms of $U^{(1)}$ and $U^{(2)}$ such that demands exhibit the myopic separable and quasilinear cases of effective consistency. However, they provided no means for being able to recover the specific utility functions generating a given set of effectively consistent demands. If sophisticated demands take the myopic separable form of effective consistency, then the unconditional period one demand function $c_1(p_1, p_2, p_3, y_1)$ must be independent of p_2 and p_3 and the full set of unconditional sophisticated demands must be rationalizable by a non-changing tastes utility U (Selden and Wei, 2015, Proposition 4). However the assumption that these demands also satisfy the properties in Corollary 1 is not sufficient to ensure that Property (S) holds and that the Hurwicz and Uzawa (1971) recovery process can employed to obtain

[24] When the functions $a(c_1)$ and $b(c_1)$ are independent of c_1, Corollary 3 can be viewed as a special case of Corollary 1.

U. But as we next show, if one additionally assumes that the conditional demand functions are affine in y_2, then the unconditional demands can be rationalized by both a non-changing tastes U and a changing tastes $(U^{(1)}, U^{(2)})$-pair taking the required forms given in Selden and Wei (2015, Corollary 1). (The generalized quasilinear case is considered in Appendix M.)

Corollary 4 *Assume that the conditions in Corollary 1 are satisfied and further assume that the unconditional demand function $c_1(p_1, p_2, p_3, y_1)$ is independent of p_2 and p_3 and the conditional demands $c_2(p_2, p_3, y_2 | \ c_1)$ and $c_3(p_2, p_3, y_2 | \ c_1)$ satisfy*

$$c_2(p_2, p_3, y_2 | \ c_1) = k_2(p_2, p_3) y_2 \quad \text{and} \quad c_3(p_2, p_3, y_2 | \ c_1) = k_3(p_2, p_3) y_2, \tag{63}$$

where $k_2(p_2, p_3)$ and $k_3(p_2, p_3)$ are arbitrary functions of (p_2, p_3), and have Property (N). Then preferences are effectively consistent in the sense of Definition 3 and sophisticated choice can be rationalized by a non-changing tastes $U(c_1, c_2, c_3)$ which takes the form

$$U(c_1, c_2, c_3) = h(g(c_1) c_2, g(c_1) c_3). \tag{64}$$

Moreover, there exists a $(U^{(1)}, U^{(2)})$-pair which generates these demands as a result of sophisticated choice, where $U^{(1)}(c_1, c_2, c_3)$ is given by

$$U^{(1)}(c_1, c_2, c_3) = f(g(c_1) c_2, g(c_1) c_3) \tag{65}$$

and $U^{(2)}(c_2, c_3)$ is homothetic and satisfies Property 1.

In Corollary 4, $g(c_1) c_2$ and $g(c_1) c_3$ are the two independent first integrals corresponding to the partial differential equation (27), where $g(c_1)$ is uniquely determined (up to an arbitrary constant of integration) by the unconditional demand function for c_1 (see the derivation of $g(c_1)$ for Example 1 below). The h function in U is uniquely determined by the c_2 and c_3 conditional demand functions and the f function in $U^{(1)}$ is arbitrary, implying that one member of $U^{(1)}$ where $f = h$ (up to a monotone transformation) corresponds to U.[25] The conditions in Corollary 4 are shown in its proof to imply that Property (S) holds. If the unconditional

[25] At first glance, the $U^{(1)}$ in Corollary 4 does not seem to be a special case of eqn. (62) in Corollary 3. However noticing that $c_2/c_3 = g(c_1) c_2/(g(c_1) c_3)$, c_2/c_3 is also a first integral, it follows that eqn. (65) can alternatively be expressed as

$$U^{(1)}(c_1, c_2, c_3) = f\left(h(g(c_1) c_2, g(c_1) c_3), \frac{c_2}{c_3}\right),$$

which is more directly consistent with (62) in Corollary 3.

demands also satisfy Property (N), then U satisfies Property 1, implying that this property is satisfied by at least one member of $\left\{U^{(1)}\right\}$.

Revisiting Example 1, one will note that the period one unconditional demand function (15) is independent of p_2 and p_3. Moreover, the full set of unconditional demands (15) - (16) satisfy each of the conditions in Corollary 4 – indeed the corresponding conditional demands are given by $c_i = y_2/(2p_i)$ $(i = 2, 3)$ ensuring that condition (63) is satisfied. Applying Corollary 4 and using eqn. (15), one obtains

$$y_1 = 3p_1c_1 + 2p_1\sqrt{c_1}. \tag{66}$$

Substituting this equation into the following ordinary differential equation derived in the proof of Corollary 4

$$p_1c_1 + p_1\frac{g(c_1)}{g'(c_1)} = y_1, \tag{67}$$

and solving for $g(c_1)$ yields

$$g(c_1) = (\sqrt{c_1} + 1)\exp K, \tag{68}$$

where K is an arbitrary constant of integration. Without loss of generality, assuming $K = 0$, one obtains

$$U^{(1)}(c_1, c_2, c_3) = f\left((\sqrt{c_1} + 1)\,c_2, (\sqrt{c_1} + 1)\,c_3\right). \tag{69}$$

Given this result we can obtain the specific utilities in Example 1. Letting $f(x, y) = \sqrt{xy}$, $U^{(1)}$ converges to the non-changing tastes U given by eqn. (17), which satisfies Property 1 over the full choice space. Alternatively assuming $f(x, y) = x^{\frac{1}{2}} + y^{\frac{1}{4}}$, we obtain the specific changing tastes form (18).

4.2 More Than Three Commodity Case

In this subsection, we return to the more general setting of Section 2, where $M > 3$. We begin by demonstrating that for this case it is possible to extend Corollary 1 and guarantee the existence of a set of period one utilities $\{U^{(1)}\}$.

Theorem 3 *Assume (i) a given set of unconditional demands* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ *which have the Properties (P), (TD), (H), (B) and (EC) and (ii) the conditional demands* $c_i(p_{K+1}, ..., p_M, y_2|\ c_1, ..., c_K)$ $(i = K + 1, ..., M)$ *are independent of* $(c_1, ..., c_K)$. *Then there exists a* $(U^{(1)}, U^{(2)})$*-pair, where* $U^{(1)}$ *is twice continuously differentiable and* $U^{(2)}$ *satisfies Property 1 and is weakly separable in*

$(c_{K+1}, ..., c_M)$, such that the given demands $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ correspond to sophisticated choice if and only if they satisfy Property (EI), the conditional demand functions $c_i(p_{K+1}, ..., p_M, y_2 | \, c_1, ..., c_K)$ $(i = K+1, ..., M)$ have the Properties (S) and (N) and the following condition is satisfied

$$\frac{\partial p_j}{\partial c_i} - p_i \frac{\partial p_j}{\partial y_2} = \frac{\partial p_i}{\partial c_j} - p_j \frac{\partial p_i}{\partial y_2} \quad (i, j = 1, ..., K), \tag{70}$$

where p_i is a function of $(c_1, ..., c_K, p_{K+1}, ..., p_M, y_2)$.[26] *In this case $U^{(1)}(c_1, ..., c_K)$ satisfies the following system of equations*

$$\frac{\partial U^{(1)}}{\partial c_i} - p_i \sum_{j=K+1}^{M} \frac{\partial c_j}{\partial y_2} \frac{\partial U^{(1)}}{\partial c_j} = 0 \quad (i = 1, ..., K). \tag{71}$$

Furthermore, $p_i \frac{\partial c_j}{\partial y_2}$ $(i = 1, ..., K, j = K+1, ..., M)$ are functions of $(p_1, ..., p_M)$ and can be transformed into functions of $(c_1, ..., c_M)$ using the inverse demand functions.

(The case with more than one change in tastes is discussed in Appendix O.)

The key difference between Theorem 3 and Corollary 1 is that once Properties (P), (TD), (H), (B), (EI) and (EC) are satisfied, the partial differential equation (27) for $U^{(1)}$ always has a solution while the group of partial differential equations (71) may not have a solution. The additional condition (70) in Theorem 3 ensures that a solution exists.[27]

Remark 2 *In attempting to also extend Theorem 2 to the more general setting where one does not assume that the conditional demands are independent of $(c_1, ..., c_K)$, the single partial differential eqn. (26) becomes the set of equations*

$$\frac{\partial U^{(1)}}{\partial c_i} + \sum_{j=K+1}^{M} \frac{\partial c_j}{\partial c_i} \frac{\partial U^{(1)}}{\partial c_j} = 0 \quad (i = 1, 2, ..., K). \tag{72}$$

[26] One will note a similarity between condition (70) and the Antonelli condition that $\forall i, j \in \{1, ..., M-1\}$

$$a_{ji} = \frac{\partial p_j}{\partial c_i} - p_i \frac{\partial p_j}{\partial c_M} = \frac{\partial p_i}{\partial c_j} - p_j \frac{\partial p_i}{\partial c_M} = a_{ij}$$

assuming $p_M = 1$ (see Katzner 1970, p. 45). If inverse demands exist and the corresponding Antonelli matrix is symmetric and negative semidefinite, it follows from Katzner (1970, Theorem 3.2-13) that Properties (S) and (N) hold. If Properties (P), (TD) and (B) are also satisfied, then there exists a twice continuously differentiable utility which rationalizes the demands.

[27] In order for the partial differential equations (71) in Theorem 3 to have a solution, one can think of (70) as ensuring that the equations in (71) are compatible. To see this point, consider a simple set of partial differential equations where there is no solution. If one equation is given by $\partial U / \partial c_1 = f(c_1, c_2, c_3, c_4)$ and another equation is given by $\partial U / \partial c_2 = g(c_1, c_2, c_3, c_4)$, then since $\partial^2 U / \partial c_1 \partial c_2 = \partial^2 U / \partial c_2 \partial c_1$, we must have $\partial f / \partial c_2 = \partial g / \partial c_1$. If this does not hold, the set of partial differential equations do not have a solution.

Although it is possible to write out the Frobenius conditions that ensure the existence of a solution for the set of partial differential equations (72), the results are cumbersome and do not seem to offer much economic insight.

We next illustrate an application of Theorem 3.

Example 7 *Assume that c_1 and c_2 are consumption goods in period one and c_3 and c_4 are goods in period two. The unconditional demands are given by*

$$c_1 = \frac{y_1}{p_1} - \frac{p_1}{p_2} - \frac{p_1}{p_3} - \frac{p_1}{p_4}, \quad c_2 = \frac{p_1^2}{p_2^2}, \quad c_3 = \frac{\frac{p_1^2}{p_3} + \frac{p_1^2}{p_4}}{2p_3} \quad \text{and} \quad c_4 = \frac{\frac{p_1^2}{p_3} + \frac{p_1^2}{p_4}}{2p_4}, \tag{73}$$

where the following is assumed to ensure that Property (P) holds

$$y_1 \geq \frac{p_1^2}{p_2} + \frac{p_1^2}{p_3} + \frac{p_1^2}{p_4}. \tag{74}$$

It is straightforward to show that these demands satisfy Properties (P), (TD), (H) and (B) but violate property (S). Following the same process as discussed in Subsection 4.1.2, one can show that Property (EC) is satisfied, derive the conditional demands $c_3(p_3, p_4, y_2|\ c_1, c_2)$ and $c_4(p_3, p_4, y_2|\ c_1, c_2)$ and verify that they are independent of (c_1, c_2) and satisfy Property (N). Then Theorem 1 holds for the conditional demands and one can recover the following period two utility

$$U^{(2)}(c_3, c_4) = \ln c_3 + \ln c_4. \tag{75}$$

It can be easily verified that (EI) holds. Combining the independent first integrals yields

$$U^{(1)}(c_1, c_2, c_3, c_4) = f\left(c_1 + 2\sqrt{c_2} + 2\sqrt{2(c_3 + c_4)}, \frac{c_3}{c_4}\right). \tag{76}$$

(Supporting computations for this example are provided in Appendix P.)

It suffices to note that the effective consistency results in Corollary 4 and Appendix M extend naturally to the case where $M > 3$ if one adds the assumption that $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ satisfy Properties (S) and (N).

4.3 Quasiconcavity of $U^{(1)}$: A Global Result

In Appendix F, both a necessary condition and a sufficient condition are given such that at least one member in $\{U^{(1)}\}$ is locally twice continuously differentiable, strictly increasing and quasiconcave. We next conjecture, but have not been able to prove, that these properties of $U^{(1)}$ will hold globally only if the period one unconditional demands $(c_1, ...c_K)$ satisfy (S) and (ND). This assertion is motivated by Example 8 below and consistent with each of the examples presented in this paper.

Conjecture 1 *Assume the conditions in Theorem 3 hold implying that there exist a set of period one utilities* $\{U^{(1)}\}$ *and a* $U^{(2)}$*, which is weakly separable in* $(c_{K+1}, \ldots, c_M)$*, and which together rationalize a given set of demands* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ *as sophisticated choice. Then at least one member in* $\{U^{(1)}\}$ *is twice continuously differentiable, strictly increasing and quasiconcave over the full choice space only if the unconditional demands* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ *have Properties (S) and (ND).*[28]

In Conjecture 1, we assume that the conditions in Theorem 3 hold, implying that the period two conditional demands $c_i(p_{K+1}, ..., p_M, y_2| \ c_1, ..., c_K)$ $(i = K + 1, ..., M)$ have the Properties (S) and (N). We also assume that the period one unconditional demands $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ have Properties (S) and (ND), where the latter implies Property (N). However, together these assumptions do not imply that the unconditional demands $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ have Properties (S) and (N) and hence there may not exist a non-changing tastes U.

Consistent with Conjecture 1, we next provide a simple three period, three commodity example in which the unconditional period one demand $c_1(p_1, p_2, p_3, y_1)$ fails to satisfy (ND). Although one can recover the generating utilities $(\{U^{(1)}\}, U^{(2)})$, none of the $U^{(1)}$ functions is quasiconcave. We also show that when there exists a $U^{(1)}$ that is twice continuously differentiable, strictly increasing and quasiconcave and a unique $U^{(2)}$ independent of c_1, the unconditional demand $c_1(p_1, p_2, p_3, y_1)$ satisfies (ND).

Example 8 *Assume three periods and one commodity in each period. The demand functions are given by*

$$c_1 = \frac{\left(\frac{p_1}{p_2^2} + \frac{p_1}{p_3^2}\right) y_1}{2 + \left(\frac{p_1}{p_2}\right)^2 + \left(\frac{p_1}{p_3}\right)^2} \quad \text{and} \quad c_i = \frac{y_1}{\left(2 + \left(\frac{p_1}{p_2}\right)^2 + \left(\frac{p_1}{p_3}\right)^2\right) p_i} \quad (i = 2, 3). \quad (77)$$

As the following demonstrates, period one consumption does not satisfy (ND)

$$\frac{\partial c_1}{\partial p_1} + c_1 \frac{\partial c_1}{\partial y_1} = \frac{2 p_2^2 p_3^2 (p_2^2 + p_3^2) y_1}{\left(2 p_2^2 p_3^2 + p_1^2 (p_2^2 + p_3^2)\right)^2} > 0. \quad (78)$$

[28]It should be noted that if the full set of unconditional demands $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ can be rationalized by a non-changing tastes utility function, they can only satisfy (S) and (N) but not (ND) since (B) and (H) imply that the Slutsky matrix is always singular and hence cannot be negative definite (see Mas-Colell, Whinston and Green 1995, Proposition 2.F.3). But it is possible for the partial demand system $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ to satisfy (S) and (ND) as shown in Example 8 below.

Since the conditional demands $c_i = y_2/(2p_i)$ $(i = 2, 3)$ *are independent of* c_1*, applying Corollary 1 and the associated recovery process one obtains*

$$U^{(1)}(c_1, c_2, c_3) = f\left(c_1^2 + c_2^2 + c_3^2, \frac{c_2}{c_3}\right) \quad \text{and} \quad U^{(2)}(c_2, c_3) = \ln c_2 + \ln c_3. \tag{79}$$

Also consistent with Conjecture 1, it can be verified that no member of the set $\{U^{(1)}\}$ *defined in (79) can be quasiconcave. Next assume a different set of unconditional demands*

$$c_1 = \frac{y_1}{p_1\left(1 + p_1\left(\frac{1}{\sqrt{2p_2}} + \frac{1}{\sqrt{2p_3}}\right)^2\right)} \text{ and } c_i = \frac{y_1 p_1\left(\frac{1}{\sqrt{2p_2}} + \frac{1}{\sqrt{2p_3}}\right)^2}{2p_i\left(1 + p_1\left(\frac{1}{\sqrt{2p_2}} + \frac{1}{\sqrt{2p_3}}\right)^2\right)} \quad (i = 2, 3). \tag{80}$$

The conditions in Conjecture 1 are satisfied including the requirement that the period one unconditional demand function satisfies (ND)

$$\frac{\partial c_1}{\partial p_1} + c_1\frac{\partial c_1}{\partial y_1} = -\frac{4p_2p_3\left(\sqrt{p_2} + \sqrt{p_3}\right)^2 y_1}{p_1\left(p_1\left(\sqrt{p_2} + \sqrt{p_3}\right)^2 + 2p_2p_3\right)^2} < 0. \tag{81}$$

Since the conditional demands $c_i = y_2/(2p_i)$ $(i = 2, 3)$ *are independent of* c_1*, applying Corollary 1 one obtains*

$$U^{(1)}(c_1, c_2, c_3) = f\left(\sqrt{c_1} + \sqrt{c_2} + \sqrt{c_3}, \frac{c_2}{c_3}\right) \quad \text{and} \quad U^{(2)}(c_2, c_3) = \ln c_2 + \ln c_3. \tag{82}$$

If $f(x, y) = x$,

$$U^{(1)}(c_1, c_2, c_3) = \sqrt{c_1} + \sqrt{c_2} + \sqrt{c_3}. \tag{83}$$

Then consistent with Conjecture 1, (83) is strictly increasing and quasiconcave over the full choice space.

One may wonder whether there is any connection between the assumption that $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ satisfy Properties (S) and (ND) and the satisfaction of Properties (P) through (EI) listed in Section 2. We next show that surprisingly, if $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ satisfy Properties (S) and (ND), then the conditions in Lemma 1 hold implying that (EC) is satisfied locally. First, denote the Slutsky matrix for the period one unconditional demands $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ as $(\sigma_{ij})_{K\times K}$.

Theorem 4 *If the unconditional demands* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ *satisfy Properties (S) and (ND), then at each point* $(p_1^0, ..., p_K^0, y_1^0) \in \mathbb{R}_{++}^K \times \mathbb{R}_+$

$$\det J_c = \det \left.\frac{\partial(c_1, ..., c_K, y_2)}{\partial(p_1, ..., p_K, y_1)}\right|_{(p_1,...,p_K,y_1)=\left(p_1^0,...,p_K^0,y_1^0\right)} \neq 0 \tag{84}$$

and Property (EC) holds in a neighborhood of $(p_1^0, ..., p_K^0, y_1^0)$.

Therefore if Conjecture 1 can be verified, then combining it with our other results, $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ satisfying Properties (S) and (ND) is necessary not only for the existence of the conditional demands but also for the possibility of a strictly increasing and quasiconcave member in the set $\{U^{(1)}\}$.

5 Rationalizing Naive Choice

This section addresses the question of integrability for naive choice in a three commodity setting. Assuming that the given demands $c_1(p_1, p_2, p_3, y_1)$, $c_2(p_1, p_2, p_3, y_1)$ and $c_3(p_1, p_2, p_3, y_1)$ correspond to naive choice, we first provide a sufficient condition for the existence of a rationalizing $(U^{(1)}, U^{(2)})$-pair. As in the case of sophisticated demands, it is possible for naive demands to correspond to a set of period one utilities $\{U^{(1)}\}$, where each member of the set differs from other members by more than an increasing transformation. Paralleling the argument in Subsections 4.1.1 and 4.1.2, one can derive the period two and three conditional demand functions $c_2(p_2, p_3, y_2)$ and $c_3(p_2, p_3, y_2)$ and use the Hurwicz and Uzawa (1971) recovery process to obtain a unique $U^{(2)}(c_2, c_3)$ (up to an increasing transformation).

To recover a $U^{(1)}$ function from naive demands, one confronts the immediate problem of having information only about the c_1 demand function (and the amount $y_2 = y_1 - p_1c_1$ available for c_2 and c_3). In the case where one has resolute demand functions corresponding to c_1, c_2 and c_3, a unique non-changing tastes utility can be recovered using Theorem 1 and the Hurwicz and Uzawa (1971) process. But for the case of changing tastes, in general one only has the c_1 demand function where naive and resolute choice agree. This problem is closely related to the question of integrability for incomplete demand systems which is often encountered in empirical static demand applications (see, for example, LaFrance and Hanemann 1989). Consider the three commodity version of the P_1 optimization problem (4). Assume that the demand function $c_1(p_1, p_2, p_3, y_1)$ is known but the resolute $c_i(p_1, p_2, p_3, y_1)$ $(i = 2, 3)$ are not known. We next give a sufficient condition for the existence of a $(U^{(1)}, U^{(2)})$-pair that rationalizes naive demands. This theorem is a direct consequence of Theorem 2 in Epstein (1982) and the results in his Table 1 adapted to our setting.[29]

[29] LaFrance and Hanemann (1989) introduce the notion of weak integrability of incomplete demand systems. They relax the condition in Epstein (1982) that $\partial c_i/\partial p_j = 0$ $(i = 1, ..., K;$ $j = K+1, ..., M)$. It is interesting to observe that this condition is equivalent to preferences being myopic separable as discussed in Kannai, Selden and Wei (2014). As will be seen below, myopic separability of period one demands not only guarantees the existence of $U^{(1)}$, but also

Theorem 5 *Assume a given set of demand functions* $c_i(p_1, p_2, p_3, y_1)$ $(i = 1, 2, 3)$ *which have the Properties (P), (TD), (H), (B) and (EC) and the corresponding conditional demand functions* $c_2(p_2, p_3, y_2|\ c_1)$ *and* $c_3(p_2, p_3, y_2|\ c_1)$ *have Property (N). Furthermore suppose that* $c_1(p_1, p_2, p_3, y_1)$ *satisfies Property (ND) and is linear in* y_1 *or independent of* (p_2, p_3)*. Then there exists a* $(U^{(1)}, U^{(2)})$*-pair which generates these demands as the result of naive choice, where* $U^{(1)}$ *is continuous, non-decreasing and quasiconcave and* $U^{(2)}$ *satisfies Property 1.*[30]

The application of Theorem 5 is quite different from that of the integrability results derived for sophisticated choice in the prior section. Rather than solving a partial differential equation such as (26) in Theorem 2, the recovery method uses the assumption in Theorem 5 that $c_1(p_1, p_2, p_3, y_1)$ is linear in y_1 or independent of (p_2, p_3) which implies the existence of an expenditure function which can then be used to recover $U^{(1)}$ following the Hurwicz and Uzawa process. This is illustrated in Example 9 in Appendix R.

For a given c_1 demand function, each specific $U^{(1)}$ derived from the expenditure function using Table 1 in Epstein (1982) may not represent the consumer's actual period one preferences generating her resolute choice demands since the full set of possible $U^{(1)}$ functions is not recovered. Next we discuss two special cases, where it is possible to characterize the full set of utilities $(\{U^{(1)}\}, U^{(2)})$ that rationalize naive choice. In these cases, the consumer's period one utility function that would rationalize resolute choice will be in the $\{U^{(1)}\}$ set. It should be emphasized that these two special cases are the only instances of which we are aware where the full set of $U^{(1)}$ functions can be recovered.

Theorem 6 *Assume a given set of demand functions* $c_i(p_1, p_2, p_3, y_1)$ $(i = 1, 2, 3)$ *which have the Properties (P), (TD), (H) and (B). Then there exists a* $(U^{(1)}, U^{(2)})$*-pair which generates these demands as the result of naive choice where* $U^{(1)}(c_1, c_2, c_3)$

the recovery of the full set of utilities $\{U^{(1)}\}$ rationalizing c_1 as naive choice. A second case where one can recover $\{U^{(1)}\}$ is when preferences are representable by the generalized quasilinear utility function (87) in Theorem 7 below. These results can be viewed as a generalization of conditions for the integrability of incomplete demand systems in Epstein (1982) and LaFrance and Hanemann (1989).

[30]Epstein (1982) assumes that the incomplete demand system satisfy Properties (S) and (ND). For the single demand function $c_1(p_1, p_2, p_3, y_1)$, (S) is automatically satisfied and (ND) becomes

$$\sigma_{11} = \frac{\partial c_1}{\partial p_1} + c_1 \frac{\partial c_1}{\partial y_1} < 0.$$

We assume c_1 satisfies (ND) instead of (N) since as Epstein (1982, Example 1) argues, for an incomplete demand system, (N) cannot ensure the existence of a quasiconcave $U^{(1)}$.

takes the form

$$U^{(1)}(c_1, c_2, c_3) = f(g(c_1) c_2, g(c_1) c_3), \tag{85}$$

where f *is an arbitrary function and* $g(c_1)$ *is uniquely determined (up to a constant) by* $c_1(p_1, p_2, p_3, y_1)$*, at least one member of* $\{U^{(1)}\}$ *is continuous, nondecreasing and quasiconcave and* $U^{(2)}$ *satisfies Property 1 if (i) the unconditional demand functions* $c_i(p_1, p_2, p_3, y_1)$ $(i = 1, 2, 3)$ *also have Property (EC), (ii) the conditional demand functions* $c_2(p_2, p_3, y_2 | c_1)$ *and* $c_3(p_2, p_3, y_2 | c_1)$ *have Property (N) and (iii) the unconditional demand function* $c_1(p_1, p_2, p_3, y_1)$ *satisifies (ND) and is independent of* p_2 *and* p_3*.*

It will be noted that in Corollary 4 we obtain exactly the same $U^{(1)}(c_1, c_2, c_3)$ for rationalizing sophisticated choice as is obtained in Theorem 6 for rationalizing naive choice. However in the latter result, unlike the former, since conditional demands are not assumed to be proportional to y_2, the preferences rationalizing naive choice may not be effectively consistent. Hence there may not be a nonchanging tastes $U(c_1, c_2, c_3)$ which rationalizes naive choice. This is consistent with the fact that in general Property (S) is not be satisfied when the conditions in Theorem 6 hold.

It follows from Theorem 6 that $U^{(2)}$ can be recovered from the conditional demands $c_2(p_2, p_3, y_2 | c_1)$ and $c_3(p_2, p_3, y_2 | c_1)$ following the Hurwicz and Uzawa (1971) recovery process. To recover $U^{(1)}(c_1, c_2, c_3)$, since f is an arbitrary function, one only needs to determine $g(c_1)$. Given that the unconditional period one demand in Theorem 6 is myopic separable, it follows from Selden and Wei (2015) that $c_1(p_1, p_2, p_3, y_1)$ is a solution to

$$p_1 c_1 + p_1 \frac{g(c_1)}{g'(c_1)} = y_1. \tag{86}$$

Therefore, one can use the same approach to recover $g(c_1)$ from the unconditional demand function $c_1(p_1, p_2, p_3, y_1)$ as shown in the proof of Corollary 4 and in the discussion of Example 1 following Corollary 4. Similar observations apply as well for Theorem 7 below.

Theorem 7 *Assume a given set of demand functions* $c_i(p_1, p_2, p_3, y_1)$ $(i = 1, 2, 3)$ *which have the Properties (P), (TD), (H) and (B). Then there exists a* $(U^{(1)}, U^{(2)})$*-pair which generates these demands as the result of naive choice where* $U^{(1)}(c_1, c_2, c_3)$ *takes the form*

$$U^{(1)}(c_1, c_2, c_3) = f(g(c_1) + c_3, c_2), \tag{87}$$

where f *is an arbitrary function and* $g(c_1)$ *is uniquely determined (up to a constant) by* $c_1(p_1, p_2, p_3, y_1)$*, at least one member of* $\{U^{(1)}\}$ *is continuous, nondecreasing and quasiconcave and* $U^{(2)}$ *satisfies Property 1 if (i) the unconditional* *demand functions* $c_i(p_1, p_2, p_3, y_1)$ $(i = 1, 2, 3)$ *also have Property (EC), (ii) the conditional demand functions* $c_2(p_2, p_3, y_2|\ c_1)$ *and* $c_3(p_2, p_3, y_2|\ c_1)$ *have Property (N) and (iii) the unconditional demand function* $c_1(p_1, p_2, p_3, y_1)$ *satisifies (ND) and is independent of* p_2 *and* y_1.[31]

It can be verified that the demand functions (15) - (16) in Example 1 satisfy the conditions in Theorem 6. As a result, these demands can be rationalized as naive choice by

$$U^{(1)}(c_1, c_2, c_3) = f\left(\left(\sqrt{c_1} + 1\right) c_2, \left(\sqrt{c_1} + 1\right) c_3\right) \tag{88}$$

and

$$U^{(2)}(c_2, c_3|\ c_1) = \ln c_2 + \ln c_3. \tag{89}$$

Given that the demands in this example are effectively consistent, it is not surprising that the rationalizing utilities (88) - (89) are the same as those obtained for sophisticated choice (see the discussion at the end of Section 4.1.3).

Clearly the integrability results for the case of naive choice are much weaker than for sophisticated choice. For the sophisticated case although one confronts incomplete information, as evidenced by the fact that demands are rationalized by a set of period one utilities $\{U^{(1)}\}$ rather than a single utility as in the classic static demand case, the full set of rationalizing utilities can be recovered. In contrast for the naive case, except for the special case of demands considered in Theorems 6 and 7, even assuming very strong conditions such as in Theorem 5, there is no known approach to recover the full set of $U^{(1)}$ functions.

(In Appendix R, we show that Theorems 5, 6 and 7 extend naturally to the case of more than three commodities. Proofs for the more general cases of Theorems 6 and 7 are given in the Appendix.)

6 Conclusion

In this paper, we extend the classic integrability results of Hurwicz and Uzawa (1971) to the case of changing tastes. Necessary and sufficient conditions are given for the existence of a rationalizing $(U^{(1)}, U^{(2)})$-pair assuming that the given demands correspond to sophisticated choice and sufficient conditions are given assuming that the demands correspond to naive choice. A number of open questions

[31] If the c_1 demand function does not depend on y_1 and only depends on p_1 and p_2, then

$$U^{(1)}(c_1, c_2, c_3) = f\left(g(c_1) + c_2, c_3\right).$$

remain. First although we can extend Corollary 1 to the case of more than three commodities, is it possible to similarly extend Theorem 3? Second whereas a classic revealed preference test (e.g., Varian 1982) based on period two demands and prices can be employed to determine whether conditional demands are consistent with maximizing $U^{(2)}$, can a revealed preference test be constructed for $U^{(1)}$?[32] Third since the integrability results for naive choice are generally weak, do stronger conditions exist in other related settings such as where instead of choosing over (c_1, c_2, c_3) vectors, the consumer chooses over current consumption and one and two period zero coupon bonds (see, for example, Selden and Wei 2015)?[33]

Appendix

A Proof of Lemma 1

First prove sufficiency. Consider the following set of equations

$$c_i = c_i\left(p_1, ..., p_M, y_1\right) \ \ (i = 1, ..., K) \quad \text{and} \quad y_2 = y_1 - \sum_{i=1}^{K} p_i c_i\left(p_1, ..., p_M, y_1\right). \tag{A.1}$$

If

$$\det \left. \frac{\partial\left(c_1, ..., c_K, y_2\right)}{\partial\left(p_1, ..., p_K, y_1\right)} \right|_{\left(p_1, ..., p_K, y_1\right) = \left(p_1^0, ..., p_K^0, y_1^0\right)} \neq 0, \tag{A.2}$$

then there is an open neighborhood containing $(p_1^0, ..., p_K^0, y_1^0)$ such that $(p_1, ..., p_K, y_1)$ can be solved for as functions of $(c_1, ..., c_K, p_{K+1}, ..., p_M, y_2)$ from the set of equations (A.1). Substituting

$$p_i\left(c_1, ..., c_K, p_{K+1}, ..., p_M, y_2\right) (i = 1, ..., K) \quad \text{and} \quad y_1\left(c_1, ..., c_K, p_{K+1}, ..., p_M, y_2\right) \tag{A.3}$$

into the period two unconditional demands, one obtains the conditional demands $c_i(p_{K+1}, ..., p_M, y_2 |\ c_1, ..., c_K)$ $(i = K + 1, ..., M)$. Next prove necessity. Since there is an open neighborhood containing $(p_1^0, ..., p_K^0, y_1^0)$ such that

$$c_i\left(p_1, ..., p_M, y_1\right) = c_i\left(p_{K+1}, ..., p_M, y_2 \middle|\ c_1, ..., c_K\right) \quad \left(i = K + 1, ..., M\right), \tag{A.4}$$

[32] Since in general $U^{(1)}$ need not be quasiconcave, it may be helpful to consider the nonparametric tests discussed in Polisson, Quah and Renou (2015) where the rationalizing utility is not required to be quasiconcave.

[33] As noted in Selden and Wei (2015, footnote 48), when considering naive and sophisticated choice for consumption and bond purchases it is useful to assume a simple transaction cost structure to ensure that the consumer avoids retrading.

$(p_1, ..., p_K, y_1)$ must be solved for as functions of $(c_1, ..., c_K, p_{K+1}, ..., p_M, y_2)$ from the set of equations (A.1). Thus the inverse function theorem implies

$$\det \left. \frac{\partial (c_1, ..., c_K, y_2)}{\partial (p_1, ..., p_K, y_1)} \right|_{(p_1,...,p_K,y_1)=\left(p_1^0,...,p_K^0,y_1^0\right)} \neq 0. \tag{A.5}$$

B Proof of Lemma 2

This result directly follows from the inverse function theorem.

C Global Tests for Properties (EC) and (EI)

To derive global tests for Properties (EC) and (EI), first consider the following global inverse function theorem.

Theorem 8 *(Gordon 1972, Theorem A) A continuously differentiable map* $\mathbf{f}(\mathbf{x})$ *from* $\mathbb{R}^N$ *to* $\mathbb{R}^N$ *is a diffeomorphism if and only if* $\mathbf{f}$ *is proper and the Jacobian* $\det(\partial f_i/\partial x_j)$ *never vanishes.*

As noted by Gordon (1972), a map is proper if and only if inverse images of compact subsets are compact. Following Wagstaff (1975), since a set in $\mathbb{R}^N$ is compact if and only if it is bounded and closed, and since the pre-image of a closed set under a continuous map is closed, the properness of any continuous $\mathbf{f} : \mathbb{R}^N \to \mathbb{R}^N$ reduces to the requirement that

$$\|\mathbf{x}\| \to \infty \Rightarrow \|\mathbf{f}(\mathbf{x})\| \to \infty, \tag{C.1}$$

i.e., if $\mathbf{x}$ goes to the boundary of $\mathbb{R}^N$, then $\|\mathbf{f}(\mathbf{x})\| \to \infty$. To apply this result in a standard demand setting, the problem is that prices typically are not allowed to be negative and hence cannot be defined on $\mathbb{R}^N$. To solve this problem, we follow Wagstaff (1975) in using the following normalization

$$\sum_{i=1}^{M} p_i = 1 \tag{C.2}$$

for a system of M commodities. Then consider the continuously differentiable excess demand function $\mathbf{z}(\mathbf{p})$ in $\mathbb{R}^{M-1}$, where $\mathbf{p}$ is a price vector in the set

$$S = \left\{ \mathbf{p} \in \mathbb{R}_{++}^{M-1} : \sum_{i=1}^{M-1} p_i < 1 \right\}. \tag{C.3}$$

Introduce the following homeomorphic map $h : \mathbf{x} \to \mathbf{p}$, where $\mathbf{x} \in \mathbb{R}^{M-1}$ and $\mathbf{p} \in S$,

$$\mathbf{h}(\mathbf{x}) = \begin{cases} \mathbf{p} + \frac{\mathbf{x}}{(1+\|\mathbf{x}\|)g\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)} & (\mathbf{x} \neq \mathbf{0}) \\ \mathbf{p} & (\mathbf{x} = \mathbf{0}) \end{cases}, \tag{C.4}$$

given that

$$g(\mathbf{x}) = \inf \left\{ t \in \mathbb{R}_{++} : \mathbf{p} + \frac{1}{t}\mathbf{x} \in S \right\}. \tag{C.5}$$

Since h is a homeomorphic map, $\mathbf{z}(\mathbf{p})$ is proper if and only if $\mathbf{z} \circ \mathbf{h}(\mathbf{x})$ is proper. Note that $\|\mathbf{x}\| \to \infty$ if and only if $\mathbf{p} \to \partial S$, where ∂S is the boundary of the set S. Therefore, Theorem 8 can be also restated as follows.

Theorem 9 *A continuously differentiable map* $\mathbf{z}$ *from* S *to* $\mathbb{R}^{M-1}$ *is a diffeomorphism if and only if for any* $i \in \{1, ..., M-1\}$

$$p_i \to 0 \Rightarrow \|\mathbf{z}(\mathbf{p})\| \to \infty \tag{C.6}$$

and the Jacobian $\det(\partial z_i / \partial p_j)$ *never vanishes.*

The condition (C.6) is often referred to as a desirability condition in equilibrium demand analyses (see, for example, Balasko 2011, p. 28).

Building on the argument in Wagstaff (1975, p. 523), in a pure exchange economy where demands and endowments are denoted respectively c_i and $\bar{c}_i$ $(i = 1, ..., M-1)$, since endowments are finite the excess demands $z_i = c_i - \bar{c}_i$ are not defined on $\mathbb{R}^{M-1}$ but on the set

$$H = \left\{ \mathbf{z} \in \mathbb{R}^{M-1} : z_i \geq -\bar{c}_i, i = 1, ..., M-1 \right\}. \tag{C.7}$$

To ensure that the map is surjective, the domain for $\mathbf{z}(\mathbf{p})$ should be modified from $\mathbb{R}^{M-1}$ to H. Since $-\bar{c}_i$ is also a boundary point, $\mathbf{z}(\mathbf{p})$ is proper if and only if for some $i \in \{1, ..., M-1\}$

$$p_i \to 0 \Rightarrow \|\mathbf{z}(\mathbf{p})\| \to \infty \text{ or } z_i(\mathbf{p}) \to -\bar{c}_i \tag{C.8}$$

for some $i \in \{1, ..., M-1\}$. Therefore, Theorem 9 can be modified as follows.

Theorem 10 *A continuously differentiable map* $\mathbf{z}$ *from* S *to* H *is a diffeomorphism if and only if for any* $i, j \in \{1, ..., M-1\}$

$$p_i \to 0 \Rightarrow \|\mathbf{z}(\mathbf{p})\| \to \infty \quad \textit{or} \quad z_i(\mathbf{p}) \to -\bar{c}_i \tag{C.9}$$

and the Jacobian $\det(\partial z_i / \partial p_j)$ *never vanishes.*

To transform this result based on excess demands to one based on unconditional demands, notice that we have the following relation between the (unconditional) demands and excess demands

$$c_i(p_1, ..., p_{M-1}) = z_i(p_1, ..., p_{M-1}) + \bar{c}_i \quad (i = 1, ..., M-1). \tag{C.10}$$

In a distribution economy such as is being assumed in the main body of this paper, it is more typical to use $y_1 = 1$ as the normalization rather than (C.2). But since we want to apply the global inverse function results of Wagstaff (1975) to our setting, we use his normalization (C.2), which will not affect the necessary and sufficient conditions for the existence of the global inverse function. However in this case, y_1 will be determined from

$$y_1 = \sum_{i=1}^{M-1} p_i \bar{c}_i + \left(1 - \sum_{i=1}^{M-1} p_i\right) \bar{c}_M. \tag{C.11}$$

Then we have the following global test for Property (EI).

Corollary 5 *For a given set of unconditional demands* $c_i(p_1, ..., p_{M-1}, y_1)$ $(i = 1, ..., M)$*, where*

$$y_1 = \sum_{i=1}^{M} p_i \bar{c}_i, \tag{C.12}$$

$(p_1, ..., p_{M-1}, y_1) \in S \times \mathbb{R}_+$ *can be expressed as continuously differentiable functions of* $(\bar{c}_1, ..., \bar{c}_M)$ *based on the set of equations*

$$c_i(p_1, ..., p_{M-1}, y_1) = \bar{c}_i \quad (i = 1, ..., M-1) \tag{C.13}$$

and

$$y_1 = \sum_{i=1}^{M} p_i \bar{c}_i \tag{C.14}$$

if and only if for any $i, j \in \{1, ..., M-1\}$

$$p_i \to 0 \Rightarrow \|(c_1(\mathbf{p}), ..., c_{M-1}(\mathbf{p}))\| \to \infty \quad \text{or} \quad c_i(\mathbf{p}) \to 0 \tag{C.15}$$

and the Jacobian $\det(\partial c_i / \partial p_j)$ *never vanishes.*

Proof. Since

$$c_i(p_1, ..., p_{M-1}, y_1) = z_i(p_1, ..., p_{M-1}, y_1) + \bar{c}_i \quad (i = 1, ..., M-1), \tag{C.16}$$

it follows from Theorem 10 that $(p_1, ..., p_{M-1}) \in S$ can be expressed as continuously differentiable functions of $(\bar{c}_1, ..., \bar{c}_M)$ based on the set of equations

$$c_i(p_1, ..., p_{M-1}, y_1) = \bar{c}_i \quad (i = 1, ..., M-1), \tag{C.17}$$

where

$$y_1 = \sum_{i=1}^{M-1} p_i \overline{c}_i + \left(1 - \sum_{i=1}^{M-1} p_i\right) \overline{c}_M, \tag{C.18}$$

if and only if for any $i, j \in \{1, ..., M-1\}$

$$p_i \to 0 \Rightarrow \|(c_1(\mathbf{p}), ..., c_{M-1}(\mathbf{p}))\| \to \infty \quad \text{or} \quad c_i(\mathbf{p}) \to 0 \tag{C.19}$$

and the Jacobian $\det(\partial c_i / \partial p_j)$ never vanishes. Noticing that

$$y_1 = \sum_{i=1}^{M} p_i \overline{c}_i = \sum_{i=1}^{M-1} p_i \overline{c}_i + \left(1 - \sum_{i=1}^{M-1} p_i\right) \overline{c}_M, \tag{C.20}$$

if $(p_1, ..., p_{M-1}) \in S$ can be expressed as continuously differentiable functions of $(\overline{c}_1, ..., \overline{c}_M)$, y_1 can be also expressed as a continuously differentiable function of $(\overline{c}_1, ..., \overline{c}_M)$, which completes the proof. ■

Paralleling (C.6) in Theorem 9, the condition

$$p_i \to 0 \Rightarrow \|(c_1(\mathbf{p}), ..., c_{M-1}(\mathbf{p}))\| \to \infty \tag{C.21}$$

in (C.15) and (C.29) in Corollary 6 below will be recognized as an assumption on the desirability of demand.

To apply Corollary 5 as a test, once the unconditional demands $c_i(p_1, ..., p_{M-1}, y_1)$ $(i = 1, ..., M)$ are given, one needs to rewrite $c_i(p_1, ..., p_{M-1}, y_1)$ $(i = 1, ..., M-1)$ as

$$c_i\left(p_1, ..., p_{M-1}, \sum_{i=1}^{M-1} p_i \overline{c}_i + \left(1 - \sum_{i=1}^{M-1} p_i\right) \overline{c}_M\right) \tag{C.22}$$

and then check whether the following conditions hold for any $\overline{c}_i \in \mathbb{R}_+$: for any $i, j \in \{1, ..., M-1\}$

$$p_i \to 0 \Rightarrow \|(c_1(\mathbf{p}), ..., c_{M-1}(\mathbf{p}))\| \to \infty \quad \text{or} \quad c_i(\mathbf{p}) \to 0 \tag{C.23}$$

and the Jacobian $\det(\partial c_i / \partial p_j)$ never vanishes.

To derive a necessary and sufficient condition for the existence of conditional demands, consider the normalization

$$\sum_{i=1}^{K} p_i = 1 \tag{C.24}$$

resulting in $(p_1, ..., p_{K-1})$ being a price vector in the set

$$S' = \left\{\mathbf{p} \in \mathbb{R}_{++}^{K-1} : \sum_{i=1}^{K-1} p_i < 1\right\}. \tag{C.25}$$

Then we have the following global test for Property (EC).

Corollary 6 *For a given set of unconditional demands* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$, *where*

$$y_1 = \sum_{i=1}^{M} p_i \overline{c}_i, \tag{C.26}$$

$(p_1, ..., p_{K-1}, y_1) \in S' \times \mathbb{R}_+$ *can be expressed as continuously differentiable functions of* $(\overline{c}_1, ..., \overline{c}_K, p_{K+1}, ..., p_M, y_2)$ *based on the set of equations*

$$c_i(p_1, ..., p_M, y_1) = \overline{c}_i \quad (i = 1, ..., K-1) \tag{C.27}$$

and

$$y_1 = \sum_{i=1}^{K} p_i \overline{c}_i + y_2 \tag{C.28}$$

if and only if for any $i, j \in \{1, ..., K-1\}$

$$p_i \to 0 \Rightarrow \|(c_1(\mathbf{p}), ..., c_{K-1}(\mathbf{p}))\| \to \infty \quad \text{or} \quad c_i(\mathbf{p}) \to 0 \tag{C.29}$$

and the Jacobian $\det(\partial c_i / \partial p_j)$ *never vanishes.*

Proof. Since

$$c_i(p_1, ..., p_M, y_1) = z_i(p_1, ..., p_M, y_1) + \overline{c}_i \quad (i = 1, ..., K-1),$$

it follows from Theorem 10 that $(p_1, ..., p_{K-1}) \in S'$ can be expressed as continuously differentiable functions of $(\overline{c}_1, ..., \overline{c}_K, p_{K+1}, ..., p_M, y_2)$ based on the set of equations

$$c_i(p_1, ..., p_M, y_1) = \overline{c}_i \quad (i = 1, ..., K-1), \tag{C.30}$$

where

$$y_1 = \sum_{i=1}^{K-1} p_i \overline{c}_i + \left(1 - \sum_{i=1}^{K-1} p_i\right) \overline{c}_K + y_2, \tag{C.31}$$

if and only if for any $i, j \in \{1, ..., K-1\}$

$$p_i \to 0 \Rightarrow \|(c_1(\mathbf{p}), ..., c_{K-1}(\mathbf{p}))\| \to \infty \quad \text{or} \quad c_i(\mathbf{p}) \to 0 \tag{C.32}$$

and the Jacobian $\det(\partial c_i / \partial p_j)$ never vanishes. Noticing that

$$y_1 = \sum_{i=1}^{K-1} p_i \overline{c}_i + \left(1 - \sum_{i=1}^{K-1} p_i\right) \overline{c}_K + y_2, \tag{C.33}$$

if $(p_1, ..., p_{K-1}) \in S'$ can be expressed as continuously differentiable functions of $(\overline{c}_1, ..., \overline{c}_K, p_{K+1}, ..., p_M, y_2)$, y_1 can be also expressed as a continuously differentiable function of $(\overline{c}_1, ..., \overline{c}_K, p_{K+1}, ..., p_M, y_2)$, which completes the proof. ■

D Proof of Theorem 2

First prove sufficiency. The sophisticated demand functions $c_1(p_1, p_2, p_3, y_1)$, $c_2(p_1, p_2, p_3, y_1)$ and $c_3(p_1, p_2, p_3, y_1)$ have the Properties (P), (TD), (H) and (B). Then if (EC) holds, the conditional demand functions $c_2(p_2, p_3, y_2|\ c_1)$ and $c_3(p_2, p_3, y_2|\ c_1)$ exist and also have the Properties (P), (TD), (H) and (B). It follows from Katzner (1970, Theorem 4.1-2) that (S) always holds for c_2 and c_3 and thus assuming (N) is satisfied, Theorem 2.6 in Jehle and Reny (2011) ensures the existence of a unique $U^{(2)}$ satisfying Property 1. Given the conditional demands $c_2(p_2, p_3, y_2|\ c_1)$ and $c_3(p_2, p_3, y_2|\ c_1)$, $U^{(1)}$ must satisfy the following partial differential equation

$$\frac{\partial U^{(1)}}{\partial c_1} + \frac{\partial c_2}{\partial c_1}\frac{\partial U^{(1)}}{\partial c_2} + \frac{\partial c_3}{\partial c_1}\frac{\partial U^{(1)}}{\partial c_3} = 0, \tag{D.1}$$

which follows from $\frac{dU^{(1)}}{dc_1} = 0$ by the chain rule. Since Property (EC) holds, (p_1, p_2, p_3) can be transformed into functions of (c_1, c_2, c_3) through inverse demands, implying that the coefficients in the first order partial differential equation (D.1) can be expressed as functions of (c_1, c_2, c_3). Therefore, $U^{(1)}$ exists, but is not uniquely determined. Next prove necessity. Assume there exists a $U^{(2)}$ satisfying Property 1. Maximizing $U^{(2)}(c_2, c_3|\ c_1)$ subject to the budget constraint

$$p_2 c_2 + p_3 c_3 = y_2 \tag{D.2}$$

yields the conditional demands $c_2(p_2, p_3, y_2|\ c_1)$ and $c_3(p_2, p_3, y_2|\ c_1)$. Therefore, the conditional demands must exist, implying that Property (EC) holds. Since the positive, continuously differentiable conditional demands $c_2(p_2, p_3, y_2|\ c_1)$ and $c_3(p_2, p_3, y_2|\ c_1)$ can be rationalized by a utility function $U^{(2)}$ satisfying Property 1, it follows from Jehle and Reny (2011) that the Properties (H) and (B) are satisfied. Since $U^{(1)}$ satisfies equation (D.1), if the inverse demands do not exist, then the coefficients of this partial differential equation cannot be expressed as functions of (c_1, c_2, c_3) and thus there is no solution for $U^{(1)}$. Therefore, Property (EI) must be satisfied.

E Four Period, Four Commodity Case

Assume there are four periods and one commodity in each period. The consumer faces the optimization problems

$$P_1: \max_{c_1, c_2, c_3, c_4} U^{(1)}(c_1, c_2, c_3, c_4) \qquad S.T.\ y_1 \geq \sum_{j=1}^{4} p_j c_j, \tag{E.1}$$

$$P_2: \max_{c_2,c_3,c_4} U^{(2)}(c_2,c_3,c_4|\ c_1) \quad S.T.\ \ y_2 = y_1 - p_1c_1 \geq \sum_{j=2}^{4} p_j c_j \tag{E.2}$$

and

$$P_3: \max_{c_3,c_4} U^{(3)}(c_3,c_4|\ c_1,c_2) \quad S.T.\ \ y_3 = y_1 - p_1c_1 - p_2c_2 \geq \sum_{j=3}^{4} p_j c_j. \tag{E.3}$$

We assume that Properties (P), (TD), (H) and (B) hold. First consider the existence of $U^{(3)}$. Suppose that (EC) holds, i.e.,[34]

$$\frac{\partial (c_1,c_2,y_3)}{\partial (p_1,p_2,y_1)} \neq 0 \quad \forall (p_1,p_2,y_1) \in \mathbb{R}^2_{++} \times \mathbb{R}_+, \tag{E.4}$$

implying that the conditional demands $c_3(p_3,p_4,y_3|\ c_1,c_2)$ and $c_4(p_3,p_4,y_3|\ c_1,c_2)$ exist. If one assumes $c_3(p_3,p_4,y_3|\ c_1,c_2)$ and $c_4(p_3,p_4,y_3|\ c_1,c_2)$ have Property (N), then there exists a $U^{(3)}$. Next consider the existence of $U^{(2)}$. Since

$$y_3 = y_2 - p_2c_2, \tag{E.5}$$

we have

$$c_i(p_3,p_4,y_3|\ c_1,c_2) = c_i(p_3,p_4,y_2 - p_2c_2|\ c_1,c_2) \quad (i = 3,4). \tag{E.6}$$

If we assume that (EC) holds, i.e.,

$$\frac{\partial (c_1,y_2)}{\partial (p_1,y_1)} \neq 0 \quad \forall (p_1,y_1) \in \mathbb{R}_{++} \times \mathbb{R}_+, \tag{E.7}$$

then we have the conditional demand function $c_2(p_2,p_3,p_4,y_2|\ c_1)$ and $U^{(2)}$ satisfies

$$\frac{\partial U^{(2)}(c_2,c_3,c_4|\ c_1)}{\partial c_2} + \frac{\partial c_3}{\partial c_2}\frac{\partial U^{(2)}(c_2,c_3,c_4|\ c_1)}{\partial c_3} + \frac{\partial c_4}{\partial c_2}\frac{\partial U^{(2)}(c_2,c_3,c_4|\ c_1)}{\partial c_4} = 0. \tag{E.8}$$

Noticing that

$$\frac{\partial c_i(p_3,p_4,y_2 - p_2c_2|\ c_1,c_2)}{\partial c_2} \quad (i = 3,4) \tag{E.9}$$

are functions of (p_2,p_3,p_4,y_2) (c_1 is treated as a constant parameter), if we assume that (EI) holds, i.e.,

$$\frac{\partial (c_2,c_3,c_4)}{\partial (p_2,p_3,p_4)} \neq 0 \quad \forall (p_2,p_3,p_4) \in \mathbb{R}^3_{++}, \tag{E.10}$$

[34] Although one can conduct the test for the global version in Appendix C, we use the local version throughout this appendix and Appendix O for simplicity.

where c_i $(i = 2, 3, 4)$ are conditional demand functions, then the coefficients of the partial differential equation (E.8) can be expressed as functions of (c_2, c_3, c_4) and hence there exists a solution $U^{(2)}$. Finally, $U^{(1)}$ satisfies

$$\frac{\partial U^{(1)}(c_1, c_2, c_3, c_4)}{\partial c_1} + \sum_{i=2}^{4} \frac{\partial c_i}{\partial c_1} \frac{\partial U^{(1)}(c_1, c_2, c_3, c_4)}{\partial c_i} = 0, \tag{E.11}$$

where $c_2 = c_2 \left(p_2, p_3, p_4, y_1 - p_1 c_1 | \; c_1\right)$ and

$$c_i = c_i \left(p_3, p_4, y_1 - p_1 c_1 - p_2 c_2 | \; c_1, c_2 \left(p_2, p_3, p_4, y_1 - p_1 c_1 | \; c_1\right)\right) \quad (i = 3, 4) . \tag{E.12}$$

If we assume that (EI) holds, i.e.,

$$\frac{\partial \left(c_1, c_2, c_3, c_4\right)}{\partial \left(p_1, p_2, p_3, p_4\right)} \neq 0 \quad \forall \left(p_1, p_2, p_3, p_4\right) \in \mathbb{R}^4_{++}, \tag{E.13}$$

where c_i $(i = 1, 2, 3, 4)$ are unconditional demand functions, then the coefficients of the partial differential equation (E.11) can be expressed as functions of (c_1, c_2, c_3, c_4) and hence there exists a solution $U^{(1)}$. In summary, we have the following theorem.

Theorem 11 *Assume a given set of demand functions* $c_i \left(p_1, p_2, p_3, p_4, y_1\right)$ $(i = 1, 2, 3, 4)$ *which have the Properties (P), (TD), (H) and (B). There exists a* $(U^{(1)}, U^{(2)}, U^{(3)})$*-triplet which generates these demands as a result of sophisticated choice, where* $U^{(1)}$ *and* $U^{(2)}$ *are twice continuously differentiable and* $U^{(3)}$ *satisfies Property 1, if and only if*

$$\frac{\partial \left(c_1, c_2, y_3\right)}{\partial \left(p_1, p_2, y_1\right)} \neq 0 \quad \forall \left(p_1, p_2, y_1\right) \in \mathbb{R}^2_{++} \times \mathbb{R}_{+}, \tag{E.14}$$

$$\frac{\partial \left(c_1, y_2\right)}{\partial \left(p_1, y_1\right)} \neq 0 \quad \forall \left(p_1, y_1\right) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$$

and

$$\frac{\partial \left(c_2, c_3, c_4\right)}{\partial \left(p_2, p_3, p_4\right)} \neq 0 \quad \forall \left(p_2, p_3, p_4\right) \in \mathbb{R}^3_{++}, \tag{E.15}$$

where

$$c_i = c_i \left(p_2, p_3, p_4, y_2 | \; c_1\right) \quad (i = 2, 3, 4) , \tag{E.16}$$

and

$$\frac{\partial \left(c_1, c_2, c_3, c_4\right)}{\partial \left(p_1, p_2, p_3, p_4\right)} \neq 0 \quad \forall \left(p_1, p_2, p_3, p_4\right) \in \mathbb{R}^4_{++}, \tag{E.17}$$

where

$$c_i = c_i \left(p_1, p_2, p_3, p_4, y_1\right) \quad (i = 1, 2, 3, 4) , \tag{E.18}$$

and the corresponding conditional demand functions

$$c_3(p_3, p_4, y_3|\ c_1, c_2) \text{ and } c_4(p_3, p_4, y_3|\ c_1, c_2) \tag{E.19}$$

have Property (N). In this case, $U^{(1)}$ *satisfies*

$$\frac{\partial U^{(1)}(c_1, c_2, c_3, c_4)}{\partial c_1} + \sum_{i=2}^{4} \frac{\partial c_i}{\partial c_1} \frac{\partial U^{(1)}(c_1, c_2, c_3, c_4)}{\partial c_i} = 0 \tag{E.20}$$

and $U^{(2)}$ *satisfies*

$$\frac{\partial U^{(2)}(c_2, c_3, c_4|\ c_1)}{\partial c_2} + \frac{\partial c_3}{\partial c_2} \frac{\partial U^{(2)}(c_2, c_3, c_4|\ c_1)}{\partial c_3} + \frac{\partial c_4}{\partial c_2} \frac{\partial U^{(2)}(c_2, c_3, c_4|\ c_1)}{\partial c_4} = 0, \tag{E.21}$$

where in eqn. (E.20),

$$\frac{\partial c_i\left(p_1, p_2, p_3, p_4, y_1\right)}{\partial c_1} \quad (i = 2, 3, 4) \tag{E.22}$$

are functions of $(p_1, p_2, p_3, p_4, y_1)$ *that can be transformed into functions of* (c_1, c_2, c_3, c_4) *using the inverse demand functions and in eqn. (E.21)*

$$\frac{\partial c_i\left(p_3, p_4, y_2 - p_2 c_2|\ c_1, c_2\right)}{\partial c_2} \quad (i = 3, 4) \tag{E.23}$$

are functions of (p_2, p_3, p_4, y_2) *(*c_1 *is treated as a constant parameter) that can be transformed into functions of* $(c_2, c_3, c_4|\ c_1)$ *using the inverse demand functions.*

F Strict Monotonicity and Quasiconcavity of $U^{(1)}$: Local Results

To simplify the notation of the conditions for the existence of at least one $U^{(1)}$ being strictly increasing and quasiconcave, denote for any function $\varphi\left(c_1, c_2, c_3\right)$

$$L\varphi = \frac{\partial \varphi}{\partial c_1} + \frac{\partial c_2}{\partial c_1}\frac{\partial \varphi}{\partial c_2} + \frac{\partial c_3}{\partial c_1}\frac{\partial \varphi}{\partial c_3} = \frac{\partial \varphi}{\partial c_1} - a\frac{\partial \varphi}{\partial c_2} - b\frac{\partial \varphi}{\partial c_3}, \tag{F.1}$$

where c_2 and c_3 are functions of $(c_1, p_2, p_3, y_1 - p_1 c_1)$ and $a = -\frac{\partial c_2}{\partial c_1}$ and $b = -\frac{\partial c_3}{\partial c_1}$ are transformed into functions of (c_1, c_2, c_3) using the inverse demand functions. Then we have the following theorem.

Theorem 12 *Assume that* $U^{(1)}$ *satisfies*

$$LU^{(1)} = \frac{\partial U^{(1)}}{\partial c_1} - a\frac{\partial U^{(1)}}{\partial c_2} - b\frac{\partial U^{(1)}}{\partial c_3} = 0. \tag{F.2}$$

Given a $(\overline{c}_1, \overline{c}_2, \overline{c}_3) \in \mathbb{R}^3_+$, a necessary condition such that at least one solution to the partial differential equation (F.2) is strictly increasing and quasiconcave in an open neighborhood of $(\overline{c}_1, \overline{c}_2, \overline{c}_3)$ is that the algebraic system

$$ax + by > 0 \qquad \text{and} \qquad xLa + yLb \leq 0 \tag{F.3}$$

has a positive solution (x, y), where a, b, La, Lb are evaluated at $(\overline{c}_1, \overline{c}_2, \overline{c}_3)$. A sufficient condition such that at least one solution to the partial differential equation (F.2) is strictly increasing and quasiconcave in an open neighborhood of $(\overline{c}_1, \overline{c}_2, \overline{c}_3)$ is that the algebraic system

$$ax + by > 0 \qquad \text{and} \qquad xLa + yLb < 0 \tag{F.4}$$

has a positive solution (x, y), where a, b, La, Lb are evaluated at $(\overline{c}_1, \overline{c}_2, \overline{c}_3)$.

Proof. First consider the necessary condition. Since $U^{(1)}$ satisfies

$$LU^{(1)} = \frac{\partial U^{(1)}}{\partial c_1} - a\frac{\partial U^{(1)}}{\partial c_2} - b\frac{\partial U^{(1)}}{\partial c_3} = 0, \tag{F.5}$$

if there exists a $U^{(1)}$ that is strictly increasing in an open neighborhood of $(\overline{c}_1, \overline{c}_2, \overline{c}_3)$, then at $(\overline{c}_1, \overline{c}_2, \overline{c}_3)$, we have

$$a\frac{\partial U^{(1)}}{\partial c_2} + b\frac{\partial U^{(1)}}{\partial c_3} = \frac{\partial U^{(1)}}{\partial c_1} > 0, \tag{F.6}$$

implying that there exists a positive solution $(x, y) = \left(\frac{\partial U^{(1)}}{\partial c_2}, \frac{\partial U^{(1)}}{\partial c_3}\right)$ satisfying

$$ax + by > 0. \tag{F.7}$$

Consider the Bordered Hessian matrix of $U^{(1)}$

$$B = \begin{bmatrix} 0 & U_2^{(1)} & U_3^{(1)} & U_1^{(1)} \\ U_2^{(1)} & U_{22}^{(1)} & U_{23}^{(1)} & U_{12}^{(1)} \\ U_3^{(1)} & U_{23}^{(1)} & U_{33}^{(1)} & U_{13}^{(1)} \\ U_1^{(1)} & U_{12}^{(1)} & U_{13}^{(1)} & U_{11}^{(1)} \end{bmatrix}, \tag{F.8}$$

where

$$U_i^{(1)} = \frac{\partial U^{(1)}}{\partial c_i} \qquad \text{and} \qquad U_{ij}^{(1)} = \frac{\partial^2 U^{(1)}}{\partial c_i \partial c_j} \quad (i, j = 1, 2, 3)\,. \tag{F.9}$$

$U^{(1)}$ being quasiconcave implies that the principal minors of the Bordered Hessian matrix B must satisfy (i)

$$\det \begin{bmatrix} 0 & U_2^{(1)} \\ U_2^{(1)} & U_{22}^{(1)} \end{bmatrix} = -\left(U_2^{(1)}\right)^2 \leq 0, \tag{F.10}$$

(ii)

$$\det \begin{bmatrix} 0 & U_2^{(1)} & U_3^{(1)} \\ U_2^{(1)} & U_{22}^{(1)} & U_{23}^{(1)} \\ U_3^{(1)} & U_{23}^{(1)} & U_{33}^{(1)} \end{bmatrix} = -U_{22}^{(1)} \left(U_3^{(1)}\right)^2 + 2U_{23}^{(1)} U_2^{(1)} U_3^{(1)} - U_{33}^{(1)} \left(U_2^{(1)}\right)^2 \geq 0 \tag{F.11}$$

and (iii) $\det B \leq 0$. It is clear that (i) is automatically satisfied. We can always assume that (ii) is satisfied with the strict inequality. The reason is as follows. One can assign arbitrary initial values for $U^{(1)}$ on the given $c_1 = \bar{c}_1$ plane (so $U^{(1)}$ is an arbitrary function of c_2 and c_3). Hence (ii) can be satisfied with strict inequality on this plane and by continuity it is satisfied in a three dimensional neighborhood of $(\bar{c}_1, \bar{c}_2, \bar{c}_3)$. Therefore, we only need a condition corresponding to (iii) $\det B \leq 0$. Differentiating

$$\frac{\partial U^{(1)}}{\partial c_1} = a\frac{\partial U^{(1)}}{\partial c_2} + b\frac{\partial U^{(1)}}{\partial c_3} \tag{F.12}$$

with respect to c_1, c_2 and c_3 respectively yields

$$U_{11}^{(1)} = a_1 U_2^{(1)} + aU_{12}^{(1)} + b_1 U_3^{(1)} + bU_{13}^{(1)}, \tag{F.13}$$

$$U_{12}^{(1)} = a_2 U_2^{(1)} + aU_{22}^{(1)} + b_2 U_3^{(1)} + bU_{23}^{(1)} \tag{F.14}$$

and

$$U_{13}^{(1)} = a_3 U_2^{(1)} + aU_{23}^{(1)} + b_3 U_3^{(1)} + bU_{33}^{(1)}, \tag{F.15}$$

where

$$a_i = \frac{\partial a}{\partial c_i} \quad \text{and} \quad b_i = \frac{\partial b}{\partial c_i} \quad (i = 1, 2, 3)\,. \tag{F.16}$$

For the matrix B, subtracting a times the second column and b times the third column from the fourth column and using eqns. (F.12) - (F.15), we have

$$\det B = \det \begin{bmatrix} 0 & U_2^{(1)} & U_3^{(1)} & 0 \\ U_2^{(1)} & U_{22}^{(1)} & U_{23}^{(1)} & a_2 U_2^{(1)} + b_2 U_3^{(1)} \\ U_3^{(1)} & U_{23}^{(1)} & U_{33}^{(1)} & a_3 U_2^{(1)} + b_3 U_3^{(1)} \\ U_1^{(1)} & U_{12}^{(1)} & U_{13}^{(1)} & a_1 U_2^{(1)} + b_1 U_3^{(1)} \end{bmatrix}. \tag{F.17}$$

For the above matrix, subtracting a times the second row and b times the third row from the fourth row and using eqns. (F.12) - (F.15), one obtains

$$\det B = \det \begin{bmatrix} 0 & U_2^{(1)} & U_3^{(1)} & 0 \\ U_2^{(1)} & U_{22}^{(1)} & U_{23}^{(1)} & a_2 U_2^{(1)} + b_2 U_3^{(1)} \\ U_3^{(1)} & U_{23}^{(1)} & U_{33}^{(1)} & a_3 U_2^{(1)} + b_3 U_3^{(1)} \\ 0 & a_2 U_2^{(1)} + b_2 U_3^{(1)} & a_3 U_2^{(1)} + b_3 U_3^{(1)} & U_2^{(1)} La + U_3^{(1)} Lb \end{bmatrix}, \tag{F.18}$$

where

$$U_2^{(1)} La + U_3^{(1)} Lb = U_2^{(1)} (a_1 - aa_2 - ba_3) + U_3^{(1)} (b_1 - ab_2 - bb_3). \tag{F.19}$$

Therefore,

$$\begin{aligned} \det B &= \left(U_2^{(1)} La + U_3^{(1)} Lb\right) \det \begin{bmatrix} 0 & U_2^{(1)} & U_3^{(1)} \\ U_2^{(1)} & U_{22}^{(1)} & U_{23}^{(1)} \\ U_3^{(1)} & U_{23}^{(1)} & U_{33}^{(1)} \end{bmatrix} + \\ &\quad \left(U_2^{(1)} \left(a_3 U_2^{(1)} + b_3 U_3^{(1)}\right) - U_3^{(1)} \left(a_2 U_2^{(1)} + b_2 U_3^{(1)}\right)\right)^2. \end{aligned} \tag{F.20}$$

Since

$$\det \begin{bmatrix} 0 & U_2^{(1)} & U_3^{(1)} \\ U_2^{(1)} & U_{22}^{(1)} & U_{23}^{(1)} \\ U_3^{(1)} & U_{23}^{(1)} & U_{33}^{(1)} \end{bmatrix} > 0 \tag{F.21}$$

is assumed, $\det B \leq 0$ implies that $U_2^{(1)} La + U_3^{(1)} Lb \leq 0$ and hence there exists a positive solution $(x, y) = \left(U_2^{(1)}, U_3^{(1)}\right)$ satisfying

$$xLa + yLb \leq 0. \tag{F.22}$$

Combining this condition with condition (F.7) above, a necessary condition such that at least one solution to the partial differential equation (F.2) is strictly increasing and quasiconcave in an open neighborhood of $(\overline{c}_1, \overline{c}_2, \overline{c}_3)$ is that the algebraic system

$$ax + by > 0 \qquad \text{and} \qquad xLa + yLb \leq 0 \tag{F.23}$$

has a positive solution (x, y), where a, b, La, Lb are evaluated at $(\overline{c}_1, \overline{c}_2, \overline{c}_3)$. Next consider the sufficient condition. Given a $(\overline{c}_1, \overline{c}_2, \overline{c}_3)$, if the algebraic system

$$ax + by > 0 \qquad \text{and} \qquad xLa + yLb < 0 \tag{F.24}$$

has a positive solution (x, y), where a, b, La, Lb are evaluated at $(\overline{c}_1, \overline{c}_2, \overline{c}_3)$, then by continuity, one can define $\phi(c_2, c_3)$ in an open neighborhood of $(\overline{c}_2, \overline{c}_3)$ such that

$$\frac{\partial \phi}{\partial c_2} = x \qquad \text{and} \qquad \frac{\partial \phi}{\partial c_3} = y \tag{F.25}$$

and thus $\phi(c_2, c_3)$ is strictly increasing in this open neighborhood. Moreover, noticing that if (x, y) is a solution to the algebraic system (F.24), then for any positive N, $\left(\frac{x}{N}, \frac{y}{N}\right)$ is also a solution and hence (x, y) can be chosen to be arbitrarily small. Since in eqn. (F.20), the first term on the right hand side is homogeneous of degree three in $\left(U_2^{(1)}, U_3^{(1)}\right)$ and the second term is homogeneous of degree four in $\left(U_2^{(1)}, U_3^{(1)}\right)$, if (x, y) is small enough, setting

$$U_2^{(1)} = \frac{\partial \phi}{\partial c_2} = x \qquad \text{and} \qquad U_3^{(1)} = \frac{\partial \phi}{\partial c_3} = y \tag{F.26}$$

results in $\det B < 0$. Therefore, solving the initial value problem

$$\frac{\partial U^{(1)}}{\partial c_1} + \frac{\partial c_2}{\partial c_1}\frac{\partial U^{(1)}}{\partial c_2} + \frac{\partial c_3}{\partial c_1}\frac{\partial U^{(1)}}{\partial c_3} = 0 \tag{F.27}$$

and

$$U^{(1)}\left(\bar{c}_1, c_2, c_3\right) = \varphi\left(c_2, c_3\right), \tag{F.28}$$

there exists a solution in an open neighborhood of $(\bar{c}_1, \bar{c}_2, \bar{c}_3)$ such that $U^{(1)}$ is strictly increasing and quasiconcave. ■

G Proof of Corollary 1

Since the conditional demands $c_2\left(p_2, p_3, y_2 \mid c_1\right)$ and $c_3\left(p_2, p_3, y_2 \mid c_1\right)$ are independent of c_1,

$$\frac{\partial c_2}{\partial c_1} = \frac{\partial c_2}{\partial y_2}\frac{\partial y_2}{\partial c_1} = -p_1\frac{\partial c_2}{\partial y_2} \qquad \text{and} \qquad \frac{\partial c_3}{\partial c_1} = \frac{\partial c_3}{\partial y_2}\frac{\partial y_2}{\partial c_1} = -p_1\frac{\partial c_3}{\partial y_2}. \tag{G.1}$$

It follows from Theorem 2 that $U^{(1)}$ satisfies

$$\frac{\partial U^{(1)}}{\partial c_1} - p_1\frac{\partial c_2}{\partial y_2}\frac{\partial U^{(1)}}{\partial c_2} - p_1\frac{\partial c_3}{\partial y_2}\frac{\partial U^{(1)}}{\partial c_3} = 0. \tag{G.2}$$

Moreover, following the discussion in Appendix F, the necessary and sufficient condition for the existence of at least one locally strictly increasing $U^{(1)}$ is that the following inequality has a positive solution

$$ax + by > 0, \tag{G.3}$$

where

$$a = -\frac{\partial c_2}{\partial c_1} = p_1\frac{\partial c_2}{\partial y_2} \qquad \text{and} \qquad b = -\frac{\partial c_3}{\partial c_1} = p_1\frac{\partial c_3}{\partial y_2}, \tag{G.4}$$

which is also equivalent to

$$\frac{\partial c_2}{\partial y_2} > 0 \qquad \text{or} \qquad \frac{\partial c_3}{\partial y_2} > 0. \tag{G.5}$$

Next we prove that $U^{(2)}$ is weakly separable in (c_2, c_3). To see this, first note that when $U^{(2)}$ is weakly separable in (c_2, c_3), the marginal rate of substitution (MRS) between c_2 and c_3 is independent of c_1, implying that the conditional demands are independent of c_1. Moreover, if $c_i(p_2, p_3, y_2 \mid c_1) = c_i(p_2, p_3, y_2)$ $(i = 2, 3)$, then the MRS between c_2 and c_3 must be independent of c_1, implying that $U^{(2)}$ is weakly separable in (c_2, c_3).

H Supporting Calculations for Example 5

To verify (EC), first calculate

$$y_2 = y_1 - p_1 c_1 = (p_1 - p_2 - p_3)\left(\frac{p_1}{p_2} + \frac{p_1}{p_3}\right) - \frac{(p_2 + p_3)\, y_1}{p_1 - p_2 - p_3}. \tag{H.1}$$

Based on eqns. (45) and (H.1),

$$\frac{\partial (c_1, y_2)}{\partial (p_1, y_1)} \neq 0 \quad \forall (p_1, y_1) \in \mathbb{R}_{++} \times \mathbb{R}_{+}, \tag{H.2}$$

and Property (EC) holds. Solving for p_1 and y_1 as functions of (c_1, y_2) and then substituting the resulting expressions into eqns. (46) - (47), one obtains the conditional demands

$$c_2 (p_2, p_3, y_2 |\ c_1) = \frac{y_2 - (p_2 - p_3)\, c_1}{2p_2} \quad \text{and} \quad c_3 (p_2, p_3, y_2 |\ c_1) = \frac{y_2 - (p_3 - p_2)\, c_1}{2p_3}. \tag{H.3}$$

Viewing c_1 as a preference parameter, the conditional demands satisfy (S) and (N). Hence one can use the Hurwicz and Uzawa (1971) recovery process to derive

$$U^{(2)} (c_2, c_3 |\ c_1) = \ln (c_2 + c_1) + \ln (c_3 + c_1). \tag{H.4}$$

To establish the existence of a $U^{(1)}$, it can be verified that

$$\det \frac{\partial (c_1, c_2, c_3)}{\partial (p_1, p_2, p_3)} \neq 0 \quad \forall (p_1, p_2, p_3) \in \mathbb{R}^3_{++} \tag{H.5}$$

and hence Property (EI) is satisfied. Solving for p_1 and y_1 as functions of (p_1, p_2, p_3, y_2) yields

$$p_1 = p_2 + p_3 + \frac{\sqrt{p_2 p_3 (p_2 + p_3)(c_1 (p_2 + p_3) + y_2)}}{p_2 + p_3} \tag{H.6}$$

and

$$y_1 = (p_2 + p_3)\, c_1 + \frac{c_1 \sqrt{p_2 p_3 (p_2 + p_3)(c_1 (p_2 + p_3) + y_2)}}{p_2 + p_3} + y_2. \tag{H.7}$$

First based on eqn. (H.3),

$$\frac{\partial c_2}{\partial c_1} = -\frac{p_1 + p_2 - p_3}{2p_2} \quad \text{and} \quad \frac{\partial c_3}{\partial c_1} = -\frac{p_1 + p_3 - p_2}{2p_3}. \tag{H.8}$$

Solving for the inverse demands from (45) - (47), one obtains

$$\begin{aligned} \frac{p_1}{y_1} &= \frac{2c_1^2 w_2 w_3^2 - 2w_2 w_3 \sqrt{2c_1^2 w_3^2 (w_2 + w_3)}}{2c_1 w_2 w_3^2 (c_1^2 - 2(w_2 + w_3))} + \\ &\quad \frac{(w_2 + w_3)\left(\sqrt{2c_1^2 w_3^2 (w_2 + w_3)} - 2w_2 w_3 - 2w_3^2\right)}{2w_2 w_3^2 (c_1^2 - 2(w_2 + w_3))}, \end{aligned} \tag{H.9}$$

$$\frac{p_2}{y_1} = \frac{2w_2w_3 + 2w_3^2 - \sqrt{2c_1^2w_3^2(w_2+w_3)}}{2w_2w_3(2(w_2+w_3) - c_1^2)} \tag{H.10}$$

and

$$\frac{p_3}{y_1} = \frac{2w_2w_3 + 2w_3^2 - \sqrt{2c_1^2w_3^2(w_2+w_3)}}{2w_3^2(2(w_2+w_3) - c_1^2)}, \tag{H.11}$$

where

$$w_2 = c_1 + c_2 \qquad \text{and} \qquad w_3 = c_1 + c_3. \tag{H.12}$$

Substituting eqns. (H.9) - (H.11) into eqn. (H.8), we have

$$\frac{\partial c_2}{\partial c_1} = \frac{-c_1^2w_2w_3 + (w_2 - c_1)\sqrt{2c_1^2w_3^2(w_2+w_3)} + c_1(2w_2w_3 + 2w_3^2)}{c_1\left(\sqrt{2c_1^2w_3^2(w_2+w_3)} - 2w_2w_3 - 2w_3^2\right)} \tag{H.13}$$

and

$$\frac{\partial c_3}{\partial c_1} = \frac{-c_1^2w_3^2 + (w_3 - c_1)\sqrt{2c_1^2w_3^2(w_2+w_3)} + c_1(2w_2w_3 + 2w_3^2)}{c_1\left(\sqrt{2c_1^2w_3^2(w_2+w_3)} - 2w_2w_3 - 2w_3^2\right)}. \tag{H.14}$$

The two characteristic equations are given by

$$\frac{dc_2}{dc_1} = \frac{-c_1^2w_2w_3 + (w_2 - c_1)\sqrt{2c_1^2w_3^2(w_2+w_3)} + c_1(2w_2w_3 + 2w_3^2)}{c_1\left(\sqrt{2c_1^2w_3^2(w_2+w_3)} - 2w_2w_3 - 2w_3^2\right)} \tag{H.15}$$

and

$$\frac{dc_3}{dc_1} = \frac{-c_1^2w_3^2 + (w_3 - c_1)\sqrt{2c_1^2w_3^2(w_2+w_3)} + c_1(2w_2w_3 + 2w_3^2)}{c_1\left(\sqrt{2c_1^2w_3^2(w_2+w_3)} - 2w_2w_3 - 2w_3^2\right)}. \tag{H.16}$$

Thus the two independent first integrals are

$$\psi_1(c_1, c_2, c_3) = c_1 + 2\sqrt{2(c_2 + c_3 + 2c_1)} \qquad \text{and} \qquad \psi_2(c_1, c_2, c_3) = \frac{c_2 + c_1}{c_3 + c_1}. \tag{H.17}$$

I Proof of Corollary 2

When the Properties (P), (TD), (B), (S) and (N) are satisfied, Theorem 1 guarantees that there exists a $U(c_1, c_2, c_3)$ to rationalize the demands. If U is twice continuously differentiable, Property (EI) holds since the inverse demands can be derived from the first order condition and Property (EC) holds since the conditional demands always exist. Moreover as discussed in footnote 16, Property (H) also holds. In this case, for the partial differential equation (26)

$$\psi_1(c_1, c_2, c_3) = U(c_1, c_2, c_3) \tag{I.1}$$

is always one of the first integrals.

J Supporting Calculations for Example 6

To recover a $\left(U^{(1)}, U^{(2)}\right)$-pair for this case, first note that

$$y_2 = \frac{y_1\left(\alpha\left(p_2+p_3\right)+p_2\left(\frac{p_1}{p_2}+\alpha\left(1+\frac{p_3}{p_2}\right)\right)^{\frac{1}{1+\delta}}+p_3\left(\frac{p_1}{p_3}+\alpha\left(1+\frac{p_2}{p_3}\right)\right)^{\frac{1}{1+\delta}}\right)}{p_1+\alpha\left(p_2+p_3\right)+p_2\left(\frac{p_1}{p_2}+\alpha\left(1+\frac{p_3}{p_2}\right)\right)^{\frac{1}{1+\delta}}+p_3\left(\frac{p_1}{p_3}+\alpha\left(1+\frac{p_2}{p_3}\right)\right)^{\frac{1}{1+\delta}}}. \tag{J.1}$$

Based on eqns. (54) and (J.1),

$$\frac{\partial\left(c_1, y_2\right)}{\partial\left(p_1, y_1\right)} \neq 0 \quad \forall (p_1, y_1) \in \mathbb{R}_{++} \times \mathbb{R}_{+}, \tag{J.2}$$

and hence Property (EC) holds. Solving for p_1 and y_1 as functions of (c_1, y_2) and substituting the resulting expressions into eqns. (55) - (56), one obtains the conditional demands

$$c_2 = \alpha c_1 + \frac{y_2 - \alpha\left(p_2+p_3\right)c_1}{p_2 + p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}} \quad \text{and} \quad c_3 = \alpha c_1 + \frac{y_2 - \alpha\left(p_2+p_3\right)c_1}{p_3 + p_2\left(\frac{p_3}{p_2}\right)^{\frac{1}{1+\delta}}}. \tag{J.3}$$

Then viewing c_1 as a preference parameter, one can verify that Property (P), (TD), (H), (B) and (N) hold. Using the Hurwicz and Uzawa (1971) recovery process, one obtains

$$U^{(2)}\left(c_2, c_3 \middle|\ c_1\right) = -\frac{\left(c_2-\alpha c_1\right)^{-\delta}}{\delta} - \frac{\left(c_3-\alpha c_1\right)^{-\delta}}{\delta}. \tag{J.4}$$

To establish the existence of a $U^{(1)}$, note that

$$\det \frac{\partial\left(c_1, c_2, c_3\right)}{\partial\left(p_1, p_2, p_3\right)} \neq 0 \quad \forall (p_1, p_2, p_3) \in \mathbb{R}^3_{++} \tag{J.5}$$

and hence Property (EI) is satisfied. Solving for p_1 and y_1 as functions of (p_1, p_2, p_3, y_2) yields

$$p_1 = p_2\left(\left(\frac{y_2/c_1 - \alpha\left(p_2+p_3\right)}{p_2 + p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}}\right)^{1+\delta} - \alpha\left(1+\frac{p_3}{p_2}\right)\right) \tag{J.6}$$

and

$$y_1 = y_2 + p_2 c_1\left(\left(\frac{y_2/c_1 - \alpha\left(p_2+p_3\right)}{p_2 + p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}}\right)^{1+\delta} - \alpha\left(1+\frac{p_3}{p_2}\right)\right). \tag{J.7}$$

Now substituting the conditional demands (J.3) into $U^{(1)}$ yields

$$\begin{aligned} & U^{(1)}\left(c_1, c_2\left(c_1\right), c_3\left(c_1\right)\right) \\ = \; & U^{(1)}\left(c_1, \alpha c_1+\frac{y_2-\alpha\left(p_2+p_3\right) c_1}{p_2+p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}}, \alpha c_1+\frac{y_2-\alpha\left(p_2+p_3\right) c_1}{p_3+p_2\left(\frac{p_3}{p_2}\right)^{\frac{1}{1+\delta}}}\right). \end{aligned} \tag{J.8}$$

Taking the derivative of the right hand side of the above equation with respect to c_1 and setting it to zero, one obtains the following partial differential equation

$$\frac{\partial U^{(1)}}{\partial c_1}+\frac{\partial c_2}{\partial c_1} \frac{\partial U^{(1)}}{\partial c_2}+\frac{\partial c_3}{\partial c_1} \frac{\partial U^{(1)}}{\partial c_3}=0, \tag{J.9}$$

where

$$\frac{\partial c_2}{\partial c_1}=\alpha-\frac{p_1+\alpha\left(p_2+p_3\right)}{p_2+p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}} \quad \text { and } \quad \frac{\partial c_3}{\partial c_1}=\alpha-\frac{p_1+\alpha\left(p_2+p_3\right)}{p_3+p_2\left(\frac{p_3}{p_2}\right)^{\frac{1}{1+\delta}}}. \tag{J.10}$$

Solving for the inverse demands yields

$$\frac{p_1}{y_1}=\frac{1}{c_1+\frac{c_2\left(c_2-\alpha c_1\right)^{-1-\delta}}{c_1^{-1-\delta}-\alpha\left(c_2-\alpha c_1\right)^{-1-\delta}-\alpha\left(c_3-\alpha c_1\right)^{-1-\delta}}+\frac{c_3\left(c_3-\alpha c_1\right)^{-1-\delta}}{c_1^{-1-\delta}-\alpha\left(c_2-\alpha c_1\right)^{-1-\delta}-\alpha\left(c_3-\alpha c_1\right)^{-1-\delta}}}, \tag{J.11}$$

$$\frac{p_2}{y_1}=\frac{\frac{\left(c_2-\alpha c_1\right)^{-1-\delta}}{c_1^{-1-\delta}-\alpha\left(c_2-\alpha c_1\right)^{-1-\delta}-\alpha\left(c_3-\alpha c_1\right)^{-1-\delta}}}{c_1+\frac{c_2\left(c_2-\alpha c_1\right)^{-1-\delta}}{c_1^{-1-\delta}-\alpha\left(c_2-\alpha c_1\right)^{-1-\delta}-\alpha\left(c_3-\alpha c_1\right)^{-1-\delta}}+\frac{c_3\left(c_3-\alpha c_1\right)^{-1-\delta}}{c_1^{-1-\delta}-\alpha\left(c_2-\alpha c_1\right)^{-1-\delta}-\alpha\left(c_3-\alpha c_1\right)^{-1-\delta}}} \tag{J.12}$$

and

$$\frac{p_3}{y_1}=\frac{\frac{\left(c_3-\alpha c_1\right)^{-1-\delta}}{c_1^{-1-\delta}-\alpha\left(c_2-\alpha c_1\right)^{-1-\delta}-\alpha\left(c_3-\alpha c_1\right)^{-1-\delta}}}{c_1+\frac{c_2\left(c_2-\alpha c_1\right)^{-1-\delta}}{c_1^{-1-\delta}-\alpha\left(c_2-\alpha c_1\right)^{-1-\delta}-\alpha\left(c_3-\alpha c_1\right)^{-1-\delta}}+\frac{c_3\left(c_3-\alpha c_1\right)^{-1-\delta}}{c_1^{-1-\delta}-\alpha\left(c_2-\alpha c_1\right)^{-1-\delta}-\alpha\left(c_3-\alpha c_1\right)^{-1-\delta}}}. \tag{J.13}$$

Therefore, we have

$$\begin{aligned} \frac{\partial c_2}{\partial c_1} & =\alpha-\frac{p_1+\alpha\left(p_2+p_3\right)}{p_2+p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}} \\ & =\alpha-\frac{\frac{c_1^{-1-\delta}-\alpha\left(c_2-\alpha c_1\right)^{-1-\delta}-\alpha\left(c_3-\alpha c_1\right)^{-1-\delta}}{\left(c_3-\alpha c_1\right)^{-1-\delta}}+\alpha\left(\frac{\left(c_2-\alpha c_1\right)^{-1-\delta}}{\left(c_3-\alpha c_1\right)^{-1-\delta}}+1\right)}{\frac{\left(c_2-\alpha c_1\right)^{-1-\delta}}{\left(c_3-\alpha c_1\right)^{-1-\delta}}+\frac{\left(c_2-\alpha c_1\right)^{-1}}{\left(c_3-\alpha c_1\right)^{-1}}} \end{aligned} \tag{J.14}$$

and

$$\begin{aligned} \frac{\partial c_3}{\partial c_1} & =\alpha-\frac{p_1+\alpha\left(p_2+p_3\right)}{p_3+p_2\left(\frac{p_3}{p_2}\right)^{\frac{1}{1+\delta}}} \\ & =\alpha-\frac{\frac{c_1^{-1-\delta}-\alpha\left(c_2-\alpha c_1\right)^{-1-\delta}-\alpha\left(c_3-\alpha c_1\right)^{-1-\delta}}{\left(c_3-\alpha c_1\right)^{-1-\delta}}+\alpha\left(\frac{\left(c_2-\alpha c_1\right)^{-1-\delta}}{\left(c_3-\alpha c_1\right)^{-1-\delta}}+1\right)}{1+\frac{\left(c_2-\alpha c_1\right)^{-\delta}}{\left(c_3-\alpha c_1\right)^{-\delta}}}. \end{aligned} \tag{J.15}$$

Thus, $U^{(1)}$ satisfies

$$0 = \frac{\partial U^{(1)}}{\partial c_1} + \left(\alpha - \frac{\frac{c_1^{-1-\delta} - \alpha(c_2-\alpha c_1)^{-1-\delta} - \alpha(c_3-\alpha c_1)^{-1-\delta}}{(c_3-\alpha c_1)^{-1-\delta}} + \alpha\left(\frac{(c_2-\alpha c_1)^{-1-\delta}}{(c_3-\alpha c_1)^{-1-\delta}} + 1\right)}{\frac{(c_2-\alpha c_1)^{-1-\delta}}{(c_3-\alpha c_1)^{-1-\delta}} + \frac{(c_2-\alpha c_1)^{-1}}{(c_3-\alpha c_1)^{-1}}} \right) \frac{\partial U^{(1)}}{\partial c_2}$$

$$+ \left(\alpha - \frac{\frac{c_1^{-1-\delta} - \alpha(c_2-\alpha c_1)^{-1-\delta} - \alpha(c_3-\alpha c_1)^{-1-\delta}}{(c_3-\alpha c_1)^{-1-\delta}} + \alpha\left(\frac{(c_2-\alpha c_1)^{-1-\delta}}{(c_3-\alpha c_1)^{-1-\delta}} + 1\right)}{1 + \frac{(c_2-\alpha c_1)^{-\delta}}{(c_3-\alpha c_1)^{-\delta}}} \right) \frac{\partial U^{(1)}}{\partial c_3}. \quad \text{(J.16)}$$

It can be verified that the two independent first integrals are

$$\psi_1(c_1, c_2, c_3) = -\frac{c_1^{-\delta}}{\delta} - \frac{(c_2 - \alpha c_1)^{-\delta}}{\delta} - \frac{(c_3 - \alpha c_1)^{-\delta}}{\delta} \quad \text{(J.17)}$$

and

$$\psi_2(c_1, c_2, c_3) = \frac{c_2 - \alpha c_1}{c_3 - \alpha c_1}. \quad \text{(J.18)}$$

K Proof of Corollary 3

When the conditional demands $c_2(p_2, p_3, y_2|\ c_1)$ and $c_3(p_2, p_3, y_2|\ c_1)$ satisfy

$$c_2(p_2, p_3, y_2|\ c_1) - a(c_1) = k_2(p_2, p_3)\, y_2 \quad \text{and} \quad c_3(p_2, p_3, y_2|\ c_1) - b(c_1) = k_3(p_2, p_3)\, y_2, \quad \text{(K.1)}$$

where $a(c_1)$ and $b(c_1)$ are arbitrary functions of c_1 and $k_2(p_2, p_3)$ and $k_3(p_2, p_3)$ are arbitrary functions of (p_2, p_3), we always have

$$\frac{c_2(p_2, p_3, y_2|\ c_1) - a(c_1)}{c_3(p_2, p_3, y_2|\ c_1) - b(c_1)} = \frac{k_2(p_2, p_3)}{k_3(p_2, p_3)}. \quad \text{(K.2)}$$

Since including the term $\frac{k_2(p_2,p_3)}{k_3(p_2,p_3)}$ in $U^{(1)}$ will not affect the optimization problem

$$\max_{c_1} U^{(1)}(c_1, c_2(c_1), c_3(c_1)) \quad S.T.\ y_1 \geq \sum_{i=1}^{3} p_i c_i, \quad \text{(K.3)}$$

$\frac{c_2 - a(c_1)}{c_3 - b(c_1)}$ is always one of the two independent first integrals for the partial differential equation (26).

Remark 3 *Since* $\frac{c_2 - a(c_1)}{c_3 - b(c_1)}$ *increases with* c_2 *and decreases with* c_3*, including this term in* $U^{(1)}$ *will in general cause* $U^{(1)}$ *to fail to be strictly increasing. But this is not always the case. For instance, the* $U^{(1)}$ *function (69) can be also written as*

$$U^{(1)}(c_1, c_2, c_3) = f\left((\sqrt{c_1} + 1)\, c_2, \frac{c_2}{c_3} \right). \quad \text{(K.4)}$$

Although the term $\frac{c_2}{c_3}$ is not increasing in both c_2 and c_3, if we use the equivalent form $f\left(\left(\sqrt{c_1}+1\right)c_2, \left(\sqrt{c_1}+1\right)c_3\right)$, it is clear that $U^{(1)}$ is strictly increasing if f is an increasing function. Moreover, in general, although the $U^{(1)}(c_1, c_2, c_3)$ form in (62) is not strictly increasing and quasiconcave in the whole space, it is possible to specify a subspace such that $U^{(1)}(c_1, c_2, c_3)$ is strictly increasing and quasiconcave as shown in Example 4.

L Proof of Corollary 4

To show preferences are effectively consistent, we need to prove that the unconditional demands $c_i(p_1, p_2, p_3, y_1)$ $(i = 1, 2, 3)$ satisfy Property (S). Note that

$$\frac{\partial c_2}{\partial p_1} = \frac{\partial c_2}{\partial y_2}\frac{\partial y_2}{\partial p_1} = \frac{\partial c_2}{\partial y_2}\frac{\partial\left(y_1 - p_1 c_1\right)}{\partial p_1} = \frac{\partial c_2}{\partial y_2}\left(-c_1 - p_1\frac{\partial c_1}{\partial p_1}\right) \tag{L.1}$$

and

$$\frac{\partial c_2}{\partial y_1} = \frac{\partial c_2}{\partial y_2}\frac{\partial y_2}{\partial y_1} = \frac{\partial c_2}{\partial y_2}\frac{\partial\left(y_1 - p_1 c_1\right)}{\partial y_1} = \frac{\partial c_2}{\partial y_2}\left(1 - p_1\frac{\partial c_1}{\partial y_1}\right), \tag{L.2}$$

implying that for the Slutsky matrix $(\sigma_{ij})_{3\times 3}$, we have

$$\begin{aligned}\sigma_{21} &= \frac{\partial c_2}{\partial p_1} + c_1\frac{\partial c_2}{\partial y_1} = -p_1 c_1\frac{\partial c_1}{\partial y_1}\frac{\partial c_2}{\partial y_2} - p_1\frac{\partial c_1}{\partial p_1}\frac{\partial c_2}{\partial y_2} \\ &= \left(y_1 - p_1 c_1\right)\frac{\partial c_1}{\partial y_1}\frac{\partial c_2}{\partial y_2} - \frac{\partial c_2}{\partial y_2}\left(y_1\frac{\partial c_1}{\partial y_1} + p_1\frac{\partial c_1}{\partial p_1}\right).\end{aligned} \tag{L.3}$$

Property (H) implies $y_1\frac{\partial c_1}{\partial y_1} + p_1\frac{\partial c_1}{\partial p_1} = 0$. Since the conditional demand $c_2(p_2, p_3, y_2 |\ c_1)$ is independent of c_1 and proportional to y_2, we have

$$\frac{\partial c_2}{\partial y_2} = \frac{c_2}{y_2}. \tag{L.4}$$

Therefore

$$\sigma_{21} = \left(y_1 - p_1 c_1\right)\frac{\partial c_1}{\partial y_1}\frac{\partial c_2}{\partial y_2} = c_2\frac{\partial c_1}{\partial y_1}. \tag{L.5}$$

Since c_1 is independent of p_2 and p_3,

$$\sigma_{12} = \frac{\partial c_1}{\partial p_2} + c_2\frac{\partial c_1}{\partial y_1} = c_2\frac{\partial c_1}{\partial y_1} = \sigma_{21}. \tag{L.6}$$

Similarly, one can also prove $\sigma_{13} = \sigma_{31}$. Moreover, it can be verified that

$$\begin{aligned}\sigma_{23} &= \frac{\partial c_2}{\partial p_3} + c_3\frac{\partial c_2}{\partial y_1} = \left(\frac{\partial c_2}{\partial p_3}\right)_{y_2 = const} - p_1\frac{\partial c_2}{\partial y_2}\frac{\partial c_1}{\partial p_3} + c_3\frac{\partial c_2}{\partial y_2}\left(1 - p_1\frac{\partial c_1}{\partial y_1}\right) \\ &= \left(\frac{\partial c_2}{\partial p_3}\right)_{y_2 = const} + c_3\frac{\partial c_2}{\partial y_2} - p_1\frac{c_2 c_3}{y_2}\frac{\partial c_1}{\partial y_1},\end{aligned} \tag{L.7}$$

where $\left(\frac{\partial c_2}{\partial p_3}\right)_{y_2=const}$ denotes the derivative of the conditional demand c_2 with respect to p_3. Similarly,

$$\begin{aligned}\sigma_{32} &= \frac{\partial c_3}{\partial p_2}+c_2\frac{\partial c_3}{\partial y_1}=\left(\frac{\partial c_3}{\partial p_2}\right)_{y_2=const}-p_1\frac{\partial c_3}{\partial y_2}\frac{\partial c_1}{\partial p_2}+c_2\frac{\partial c_3}{\partial y_2}\left(1-p_1\frac{\partial c_1}{\partial y_1}\right)\\ &= \left(\frac{\partial c_3}{\partial p_2}\right)_{y_2=const}+c_2\frac{\partial c_3}{\partial y_2}-p_1\frac{c_2c_3}{y_2}\frac{\partial c_1}{\partial y_1}. \qquad \text{(L.8)}\end{aligned}$$

Since the conditional demands $c_2\left(p_2,p_3,y_2|\ c_1\right)$ and $c_3\left(p_2,p_3,y_2|\ c_1\right)$ satisfy Property (S),

$$\left(\frac{\partial c_2}{\partial p_3}\right)_{y_2=const}+c_3\frac{\partial c_2}{\partial y_2}=\left(\frac{\partial c_3}{\partial p_2}\right)_{y_2=const}+c_2\frac{\partial c_3}{\partial y_2}, \qquad \text{(L.9)}$$

implying that $\sigma_{23}=\sigma_{32}$. Therefore the unconditional demands $c_i\left(p_1,p_2,p_3,y_1\right)$ $(i=1,2,3)$ satisfy Property (S) and preferences are effectively consistent. It follows from Selden and Wei (2015) that there exists a $U\left(c_1,c_2,c_3\right)$ that takes the form

$$U\left(c_1,c_2,c_3\right)=h\left(g\left(c_1\right)c_2,g\left(c_1\right)c_3\right) \qquad \text{(L.10)}$$

to rationalize the demands, where $g\left(c_1\right)$ is uniquely determined (up to an arbitrary constant of integration) from the c_1 unconditional demand function. To see this, note that the c_1 unconditional demand function is a solution to the following ordinary differential equation

$$p_1c_1+\frac{p_1g\left(c_1\right)}{g'\left(c_1\right)}=y_1. \qquad \text{(L.11)}$$

To recover $g\left(c_1\right)$ from the c_1 demand function, express y_1 as a function of $\left(p_1,c_1\right)$. Since Property (H) holds, one must have

$$y_1=p_1\varphi\left(c_1\right), \qquad \text{(L.12)}$$

implying that

$$c_1+\frac{g\left(c_1\right)}{g'\left(c_1\right)}=\varphi\left(c_1\right), \qquad \text{(L.13)}$$

or equivalently,

$$\left(\ln g\left(c_1\right)\right)'=\frac{1}{\varphi\left(c_1\right)-c_1}. \qquad \text{(L.14)}$$

Solving the above ordinary differential equation yields a unique function $g\left(c_1\right)$ up to an arbitrary constant of integration. Following from Selden and Wei (2015), the demands $c_i\left(p_1,p_2,p_3,y_1\right)$ $(i=1,2,3)$ also correspond to the sophisticated choice of a $\left(U^{(1)},U^{(2)}\right)$-pair, where

$$U^{(1)}\left(c_1,c_2,c_3\right)=f\left(g\left(c_1\right)c_2,g\left(c_1\right)c_3\right) \qquad \text{(L.15)}$$

and $U^{(2)}$ is homothetic.

M Generalized Quasilinear Effective Consistency Case

In this appendix, we discuss the integrability problem associated with sophisticated choice for the generalized quasilinear effective consistency case. A sufficient condition for effective consistency given by Selden and Wei (2015) assumes that $U^{(1)}(c_1, c_2, c_3)$ and $U^{(2)}(c_2, c_3)$ take the following quasilinear forms, i.e.,

$$U^{(1)}(c_1, c_2, c_3) = g(c_1) + f(c_2) + c_3 \quad \text{and} \quad U^{(2)}(c_2, c_3) = h(c_2) + c_3. \tag{M.1}$$

However for effective consistency, $U^{(1)}$ can take the more general form

$$U^{(1)}(c_1, c_2, c_3) = f(g(c_1) + c_3, c_2), \tag{M.2}$$

which will be referred to as the generalized quasilinear form. The reason is as follows. For naive choice, c_1^* is determined by the first order condition $g'(c_1) = p_1/p_3$. For sophisticated choice, we can rewrite $U^{(1)}$ as

$$U^{(1)}(c_1, c_2, c_3) = f\left(g(c_1) + \frac{y_1 - p_1 c_1 - p_2 c_2}{p_3}, c_2\right). \tag{M.3}$$

Since c_2^{**} is independent of y_2, the optimal c_1^{**} must satisfy

$$g'(c_1) - \partial\left(\frac{y_1 - p_1 c_1 - p_2 c_2}{p_3}\right)/\partial c_1 = g'(c_1) - \frac{p_1}{p_3} = 0. \tag{M.4}$$

Therefore, $c_1^* = c_1^{**}$ and preferences are effectively consistent. Then we have the following corollary.

Corollary 7 *Assume the conditions in Corollary 1 are satisfied and further assume that the demand function $c_1(p_1, p_2, p_3, y_1)$ is independent of p_2 and y_1, the conditional demand $c_2(p_2, p_3, y_2|\ c_1)$ is independent of y_2 and the conditional demands $c_2(p_2, p_3, y_2|\ c_1)$ and $c_3(p_2, p_3, y_2|\ c_1)$ have Property (N). Then preferences are effectively consistent in the sense of Definition 3 and sophisticated choice can be rationalized by a non-changing tastes $U(c_1, c_2, c_3)$ which takes the form*

$$U(c_1, c_2, c_3) = g(c_1) + h(c_2) + c_3. \tag{M.5}$$

Moreover, there exists a $(U^{(1)}, U^{(2)})$-pair which generates these demands as a result of sophisticated choice, where $U^{(1)}(c_1, c_2, c_3)$ is given by

$$U^{(1)}(c_1, c_2, c_3) = f(g(c_1) + c_3, c_2) \tag{M.6}$$

and $U^{(2)}(c_2, c_3)$ is given by

$$U^{(2)}(c_2, c_3) = h(c_2) + c_3 \tag{M.7}$$

and satisfies Property 1.

Proof. To show preferences are effectively consistent, we need to prove that the unconditional demands $c_i(p_1, p_2, p_3, y_1)$ $(i = 1, 2, 3)$ satisfy Property (S). Notice that

$$\frac{\partial c_2}{\partial p_1} = \frac{\partial c_2}{\partial y_2}\frac{\partial y_2}{\partial p_1} = 0 \quad \text{and} \quad \frac{\partial c_2}{\partial y_1} = \frac{\partial c_2}{\partial y_2}\frac{\partial y_2}{\partial y_1} = 0, \tag{M.8}$$

implying that

$$\sigma_{21} = \frac{\partial c_2}{\partial p_1} + c_1\frac{\partial c_2}{\partial y_1} = 0.$$

Since c_1 is independent of p_2 and y_1,

$$\sigma_{12} = \frac{\partial c_1}{\partial p_2} + c_2\frac{\partial c_1}{\partial y_1} = 0 = \sigma_{21}. \tag{M.9}$$

Moreover, since $c_3(p_2, p_3, y_2|\ c_1)$ is independent of c_1,

$$\frac{\partial c_3}{\partial p_1} = \frac{\partial c_3}{\partial y_2}\frac{\partial y_2}{\partial p_1} = \frac{\partial(y_2 - p_2c_2)}{p_3\partial y_2}\frac{\partial y_2}{\partial p_1} = \frac{1}{p_3}\left(-c_1 - p_1\frac{\partial c_1}{\partial p_1}\right) \tag{M.10}$$

and

$$\frac{\partial c_3}{\partial y_1} = \frac{\partial c_3}{\partial y_2}\frac{\partial y_2}{\partial y_1} = \frac{\partial(y_2 - p_2c_2)}{p_3\partial y_2}\frac{\partial y_2}{\partial y_1} = \frac{1}{p_3}\left(1 - p_1\frac{\partial c_1}{\partial y_1}\right), \tag{M.11}$$

implying that

$$\sigma_{31} = \frac{\partial c_3}{\partial p_1} + c_1\frac{\partial c_3}{\partial y_1} = -\frac{p_1}{p_3}\frac{\partial c_1}{\partial p_1}.$$

Since c_1 is independent of p_2 and y_1,

$$\sigma_{13} = \frac{\partial c_1}{\partial p_3} + c_3\frac{\partial c_1}{\partial y_1} = \frac{\partial c_1}{\partial p_3}. \tag{M.12}$$

Property (H) implies $p_1\frac{\partial c_1}{\partial p_1} + p_3\frac{\partial c_1}{\partial p_3} = 0$. Therefore, $\sigma_{13} = \sigma_{31}$. Finally, it can be verified that

$$\sigma_{23} = \left(\frac{\partial c_2}{\partial p_3}\right)_{y_2=const} - p_1\frac{\partial c_2}{\partial y_2}\frac{\partial c_1}{\partial p_3} + c_3\frac{\partial c_2}{\partial y_1} = \left(\frac{\partial c_2}{\partial p_3}\right)_{y_2=const} \tag{M.13}$$

and

$$\begin{aligned}\sigma_{32} &= \left(\frac{\partial c_3}{\partial p_2}\right)_{y_2=const} - p_1\frac{\partial c_2}{\partial y_2}\frac{\partial c_1}{\partial p_2} + c_2\frac{\partial c_3}{\partial y_1} \\ &= \left(\frac{\partial c_3}{\partial p_2}\right)_{y_2=const} + c_2\frac{\partial c_3}{\partial y_2}.\end{aligned} \tag{M.14}$$

Since the conditional demands $c_2(p_2, p_3, y_2|\ c_1)$ and $c_3(p_2, p_3, y_2|\ c_1)$ satisfy Property (S) and

$$\left(\frac{\partial c_2}{\partial p_3}\right)_{y_2=const} + c_3\frac{\partial c_2}{\partial y_2} = \left(\frac{\partial c_2}{\partial p_3}\right)_{y_2=const} = \left(\frac{\partial c_3}{\partial p_2}\right)_{y_2=const} + c_2\frac{\partial c_3}{\partial y_2}, \tag{M.15}$$

this implies that $\sigma_{23} = \sigma_{32}$. Therefore the unconditional demands $c_i(p_1, p_2, p_3, y_1)$ $(i = 1, 2, 3)$ satisfy Property (S) and hence preferences are effectively consistent. It follows from Selden and Wei (2015) that there exists a $U(c_1, c_2, c_3)$ that takes the form

$$U(c_1, c_2, c_3) = g(c_1) + h(c_2) + c_3 \tag{M.16}$$

to rationalize the demands, where $g(c_1)$ and $h(c_2)$ are uniquely determined (up to an arbitrary constant of integration) from the c_1 and c_2 unconditional demand function, respectively. To recover $g(c_1)$ from $c_1(p_1, p_2, p_3, y_1)$, noticing that the unconditional demand function $c_1(p_1, p_2, p_3, y_1)$ is independent of p_2 and y_1 and Property (H) holds, one must have

$$c_1(p_1, p_2, p_3, y_1) = \varphi\left(\frac{p_1}{p_3}\right). \tag{M.17}$$

It follows from the first order condition of the utility (M.16) that

$$g'(c_1) = \frac{p_1}{p_3} = \varphi^{-1}(c_1). \tag{M.18}$$

$h(c_2)$ can be recovered following a similar argument. Since it is obvious that $g(c_1) + c_3$ and c_2 are two first integrals, the demands also correspond to the sophisticated choice of a $\left(U^{(1)}, U^{(2)}\right)$-pair, where

$$U^{(1)}(c_1, c_2, c_3) = f(g(c_1) + c_3, c_2) \qquad \text{and} \qquad U^{(2)}(c_2, c_3) = h(c_2) + c_3, \tag{M.19}$$

where f is an arbitrary function. ■

Remark 4 *For the generalized quasilinear case in Corollary 7, $c_2(p_2, p_3, y_2|\ c_1)$ being independent of y_2 and c_1 implies that c_3 is linear in y_2. Moreover, since $c_2(p_2, p_3, y_2|\ c_1)$ is independent of y_2 and c_1, it is one of the first integrals in $U^{(1)}$, which can be viewed as a degenerate case for $\frac{c_2 - a(c_1)}{c_3 - b(c_1)}$.*

N Proof of Theorem 3

First prove sufficiency. Since the demand functions $c_i(p_1, ..., p_M, y_1)$ $(i = 1, 2, ..., M)$ satisfy the Properties (P), (TD), (H), (B) and (EC), the conditional demand functions $c_2(p_{K+1}, ..., p_M, y_2|\ c_1, ..., c_K)$ and $c_3(p_{K+1}, ..., p_M, y_2|\ c_1, ..., c_K)$ exist due to (EC) and also have the Properties (P), (TD), (H) and (B). Since the conditional demands $c_i(p_{K+1}, ..., p_M, y_2|\ c_1, ..., c_K)$ $(i = K + 1, ..., M)$ are independent of $(c_1, ..., c_K)$ and also have the Properties (S) and (N), Theorem 2.6 in Jehle and

Reny (2011) ensures the existence of a unique $U^{(2)}$ satisfying Property 1. Given the conditional demands

$$c_j\left(p_{K+1},...,p_M,y_1-\sum_{i=1}^{K}p_ic_i\right)\quad(j=K+1,...,M)\,, \tag{N.1}$$

$U^{(1)}$ must satisfy the following system of partial differential equations

$$\frac{\partial U^{(1)}}{\partial c_i}+\sum_{j=K+1}^{M}\frac{\partial c_j}{\partial c_i}\frac{\partial U^{(1)}}{\partial c_j}=0\quad(i=1,2,...,K)\,. \tag{N.2}$$

Noticing that

$$\frac{\partial c_j}{\partial c_i}=\frac{\partial c_j\left(p_{K+1},...,p_M,y_1-\sum_{i=1}^{K}p_ic_i\right)}{\partial c_i}=-p_i\frac{\partial c_j}{\partial y_2}, \tag{N.3}$$

eqn. (N.2) can be rewritten as

$$\frac{\partial U^{(1)}}{\partial c_i}-p_i\sum_{j=K+1}^{M}\frac{\partial c_j}{\partial y_2}\frac{\partial U^{(1)}}{\partial c_j}=0\quad(i=1,2,...,K)\,. \tag{N.4}$$

Since Property (EI) holds, p_i $(i=1,2,...,K)$ can be transformed into functions of c_i $(i=1,2,...,M)$ through the inverse demands, implying that the coefficients in the first order partial differential equation (N.4) can be expressed as functions of $(c_1,...,c_M)$. Setting

$$\begin{aligned}&\widetilde{U}^{(1)}\left(c_1,..,c_K,p_{K+1},..,p_M,y_2\right)\\ =\;&U^{(1)}(c_1,...,c_K,c_{K+1}\left(p_{K+1},...,p_M,y_2\right),...,c_M\left(p_{K+1},...,p_M,y_2\right)),\end{aligned} \tag{N.5}$$

eqn. (N.4) can be rewritten as

$$\frac{\partial \widetilde{U}^{(1)}}{\partial c_i}-p_i\frac{\partial \widetilde{U}^{(1)}}{\partial y_2}=0\quad(i=1,2,...,K)\,. \tag{N.6}$$

Defining

$$L_i\widetilde{U}^{(1)}=\frac{\partial \widetilde{U}^{(1)}}{\partial c_i}-p_i\frac{\partial \widetilde{U}^{(1)}}{\partial y_2}\quad(i=1,2,...,K)\,, \tag{N.7}$$

eqn. (N.4) can be rewritten as

$$L_i\widetilde{U}^{(1)}=0\quad(i=1,2,...,K)\,. \tag{N.8}$$

It follows from the Frobenius theorem for integrability that the above system of partial differential equations have a solution if

$$L_iL_j\widetilde{U}^{(1)}=L_jL_i\widetilde{U}^{(1)}, \tag{N.9}$$

or equivalently

$$\frac{\partial p_j}{\partial c_i} - p_i \frac{\partial p_j}{\partial y_2} = \frac{\partial p_i}{\partial c_j} - p_j \frac{\partial p_i}{\partial y_2} \quad (i, j = 1, 2, ..., K). \tag{N.10}$$

Next prove necessity. Since the conditional demands $c_2\left(p_{K+1}, ..., p_M, y_2 \middle| \; c_1, ..., c_K\right)$ and $c_3\left(p_{K+1}, ..., p_M, y_2 \middle| \; c_1, ..., c_K\right)$ (i) satisfy the Properties (P), (TD) and (H), (ii) are independent of $(c_1, ..., c_K)$ and (iii) can be rationalized by a utility function $U^{(2)}$ satisfying Property 1, it follows from Theorem 2.6 in Jehle and Reny (2011) that the Properties (B), (S) and (N) are satisfied. Set

$$\widetilde{U}^{(1)}\left(c_1, .., c_K, p_{K+1}, .., p_M, y_2\right) = U^{(1)}(c_1\left(p_{K+1}, ..., p_M, y_2\right), ..., c_M\left(p_{K+1}, ..., p_M, y_2\right)). \tag{N.11}$$

Since $\widetilde{U}^{(1)}$ satisfies

$$\frac{\partial \widetilde{U}^{(1)}}{\partial c_i} - p_i \frac{\partial \widetilde{U}^{(1)}}{\partial y_2} = 0 \quad (i = 1, 2, ..., K), \tag{N.12}$$

if the inverse demands do not exist, then the coefficients of the above partial differential equation cannot be expressed as functions of $(c_1, ..., c_K, p_{K+1}, .., p_M, y_2)$ and thus there is no solution for $U^{(1)}$. Therefore, Property (EI) must be satisfied. Moreover, it follows from the Frobenius theorem for integrability, eqn. (N.12) has a solution only if

$$L_i L_j \widetilde{U}^{(1)} = L_j L_i \widetilde{U}^{(1)}, \tag{N.13}$$

or equivalently

$$\frac{\partial p_j}{\partial c_i} - p_i \frac{\partial p_j}{\partial y_2} = \frac{\partial p_i}{\partial c_j} - p_j \frac{\partial p_i}{\partial y_2} \quad (i, j = 1, 2, ..., K), \tag{N.14}$$

where p_i is a function of $c_1, ..., c_K, p_{K+1}, ..., p_M, y_2$.

O Multiple Changes in Tastes: General Case

Assume there are three periods and there are K commodities in the first period, $H - K$ commodities in the second period and $M - H$ commodities in the third period. The consumer faces the optimization problems

$$P_1 : \quad \max_{c_1, ..., c_M} U^{(1)}(c_1, ..., c_M) \qquad S.T. \; y_1 \geq \sum_{j=1}^{M} p_j c_j, \tag{O.1}$$

$$P_2 : \quad \max_{c_{K+1}, ..., c_M} U^{(2)}(c_{K+1}, ..., c_M | \; c_1, ..., c_K) \quad S.T. \;\; y_2 = y_1 - \sum_{j=1}^{K} p_j c_j \geq \sum_{j=K+1}^{M} p_j c_j \tag{O.2}$$

and

$$P_3: \quad \max_{c_{H+1},\dots,c_M} U^{(3)}(c_{H+1},...,c_M|\ c_1,...,c_H) \quad S.T.\ \ y_3 = y_1 - \sum_{j=1}^{H} p_j c_j \geq \sum_{j=H+1}^{M} p_j c_j. \tag{O.3}$$

First consider the existence of $U^{(3)}$. We assume that the demand functions $c_i(p_1,...,p_M,y_1)$ $(i = 1, 2, ..., M)$ satisfy the Properties (P), (TD), (H) and (B). If we also assume (EC) holds, i.e.,

$$\frac{\partial(c_1,...,c_H,y_3)}{\partial(p_1,...,p_H,y_1)} \neq 0 \quad \forall(p_1,...,p_H,y_1) \in \mathbb{R}^H_{++} \times \mathbb{R}_+, \tag{O.4}$$

then the conditional demands $c_i(p_{H+1},...,p_M,y_3|\ c_1,...,c_H)$ $(i = H+1,...,M)$ exist and also have the Properties (P), (TD), (H) and (B). If the conditional demands $c_i(p_{H+1},...,p_M,y_3|\ c_1,...,c_H)$ $(i = H+1,...,M)$ are independent of $(c_1,...,c_H)$ and also have the Properties (S) and (N), Theorem 2.6 in Jehle and Reny (2011) ensures the existence of a unique $U^{(3)}$ satisfying Property 1. Next consider the existence of $U^{(2)}$. Since

$$y_3 = y_2 - \sum_{j=K+1}^{H} p_j c_j, \tag{O.5}$$

we have $\forall i \in \{H+1,...,M\}$

$$c_i(p_{H+1},...,p_M,y_3|\ c_1,...,c_H) = c_i\left(p_{H+1},...,p_M,y_2 - \sum_{j=K+1}^{H} p_j c_j \middle|\ c_1,...,c_H\right). \tag{O.6}$$

If we assume that (EC) holds, i.e.,

$$\frac{\partial(c_1,...,c_K,y_2)}{\partial(p_1,...,p_K,y_1)} \neq 0 \quad \forall(p_1,...,p_K,y_1) \in \mathbb{R}^K_{++} \times \mathbb{R}_+, \tag{O.7}$$

then the conditional demands $c_i(p_{K+1},...,p_M,y_2|\ c_1,...,c_K)$ $(i = K+1,...,M)$ exist. Further assume that the conditional demands $c_i(p_{K+1},...,p_M,y_2|\ c_1,...,c_K)$ $(i = K+1,...,M)$ are independent of $(c_1,...,c_K)$. Then $U^{(2)}(c_{K+1},...,c_M)$ satisfies $\forall i \in \{K+1,...,H\}$

$$\frac{\partial U^{(2)}}{\partial c_i} + \sum_{j=H+1}^{M} \frac{\partial c_j}{\partial c_i}\frac{\partial U^{(2)}}{\partial c_i} = 0. \tag{O.8}$$

Noticing that $c_j(p_{H+1},...,p_M,y_3|\ c_1,...,c_H)$ $(j = H+1,...,M)$ are independent of $(c_1,...,c_H)$, $\forall i \in \{K+1,...,H\}$ and $\forall j \in \{H+1,...,M\}$,

$$\frac{\partial c_j}{\partial c_i} = \frac{\partial c_j\left(p_{H+1},...,p_M,y_2 - \sum_{l=K+1}^{H} p_l c_l\right)}{\partial c_i} = -p_i\frac{\partial c_j}{\partial y_3} \tag{O.9}$$

and eqn. (O.8) can be rewritten as

$$\frac{\partial U^{(2)}}{\partial c_i} - p_i \sum_{j=H+1}^{M} \frac{\partial c_j}{\partial y_3} \frac{\partial U^{(2)}}{\partial c_j} = 0 \quad (i = K+1, ..., H). \tag{O.10}$$

If we assume that (EI) holds, i.e.,

$$\frac{\partial (c_{K+1}, ..., c_M)}{\partial (p_{K+1}, ..., p_M)} \neq 0 \quad \forall (p_{K+1}, ..., p_M) \in \mathbb{R}^{M-K}_{++}, \tag{O.11}$$

where c_i $(i = K+1, ..., M)$ are conditional demand functions and the following Frobenius condition holds

$$\frac{\partial p_j}{\partial c_i} - p_i \frac{\partial p_j}{\partial y_3} = \frac{\partial p_i}{\partial c_j} - p_j \frac{\partial p_i}{\partial y_3} \quad (i, j = K+1, ..., H), \tag{O.12}$$

where p_i is a function of $c_{K+1}, ..., c_H, p_{H+1}, ..., p_M, y_3$, then there exists a solution to the set of partial differential equations (O.10) implying that $U^{(2)}$ exists. Finally, $U^{(1)}$ satisfies $\forall i \in \{1, ..., K\}$

$$\frac{\partial U^{(1)}}{\partial c_i} + \sum_{j=K+1}^{M} \frac{\partial c_j}{\partial c_i} \frac{\partial U^{(1)}}{\partial c_i} = 0, \tag{O.13}$$

where $c_j = c_j \left(p_{K+1}, ..., p_M, y_1 - \sum_{l=1}^{K} p_l c_l \right)$ $(j = K+1, ..., H)$ and $\forall j \in \{H+1, ..., M\}$

$$c_j = c_j \left(p_{H+1}, ..., p_M, y_1 - \sum_{l=K+1}^{H} p_l c_l - \sum_{l=1}^{K} p_l c_l \right) = c_j \left(p_{K+1}, ..., p_M, y_1 - \sum_{l=1}^{K} p_l c_l \right). \tag{O.14}$$

Therefore, $\forall i \in \{1, ..., K\}$ and $\forall j \in \{K+1, ..., M\}$,

$$\frac{\partial c_j}{\partial c_i} = \frac{\partial c_j \left(p_{K+1}, ..., p_M, y_1 - \sum_{l=1}^{K} p_l c_l \right)}{\partial c_i} = -p_i \frac{\partial c_j}{\partial y_2} \tag{O.15}$$

and eqn. (O.13) can be rewritten as

$$\frac{\partial U^{(1)}}{\partial c_i} - p_i \sum_{j=K+1}^{M} \frac{\partial c_j}{\partial y_2} \frac{\partial U^{(1)}}{\partial c_j} = 0 \quad (i = 1, ..., K). \tag{O.16}$$

If we assume that (EI) holds, i.e.,

$$\frac{\partial (c_1, ..., c_M)}{\partial (p_1, ..., p_M)} \neq 0 \ \ \forall (p_1, ..., p_M) \in \mathbb{R}^{M}_{++}, \tag{O.17}$$

where c_i $(i = 1, ..., M)$ are unconditional demand functions and the following Frobenius condition holds

$$\frac{\partial p_j}{\partial c_i} - p_i \frac{\partial p_j}{\partial y_2} = \frac{\partial p_i}{\partial c_j} - p_j \frac{\partial p_i}{\partial y_2} \quad (i, j = 1, ..., K), \tag{O.18}$$

where p_i is a function of $c_1, ..., c_K, p_{K+1}, ..., p_M, y_2$, then there exists a solution to the set of partial differential equations (O.16) implying that $U^{(1)}$ exists.

P Supporting Calculations for Example 7

It can be verified that

$$\det \frac{\partial \left(c_1, c_2, c_3, c_4\right)}{\partial \left(p_1, p_2, p_3, p_4\right)} \neq 0 \quad \forall (p_1, p_2, p_3, p_4) \in \mathbb{R}^4_{++}, \tag{P.1}$$

implying that (EI) holds. Using the conditional demands and inverse demand functions, eqn. (71) in Theorem 3 can be expressed as

$$2\sqrt{c_3 + c_4}\frac{\partial U^{(1)}}{\partial c_1} - \sqrt{2}c_3\frac{\partial U^{(1)}}{\partial c_3} - \sqrt{2}c_4\frac{\partial U^{(1)}}{\partial c_4} = 0 \tag{P.2}$$

and

$$2\sqrt{\left(c_3 + c_4\right) c_2}\frac{\partial U^{(1)}}{\partial c_2} - \sqrt{2}c_3\frac{\partial U^{(1)}}{\partial c_3} - \sqrt{2}c_4\frac{\partial U^{(1)}}{\partial c_4} = 0. \tag{P.3}$$

Applying eqn. (70) in Theorem 3, the necessary and sufficient condition for the existence of a solution to eqns. (P.2) - (P.3) is

$$\frac{\partial p_2}{\partial c_1} - p_1\frac{\partial p_2}{\partial y_2} = \frac{\partial p_1}{\partial c_2} - p_2\frac{\partial p_1}{\partial y_2}. \tag{P.4}$$

Expressing p_1 and p_2 as functions of $(c_1, c_2, p_3, p_4, y_2)$ and substituting them into (P.4), it follows that

$$\begin{aligned} \frac{\partial p_2}{\partial c_1} - p_1\frac{\partial p_2}{\partial y_2} &= 0 - \frac{1}{2\left(\frac{1}{p_3} + \frac{1}{p_4}\right)\sqrt{c_2}} = -\frac{1}{2\left(\frac{1}{p_3} + \frac{1}{p_4}\right)\sqrt{c_2}} \\ &= \frac{\partial p_1}{\partial c_2} - p_2\frac{\partial p_1}{\partial y_2}, \end{aligned} \tag{P.5}$$

implying that eqn. (P.4) holds and a solution to the partial differential eqns. (71) exists. Therefore, eqns. (P.2) - (P.3) have a solution. To solve these partial differential equations for $U^{(1)}$, note that the two independent first integrals for eqn. (P.2) are given by

$$\psi_1\left(c_1, c_2, c_3, c_4\right) = c_1 + g\left(c_2\right) + 2\sqrt{2\left(c_3 + c_4\right)} \quad \text{and} \quad \psi_2\left(c_1, c_2, c_3, c_4\right) = \frac{c_3}{c_4} \tag{P.6}$$

and the two independent first integrals for eqn. (P.3) are given by

$$\psi_1\left(c_1, c_2, c_3, c_4\right) = h\left(c_1\right) + 2\sqrt{c_2} + 2\sqrt{2\left(c_3 + c_4\right)} \quad \text{and} \quad \psi_2\left(c_1, c_2, c_3, c_4\right) = \frac{c_3}{c_4}. \tag{P.7}$$

Therefore,

$$U^{(1)}\left(c_1, c_2, c_3, c_4\right) = f\left(c_1 + 2\sqrt{c_2} + 2\sqrt{2\left(c_3 + c_4\right)}, \frac{c_3}{c_4}\right). \tag{P.8}$$

Q Proof of Theorem 4

Note that

$$
\begin{aligned}
& J_c \\
= & \frac{\partial\left(c_1, \ldots, c_K, y_2\right)}{\partial\left(p_1, \ldots, p_K, y_1\right)} \\
= & \left[\begin{array}{cccc}
\frac{\partial c_1}{\partial p_1} & \ldots & \frac{\partial c_1}{\partial p_K} & \frac{\partial c_1}{\partial y_1} \\
\vdots & \ldots & \vdots & \vdots \\
\frac{\partial c_K}{\partial p_1} & \ldots & \frac{\partial c_K}{\partial p_K} & \frac{\partial c_K}{\partial y_1} \\
-c_1-\sum_{i=1}^K p_i \frac{\partial c_i}{\partial p_1} & \ldots & -c_K-\sum_{i=1}^K p_i \frac{\partial c_i}{\partial p_K} & 1-\sum_{i=1}^K p_i \frac{\partial c_i}{\partial y_1}
\end{array}\right] . \quad (Q.1)
\end{aligned}
$$

Since the matrix determinant is invariant under elementary transformations, multiplying the i^{th} line of the matrix J_c by p_i $(i=1, \ldots, K)$ and adding them to the last line of J_c yields

$$
\det J_c=\det\left[\begin{array}{cccc}
\frac{\partial c_1}{\partial p_1} & \ldots & \frac{\partial c_1}{\partial p_K} & \frac{\partial c_1}{\partial y_1} \\
\vdots & \ldots & \vdots & \vdots \\
\frac{\partial c_K}{\partial p_1} & \ldots & \frac{\partial c_K}{\partial p_K} & \frac{\partial c_K}{\partial y_1} \\
-c_1 & \ldots & -c_K & 1
\end{array}\right] . \quad (Q.2)
$$

Multiplying the last line of J_c by $-\frac{\partial c_i}{\partial y_1}$ $(i=1, \ldots, K)$ and adding them respectively to the i^{th} line of J_c, one obtains

$$
\begin{aligned}
\det J_c & =\det\left[\begin{array}{cccc}
\frac{\partial c_1}{\partial p_1}+c_1 \frac{\partial c_1}{\partial y_1} & \ldots & \frac{\partial c_1}{\partial p_K}+c_K \frac{\partial c_1}{\partial y_1} & 0 \\
\vdots & \ldots & \vdots & \vdots \\
\frac{\partial c_K}{\partial p_1}+c_1 \frac{\partial c_K}{\partial y_1} & \ldots & \frac{\partial c_K}{\partial p_K}+c_K \frac{\partial c_K}{\partial y_1} & 0 \\
-c_1 & \ldots & -c_K & 1
\end{array}\right] \\
& =\det\left[\begin{array}{ccc}
\frac{\partial c_1}{\partial p_1}+c_1 \frac{\partial c_1}{\partial y_1} & \ldots & \frac{\partial c_1}{\partial p_K}+c_K \frac{\partial c_1}{\partial y_1} \\
\vdots & \ldots & \vdots \\
\frac{\partial c_K}{\partial p_1}+c_1 \frac{\partial c_K}{\partial y_1} & \ldots & \frac{\partial c_K}{\partial p_K}+c_K \frac{\partial c_K}{\partial y_1}
\end{array}\right], \quad (Q.3)
\end{aligned}
$$

which is the determinant of the Slutsky matrix of the unconditional demands $c_i\left(p_1, \ldots, p_M, y_1\right)$ $(i=1, \ldots, K)$, i.e.,

$$
\det\left(\sigma_{ij}\right)_{K \times K}=\det \frac{\partial\left(c_1, \ldots, c_K, y_2\right)}{\partial\left(p_1, \ldots, p_K, y_1\right)} . \quad (Q.4)
$$

If the period one demand functions $c_i\left(p_1, \ldots, p_M, y_1\right)$ $(i=1, \ldots, K)$ have the Properties (S) and (ND), then

$$
\det\left(\sigma_{ij}\right)_{K \times K}\left\{\begin{array}{l}
<0 \text { if } K \text { is odd } \\
>0 \text { if } K \text { is even }
\end{array}\right. , \quad (Q.5)
$$

implying that at each point $(p_1^0, ..., p_K^0, y_1^0) \in \mathbb{R}_{++}^K \times \mathbb{R}_+$

$$\det J_c = \det \left.\frac{\partial (c_1, ..., c_K, y_2)}{\partial (p_1, ..., p_K, y_1)}\right|_{(p_1,...,p_K,y_1)=\left(p_1^0,...,p_K^0,y_1^0\right)} \begin{cases} < 0 & \text{if } K \text{ is odd} \\ > 0 & \text{if } K \text{ is even} \end{cases}. \quad \text{(Q.6)}$$

It follows from Lemma 1 that Property (EC) holds in a neighborhood of $(p_1^0, ..., p_K^0, y_1^0)$.

R Rationalizing Naive Choice

In this appendix, we discuss how to rationalize naive choice in more detail. In the next example, the given demands satisfy Slutsky symmetry and hence they correspond to a case where naive and sophisticated choice agree and one can use the Hurwicz and Uzawa (1971) process to recover a non-changing tastes U and thus also a $U^{(1)}$ and a $U^{(2)}$. Moreover, since the c_1 demand function satisfies the condition in Theorem 5,[35] we show that these demands can be also viewed as corresponding to naive choice and derive a $(U^{(1)}, U^{(2)})$-pair that rationalizes the demands.

Example 9 *Assume that*

$$c_1 = \frac{y_1}{p_1 + p_2\left(\frac{p_1}{p_2}\right)^2 + p_3\left(\frac{p_1}{p_3}\right)^2}, \qquad c_2 = \frac{\left(\frac{p_1}{p_2}\right)^2 y_1}{p_1 + p_2\left(\frac{p_1}{p_2}\right)^2 + p_3\left(\frac{p_1}{p_3}\right)^2} \quad \text{(R.1)}$$

and

$$c_3 = \frac{\left(\frac{p_1}{p_3}\right)^2 y_1}{p_1 + p_2\left(\frac{p_1}{p_2}\right)^2 + p_3\left(\frac{p_1}{p_3}\right)^2}. \quad \text{(R.2)}$$

It can be verified that the corresponding Slutsky matrix is symmetric and negative semidefinite. Following the Hurwicz and Uzawa (1971) recovery process, these demands can be rationalized by

$$U(c_1, c_2, c_3) = \sqrt{c_1} + \sqrt{c_2} + \sqrt{c_3}. \quad \text{(R.3)}$$

Next assume that the given demands correspond to naive choice. Since the conditional demands for c_2 and c_3 are

$$c_2 = \frac{y_1 - p_1 c_1}{p_2 + p_3\left(\frac{p_2}{p_3}\right)^2} = \frac{y_2}{p_2 + p_3\left(\frac{p_2}{p_3}\right)^2} \quad \text{(R.4)}$$

[35] The c_1 demand function will be seen not to take the form in Table 1 below associated with effective consistency.

and

$$c_3 = \frac{(y_1 - p_1 c_1)\left(\frac{p_2}{p_3}\right)^2}{p_2 + p_3\left(\frac{p_2}{p_3}\right)^2} = \frac{y_2\left(\frac{p_2}{p_3}\right)^2}{p_2 + p_3\left(\frac{p_2}{p_3}\right)^2}, \tag{R.5}$$

we have

$$U^{(2)}(c_2, c_3) = \sqrt{c_2} + \sqrt{c_3}. \tag{R.6}$$

Next we apply the process in Epstein (1982) to recover $U^{(1)}$. Since the c_1 demand function is

$$c_1 = \frac{y_1}{p_1 + p_2\left(\frac{p_1}{p_2}\right)^2 + p_3\left(\frac{p_1}{p_3}\right)^2}, \tag{R.7}$$

following Table I in Epstein (1982), we can assume that the indirect utility function is given by

$$\begin{aligned} V(p_1, p_2, p_3, y_1) &= \frac{y_1}{\exp\left(\int^{p_1} \frac{1}{t + p_2\left(\frac{t}{p_2}\right)^2 + p_3\left(\frac{t}{p_3}\right)^2} dt\right)} && \text{(R.8)} \\ &= \frac{y_1}{\exp\left(\ln p_1 - \ln\left(p_1 p_2 + p_1 p_3 + p_2 p_3\right) + C\right)} && \text{(R.9)} \\ &= \frac{\left(p_1 p_2 + p_1 p_3 + p_2 p_3\right) y_1}{p_1 \exp C}, && \text{(R.10)} \end{aligned}$$

where C is a constant ensuring that Property (B) is satisfied. Following Roy's identity,

$$c_2 = \frac{-\partial V/\partial p_2}{\partial V/\partial y_1} \quad \text{and} \quad c_3 = \frac{-\partial V/\partial p_3}{\partial V/\partial y_1}. \tag{R.11}$$

It can be verified that if

$$C = \ln(p_2 p_3), \tag{R.12}$$

then

$$p_1 c_1 + p_2 c_2 + p_3 c_3 = y_1. \tag{R.13}$$

Therefore, the indirect utility function is given by

$$V(p_1, p_2, p_3, y_1) = \frac{\left(p_1 p_2 + p_1 p_3 + p_2 p_3\right) y_1}{p_1 p_2 p_3}. \tag{R.14}$$

Then we have

$$c_2 = \frac{-\partial V/\partial p_2}{\partial V/\partial y_1} = \frac{\left(\frac{p_1}{p_2}\right)^2 y_1}{p_1 + p_2\left(\frac{p_1}{p_2}\right)^2 + p_3\left(\frac{p_1}{p_3}\right)^2} \tag{R.15}$$

and

$$c_3 = \frac{-\partial V/\partial p_3}{\partial V/\partial y_1} = \frac{\left(\frac{p_1}{p_3}\right)^2 y_1}{p_1 + p_2\left(\frac{p_1}{p_2}\right)^2 + p_3\left(\frac{p_1}{p_3}\right)^2}. \tag{R.16}$$

Following the Hurwicz and Uzawa (1971) recovery process, these demands can be rationalized by

$$U^{(1)}(c_1, c_2, c_3) = \sqrt{c_1} + \sqrt{c_2} + \sqrt{c_3}, \tag{R.17}$$

which coincides with eqn. (R.3).[36] *However, since this process assumes that the preferences are homothetic, it cannot give all possible forms of* $U^{(1)}$.

The following provides a simple application of Theorem 6, where the demands also satisfy effective consistency.

Example 10 *Assume that*

$$c_1 = \frac{y_1}{2p_1}, \quad c_2 = \frac{y_1}{4p_2} \quad \text{and} \quad c_3 = \frac{y_1}{4p_3}. \tag{R.18}$$

It can be verified that the corresponding Slutsky matrix is symmetric and negative semidefinite. Following the Hurwicz and Uzawa (1971) recovery process, these demands can be rationalized by

$$U(c_1, c_2, c_3) = 2\ln c_1 + \ln c_2 + \ln c_3. \tag{R.19}$$

Next assume that the given demands correspond to naive choice. Since the conditional demands for c_2 *and* c_3 *are*

$$c_2 = \frac{y_1 - p_1 c_1}{2p_2} = \frac{y_2}{2p_2} \qquad \text{and} \qquad c_3 = \frac{y_1 - p_1 c_1}{2p_3} = \frac{y_2}{2p_3}, \tag{R.20}$$

we have

$$U^{(2)}(c_2, c_3) = \ln c_2 + \ln c_3. \tag{R.21}$$

Since the unconditional demands (R.18) satisfy the requirements in Theorem 6, we have

$$U^{(1)}(c_1, c_2, c_3) = f(g(c_1)c_2, g(c_1)c_3). \tag{R.22}$$

Combining the first order condition with the budget constraint, one can verify that the optimal c_1 *satisfies*

$$p_1 c_1 + p_1 \frac{g(c_1)}{g'(c_1)} = y_1. \tag{R.23}$$

[36] To see the direct utility function (R.17) is consistent with the indirect utility function (R.14), substitute the optimal demands (R.7), (R.15) and (R.16) into (R.17) and simplify yielding

$$V(p_1, p_2, p_3, y_1) = \sqrt{\frac{(p_1 p_2 + p_1 p_3 + p_2 p_3) y_1}{p_1 p_2 p_3}},$$

which is ordinally equivalent to the indirectly utility function (R.14).

Since the solution to the above equation is $c_1 = \frac{y_1}{2p_1}$*, we have*

$$\frac{g(c_1)}{g'(c_1)} = c_1, \tag{R.24}$$

implying that

$$g(c_1) = Kc_1, \tag{R.25}$$

where K *is a constant. Without loss of generality, assume* $K = 1$ *and then we have*

$$U^{(1)}(c_1, c_2, c_3) = f(c_1c_2, c_1c_3). \tag{R.26}$$

In summary, the demand functions (R.18) can be generated as the result of naive choice using the period one and two utilities

$$U^{(1)}(c_1, c_2, c_3) = f(c_1c_2, c_1c_3) \qquad \text{and} \qquad U^{(2)}(c_2, c_3) = \ln c_2 + \ln c_3, \tag{R.27}$$

where $f(x, y)$ *is an arbitrary function and defines the complete set of period one utilities* $\{U^{(1)}\}$ *for naive choice associated with the unconditional demands (R.18). Next we show that if one follows the Epstein (1982) process to recover* $U^{(1)}$*, then only a subset of the set of all possible* $U^{(1)}$ *functions in (R.26) can be obtained. Since the* c_1 *demand function is given by*

$$c_1 = \frac{y_1}{2p_1}, \tag{R.28}$$

it follows from Table I in Epstein (1982) that the indirect utility function is given by

$$V(p_1, p_2, p_3, y_1) = \frac{y_1}{\exp\left(\int^{p_1} \frac{1}{2t} dt\right)} = \frac{y_1}{\sqrt{p_1} \exp C}, \tag{R.29}$$

where C *is a constant which ensures that Property (B) is satisfied. Following Roy's identity,*

$$c_2 = \frac{-\partial V / \partial p_2}{\partial V / \partial y_1} \qquad \text{and} \qquad c_3 = \frac{-\partial V / \partial p_3}{\partial V / \partial y_1}. \tag{R.30}$$

It can be verified that if

$$C = \ln\left(p_2^{\alpha} p_3^{1/2-\alpha}\right), \tag{R.31}$$

where $0 < \alpha < \frac{1}{2}$ *is some constant, then*

$$p_1c_1 + p_2c_2 + p_3c_3 = y_1. \tag{R.32}$$

Therefore, the indirect utility function is given by

$$V(p_1, p_2, p_3, y_1) = \frac{y_1}{\sqrt{p_1} p_2^{\alpha} p_3^{1/2-\alpha}}. \tag{R.33}$$

$c_1(p_1, p_2, p_3, y_1)$	$U^{(1)}(c_1, c_2, c_3)$
$c_1(p_1, p_2, p_3, y_1) = h_1(p_1, y_1)$	$U^{(1)}(c_1, c_2, c_3) = f(g(c_1)c_2, g(c_1)c_3)$
$c_1(p_1, p_2, p_3, y_1) = h_1(p_1, p_3)$	$U^{(1)}(c_1, c_2, c_3) = f(g(c_1) + c_3, c_2)$
$c_1(p_1, p_2, p_3, y_1) = h_1(p_1, p_2)$	$U^{(1)}(c_1, c_2, c_3) = f(g(c_1) + c_2, c_3)$
$c_1(p_1, p_2, p_3, y_1) = a(p_1, p_2, p_3)y_1$	There exists a homothetic $U^{(1)}$
$c_1(p_1, p_2, p_3, y_1) = a(p_1, p_2, p_3)y_1 + b(p_1, p_2, p_3)$	There exists a quasihomothetic $U^{(1)}$

Table 1:

Then we have

$$c_2 = \frac{-\partial V/\partial p_2}{\partial V/\partial y_1} = \frac{\alpha y_1}{p_2} \quad \textit{and} \quad c_3 = \frac{-\partial V/\partial p_3}{\partial V/\partial y_1} = \frac{(1/2-\alpha)\, y_1}{p_3}. \tag{R.34}$$

Following the Hurwicz and Uzawa (1971) recovery process, these demands can be rationalized by

$$U^{(1)}(c_1, c_2, c_3) = \sqrt{c_1} c_2^{\alpha} c_3^{1/2-\alpha}. \tag{R.35}$$

This utility will be recognized to be a special case of the general form (R.27) where $f(x, y) = x^{\alpha} y^{1/2-\alpha}$. *To see that the set* $\{U^{(1)}\}$ *includes other well-behaved period one utilities satisfying Property 1, let* $f(x, y) = x^{\frac{1}{2}} + y^{\frac{1}{4}}$. *In this case the specific member is given by*

$$U^{(1)}(c_1, c_2, c_3) = (c_1 c_2)^{\frac{1}{2}} + (c_1 c_3)^{\frac{1}{4}}, \tag{R.36}$$

which satisfies Property 1. Clearly (R.36) is not homothetic and not ordinally equivalent to (R.35) and would generate very different resolute demands even though it together with $U^{(2)}$ *would generate exactly the same set of unconditional naive demands (R.18).*

This example illustrates that when effective consistency holds an infinite number of $U^{(1)}$ functions can be recovered, whereas in Example 9 where effective consistency does not hold only a single $U^{(1)}$ can be recovered (although in principle other period one utilities may exist). It is an open question whether $\{U^{(1)}\}$ can be other than a singleton for any case of naive choice not satisfying effective consistency.

Table 1 summarizes the relationship between naive c_1 demand functions and the rationalizing $U^{(1)}$ functions. It is assumed that Properties (P), (TD), (H) and (EC) hold and the c_1 (unconditional) demand function satisfies $p_1 c_1 < y_1$ and Property (ND). The first line in the table follows from Theorem 6 and the second and third lines follow from Theorem 7. For these three cases, the full set of possible $U^{(1)}$ functions can be determined. The fourth and fifth lines follow from Epstein

(1982, Table 1). It should be emphasized that for these latter two cases, Epstein (1982) only proves the existence of $U^{(1)}$ and gives an approach for recovering a $U^{(1)}$ by assuming c_2 and c_3 are linear in income. Therefore although $U^{(1)}$ may not be unique (up to an increasing transformation), following Epstein's (1982) approach, one can only recover the homothetic or quasihomothetic members from the set of possible functions.

Since the results in Epstein (1982) can be applied to the case where $M > 3$, we next extend Theorem 5 to the $M > 3$ case.

Theorem 13 *Assume that a given set of demand functions* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ *have the Properties (P), (TD), (H), (B) and (EC). If the demand functions* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ *(i) have the Properties (S) and (ND), (ii) are linear in* y_1 *or (iii) are independent of* $(p_{K+1}, ..., p_M)$ *and the conditional demand functions* $c_i(p_{K+1}, ..., p_M, y_2 | \ c_1, ..., c_K)$ $(i = K+1, ..., M)$ *have the Properties (S) and (N), then there exists a* $(U^{(1)}, U^{(2)})$*-pair which generates these demands as the result of naive choice, where* $U^{(1)}$ *is continuous, non-decreasing and quasiconcave and* $U^{(2)}$ *satisfies Property 1.*

Proof. Since the naive demands $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ satisfy Properties (P), (TD), (H), (B) and (EC), it follows that $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ also have Properties (P), (TD), (H) and satisfy

$$\sum_{i=1}^{K} p_i c_i < y_1. \tag{R.37}$$

If we further assume that $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ (i) have the Properties (S) and (ND) and (ii) are either linear in y_1 or independent of $(p_{K+1}, ..., p_M)$, then it follows from Epstein (1982) that there exists a continuous, non-decreasing and quasiconcave $U^{(1)}(c_1, ..., c_M)$ to rationalize the incomplete demand system $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$. Moreover, the conditional demands $c_i(p_{K+1}, ..., p_M, y_2 | \ c_1, ..., c_K)$ $(i = K+1, ..., M)$ exist due to (EC) and also satisfy Properties (P), (TD), (H) and

$$\sum_{i=K+1}^{M} p_i c_i = y_2. \tag{R.38}$$

Since we further assume that $c_i(p_{K+1}, ..., p_M, y_2 | \ c_1, ..., c_K)$ $(i = K+1, ..., M)$ satisfy the Properties (S) and (N), it follows from Hurwicz and Uzawa (1971) that there exists a $U^{(2)}(c_{K+1}, ..., c_M | \ c_1, ..., c_K)$ satisfying Property 1 which rationalizes the conditional demands. Thus there exists a $(U^{(1)}, U^{(2)})$-pair which rationalizes the demands $c_i(p_1, ..., p_M, y_1)$ $(i = 1, 2, ..., M)$ as a result of naive choice. ■

Moreover since the sufficient conditions for effective consistency can naturally be extended to the $M > 3$ case, we next generalize Theorems 6 and 7 (the proof of Theorem 15 is similar to that of Theorem 14 and hence is omitted).

Theorem 14 *Assume that a given set of the demand functions* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ *have the Properties (P), (TD), (H) and (B). Then there exists a* $(U^{(1)}, U^{(2)})$*-pair such that the demands* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ *correspond to naive choice where* $U^{(1)}(c_1, ..., c_M)$ *takes the form*

$$U^{(1)}(c_1, ..., c_M) = f(g(c_1, .., c_K)c_{K+1}, ..., g(c_1, .., c_K)c_M), \tag{R.39}$$

where f *is an arbitrary function and* $g(c_1, .., c_K)$ *is uniquely determined (up to an arbitrary constant of integration) by* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$*, at least one member of* $\{U^{(1)}\}$ *is continuous, non-decreasing and quasiconcave and* $U^{(2)}$ *satisfies Property 1 if (i) the unconditional demand functions* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ *have Property (EC), (ii) the corresponding conditional demands* $c_i(p_{K+1}, ..., p_M, y_2 | \, c_1, ..., c_K)$ $(i = K+1, ..., M-1)$ *have the Properties (S) and (N) and (iii) the unconditional demand functions* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ *have the Properties (S) and (ND) and are independent of* $(p_{K+1}, ..., p_M)$.

Proof. Since $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ have the Properties (P), (TD), (H) and (B), it follows from Theorem 1 that there exists a $U^{(2)}$ satisfying Property 1 if and only if (i) the unconditional demand functions $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ have Property (EC) and (ii) the corresponding conditional demands $c_i(p_{K+1}, ..., p_M, y_2 | \, c_1, ..., c_K)$ $(i = K+1, ..., M-1)$ have the Properties (S) and (N). It follows from Theorem 2 in Epstein (1982) that if the unconditional demand functions $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ have the Properties (P), (TD), (H), (S), (ND) and are independent of $(p_{K+1}, ..., p_M)$, then there exists a continuous, non-decreasing and quasiconcave $U^{(1)}$ to rationalize this partial demand system. Then following Kannai, Selden and Wei (2014), $U^{(1)}(c_1, ..., c_M)$ takes the form

$$U^{(1)}(c_1, ..., c_M) = f(g(c_1, .., c_K)c_{K+1}, ..., g(c_1, .., c_K)c_M) \tag{R.40}$$

if and only if $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ are independent of $(p_{K+1}, ..., p_M)$. ■

Theorem 15 *Assume that a given set of demand functions* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ *have the Properties (P), (TD), (H), (B) and (EC). Then there exists a* $(U^{(1)}, U^{(2)})$*-pair such that the demands* $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ *correspond to naive choice where* $U^{(1)}(c_1, ..., c_M)$ *takes the form*

$$U^{(1)}(c_1, ..., c_M) = f(g(c_1, .., c_K) + c_M, c_{K+1}, ..., c_{M-1}), \tag{R.41}$$

where f is an arbitrary function and $g(c_1, .., c_K)$ is uniquely determined (up to an arbitrary constant of integration) by $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$, at least one member of $\{U^{(1)}\}$ is continuous, non-decreasing and quasiconcave and $U^{(2)}$ satisfies Property 1 if (i) the unconditional demand functions $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., M)$ have Property (EC), (ii) the corresponding conditional demands $c_i(p_{K+1}, ..., p_M, y_2 | \; c_1, ..., c_K)$ $(i = K+1, ..., M-1)$ have Properties (S) and (N) and (iii) the unconditional demand functions $c_i(p_1, ..., p_M, y_1)$ $(i = 1, ..., K)$ have the Properties (S) and (ND) and are independent of $(p_{K+1}, ..., p_{M-1}, y_1)$.

Remark 5 *As in the three period, three commodity case, it remains unresolved whether (i) the sufficient conditions in Theorem 13 can be weakened and (ii) there are other cases in addition to those in Theorems 14 and 15 where one can recover the full set of $U^{(1)}$ functions.*

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