Generalizing the permanent-income hypothesis: Revisiting Friedman's conjecture on consumption ☆

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Abstract

Friedman's contribution to the consumption literature goes well beyond the seminal permanent-income hypothesis. He conjectured that the marginal propensity to consume out of financial wealth shall be larger than out of ''human wealth'', the present discounted value of future labor income. I present an explicitly solved model to deliver this widely noted consumption property by specifying that the conditional variance of changes in income increases with its level. A larger realization of income not only implies a higher level of human wealth, but also signals a riskier stream of future labor income, inducing a higher precautionary saving, and thus giving rise to Friedman's conjecture. Appropriately adjusting human wealth for income risk, I show that Friedman's conjecture may be formulated as a ''generalized'' permanent income hypothesis. I further show that Friedman's conjecture captures the first-order effect of stochastic precautionary savings. Finally, I propose a natural decomposition of the optimal saving rule to formalize various motives for holding wealth as emphasized in [Friedman, M., 1957. A Theory of the Consumption Function. Princeton University Press, Princeton].

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1. Introduction

The permanent-income hypothesis (PIH) of Friedman (1957) states that consumption is equal to the annuity value of total wealth given by the sum of financial wealth (cumulative savings) and "human" wealth, the discounted expected value of future income, using the risk-free rate. This in turn implies that changes in consumption are not predictable, thus delivering the well known martingale consumption result (Hall, 1978). Phrased in terms of saving, the PIH states that the agent only saves in anticipation of possible future declines in his labor income. Campbell (1987) dubbed the PIH-implied saving motive as saving for "rainy days."

While Friedman's permanent-income hypothesis serves as the cornerstone of the consumption literature, his contribution to this body of work goes well beyond the PIH. For example, on page 16 of Friedman (1957), he wrote that "current consumption may be expected to depend not only on total permanent income and the interest rate, but also on the fraction of permanent income derived from nonhuman wealth, or—what is equivalent for a given interest rate, on the ratio of nonhuman wealth to permanent income. The higher this ratio, the less need there is for an additional reserve, and the higher current consumption may be expected to be. The crucial variable is the ratio for nonhuman wealth to permanent income, not the absolute amount of nonhuman wealth". Friedman defined the permanent income as the annuity value of the sum of "human" and nonhuman (financial) wealth. Thus, he suggested a consumption rule with a lower marginal propensity to consume (MPC) out of human wealth than out of nonhuman wealth.

Despite its overshadowing by the PIH hypothesis, Friedman's lesser known conjecture on the lower MPC out of human wealth has found widespread empirical support in various papers, particularly after the publication of Hall's (1978) groundbreaking work. Flavin (1981) tested and demonstrated, for example, that changes in consumption are predicted by current income, which is inconsistent with the random-walk consumption prediction of Friedman's PIH (Hall, 1978). This empirical regularity on the predictability of changes in consumption by variables such as income is known as the "excess-sensitivity" puzzle (Flavin, 1981). I show that the excess sensitivity of consumption may be explained by a lower MPC out of human wealth than out of financial wealth. Moreover, I also demonstrate that the "excess-smoothness" puzzle of Deaton (1987) may also be reconciled by using this conjectured consumption rule.

This paper formalizes Friedman's conjecture on the consumption rule stated above by constructing an intertemporal precautionary savings model.[1] I argue that a "lower" MPC out of human wealth than out of financial wealth is among the most important features of a sensible consumption rule, because this feature captures the first-order effect of precautionary saving. A precautionary agent rationally values a unit of human wealth less than a unit of financial wealth. This translates into a lower MPC out of human wealth than out of financial wealth, in terms of the consumption rule.

An essential ingredient of the model is that the conditional variance of income changes, as measured in absolute magnitude (i.e., in levels and not in logarithms), increases with the

[1]Leland (1968) is among the earliest works on precautionary savings. Also see Sandmo (1970), Drèze and Modigliani (1972), Zeldes (1989), Caballero (1991), Deaton (1991), Carroll (1997), and Gourinchas and Parker (2002), among others. Deaton (1992) and Attanasio (1999) survey the literature.

level of income.[2] There is much empirical evidence in support of a conditionally heteroskedastic income process in levels. A common specification of the income process in the literature is a conditionally homoskedastic income process in logarithm (MaCurdy, 1982). A conditionally homoskedastic process in logarithm implies that the conditional variance of income changes in levels must increase with the level of labor income. This paper applies a widely used class of stochastic processes, known as affine processes,[3] to model the dynamics of the income process. A key feature of affine processes is that the conditional variance of changes in income is an increasing linear function of the level of current income.

The intuitive reasoning behind using conditionally heteroskedastic processes to model income is as follows. Consider the effect of an increase in current labor-income. Since higher income also signals higher future labor-income uncertainty, precautionary saving shall increase. As a result, consumption will increase less out of a unit increase in ‘‘human’’ wealth than out of a unit increase in financial wealth. This in turn implies an optimal consumption rule with a lower MPC out of human wealth than out of financial wealth. In addition to capturing these empirical features, the affine process is also analytically tractable.

My work is closely related to Zeldes (1989) and Caballero (1990, 1991). Zeldes (1989) derived an incomplete-markets consumption model with constant-relative-risk-averse (CRRA) utility, by using numerical dynamic programming.[4] Based on his numerical consumption policy rule, Zeldes concluded that ‘‘one possible remedy to this problem would be to put a weight of less than one on human wealth before adding it to financial wealth, or to discount expected future income at a higher discount rate.’’[5] Zeldes’ work provides a justification for Friedman’s conjecture on a lower MPC out of human wealth than out of financial wealth. In this paper, I show that a model with conditionally heteroskedastic income not only generates a consumption rule with a lower MPC out of human wealth than out of financial wealth, as stated by Friedman (1957) and Zeldes (1989), but also formalizes Zeldes’ statement that another way to ‘‘fix’’ the PIH rule is to use a ‘‘risk-adjusted’’ measure for human wealth by calculating expected future income at a higher discount rate. Appropriately accounting for risk puts the ‘‘risk-adjusted’’ human wealth and financial wealth on equal footing. These two forms of wealth, therefore, offer the same capacity in terms of financing consumption, when based on the agent’s optimality.

Caballero (1991) proposed an analytically tractable optimal consumption rule based on the additively separable constant-absolute-risk-averse (CARA) utility and an autoregressive moving average income process.[6] He showed that the optimal consumption level is

[2]This assumption does not yet restrict how the conditional variance of *percentage* changes of income depends on income.

[3]Affine models are widely used in finance. See Vasicek (1977), Cox et al. (1985), Duffie and Kan (1996), and Dai and Singleton (2000), for examples of affine models.

[4]Hayashi (1982) tested a generalized version of the permanent-income model, using a higher discount rate for human wealth than the risk-free rate. However, his model is not based on optimality.

[5]The problem Zeldes refers to in this quote relates to the inconsistency between the CRRA-utility-based optimal consumption that he solved numerically, and the martingale consumption result implied by the PIH.

[6]Motivated by the empirical observation that the individual agent faces both transitory and persistent shocks and moreover, by the fact that the agent often only observes his total income, Wang (2004) extends Caballero’s precautionary saving model to incorporate the effect of the agent’s learning about his partially observed income on his consumption-saving decision. Wang (2004) shows that the agent’s precautionary savings demand is further enhanced because of the estimation risk. Intuitively, the agent needs to forecast the individual components of his income in order to form his consumption saving decision. When he forecasts, he inevitably incurs estimation risk in doing so. Therefore, he rationally shades his current consumption in order to build assets to smooth his future consumption.

lower than that under certainty equivalence by a term that captures a precautionary premium. Because of conditionally homoskedastic income shocks, his model predicts constant precautionary saving demand, which in turn implies that the MPC out of human wealth is equal to that out of financial wealth. In a related paper, Caballero (1990) mentioned the potential importance of conditional heteroskedasticity of income shocks in explaining consumption puzzles.

Following Merton (1971), Kimball and Mankiw (1989), and Caballero (1991), I assume CARA utility for technical convenience.[7] This is not surprising. After all, as Zeldes (1989) noted in the abstract, "No one has derived closed-form solutions for consumption with stochastic labor income and constant relative risk aversion utility." Both Zeldes (1989) and this paper conclude that a lower MPC out of human wealth is crucial for a realistic consumption rule. Unlike Caballero (1991), this paper delivers a lower MPC out of human wealth than out of financial wealth.

If we take the PIH rule as the first-order approximation of a realistic optimal consumption rule, then a linear consumption rule (in financial wealth and human wealth) with a lower MPC out of human wealth is the natural second-order approximation of a realistic consumption rule. What the second-order linear approximation captures and the PIH rule does not, is the "stochastic" precautionary saving. It is precisely this stochastic precautionary saving that generates empirical predictions consistent with "excess-sensitivity" and "excess-smoothness" puzzles. Nevertheless, it is worth noting that the second-order approximation does not capture the non-linearity feature of the consumption rule (Carroll and Kimball, 1996). However, the key result that the MPC out of human wealth is *lower* than out of financial wealth remains valid even after incorporating the non-linearity of the consumption rule, as shown via numerical methods by Zeldes (1989). Moreover, the concavity of the consumption rule is quantitatively insignificant when the agent is not liquidity constrained. Finally, I use the explicitly derived consumption rule to decompose saving motives and formalize Friedman's insights on various motives for holding wealth. The decomposition of saving further sheds light on the determinants of the consumption rule.

The remainder of the paper is organized as follows. Section 2 describes the setup of the model and solves the optimal consumption rule. In Section 3, I interpret the model's implication for the MPCs, decompose the saving rate, and discuss the robustness of the main result. Section 4 concludes. Appendices contain relevant technical details.

2. The model

An infinitely lived agent receives an exogenously given perpetual stream of stochastic labor income. He can borrow or lend at a constant positive risk-free interest rate r. There exist no other financial assets, hence markets are incomplete with respect to labor-income uncertainty. For technical convenience, the model is cast in continuous time. I first

[7] Merton (1971) derived an explicitly solved consumption rule using Poisson processes to model the income dynamics. Kimball and Mankiw (1989) used a continuous-time Markov chain to model the income process and solved the optimal consumption rule in closed form. Svensson and Werner (1993) solves an optimal consumption and portfolio choice problem with non-tradable labor income and CARA utility. Weil (1993) uses an AR1 income process and a recursive utility specification (with constant elasticity of intertemporal substitution and constant absolute risk aversion coefficient) to derive an analytical consumption rule. In his model, the MPCs out of financial wealth and out of internationally defined human wealth are equal.

introduce a natural parametric model for the labor-income process, one leading to a *lower* MPC out of human wealth than out of financial wealth.

The macroeconomic consumption literature often postulates a Gaussian autoregressive income process,[8] which has a few drawbacks. It is unbounded from below and symmetrically distributed, with no excess kurtosis. Empirically, labor income is positively skewed, fat-tailed, and bounded from below. Moreover, the conditional variance of changes in labor income, in a Gaussian setting, is deterministic, and thus cannot depend on income outcomes. Another frequently adopted model assumes that the logarithm of income, rather than its level, is a conditionally homoskedastic Markov process,[9] which implies that the conditional variance of changes in income increases in the *level* of income.

The affine income process introduced here is also conditionally heteroskedastic, in that a higher level of income implies a higher conditional volatility of changes in income, signaling a riskier stream of future labor income. Furthermore, the affine process is shown to be more tractable. Specifically, I propose a conditionally heteroskedastic affine income process, allowing for positive skewness, excess kurtosis and boundedness (from below). Suppose that the agent's labor income y is given by the following dynamics:

$$\mathrm{d}y_t = \mu(y_t)\,\mathrm{d}t + \sigma(y_t)\,\mathrm{d}W_t, \quad t \geqslant 0, \ \text{given } y_0, \tag{1}$$

where W is a standard Brownian motion. The drift and volatility functions $\mu(\cdot)$ and $\sigma(\cdot)$ in (1) are defined by

$$\mu(y) = \theta - \kappa y, \tag{2}$$

$$\sigma(y) = \sqrt{l_0 + l_1 y}, \tag{3}$$

respectively, where θ, κ, l_0 and l_1 are constant parameters. This is an example of an affine diffusion, because the drift $\mu(y)$ and the conditional variance function $\sigma^2(y)$ are affine in the income level y. A positive coefficient l_1 captures a monotonically increasing relationship between the conditional variance of and the level of labor income. I split this class of processes into two groups.

One group has conditionally homoskedastic shocks ($l_1 = 0$), wherein the coefficient $\sigma_0 \equiv \sqrt{l_0}$ is the volatility of the income process. The process need not be stationary. The conventional autoregressive Gaussian process, possibly unit root ($\kappa = 0$), with or without drift, is a special case. The focus of this paper is the second group, for which innovations are conditionally heteroskedastic ($l_1 > 0$), and for which a higher income signals a more volatile future labor income.

Having described the agent's income process, I next specify the agent's preference and his optimization problem. For tractability reasons, I assume that the agent is endowed with CARA utility, following Merton (1971), Kimball and Mankiw (1989), and Caballero (1991). That is, the agent has the following expected utility function:

$$U(c) = \mathbb{E}\left(\int_0^\infty \mathrm{e}^{-\beta s} u(c_s)\,\mathrm{d}s\right), \tag{4}$$

where $\beta > 0$ is the subjective discount rate and $\gamma > 0$ is the coefficient of absolute risk aversion ($u(c) = -\mathrm{e}^{-\gamma c}/\gamma$).

[8]See Deaton (1992) and Attanasio (1999) for reviews.
[9]See MaCurdy (1982) and Deaton (1991).

I now state the agent's optimization problem. The agent maximizes his utility given in (4) subject to his endowed income process (1)–(3), his wealth accumulation process

$$\mathrm{d}x_t = (rx_t + y_t - c_t)\,\mathrm{d}t, \quad t \geqslant 0, \text{ given } x_0, \tag{5}$$

and the transversality condition (A.6) specified in the Appendix. Since the objective of this paper is to understand the effect of incomplete markets on the MPCs, I have intentionally chosen to allow the agent only the opportunity to invest in risk-free assets. See Davis and Willen (2002) for an analysis with conditionally homoskedastic income shocks and options to invest in correlated risky assets.

Let $J(x, y)$ denote the agent's value function when his wealth is x and his income is y. The agent's value function $J(x, y)$ solves the Hamilton–Jacobi–Bellman (HJB) equation:

$$\beta J(x, y) = \sup_{\bar{c}} \{u(\bar{c}) + \mathscr{D}^{\bar{c}} J(x, y)\}, \tag{6}$$

where

$$\mathscr{D}^{\bar{c}} J(x, y) = (rx + y - \bar{c}) J_x(x, y) + (\theta - \kappa y) J_y(x, y) + \tfrac{1}{2}(l_0 + l_1 y) J_{yy}(x, y). \tag{7}$$

The left side of (6) is the flow value of the agent's value function. The right side of (6) is the sum of his instantaneous utility from current consumption and the instantaneous expected changes of his value function. The agent optimally chooses his consumption by equating the two sides of (6).

The following proposition summarizes the solution to the optimization problem stated above.

Proposition. *The optimal consumption rule c^* is affine in financial wealth x and current income y, in that,*

$$c_t^* = r(x_t + a_y y_t + a_0) \quad \text{for all } t, \tag{8}$$

where

$$a_y = \frac{a_h}{r + \kappa}, \tag{9}$$

$$a_h = \begin{cases} \frac{1}{\Delta_1}\left(\sqrt{1 + 2\Delta_1} - 1\right) < 1 & \text{if} \quad l_1 > 0, \\ 1 & \text{if} \quad l_1 = 0, \end{cases} \tag{10}$$

$$a_0 = \frac{1}{r}\left(\frac{\beta - r}{\gamma r} + \frac{\theta}{r + \kappa} a_h - \frac{1}{2}\Delta_0 a_h^2\right), \tag{11}$$

$$\Delta_1 = \frac{\gamma r l_1}{(r + \kappa)^2} \geqslant 0, \tag{12}$$

$$\Delta_0 = \frac{\gamma r l_0}{(r + \kappa)^2}. \tag{13}$$

Appendix A contains the derivation of the optimal consumption rule. The next section derives and analyzes the implications of the consumption rule (8) and how it relates to Friedman's original conjecture on consumption.

3. Model implications

In this section, I first show that the proposed model provides a theoretical justification of Friedman's conjecture on a lower MPC out of "human" wealth than out of financial wealth. Second, I construct a generalized permanent-income hypothesis, which provides an alternative formulation and interpretation of Friedman's conjecture on the consumption rule. Finally, I propose a saving decomposition and use it to analyze the three saving motives layed out in Friedman (1957).

3.1. The marginal propensities to consume

The optimal consumption rule (8) relates the agent's consumption to his wealth, a stock variable, and his current income, a flow variable. Following the insight of Friedman's permanent income hypothesis, I convert current income y to a "stock" measure of wealth for income. This leads us to define "human" wealth.

I follow Friedman (1957) and Hall (1978) to define human wealth h as the expected present value of future labor income, discounted at the risk-free interest rate r, in that

$$h_t = \mathbb{E}_t\left(\int_t^\infty e^{-r(s-t)} y_s \, ds\right), \tag{14}$$

conditioning on $\mathscr{F}_t$, the agent's information set at time t. Note that $\mathbb{E}_t$ denotes $\mathscr{F}_t$-conditional expectation. Assume $r+\kappa>0$ to ensure that human wealth is finite. For the affine income process given by (1)–(3), human wealth is affine in current labor income, in that

$$h_t = \frac{1}{r+\kappa}\left(y_t + \frac{\theta}{r}\right). \tag{15}$$

This definition of human wealth, however, ignores risk and therefore overstates wealth associated with stochastic uninsurable labor income. Note that a larger degree of mean reversion (higher κ) lowers the response of human wealth to a unit increase of current income, *ceteris paribus*. Intuitively, when income is more transitory (higher κ), current income y carries less weight in "human" wealth, *ceteris paribus*.

Expressing the optimal consumption rule (8) in terms of stock variables (financial and human wealth) gives

$$c_t^* = r(x_t + a_h h_t - b_0), \tag{16}$$

where

$$b_0 = \frac{1}{r}\left(\frac{1}{2}\Delta_0 a_h^2 - \frac{\beta - r}{\gamma r}\right). \tag{17}$$

The MPC out of human wealth $\omega_h \equiv r a_h$ is always less than that out of financial wealth. This inequality holds *strictly* when the conditional variance of labor income depends directly on its level ($l_1>0$). To understand the intuition behind the results, consider the effect of a dollar increase in current income. A dollar increase of current income not only raises human wealth by the amount of $1/(r+\kappa)$ (see (15)), but also signals a more volatile stream of future labor income ($l_1>0$). An agent will therefore increase his consumption by an amount *less* than $r/(r+\kappa)$, the annuity value of the increase in human wealth, because the agent's precautionary demand also increases with income ($l_1 > 0$). On the other hand, a unit increase in financial wealth increases consumption by r per unit of time. Therefore, consumption responds less to a unit increase in human wealth than a unit increase in financial wealth. I next take the model's key prediction on the MPCs and re-evaluate the following conventional wisdom in the consumption literature: consumption responds "one to one" to permanent shocks, because there is no way to "diversify" and "smooth" away permanent shocks.

When shocks are permanent (such as a unit-root income process ($\kappa = 0$)), human wealth increases by the perpetuity amount $1/r$ for a unit increase in current income. However, consumption increases by less than unity, the annuity value of the corresponding increase in human wealth, because the precautionary savings demand π_t increases in the level of income. This implies that the agent saves some portion out of his income for precautionary reasons, even for permanent shocks. This differs from the conventional wisdom that consumption responds "one to one" with respect to permanent shocks.

I now quantify the agent's precautionary savings motive. To do so, I first introduce a natural benchmark: certainty-equivalence consumption level c^{p}, obtained by setting $l_0 = l_1 = 0$ in (16). This gives

$$c_t^{\mathrm{p}} = r\left(x_t + h_t + \frac{\beta - r}{\gamma r^2}\right) \equiv w_t^{\mathrm{p}} + \frac{\beta - r}{\gamma r}, \tag{18}$$

where $w_t^{\mathrm{p}} = r(x_t + h_t)$ is "permanent income," as defined in Friedman (1957). The precautionary saving premium[10] π is measured as the difference between the certainty-equivalence consumption and optimal consumption c^*, in that $\pi \equiv c^{\mathrm{p}} - c^*$. From (16) and (18), the precautionary saving premium is given by

$$\pi_t = \frac{1}{2} a_h^2 (\varDelta_0 + r\varDelta_1 h_t) = \frac{\gamma r}{2} a_y^2 \left(l_0 + l_1 y_t\right). \tag{19}$$

If shocks are conditionally heteroskedastic, then π_t increases in labor income y and thus also increases in human wealth h.

A special case of (19) is a continuous-time counterpart of Caballero (1991), obtained by setting $l_1 = 0$. The precautionary savings premium in this case is $\pi = \varDelta_0/2 = 0.5\gamma r l_0/(r + \kappa)^2$, a constant, independent of the agent's financial and human wealth. All agents, regardless of differences in their current labor incomes, have the same precautionary saving premium. Wang (2003) shows that Caballero-type agents[11] behave effectively as Friedman-type permanent-income consumers, in incomplete-markets equilibrium models, known as Bewley models.[12] The market-clearing condition in the risk-free asset leads to an equilibrium interest rate that is lower than the subjective discount rate. As a result, the constant dissaving due to impatience is *exactly* offset by the constant precautionary saving premium in equilibrium, leaving the consumer effectively only to save in anticipation of

[10]Kimball (1990) introduced the concept of prudence and measures precautionary premium, based on the convexity of marginal utility.

[11]A Caballero-type agent is defined with time-additive separable CARA utility and conditionally homoskedastic uninsurable income process, a special case of the proposed model here.

[12]See Bewley (1986), Aiyagari (1994), Huggett (1993), and Ljungqvist and Sargent (2004) for a textbook treatment on Bewley models. Bewley models extend the permanent income/precautionary saving models to equilibrium settings with a continuum of ex ante identical, but ex post heterogeneous agents. See Krusell and Smith (1998) for wealth distribution with aggregate uncertainty.

possible future changes in income (Campbell, 1987). This results in the agent essentially behaving as a permanent-income consumer.

With conditionally heteroskedastic income innovations ($l_1 > 0$) and CARA utility, the agent's precautionary savings demand increases linearly in the level of income. Agents accumulate wealth at different rates for precautionary reasons, depending upon their income levels. They do not behave in accordance with the PIH even in equilibrium Bewley models, provided that income shocks are heteroskedastic.

I next provide an alternative way of phrasing Friedman's conjecture by incorporating income risk via "risk-adjusted human wealth."

3.2. *Generalized permanent income hypothesis*

The traditional definition of human wealth as given in (14) ignores risk and consequently overstates the value that a precautionary agent attaches to his future stochastic labor income. A conventional wisdom in the consumption literature is that if the risk associated with stochastic income is "appropriately" incorporated, then the certainty-equivalence consumption rule will hold. Friedman (1957) wrote "the rate of interest at which an individual can borrow on the basis of his future earnings may be different from the rate at which he can borrow on the basis of financial capital." More recently, Zeldes (1989) suggested that a remedy to the failure of the certainty-equivalence-based permanent-income model is "to discount expected future income at a higher discount rate" than the interest rate.

First, I propose the following definition of a "risk-adjusted" human wealth h^{R}:

$$h_t^{\mathrm{R}} = \mathbb{E}_t^{\mathrm{R}}\left(\int_t^{\infty} \mathrm{e}^{-r(s-t)} y_s \,\mathrm{d}s\right), \tag{20}$$

where $\mathbb{E}_t^{\mathrm{R}}$ denotes the $\mathscr{F}_t$-conditional expectation under an alternative measure R. Under this alternative measure R, the income dynamics given in (1)–(3) under the income generating measure P have the following dynamics:

$$\mathrm{d}y_t = (\theta^{\mathrm{R}} - \kappa^{\mathrm{R}} y_t)\,\mathrm{d}t + \sigma(y_t)\,\mathrm{d}W_t^{\mathrm{R}}, \tag{21}$$

where W_t^{R} is the Brownian motion under measure R and is linked to measure P by (B.8), and the parameters in the drift function are given by

$$\theta^{\mathrm{R}} = \theta - \tfrac{1}{2}\gamma r a_y l_0, \tag{22}$$

$$\kappa^{\mathrm{R}} = \kappa + \tfrac{1}{2}\gamma r a_y l_1 \geqslant \kappa. \tag{23}$$

Note that the volatility function is the same under measure R and the original income generating measure P. However, the drift of the income process is "lower" for positive y under measure R than under measure P, in that $\theta^{\mathrm{R}} \leqslant \theta$ and $\kappa^{\mathrm{R}} \geqslant \kappa$. Using (21)–(23) allows us to compute the "risk-adjusted" human wealth as follows:

$$h_t^{\mathrm{R}} = \frac{1}{r + \kappa^{\mathrm{R}}}\left(y_t + \frac{\theta^{\mathrm{R}}}{r}\right). \tag{24}$$

For positive income, it is straightforward to conclude that the risk-adjusted human wealth h^{R} is lower than the conventionally defined "risk-neutral" human wealth, in that $h_t^{\mathrm{R}} < h_t$.

We may now re-phrase the optimal consumption rule (8) in terms of this newly introduced "risk-adjusted" human wealth h^{R} as follows:

$$c_t^* = r\left(x_t + h_t^{\mathrm{R}} + \frac{\beta - r}{\gamma r^2}\right). \tag{25}$$

Note that the optimal consumption rule takes the certainty equivalence PIH form under measure R. Although the consumption rule (25) looks identical to the PIH consumption rule (18), these two rules are different because the underlying probability measures are different. Put differently, the (generalized) "permanent-income" hypothesis would hold at an elevated discount rate of $(r + \gamma r a_y l_1/2)$ for the income process. Thus, we have formalized the suggestion that Zeldes (1989) made, in that we have a remedy to fix the certainty-equivalence-based permanent-income model by discounting expected future income at $(r + \gamma r a_y l_1/2)$, a higher discount rate than the interest rate r. Intuitively, a higher degree of conditional heteroskedasticity of income or a larger γ implies a larger premium over the interest rate, $\gamma r a_y l/2$. I dub this version of the certainty equivalence consumption rule (25) under the alternative measure R as the "generalized" permanent-income hypothesis.

This paper has so far focused on precautionary savings. However, saving out of precaution is only one of the motives. Informal discussions of different motives for saving have a long tradition in the consumption literature, at least dating back to Friedman (1957). The next section formalizes Friedman's insights on various motives for holding wealth, by providing a saving decomposition. Decomposition analysis allows us to compare my model with other consumption models along various saving motives. I show that my analytical model approximates numerically solved CRRA-based consumption models up to the second order.

3.3. Motives for saving: a decomposition analysis

I offer a natural and explicitly solved decomposition of the agent's saving. Gourinchas and Parker (2001) offered a decomposition analysis of savings motive. Their decomposition is based on the second-order Taylor expansion of the Euler equation, not on the consumption-saving rule. This is primarily due to the numerical nature of the consumption rule in their model. Unlike their work, the decomposition proposed here is exact without using any approximation. Moreover, it relates to Friedman's original insights on the three different saving motives to be discussed in detail below.

The optimal saving rate may be obtained by plugging the optimal consumption rule (8) into the wealth accumulation (5). This gives

$$s_t = r x_t + y_t - c_t^* = (1 - r a_y) y_t - r a_0, \tag{26}$$

where a_y and a_0 are given in (9) and (11), respectively. A simple saving rule (26) has rich implications for different saving motives. I decompose the total saving rate into the three motives stated in Friedman (1957): (i) the straightening out of the consumption stream; (ii) the earning of interests on assets; and (iii) the saving for unexpected low incomes. The saving rate s_t may be written as

$$s_t = \psi_t + \pi_t - \phi_t, \tag{27}$$

where

$$\psi_t = \frac{\kappa y_t - \theta}{r + \kappa}, \tag{28}$$

$$\phi_t = \frac{\beta - r}{\gamma r} \tag{29}$$

and π_t is given by (19). The first term ψ_t formalizes the agent's motive to "straighten out of consumption stream," as stated in Friedman (1957). It measures the portion of the saving that is due to "expected" future declines in labor income. Campbell (1987) dubbed this "saving for a rainy day," an equivalent way of phrasing Friedman's PIH in terms of saving. If $\kappa \leqslant 0$, the agent's income grows over time in expectation and, hence, he borrows against future income ($\psi_t < 0$). For stationary income ($\kappa > 0$), the "saving for a rainy day" ψ_t is given by

$$\psi_t = \frac{\kappa}{r + \kappa}\left(y_t - \bar{y}\right) = y_t - rh_t, \tag{30}$$

where $\bar{y} = \theta/\kappa$ is the long-run mean of income. When $y_t > \bar{y}$, the agent expects that his income will fall in the long run (due to mean reversion), and thus he saves a portion of his current income in excess of its long-run mean $\bar{y}$ in anticipation of future "rainy" days. A higher rate κ of mean reversion or a larger difference between current income and its long-run mean, $\left(y_t - \bar{y}\right)$, induces a higher saving rate, *ceteris paribus*. The saving ψ_t is positive if current income is above the annuity value of human wealth (Friedman, 1957). Eq. (30) completely summarizes the PIH in terms of saving.

If ψ_t is the only saving component, then changes in consumption are not predictable (Hall, 1978). Note that "saving for a rainy day" is independent of the agent's utility function. Any forward-looking consumption model (whether it is based on certainty equivalence or CRRA utility) contains and has the same piece of the PIH-implied component of saving for "rainy" days (Campbell, 1987). While I have focused on the individual's optimal saving rule, it is worth noting that PIH-implied saving ψ plays no role in the determination of the *equilibrium* interest rate in the standard heterogeneous-agent Bewley models.[13]

Empirically, changes in consumption are predicted by variables such as labor income. This is known as the *excess-sensitivity* puzzle (Flavin, 1981). My model shows that a direct dependence of the conditional variance of labor income on its own level ($l_1 > 0$) explains the "excess sensitivity" of consumption. In order to highlight the mechanism, consider the implications of (8) on the time-series properties of consumption. Eq. (B.4) implies that

$$\mathbb{E}_t(\Delta c^*_{t+1}) = \begin{cases} r(1 - a_h)[(y_t + l_0/l_1)n(1;\kappa) + \tilde{\theta}\, m(1;\kappa)] - (\beta - r)/\gamma, & l_1 > 0, \\ r\Delta_0/2 - (\beta - r)/\gamma, & l_1 = 0, \end{cases} \tag{31}$$

where $\Delta c^*_{t+1} \equiv c^*_{t+1} - c^*_t$, and $\tilde{\theta} = \theta + \kappa l_0/l_1$, for $l_1 > 0$. When the labor-income innovation is conditionally homoskedastic, as in Caballero (1991), (31) implies that current income does *not* predict future movements in consumption. That is, the precautionary savings motive together with stochastic labor income are not sufficient for the excess sensitivity of

[13]By appealing to the law of large numbers, I sum (30) across the continuum of agents and thus obtain zero net saving for "rainy" days (in flow terms). Note that these Bewley economies have the properties of cross-sectional stationarity and aggregate constancy.

consumption. With $l_1 > 0$, however, the model predicts a strictly positive regression coefficient $r(1-a_h)(1-e^{-\kappa})/\kappa$ of Δc^*_{t+1} on current income y_t, supporting the empirical finding of the excess-sensitivity. Eq. (31) also implies that consumption is a submartingale,[14] expected to grow over time, consistent with the empirical findings of *excess-growth* (Deaton, 1987).

The decomposition analysis shows that utility-based models can only differ from the PIH in the other two components of saving: (i) dissaving ϕ_t due to being relatively impatient ($\beta > r$), and (ii) precautionary savings demand π_t. The dissaving ϕ_t corresponds to the motive of paying interests, the second motive pointed out in Friedman (1957). Dissaving ϕ_t is driven by $(\beta - r)$, the difference between the individual's discount rate and the interest rate. If the subjective discount rate β is larger than the interest rate r, then the agent dissaves because he is relatively impatient. In this model, the dissaving ϕ_t is *constant*, because the CARA utility lacks wealth effect. If we set $\beta = r$, as in Hall (1978) and Zeldes (1989), then the dissaving due to impatience is the same and is equal to zero for all consumption models including CRRA utility-based models. Then, the only difference between the proposed model and existing CRRA-utility-based models is on precautionary savings. The precautionary saving term π_t captures Friedman's insight that the agent may want to save in order to build a reserve for an emergency. In Subsection 3.1, I show that the precautionary saving term π_t is stochastic and offer the second-order approximation of an optimal consumption rule in CRRA-utility-based models. To sum up, my model generates a realistic consumption rule (up to the second order) and provides insights on how various parameters affect the agent's consumption behavior by exploiting the analytical tractability of the policy rule. The new model complements the existing results derived from CRRA-utility-based models.

4. Conclusions

This paper provides the first explicitly solved optimal consumption model with a *lower* marginal propensity to consume out of human wealth than out of financial wealth, a widely noted desirable property of the consumption rule (Friedman, 1957; Zeldes, 1989). The model's implications are consistent with empirical regularities such as excess sensitivity, excess growth, and excess smoothness of consumption. The main assumption is that the conditional variance of changes in income is an increasing function of current income. There is much empirical evidence in support of the conditional heteroskedasticity of income shocks in levels. Specifically, I model the income process with the widely used affine process for technical convenience. A higher level of current income not only implies a higher level of human wealth, but also signals a riskier stream of future income. As a result, a precautionary agent saves more than the PIH-implied saving "for rainy days" (Campbell, 1987). Moreover, the additional precautionary saving increases with the agent's current income. Finally, I propose a formal saving decomposition, which provides a theoretical foundation for the various motives for holding wealth pointed out by Friedman (1957).

[14] This is under the commonly made assumption $\beta = r$. Otherwise, $\{z_t \equiv c^*_t + (\beta - r)t/r\}_{t \geqslant 0}$ is a submartingale, as shown in Appendix B.

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Appendix A. Proof of the Proposition

I derive the optimal consumption rule using dynamic programming and then verify the transversality condition implied by the policy function.

First, I conjecture that the value function takes an exponential-affine form:

$$J(x,y) = -\frac{1}{\gamma r}\exp[-\gamma r(x + a_y y + a_0)]. \tag{A.1}$$

The first-order condition for the HJB (6) is $u'(\bar{c}) = J_x(x,y)$. Plugging the implied values for $\bar{c}$, J_x, J_y, and J_{yy} into (6) gives

$$0 = -\frac{1}{\gamma} + \frac{\beta}{\gamma r} + (1 - ra_y)y - ra_0 + (\theta - \kappa y)a_y - \frac{1}{2}(l_0 + l_1 y)\gamma r a_y^2. \tag{A.2}$$

Note that the above equation holds for any income level y. Thus, matching coefficients for both the linear term in y and the constant term gives

$$0 = 1 - ra_y - \kappa a_y - \tfrac{1}{2}\gamma r l_1 a_y^2, \tag{A.3}$$

$$a_0 = \frac{1}{r}\left(\frac{\beta - r}{\gamma r} + \theta a_y - \frac{1}{2}\gamma r l_0 a_y^2\right). \tag{A.4}$$

Eq. (A.3) may be written as

$$\tfrac{1}{2}\Delta_1 a_h^2 + a_h - 1 = 0, \tag{A.5}$$

where a_h and Δ_1 are given in (9) and (12), respectively. For conditionally homoskedastic shocks ($l_1 = 0$), we have $a_h = 1$. For conditionally heteroskedastic shocks ($l_1 > 0$), there are two candidate roots for a_h. I discard the negative root, since it implies a negative MPC out of current income. The positive root, between zero and one, is given in (10).

Next, I check the agent's transversality condition given below:

$$\lim_{t\to\infty} \mathbb{E}[e^{-\beta t}|J(x_t, y_t)|] = 0, \tag{A.6}$$

where x is the wealth process associated with the consumption process c. Condition (A.6) restricts the rate at which the debt is allowed to grow. Using the optimal saving rule (26) to evaluate $e^{-\beta\tau}|J(x_\tau, y_\tau)|$ gives

$$e^{-\beta\tau}J(x_\tau, y_\tau) = -\frac{e^{-\gamma r(x_0 + a_0)}}{\gamma r}\mathbb{E}\left[\exp\left(-\int_0^\tau (\beta - \gamma r^2 a_0 + \gamma r(1 - ra_y)y_s)\,ds\right)e^{-\gamma r a_y y_\tau}\right].$$

Duffie et al. (2000) showed that under technical regularity conditions,

$$\mathbb{E}\left[\exp\left(-\int_0^\tau (\rho_0+\rho_1 y_s)\,\mathrm{d}s\right)\mathrm{e}^{\rho_2 y_\tau}\right] = \exp[A(\tau)+B(\tau)y_0],$$

where coefficients $A(\tau)$ and $B(\tau)$ solve the following Riccati equations:

$$\dot{B}(\tau) = -\kappa B(\tau) + \tfrac{1}{2} l_1 B^2(\tau) - \rho_1, \tag{A.7}$$

$$\dot{A}(\tau) = \theta B(\tau) + \tfrac{1}{2} l_0 B^2(\tau) - \rho_0, \tag{A.8}$$

with initial conditions $B(0)=\rho_2$ and $A(0)=0$. Setting $\rho_0 = \left(\beta - \gamma r^2 a_0\right)$, $\rho_1 = \gamma r(1-ra_y)$ and $\rho_2 = -\gamma r a_y$ gives

$$-\kappa u + \frac{l_1}{2}u^2 - \rho_1 = \gamma r\left(\frac{\gamma r l_1}{2}a_y^2 + (r+\kappa)a_y - 1\right) = 0.$$

Therefore, the solution of the ODE (A.7) is constant, in that $B(\tau)=\rho_2=-\gamma r a_y$, for all $\tau \geqslant 0$. The solution of (A.8) is thus given by

$$A(\tau) = \left(\theta u + \frac{l_0}{2}u^2 - \rho_0\right)\tau = -r\tau. \tag{A.9}$$

Therefore, the transversality condition (A.6) is satisfied if and only if the following holds:

$$\lim_{t\to\infty}\mathbb{E}[\mathrm{e}^{-\beta t}|J(x_t,y_t)|] = \lim_{t\to\infty}\frac{1}{\gamma r}\exp(-rt-\gamma r(x_0+a_y y_0+a_0)) = 0. \tag{A.10}$$

In summary, a positive interest rate ($r>0$) suffices the transversality condition (A.6) to hold. Additional technical conditions associated with the standard verification procedure may be found in Duffie (2001) and Karatzas and Shreve (1991). □

Appendix B. A risk-adjusted measure of wealth for income

This appendix provides some details in leading to the calculation of h^{R}, the "risk-adjusted" human wealth.

First, consider the model implied consumption dynamics. Eqs. (8) and (5) together imply that

$$\mathrm{d}c_t^* = \frac{1}{\gamma}\left[\frac{1}{2}\eta_t^2 - (\beta - r)\right]\mathrm{d}t + \frac{1}{\gamma}\eta_t\,\mathrm{d}W_t, \tag{B.1}$$

where

$$\eta_t = \gamma r a_y \sigma(y_t) \geqslant 0. \tag{B.2}$$

To control for the difference between the agent's subjective discount rate and the interest rate, I denote $z_t \equiv c_t^* + (\beta - r)t/\gamma$. Integrating (B.1) on both sides and re-arranging gives

$$\mathbb{E}_t(z_s - z_t) = \mathbb{E}_t\left(\int_t^s \frac{\eta_v^2}{2\gamma}\,\mathrm{d}v\right) \tag{B.3}$$

$$= \begin{cases} r(1-a_h)[(y_t + l_0/l_1)n(s-t;\kappa) + \tilde{\theta}m(s-t;\kappa)], & l_1>0, \\ r\Delta_0(s-t)/2, & l_1 = 0, \end{cases} \tag{B.4}$$

for $s > t$, where $\tilde{\theta} = \theta + \kappa l_0/l_1$, for $l_1 > 0$, and

$$n(v;\delta) = (1 - e^{-\delta v})/\delta, \tag{B.5}$$

$$m(v;\delta) = (v - n(v;\delta))/\delta, \tag{B.6}$$

Eq. (B.4) implies that z is a submartingale.[15]

Our objective is thus to construct a new measure R such that $\{z_t = c_t^* + (\beta - r)t/\gamma : t \geqslant 0\}$ is a martingale under this measure R. That is, consumption (adjusting for $\beta - r$) is a martingale under this measure R.

The Girsanov's Theorem (Karatzas and Shreve, 1991) is the key to achieve this change of measure. The consumption dynamics (more precisely z dynamics) may be written as follows:

$$\mathrm{d}z_t = \frac{1}{\gamma}\eta_t\left(\mathrm{d}W_t + \frac{\eta_t}{2}\,\mathrm{d}t\right) = \frac{1}{\gamma}\eta_t\,\mathrm{d}W_t^{\mathrm{R}}. \tag{B.7}$$

Thus, z is a martingale under measure R if we choose this alternative measure R by linking the Brownian motion W_t^{R} under this measure R to the Brownian motion W under the income generating measure P as follows:

$$\mathrm{d}W_t^{\mathrm{R}} = \mathrm{d}W_t + \frac{\eta_t}{2}\,\mathrm{d}t = \mathrm{d}W_t + \frac{\gamma r a_y}{2}\sqrt{l_0 + l_1 y_t}\,\mathrm{d}t. \tag{B.8}$$

Then, by construction, z is a martingale under this newly constructed measure R. The main text continues with the analysis of "risk-adjusted" human wealth, the reformulation of the consumption rule, and the interpretation of Friedman's conjecture.

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[15] Note that $n(v;0) = v$ and $m(v;0) = v^2/2$.

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