

**A CONTINUOUS TIME EQUILIBRIUM MODEL OF
FORWARD PRICES AND
FUTURES PRICES IN A MULTIGOOD ECONOMY***

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This paper is a theoretical investigation of equilibrium forward and futures prices. We construct a rational expectations model in continuous time of a multigood, identical consumer economy with constant stochastic returns to scale production. Using this model we find three main results. First, we find formulas for equilibrium forward, futures, discount bond, commodity bond and commodity option prices. Second, we show that a futures price is actually a forward price for the delivery of a random number of units of a good; the random number is the return earned from continuous reinvestment in instantaneously riskless bonds until maturity of the futures contract. Third, we find and interpret conditions under which normal backwardation or contango is found in forward or futures prices; these conditions reflect the usefulness of forward and futures contracts as consumption hedges.

1. Introduction

The purpose of this paper is to examine forward and futures prices in a general equilibrium model. We wish to understand how forward and futures prices are determined; why they differ; and how they are related to each other and to other prices in the economy, such as spot prices. We also wish to understand what economic risks are reflected in forward and futures prices. First we review the distinction between forward and futures contracts and forward and futures prices.

Forward and futures prices differ in a fundamental way because forward and futures contracts promise fundamentally different payoffs. An investor

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who buys a forward contract agrees to buy one unit of a good at a specified future time, called the maturity time. The price at which the purchase will be made is called the forward price. The forward price is determined when the contract is written, is specified in the contract and does not change over the life of the contract. The forward price is chosen so that the purchaser of the forward contract, who is in the 'long' position, pays or receives nothing when the contract is written. At the time of maturity the long investor receives one unit of the good, which is delivered by the seller of the forward contract, who is in the 'short' position; at maturity time the short investor receives the forward price specified in the contract.

The value of a forward contract fluctuates between the time it is written and the time it matures. When the contract is written it has no value, but on the maturity date the long investor realizes a profit (or loss) equal to the difference between the spot price of one unit of the good and the contracted forward price. Between writing and maturity the value of a forward contract will fluctuate because the value of the right to buy at the forward price written in the contract changes as the spot price changes.

A futures contract is like a forward contract paid for on a unique installment plan. The investor who buys a futures contract agrees to buy one unit of a good at a specified maturity time at a specified futures price. The futures price is determined when the contract is written and is specified in the contract. The futures price is chosen so that no payment is made when the contract is written, i.e., at initiation the futures contract has zero market value. But as the contract matures, the investor must make or receive daily installment payments toward the eventual purchase of the good. The total of the daily installments and the payment at maturity will equal the futures price specified when the contract was initiated.

What makes futures contracts unique among installment plans is that the daily installments are not specified in advance in the contract, but are determined by the daily change in the futures price. If the futures price rises then the investor who is long the futures contract receives a payment from the investor who is short. The payment is the rise in the futures price from the previous day. On the other hand, if the futures price falls then the long pays the short the change in the daily futures price. This process is called 'marking-to-market' on futures exchanges.

The effect of marking-to-market is to rewrite the futures contract each day at the new futures price. (What the investor does with the installment payment he has received is a separate issue that we discuss later in section 5.) Hence the value of the futures contract after the daily settlement will always be zero since the value of a newly written futures contract is zero. When the contract matures the long investor will have already paid or received the difference between the initial futures price and the futures price at the maturity time. With these payments to his credit he will have a balance due

equal to the futures price at the maturity time. But the value of a futures contract written at the maturity time for immediate delivery must be zero. Therefore at maturity the futures price must equal the current spot price. Thus the balance due is simply the current spot price at the maturity time. Unlike a forward contract, the value of a futures contract — after settlement — is always zero.

The distinction between the payoffs from forward and futures contracts was made first by Black (1976). Jarrow and Oldfield (1981) further clarified this distinction by showing that when the daily interest rate is constant, forward and futures prices are identical.

In order to examine forward and futures prices we construct a rational expectations general equilibrium model in continuous time of a multigood, identical consumer economy with constant stochastic returns to scale production. This model is developed in sections 2, 3 and 4. In section 2 we describe consumer's preferences, information and endowments and we specify the production and investment possibilities. In the third section we find the consumer's budget constraint and set up his lifetime consumption-investment problem. In section 4 we define a rational expectations equilibrium in which the price functions which the consumers use to make their optimal decisions are precisely the equilibrium price functions which continuously clear the markets.

Sections 5 and 6 contain our main results. In section 5 we state our main pricing formulas, which are derived in the appendix. We find the prices of discount bonds, commodity discount bonds, commodity options, forward contracts, forward prices and futures prices.

In section 5 we show that a futures price is actually a forward price for the delivery of a random number of units of a good. The random number is independent of the good and is equal to the return earned from continuous reinvestment in instantaneously riskless bonds until maturity of the futures contract. The equivalent quantity in discrete time is the return earned from rolling over daily riskless bonds until the maturity of the futures contract. By using a portfolio strategy similar to one first used by Cox, Ingersoll and Ross (1981), we show how to use futures contract to replicate the payoff from a forward contract on the random number of units of a good. We also show that the futures price is the present value of the same random number of units of a good to be delivered at the maturity time.

In section 6 we discuss the implications of our model for the theories of normal backwardation and contango. Normal backwardation is the hypothesis advanced by Keynes (1923) and Hicks (1939) who argue that futures prices must rise on average over the life of a futures contract in order to induce speculators to hold long positions. Hence the futures price is a downward biased estimate of the future spot price since we know at maturity that the futures price equals the spot price. The converse hypothesis called

contango is advanced by Telser (1958) who argues that speculators are essentially gambling in the futures market and will hold long positions even when futures prices are expected to fall on average. Under contango the futures price is an upward biased estimate of the future spot price.¹ The discussion of normal backwardation and contango usually is in terms of futures prices, but the issue applies equally well to forward prices and we shall examine them both.

The ability of investors to hedge against welfare losses by hedging their future consumption determines whether normal backwardation or contango will characterize these prices. In brief we will show that normal backwardation is found when futures and forward contracts are poor consumption hedges. By a poor consumption hedge we mean that these contracts are more profitable than average when the marginal utility of consumption is lower than average and vice versa. On the other hand, contango prevails if forward or futures contracts are good instruments to insure against adverse consumption opportunities. That is, we find contango when futures or forward contracts are more profitable than average when the marginal utility of consumption is higher than average. In our discussion we shall emphasize the usefulness of these contracts as insurance (or hedges) against welfare losses.

Our approach to asset pricing is to use a general equilibrium model in which only the preferences, endowments and information of economic agents and the technology of production are specified exogenously and all prices are determined endogenously in equilibrium. Thus the equilibrium prices are directly related to the preferences and underlying stochastic production in the economy. Our model is related by this approach to several single-good models. The pure exchange models of Rubinstein (1976) and Lucas (1978) and the extension of Lucas' model by Johnsen (1978) all demonstrate the key role of the marginal utility of consumption in asset pricing; Brock (1978) extends these pure trade models to a production economy. Prescott and Mehra (1980) use an identical consumer model of production with many

¹The term 'normal backwardation' was originally used by Keynes (1923) and Hicks (1939) to refer to the situation where the current futures price is less than the current spot price. Hicks (1939, p. 138) explains the term:

In 'normal' conditions, when demand and supply conditions are expected to be about the same in a month's time as it is today, the futures price for one month's delivery is bound to be below the spot price now ruling. The difference between these two prices (the current spot price and the currently fixed futures price) is called by Mr. Keynes 'normal backwardation'.

The reason the futures price is 'bound to be below the spot price' is that 'speculators' must — on average — receive a profit to induce them to bear the risk of taking long positions. A rising price trend is the reward for this risk bearing. The modern usage of the term normal backwardation (and contango) refer to the price trend or, equivalently, the bias in futures prices as predictors of spot prices. We thank Jeffrey Williams and Stanley Black for bringing the original definition to our attention.

commodities to examine firm share prices in a recursive competitive equilibrium; they do not, however examine commodity contract prices.

The continuous-time, single-good, identical consumer, production economy model of Cox, Ingersoll and Ross (1978) is the prototype for our multigood model; our model generalizes theirs to an economy with many goods so that commodity prices can be studied. In an earlier draft, Cox, Ingersoll and Ross (1977) found a formula for futures prices of discount bonds, but removed it from Cox, Ingersoll and Ross (1978). By combining this futures price formula from Cox, Ingersoll and Ross (1977) with valuation formulas in Cox, Ingersoll and Ross (1978), their technique for finding futures prices can be shown to be equivalent to the one we find.²

A different approach to asset pricing is to price one asset relative to a second, which has an exogenously specified price or return. [This is called partial equilibrium; a well-known model of this sort is the Black-Scholes (1973) option pricing model.] While this approach cannot relate prices to preferences and the underlying stochastic production in the economy, it can specify important restrictions on the relative prices or returns of two assets. Breeden (1979) examines the relative structure of security returns and shows that under certain conditions a securities expected return is a linear function of the covariance of its return with aggregate consumption. He uses his model in Breeden (1980) to relate the equilibrium returns on futures contracts to the stochastic process for aggregate consumption. Grauer and Litzenberger (1979) study the pricing of futures contracts and other risky assets in a two-period exchange economy where individuals have homothetic utility functions.

2. Description of the economy

We consider an economy with a large number, I , of identical consumers.³ Each consumer seeks to maximize his lifetime expected utility

$$E \left[\int_0^{\infty} e^{-\rho t} u(c(t)) dt \right], \quad (1)$$

²This equivalence is not obvious. We thank Doug Breeden for telling us of the equivalence and supplying the proof found in footnote 8. Also, since completing an earlier draft of this paper we have received papers by Cox, Ingersoll and Ross (1981) and French (1981), who, using arbitrage arguments, derived forward and futures prices equivalent to the prices we find for our model in section 5.

³The assumption that all consumers have identical preferences and endowments is much less restrictive than it first appears to be. We could, as Cox, Ingersoll and Ross (1978) do, introduce firms which invest in the production, buy and sell commodity contracts and, in general, act as competitive economic agent. These firms would, of course, issue shares which would be owned by the consumers and traded in a competitive capital market. As Cox, Ingersoll and Ross show, the existence of these firms would not alter any prices or quantities invested — even though the firms are not identical.

where $c(t) = (c_1(t), \dots, c_N(t))$ is a vector of stochastic processes representing the time t rate of consumption of N goods, ρ is a discount factor, $u(\cdot)$ is an instantaneous utility function and $E[\cdot]$ is an expectation operator. The utility function u is assumed to be strictly concave, increasing and twice continuously differentiable in all of its arguments. To ensure internal solutions to the consumer's optimization problem we assume that the marginal utility of consumption in each good approaches infinity as consumption of that good approaches zero.

Each consumer is identically endowed. Let $Q(t) = (Q_1(t), \dots, Q_N(t))'$ be the time t aggregate stock of goods in the economy. At time zero each consumer has an endowment of $(1/I)Q(0)$ of the N goods. Because consumers are identical and production is constant stochastic returns to scale, each consumer will continue to own $1/I$ of the aggregate stock of goods at all future times in equilibrium.

The consumption goods are produced by N distinct production process, one for each consumption good.⁴ The production process for good i , in general, uses all of the N goods as inputs and produces an uncertain quantity of good i . (It is possible to specialize production so that some of the goods are not required as input for the production of good i .) Each production process has constant stochastic returns to scale. The production processes themselves change over time in response to random 'technological shocks', which are exogenously determined. All consumers have equal access to the production processes.

Let $k_i(t) = (k_{i1}(t), \dots, k_{iN}(t))$ be the quantities of each good invested by the typical consumer in the production process for good i at time t . The total amount of good j invested in production is $q_j = \sum_{i=1}^N k_{ij}$ for $j = 1, \dots, N$. Finally, let $X(t) = (X_1(t), \dots, X_M(t))'$ be a set of M state variables which summarize the current state of technology. Output of good i net of consumption, $dq_i(t)$, is then given by the stochastic differential equation⁵

$$dq_i = [g_i(k_i, X) - c_i]dt + G_i(k_i, X)dZ \quad \text{for } i = 1, \dots, N, \quad (2)$$

where g_i is the mean instantaneous gross output, G_i is an $1 \times (N + M)$ vector of diffusion coefficient functions, and $Z(t)$ is a vector of $N + M$ independent Wiener processes. For each $i = 1, \dots, N$ the functions g_i and G_i are homogeneous of degree one in k_i making production constant stochastic returns to scale,

$$\lambda g_i(k_i, X) = g_i(\lambda k_i, X) \quad \text{and} \quad \lambda G_i(k_i, X) = G_i(\lambda k_i, X).$$

⁴We have allowed for only one production process for each good. This is, however, not a restrictive assumption. When many production processes are available in each good, risk averse consumers will diversify their investment across the production processes. This result is well known in the portfolio theory literature. See Merton (1971) or Cox, Ingersoll and Ross (1978).

⁵See Fleming and Rishel (1975).

For notational convenience we rewrite (2) in vector form

$$dq = [g - c]dt + G dZ, \quad (3)$$

where $dq = (dq_1, \dots, dq_N)'$, $g = (g_1, \dots, g_N)'$ and $G = (G_1, \dots, G_N)'$. We assume that the instantaneous covariance matrix of output is positive, i.e., $GG' > 0$.

The state of technology is stochastic and independent of any decisions made by the consumers. Each consumer knows that X has dynamics given by

$$dX(t) = \mu(X)dt + S(X)dZ. \quad (4)$$

In (4) $\mu(X)$ is an $M \times 1$ drift vector which is the expected instantaneous rates of change in the state variables and $S(X)$ is an $M \times (N + M)$ diffusion matrix such that $S'S > 0$ is the instantaneous covariance matrix of the state variables. Output and changes in state are correlated with covariance matrix GS' .

To each consumer the state of the economy is characterized by the vector X of technological state variables and the vector $Q(t)$ of aggregate stocks of goods. By this we mean that if at two different points in calendar time the economy arrives at a particular state (Q, X) , it will behave the same way both times, regardless of the route by which the state was attained each time. All consumers assume that the dynamics for Q are given by

$$dQ = h(Q, X)dt + H(Q, X)dZ, \quad (5)$$

where h is an $N \times 1$ drift vector and H is an $N \times (N + M)$ diffusion matrix such that $HH' > 0$ is the instantaneous covariance matrix of aggregate output.

It is important to emphasize that each consumer acts as if the dynamics of the aggregate quantities are independent of his own actions. When making consumption or investment decisions the consumer, as part of his price taking behavior, takes the functions $h(Q, X)$ and $H(Q, X)$ as given. Different consumer beliefs about h and H will produce different equilibrium allocations of investment and different equilibrium prices. Later, when we define a rational expectations equilibrium we will find h and H by requiring that the observed aggregate dynamics actually result from the aggregation of consumer decisions. But in the meantime we can regard h and H as arbitrary so the covariance between aggregate stocks and individual consumer output is $H'G$.

Since all consumers are identical in this economy, it is easy to see that autarky is Pareto-optimal. Hence we need not establish any markets to support the Pareto-optimal allocations. Nevertheless, we will assume the

existence of competitive markets where specific types of goods or contracts are traded. Since we know the equilibrium volume of trading in these (and any other) markets will be zero, the main issue then is to determine the equilibrium price functions.

A short digression on the determination of price dynamics will define a useful notational convention. Since the state variables summarize the consumer's knowledge about the economy, all equilibrium prices can be expressed as functions of the state variables only. Let $\pi(t) = \pi(Q(t), X(t), t)$ be the time t price of some good or contract defined below. By Ito's Lemma the dynamics of $\pi(t)$ are

$$d\pi(t) = (L\pi)dt + (\pi_X S + \pi_Q H)dZ,$$

where L is a differential operator defined by

$$\begin{aligned} L \equiv & \sum_{i=1}^M \mu_i \frac{\partial}{\partial X_i} + \sum_{i=1}^N h_i \frac{\partial}{\partial Q_i} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N (HH')_{ij} \frac{\partial^2}{\partial Q_i \partial Q_j} \\ & + \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^M (SS')_{ij} \frac{\partial^2}{\partial X_i \partial X_j} + \sum_{i=1}^N \sum_{j=1}^M (HS')_{ij} \frac{\partial^2}{\partial Q_i \partial X_j} + \frac{\partial}{\partial t}, \end{aligned}$$

and $\pi_X \equiv (\partial\pi/\partial X_1, \dots, \partial\pi/\partial X_M)$ and $\pi_Q \equiv (\partial\pi/\partial Q_1, \dots, \partial\pi/\partial Q_N)$.

As a notational convention we define for any price π the operators β_π and σ_π given by

$$\beta_\pi \equiv L\pi, \tag{6}$$

and

$$\sigma_\pi \equiv \pi_X S + \pi_Q H. \tag{7}$$

Thus β_π is the expected instantaneous rate of change in π and $\sigma_\pi \sigma'_\pi$ is the instantaneous variance of the change in π .

We now specify which markets exist and define what goods or contracts are traded in these markets. We are interested in the relative spot prices of all goods and the prices of two general types of contracts.

We assume that competitive spot market exists in which currently available goods are continuously traded. Good one is arbitrarily chosen as the numeraire good so that its time t price, denoted by $P_1(t)$, is identically one for all t . The relative spot prices at time t of the other goods are denoted by $P_i(t)$ for $i = 2, \dots, N$. By Ito's Lemma the dynamics of $P_i(t)$ are

$$dP_i(t) = \beta_{P_i} dt + \sigma_{P_i} dZ \quad \text{for } i = 1, \dots, N, \tag{8}$$

where β_{P_i} and σ_{P_i} are found by substituting $P_i = \pi$ into (6) and (7), respectively. It will be convenient to rewrite (8) in vector notation,

$$dP = \beta_P dt + \sigma_P dZ, \quad (9)$$

where $P = (P_1, \dots, P_N)'$, $\beta_P = (\beta_{P_1}, \dots, \beta_{P_N})'$ and $\sigma_P = (\sigma_{P_1}, \dots, \sigma_{P_N})'$.

Consumers may also create and trade in competitive markets securities which have payoffs specified as functions of the state of the economy $(Q(s), X(s))$ for $s \in [t, T]$, where the time between t and T is the 'lifetime' of the contract. We will consider two types of contracts, both of which are traded continuously. Contracts of the first type have a non-zero market value and have payoffs only at the time of maturity, T . We call these contracts 'maturity-payoff contracts'. Examples of this type of contract are commodity discount bonds, forward contracts, and unit discount bonds. Contracts of the second type have a market value of zero at all times, but these contracts have continuous payoffs. We call these contracts 'continuous-payoff contracts'. An example is the futures contract which has a value of zero but the owner gets (or pays) the difference between the futures prices at every instant.

The unit discount bond price, $B_1(t, T)$, is the price to be paid at time t in the numeraire good for default-free delivery of one unit of the numeraire good at some future time T . The currently maturing unit discount bond defines the current instantaneous risk-free interest rate,⁶ $r(t)$, by $r(t) = -\partial B_1(t, t)/\partial T$. We assume that consumers can instantaneously borrow or lend the numeraire good at the sure interest rate r . The commodity discount bond price, $B_i(t, T)$, of good i is the price to be paid at time t in numeraire good for delivery of one unit of good i at some future time T .

Forward contracts were defined in section 1. Let T denote the maturity time, t the current time, and $F_i(t, T)$ the current forward price for one unit of good i . We denote by $V_i(s, T, F_i)$ the time s market value of a forward contract written at forward price F_i . Since a forward contract has zero market value when it is written we have $V_i(t, T, F_i(t, T)) = 0$. At maturity $V_i(T, T, F_i(t, T)) = P_i(T) - F_i(t, T)$, the profit (or loss) from buying one unit of good at price $F_i(t, T)$ when the spot price is $P_i(T)$.

A final type of maturity-payoff contract is a commodity option. A commodity option is an 'European' option. The holder of this contract on good i has an option to buy one unit of good i at a fixed time, T , in the future at a specified exercise price, D_i , determined now, t , but paid at T . We denote by $O_i(t, T, D_i)$ the price of the commodity option quoted at t and due to expire at $T > t$.

⁶Cox, Ingersoll and Ross (1978) show that instantaneous risk-free borrowing and lending is feasible for consumers with risky portfolios when all price processes are continuous and trading is continuous.

The only type of continuous-payoff contract we consider is a futures contract, which was defined in section 1. Again let t denote the current time, T the maturity time, and let $f_i(t, T)$ denote the futures price for good i . In continuous time the futures contract is rewritten at every instant in time at the new futures price $f_i(t, T)$ and thus always has a market value of zero. The continuous payoff is determined by the instantaneous changes in the futures price calculated below.

Without loss of generality we assume that there is one maturity-payoff contract with market value denoted by v and there is one continuous-payoff contract which has zero market value but provides a stochastic payoff df over the next dt time interval. By Ito's Lemma the dynamics of v are

$$dv = \beta_v dt + \sigma_v dZ, \quad (10)$$

where β_v and σ_v are found by substituting $v = \pi$ in (6) and (7), respectively. Similarly the stochastic payoff

$$df = \beta_f dt + \sigma_f dZ, \quad (11)$$

where β_f and σ_f are again found using (6) and (7). The important difference between the two types of contracts is that a continuous-payoff contract requires no investment, but provides continuous stochastic payoffs, while a maturity-payoff contract has a non-zero market value in general, but provides a payoff only at the maturity date.

3. The consumer's optimization problem

Faced with the opportunity set described in section two, the consumer at each point in time must allocate his wealth among investments in the production technologies, the two contracts and riskless borrowing or lending. He must also choose a consumption vector, $c(t)$. Let $w(t)$ be the consumer's wealth at time t , i.e., the market value of all goods and contracts held by the consumer at time t . Let $q(t) = (q_1(t), \dots, q_N(t))'$ be the vector of quantities invested in each of the N production processes at time t . Let $m(t)$ be the number of maturity-payoff contracts held and $n(t)$ be the number of continuous-payoff contracts held. His total wealth, w , must be fully invested at each time t . Hence the wealth invested in riskless borrowing or lending is given by $[w - P'q - mv]$. The consumer's wealth change over the next ' dt ' time interval is the sum of his gain (or loss) from investment in production, maturity-payoff contracts, continuous-payoff contracts and riskless borrowing

(or lending),

$$\begin{aligned} dw &= d(P'q) + m dv + n df + (w - P'q - mv) r dt \\ &= P' dq + q' dP + dP \cdot dq + m dv + n df + (w - P'q - mv) r dt, \end{aligned}$$

by Ito's Lemma. The inner product $dP \cdot dq = \text{tr}(\sigma_P G') dt$. Hence

$$dw = \beta_w dt + \sigma_w dZ, \quad (12)$$

where

$$\begin{aligned} \beta_w &= P'(g - c) + q' \beta_P + \text{tr}(\sigma_P G') \\ &\quad + m(\beta_v - rv) + n\beta_f + r(w - P'q), \end{aligned} \quad (13)$$

$$\sigma_w = P'G + q'\sigma_P + m\sigma_v + n\sigma_f. \quad (14)$$

In (12) β_w is the instantaneous expected rate of change in wealth and $\sigma_w \sigma_w'$ is the variance of this change.

The consumer seeks to maximize his lifetime expected utility (1) subject to his budget constraint (12). In doing so he takes as given the price functions P , v , f and r and the state dynamics (4) and (5). Standard optimal control theory [see Fleming and Rishel (1975)] proves that if the functions⁷ $k_i(w, Q, X)$ for $i = 1, \dots, N$, $m(w, Q, X)$, $n(w, Q, X)$, $c(w, Q, X)$ and $J(w, Q, X)$ satisfy the Bellman equation for this problem, then $\{k_i\}$, m , n and c are optimal consumer decisions and J is the optimum value function. That is, $J(w, Q, X)$ is the consumer's expected lifetime utility derived from having wealth w when the state is (Q, X) and proceeding optimally into the future. The Bellman equation for the consumer's problem is

$$\begin{aligned} 0 = \max_{\{k, m, n, c\}} \{ & u(c) - \rho J + LJ + \beta_w J_w + \frac{1}{2} J_{ww} \sigma_w \sigma_w' \\ & + \sigma_w S' J_{wX} + \sigma_w H' J_{wQ} \}. \end{aligned} \quad (15)$$

Here $J_{wX} = (\partial^2 J / \partial w \partial X_1, \dots, \partial^2 J / \partial w \partial X_M)'$ and $J_{wQ} = (\partial^2 J / \partial w \partial Q_1, \dots, \partial^2 J / \partial w \partial Q_N)'$. In the appendix we find the necessary and sufficient first-order condition for the maximization in (15).

The first-order conditions have some by now well-known implications for the consumer's asset demands in partial equilibrium. His demand for an asset

⁷There are regularity conditions for q , m , n , c and J . See Fleming and Rishel (1975). We assume sufficient regularity of all functions for the appropriate theorems to apply.

will reflect not only its instantaneous risk–return tradeoff, but also the asset’s usefulness as a hedge against changes in the investment opportunity set. Hence it is possible to derive Breeden’s (1979) ‘single beta’ capital asset pricing model or a multifactor capital asset pricing model as in Merton (1973) or Richard (1979).

4. Market equilibrium

In the previous section we saw that given the price functions and the state dynamics, each consumer determined his optimal consumption and investment decisions by solving (15). In this section we specify conditions for market equilibrium which allow us to determine the price functions and the state dynamics taken as given by consumers. There are two types of conditions that define market equilibrium.

The first set of conditions for market equilibrium is that markets continuously clear in the ordinary sense that supply equals demand. These equilibrium conditions are straightforward in this economy because all consumers are identical. Hence each consumer must own and invest an equal share of the aggregate stocks of goods at every t ,

$$q = (1/I)Q. \quad (16)$$

Since consumers are identical, equilibrium demand for the maturity-payoff contract, the continuous-payoff contract and riskless borrowing or lending are all zero:

$$m = 0, \quad (17)$$

$$n = 0, \quad (18)$$

and

$$w = P'q. \quad (19)$$

Eqs. (16) and (19) imply that in equilibrium all wealth is held in stocks of commodities, i.e., $Iw = P'Q$.

The second condition for market equilibrium is that consumers have rational expectations. By this we mean that the price functions and state dynamics assumed by consumers in solving (15) be the actual price functions and state dynamics which are implied by the aggregation of the consumers optimal decisions. We have already assumed that consumers know the correct dynamics, eq. (4), for the technological state variables, X .

Aggregation of the consumer's stock dynamic, eq. (3), must equal the assumed aggregate stock dynamics:

$$h = I(g - c), \quad (20)$$

and

$$H = IG. \quad (21)$$

A rational expectations equilibrium boils down to finding price functions P , v , f and r , a value of function J and optimal controls $\{k_i\}$, m , n and c which simultaneously satisfy (15)–(21).

In the appendix we find the general formulas for the equilibrium price functions. In the next section we apply these general formulas to find the prices of commodity discount bonds, unit discount bonds, forward contracts and commodity options. We also characterize the equilibrium forward and futures prices.

5. Equilibrium prices of claims

In this section we present general formulas for the equilibrium prices of commodities and contracts for commodities. In the appendix we derive formulas for generic maturity-payoff and continuous-payoff contracts in Theorem A.1 and Theorem A.2, respectively. We then apply these theorems to derive the prices of the particular claims discussed in this section. Our discussion emphasizes the economic intuition behind the formulas.

The prices of all the commodity contracts found in this section, except for the futures price, are applications of a general formula for finding the present value of a risky commodity flow. This general formula for type 1 claims is given by Theorem A.1 of the appendix; we state a version of that theorem here:

Theorem 1. Let $\alpha_i(Q(T), X(T)) = \alpha_i(T)$ denote a (possibly random) state dependent quantity of good i to be received at time T . The time t market value of α_i is

$$v_i(t) = E_t \left[\frac{u_1(c(T))}{u_1(c(t))} \exp \{ -\rho(T-t) \} P_i(T) \alpha_i(T) \right],$$

where $c(t)$ is per capita consumption at time t . Notice that $v_i(t)$ is implicitly a function of the current state $(Q(t), X(t), t)$ but this dependence has been suppressed for notational convenience.

Theorem 1 says that the present value of a random commodity flow at time T is found by the following steps. First convert the random quantity, $\alpha_i(T)$, of good i into a random quantity of good one by multiplying by $P_i(T)$. Next convert the time T random numeraire good payoff, $P_i(T)\alpha_i(T)$, into a random time t numeraire good payoff by multiplying by the marginal rate of substitution of good one at time T for good one at time t , $u_1(c(T))e^{-\rho(T-t)}/u_1(c(t))$. Finally convert the random time t numeraire good payoff to a deterministic quantity by taking expectations. Theorem 1 is a natural extension to our continuous time, multigood model of the present value formulas found by, among others, Nielsen (1974), Rubinstein (1976), Johnsen (1978) and Cox, Ingersoll and Ross (1978).

The equilibrium price at time t of a default-free discount bond on any good i due to mature at time T is given by

$$B_i(t, T) = E_t \left[\frac{u_1(c(T))P_i(T)}{u_1(c(t))} \exp \{ -\rho(T-t) \} \right]. \quad (22)$$

This follows from Theorem 1 with $\alpha_i \equiv 1$. In the appendix we show that

$$P_i(t) = \frac{u_i(c(t))}{u_1(c(t))}, \quad i = 1, \dots, N, \quad (23)$$

so that the spot price at time t of good i is, of course, the marginal rate of substitution of good i for good one, the numeraire good. Making use of (23) we can rewrite (22) as

$$B_i(t, T) = E_t \left[\frac{u_i(c(T))}{u_1(c(t))} \exp \{ -\rho(T-t) \} \right]. \quad (24)$$

The interpretation as the expected marginal rate of substitution is obvious.

In the special case of the numeraire good we have $P_i(T) \equiv 1$, so (22) specializes to

$$B_1(t, T) = E_t \left[\frac{u_1(c(T))}{u_1(c(t))} \exp \{ -\rho(T-t) \} \right]. \quad (25)$$

This formula describes real discount bond prices in the numeraire good and hence can be used to find the term structure of interest rates. In fact this solution reduces to that found by Cox, Ingersoll and Ross (1978) in a single-good economy.

We now turn our attention to forward contracts. The equilibrium value of a forward contract on good i written at time t when the forward price was

$F_i(t, T)$, due to mature at time T and quoted at s where $t \leq s \leq T$ is given by

$$\begin{aligned} V_i(s, T, F_i(t, T)) &= E_s \left[\frac{u_1(c(T))}{u_1(c(s))} \{P_i(T) - F_i(t, T)\} \exp \{-\rho(T-s)\} \right] \\ &= B_i(s, T) - F_i(t, T) B_1(s, T). \end{aligned} \quad (26)$$

This follows from Theorem 1 with $\alpha_i P_i = P_i(T) - F_i(t, T)$. The value of the contract at time T can be positive, zero or negative as $P_i(T) \geq F_i$.

In equilibrium the value of a newly written forward contract must be zero, i.e., $V_i(t, T, F_i(t, T)) = 0$. This implies that

$$F_i(t, T) = B_i(t, T) / B_1(t, T).^8 \quad (27)$$

Once again the formula for the equilibrium forward price has an intuitive interpretation: it is equal to the time t cost of one unit of good i to be delivered at time T , but with payment deferred until time T .

We can derive an alternative expression for forward prices by using the currently observed forward rates of interest, $R(t, s)$, in the numeraire good. The forward interest rates are defined by

$$B_1(t, T) \equiv \exp \left\{ - \int_t^T R(t, s) ds \right\}. \quad (28)$$

Substituting (22) and (28) into (27) we find that the forward prices are

$$F_i(t, T) = E_t \left[\frac{u_1(c(T))}{u_1(c(t))} P_i(T) \exp \left\{ \int_t^T (R(t, s) - \rho) ds \right\} \right]. \quad (29)$$

Hence the forward prices depend on the forward rates of interest. The expression given by (29) is particularly useful in comparing forward and futures prices.

The equilibrium futures price quoted on a futures contract at time t for good i to be delivered at time T is given by⁹

$$f_i(t, T) = E_t \left[\frac{u_1(c(T))}{u_1(c(t))} P_i(T) \exp \left\{ \int_t^T (r(s) - \rho) ds \right\} \right]. \quad (30)$$

⁸Substituting (27) into (26) gives $V_i(s, T, F_i(t, T)) = B_i(s, T)[F_i(s, T) - F_i(t, T)]$ the value of a forward contract derived by Jarrow and Oldfield (1981) using an arbitrage argument.

⁹Cox, Ingersoll and Ross (1977) find a formula for the futures price of a discount bond that can be shown to be equivalent to our solution. Let $B(t, T)$ denote the time t price of a discount bond maturing at time $T > t$. Cox, Ingersoll and Ross (1977, p. 33) state that the time $s < t$ futures price for this bond is

$$f(s, t, T) = \hat{E}_s[B(t, T)] \quad (F.1)$$

This formula is derived in the appendix. In (30), $r(\cdot)$ is the instantaneous riskless rate of interest in the numeraire good.¹⁰

Eqs. (29) and (30) provide a striking contrast as to why forward prices and futures prices, in general, differ. At time t , the forward price is the present value of a known number, $\exp\{\int_t^T R(t,s)ds\} = 1/B_1(t, T)$, of units of good i to be delivered at time T . The future price is the present value of a random number, $\exp\{\int_t^T r(s)ds\}$, of units of good i to be delivered at time T . Thus the futures price represents a simultaneous speculation on both the future spot price and the number of units to be delivered, which in turn is determined by the future instantaneous interest rates. In contrast the forward price represents a speculation only about the future spot price.

In fact the futures price, $f_i(t, T)$, is equal to the forward price, $\phi_i(t, T)$, for the delivery of the random quantity, $\exp\{\int_t^T r(s)ds\}$, of good i . To see this consider a forward contract written at time $t < T$ for the time T delivery of $\exp\{\int_t^T r(s)ds\}$ units of good i . Note that the quantity $\exp\{\int_t^T r(s)ds\}$ is independent of the good on which the contract is written and is known at time T . This forward contract requires no payment at time t , but specifies the price per unit of good i , $\phi_i(t, T)$, to be paid at time T . To show that $\phi_i(t, T) = f_i(t, T)$ first note that the profits, $\pi_i(t, T)$, realized at time T from going long in the forward contract written at time t are

$$\pi_i(t, T) = \exp\left\{\int_t^T r(s)ds\right\} [P_i(T) - \phi_i(t, T)]. \quad (31)$$

where \hat{E}_s denotes the conditional expectation at time s relative to a certain risk-adjusted diffusion. Using Lemma 4 and Theorem 4 of Cox, Ingersoll and Ross (1978), (F.1) yields

$$f(s, t, T) = E_s \left\{ \frac{J_w(t)}{J_w(s)} \exp\left(\int_s^t r(u)du\right) B(t, T) \right\}, \quad (F.2)$$

where $J_w(\cdot)$ is the marginal indirect utility of wealth. But from the first-order condition [Cox, Ingersoll and Ross (1978, eq. (10a))],

$$J_w(t) = U_c(c(t), t) = u'(c(t))e^{-\rho t}, \quad (F.3)$$

for the discounted utility we use. Substituting (F.3) into (F.2) gives

$$f(s, t, T) = E_s \left\{ \frac{u'(c(t))}{u'(c(s))} \exp\left[\int_s^t (r(u) - \rho)du\right] B(t, T) \right\}, \quad (F.4)$$

which is the single-good equivalent of our eq. (30). We thank Doug Breeden for supplying this derivation.

¹⁰These formulas have been tested empirically by two different methods. Sundaresan (1980) chooses a particular class of utility functions and production function and derives closed form solutions to (29) and (30). French (1981) compares observed forward and futures prices under the assumption that the marginal utility of wealth is constant. Both Sundaresan and French reject the model under their assumptions. A new technique, developed by Hansen and Singleton (1981), assumes a particular class of utility functions and uses consumption data to test formulas like (29) and (30). This technique has not yet been applied to futures or forward price data.

Since no investment is required we must have in equilibrium that the time t present value of the profit is zero. Using $P_i(t)\alpha_i(T) = \pi_i(t, T)$ in Theorem 1 and multiplying by $u_1(c(t))\exp\{\rho(T-t)\}$ we get

$$0 = E_t[u_1(c(T))\pi_i(t, T)]. \quad (32)$$

Eq. (32) says that the forward price, $\phi_i(t, T)$, is determined so that the expected marginal utility of receiving a profit, $\pi_i(t, T)$, of $P_i(T) - \phi_i(t, T)$ on $\exp\{\int_t^T r(s)ds\}$ units of good i is zero. From (32) we find that

$$\phi_i(t, T) = \frac{E_t\left[u_1(c(T))P_i(T)\exp\left\{\int_t^T r(s)ds\right\}\right]}{E_t\left[u_1(c(T))\exp\left\{\int_t^T r(s)ds\right\}\right]}. \quad (33)$$

In the appendix, eq. (A.16), we show that

$$1 = \frac{\exp\{-\rho(T-t)\}}{u_1(c(t))} E_t\left[u_1(c(T))\exp\left\{\int_t^T r(s)ds\right\}\right]. \quad (34)$$

Eq. (34) says that the time t present value of the proceeds of an investment of one unit of numeraire good in rolling over instantaneous riskless bonds from time t to time T must be one. Substituting (34) into (33) shows that

$$\phi_i(t, T) = E_t\left[\frac{u_1(c(T))}{u_1(c(t))} P_i(T)\exp\left\{\int_t^T (r(s) - \rho)ds\right\}\right]. \quad (35)$$

Comparing (30) and (35) shows that $\phi_i(t, T) = f_i(t, T)$.

Using a portfolio strategy similar to the one first used in Cox, Ingersoll and Ross (1981)¹¹ we show that with no investment at time t , we can use futures contracts to replicate the time T payoff, $\pi_i(t, T)$, of the forward contract described above. Consider the plan, which we call the 'forward plan', of going long in $\exp\{\int_t^T r(s)ds\}$ futures contracts maturing at time T at each time τ between t and T . The plan generates at each time τ an instantaneous payoff $\exp\{\int_t^\tau r(s)ds\}df_i(\tau, T)$. Continually reinvest the past payoffs and accumulated interest at the instantaneous riskless interest rate, $r(\tau)$. Let $\pi_i(t, \tau)$ denote the time τ value of this plan; since no investment was required $\pi_i(t, t) = 0$. The instantaneous change in value, $d\pi_i(t, \tau)$, is the sum of

¹¹Cox, Ingersoll and Ross (1981) first used a variation of the forward-plan, requiring a positive investment, in their proof of their Proposition 2.

the proceeds from the futures contracts and the interest on $\pi_i(t, \tau)$,

$$d\pi_i(t, \tau) = \exp \left\{ \int_t^\tau r(s) ds \right\} df_i(\tau, T) + \pi_i(t, \tau) r(\tau) d\tau. \quad (36)$$

The solution to (36) with $\pi_i(t, t) = 0$ is

$$\pi_i(t, \tau) = \exp \left\{ \int_t^\tau r(s) ds \right\} [f_i(\tau, T) - f_i(t, T)]. \quad (37)$$

Hence at time $\tau = T$, eqs. (31) and (37) give identical expressions for $\pi_i(t, T)$ since $f_i(T, T) = P_i(T)$. The fact that the forward plan for investment in futures contracts and the forward contract for $\exp \left\{ \int_t^T r(s) ds \right\}$ units of a good results in the identical payoff at time T will prove to be the key to interpreting futures prices in the next section.

When the instantaneous spot rate r is deterministic, forward prices and futures prices coincide, i.e., $F_i(t, T) = f_i(t, T)$. This result extends slightly the work of Jarrow and Oldfield (1981) who show that $F_i(t, T) = f_i(t, T)$ when r is constant.

Lastly we find the price formula for a commodity option. The equilibrium price, $O_i(t, T, D_i)$, of a commodity option with an exercise price D_i is given by

$$O_i(t, T, D_i) = E_t \left[e^{-\rho(T-t)} \frac{u_1(c(T))}{u_1(c(t))} \max \{0, P_i(T) - D_i\} \right]. \quad (38)$$

This follows from Theorem 1 with $\alpha_i P_i = \max \{0, P_i(T) - D_i\}$. Intuitively, the price is the discounted value of the payments to be received at T . The discount factor is again the marginal rate of substitution of consumption of numeraire good at time T for numeraire good at time t .

6. Normal backwardation, contango and forward and futures prices as unbiased forecasters of future spot prices

In this section we discuss the implications of our model for the theories of normal backwardation and contango in both forward and futures prices. Recall from the introduction that normal backwardation occurs when the futures price is a downward biased estimate of the future spot price, while contango occurs if it is an upward biased estimate. The issues of normal backwardation and contango apply equally well to forward prices. For both prices the direction of the bias hinges on the usefulness of the corresponding contract as a hedge against welfare losses.

We examine forward prices first since they are simpler to analyze than futures prices. Recall that a forward contract on good i written at time t

requires the long position to buy one unit of good i at time T for the contracted forward price, $F_i(t, T)$, paid at time T ; no goods change hands at the time of writing. (All prices and profits are in terms of good one which is the numeraire.) The profit (or loss) realized at time T from holding a forward contract is $P_i(T) - F_i(t, T)$. Substituting (22) and (25) into (27) and rearranging gives

$$0 = E_t[u_1(c(T))(P_i(T) - F_i(t, T))]. \quad (39)$$

Eq. (39) says that the expected marginal utility of buying a newly written forward contract must be zero since the required investment is zero.

Normal backwardation will characterize a forward price when forward contracts are relatively poor consumption hedges. To see this rewrite (39) as

$$F_i(t, T) = E_t[P_i(T)] + \frac{\text{cov}_t[u_1(c(T)), (P_i(T) - F_i(t, T))]}{E_t[u_1(c(T))]} \quad (40)$$

The forward price is a downward biased estimate of the future spot price — normal backwardation — when the profits from a forward contract are negatively correlated with the marginal utility of consumption of the numeraire good. If the gains from holding a long position in a forward contract are higher than average when the marginal utility of consumption is lower than average, then the forward contract is not a very good hedge against welfare losses. Hence an investor holding the long position in a forward contract is in a sense issuing insurance against a welfare loss to the investor holding the short position. Therefore $F_i(t, T) < E_t[P_i(T)]$ to induce the investor to issue the insurance. Conversely, if higher than average profits tend to occur when the marginal utility of consumption is higher than average, then long investors have bought a hedge against welfare losses. Hence $F_i(t, T) > E_t[P_i(T)]$ since the investor is buying insurance.

The analysis of normal backwardation in futures prices is similar to the analysis in forward prices, but more difficult. The difficulty arises from the fact that futures contracts have continuous payoffs until maturity in contrast to a forward contract, which has a single payoff at maturity. In order to analyze the usefulness of futures contracts as hedges for consumption at time T , we must first calculate the total time T payoff derived from an investment strategy in futures contracts. But as we showed in section 5 the payoff realized at time T from an investment in futures contracts following the forward plan is equal to the profit, $\pi_i(t, T)$ given by eq. (31), realized from investing in a forward contract for $\exp\{\int_t^T r(s)ds\}$ units of good i to be delivered at time T .

The key then to analyzing the relationship between the current futures price and the expected future spot price is to see that the futures price is

actually a forward price for the delivery of a random quantity of good at the time of maturity. Hence we can determine the sources of bias in the futures price as a predictor of future spot prices by analyzing the sources of bias in the equivalent forward price for the delivery of $\exp\left\{\int_t^T r(s)ds\right\}$ units of a good.

Whether the futures price is a downward biased (normal backwardation) or an upward biased (contango) predictor of the future spot price depends on the sum of two factors. The first factor is a measure of the usefulness of the futures contract or the equivalent forward contract as a consumption hedge. The second factor measures the riskiness of the time T payoff from the futures contract or the equivalent forward contract. To see this substitute (31) and $f_i(t, T) = \phi_i(t, T)$ into (32) to get

$$0 = E_t \left[u_1(c(T)) \exp \left\{ \int_t^T r(s)ds \right\} (P_i(T) - f_i(t, T)) \right]. \quad (41)$$

Expand (41) to get

$$\begin{aligned} 0 = & E_t[u_1(c(T))] E_t \left[\exp \left\{ \int_t^T r(s)ds \right\} (P_i(T) - f_i(t, T)) \right] \\ & + \text{cov}_t \left[u_1(c(T)), \exp \left\{ \int_t^T r(s)ds \right\} (P_i(T) - f_i(t, T)) \right]. \end{aligned} \quad (42)$$

Substituting (31) into the covariance term in (42) and solving for $f_i(t, T)$ we get

$$\begin{aligned} f_i(t, T) = & E_t[P_i(T)] + \frac{\text{cov}_t[u_i(c(T)), \pi_i(t, T)]}{E_t[u_i(c(T))] E_t \left[\exp \left\{ \int_t^T r(s)ds \right\} \right]} \\ & + \frac{\text{cov}_t \left[\exp \left\{ \int_t^T r(s)ds \right\}, P_i(T) - f_i(t, T) \right]}{E_t \left[\exp \left\{ \int_t^T r(s)ds \right\} \right]}. \end{aligned} \quad (43)$$

The first factor determining the bias in the futures price is $\text{cov}_t[u_1(c(T)), \pi_i(t, T)]$, which reflects the usefulness of the futures contract or the equivalent forward contract as a consumption hedge. This factor is similar to the consumption hedging factor in the forward price formula, eq. (40), which was discussed above and needs no further interpretation.

The second factor is $\text{cov}_t[\exp\{\int_t^T r(s)ds\}, P_i(T) - F_i(t, T)]$ which reflects the risk of the futures contract's profit at time T . Suppose the spot price and the instantaneous interest rate are positively correlated. To a long investor, the number of goods delivered by the forward plan, $\exp\{\int_t^T r(s)ds\}$, tends to be higher than average when the profit per unit delivered, $P_i(T) - F_i(t, T)$, is higher than average. Conversely, the number of units of goods delivered is small when there is a loss on each unit. Thus the long investor is buying insurance from the short investor, and he must pay for this insurance through a higher futures price. On the other hand, if the spot price and the instantaneous interest rate are negatively correlated, the short investor is purchasing insurance from the long investor, resulting in a lower futures price, *ceteris paribus*.

In general futures prices are biased predictors of future spot prices.¹² Examination of eq. (43) shows that the magnitude and direction of the bias are difficult to determine, but easy to understand. The balance of two insurance factors — one for consumption risk and one for profit risk — determine the bias. These factors can be reinforcing or offsetting. Hence, even if interest rates and prices are generally positively correlated the futures price can nevertheless be a downward biased estimate of the future spot price because the futures contract may not be well suited to hedging consumption risks.

In conclusion we see that in equilibrium both forward prices and futures prices are determined by the usefulness of forward and futures contracts in hedging against welfare losses. If an investor can successfully use these contracts to hedge consumption risks, then we will observe contango; if these contracts are poor consumption hedges we will observe normal backwardation. In addition futures price reflect the riskiness of the comovement of prices and interest rates.

Appendix

We establish the pricing formulas in section 5. We can considerably simplify notation by defining the differential operator

$$\hat{L} = L + \beta_w \frac{\partial}{\partial w} + \frac{1}{2} \sigma_w \sigma'_w \frac{\partial^2}{\partial w^2} + \sum_{i=1}^m (\sigma_w S'_i) \frac{\partial^2}{\partial w \partial X_i} + \sum_{i=1}^N (\sigma_w H'_i) \frac{\partial^2}{\partial w \partial Q_i}. \quad (\text{A.1})$$

¹²Grauer and Litzenberger (1978) draw a similar conclusion for their model. There is some problem in the interpretation of Grauer and Litzenberger's conclusion since they use a two-period model. As Jarrow and Oldfield (1980) have shown, in a two-period model forward and futures prices are identical.

In (A.1), L is the differential operator defined in section 2 of the text. The operator \hat{L} when applied to some function $\pi(w, Q, X, t)$ gives the expected rate of change of π which is $\hat{L}\pi$.

The Bellman equation (15) can be rewritten as

$$0 = u(c) + \hat{L}J - \rho J. \quad (\text{A.2})$$

The first-order conditions are found by differentiating (A.2),

$$0 = \frac{\partial u}{\partial c_i} + \frac{\partial(\hat{L}J)}{\partial c_i} = u_i - P_i J_w, \quad i = 1, \dots, N, \quad (\text{A.3})$$

$$\begin{aligned} 0 &= \frac{\partial(\hat{L}J)}{\partial k_{ij}} \\ &= \left[P_i \frac{\partial g_i}{\partial k_{ij}} + \beta_{P_j} + \frac{\partial G_i}{\partial k_{ij}} \sigma'_{P_i} - r P_j \right] J_w \\ &\quad + \left[P_i \frac{\partial G_i}{\partial k_{ij}} + \sigma_{P_j} \right] \left[\sigma'_w J_w + S' J_{wX} + H' J_{wQ} \right], \quad i, j = 1, \dots, N, \end{aligned} \quad (\text{A.4})$$

$$0 = \frac{\partial(\hat{L}J)}{\partial m} = (\beta_v - rv) J_w + \sigma_v \sigma'_w J_{ww} + \sigma_v S' J_{wX} + \sigma_v H' J_{wQ}, \quad (\text{A.5})$$

$$0 = \frac{\partial(\hat{L}J)}{\partial n} = \beta_f J_w + \sigma_f \sigma'_w J_{ww} + \sigma_f S' J_{wX} + \sigma_f H' J_{wQ}. \quad (\text{A.6})$$

In (A.3)–(A.6), $u_i = \partial u / \partial c_i$, $J_{wX} = [\partial J_w / \partial X_1, \dots, \partial J_w / \partial X_M]'$, $J_{wQ} = [\partial J_w / \partial Q_1, \dots, \partial J_w / \partial Q_N]'$ and $\partial G_i / \partial k_{ij} = (\partial G_{i1} / \partial k_{ij}, \dots, \partial G_{i,N+M} / \partial k_{ij})$. The price function P , v , f and r , the value function J and the optimal control functions $\{k_{ij}\}$, m , n and c , which simultaneously satisfy the consumer's optimization problem, (A.2)–(A.6), and the equilibrium conditions, (16)–(21), comprise a rational expectations equilibrium.

Maturity-payoff contracts with arbitrary boundary conditions are evaluated according to Theorem A.1.

Theorem A.1. Let $v(Q(t), X(t), t, T) = v(t)$ denote the time t price of a maturity-payoff contract with boundary conditions at T ,

$$v(Q(T), X(T), T, T) = \zeta(Q(T), X(T)). \quad (\text{A.7})$$

Then

$$v(t) = E_t \left[\frac{u_1(c(T))}{u_1(c(t))} \exp[-\rho(T-t)] \zeta \right], \quad (\text{A.8})$$

where $c(t)$ is per capita consumption at time t .

Proof. Take the total derivative of (A.2) with respect to w using the first order conditions to get

$$0 = \frac{\partial(\hat{L}J)}{\partial w} - \rho J_w = \left(\frac{\partial \hat{L}}{\partial w} \right) J + \hat{L} J_w - \rho J_w = r J_w + \hat{L} J_w - \rho J_w, \quad (\text{A.9})$$

since $\partial \hat{L} / \partial w = r(\partial / \partial w)$. Let $J(t)$ denote $J(w(t), Q(t), X(t))$, but note that $J(t)$ is not an explicit function of t . Examining (A.5) we see that it can be rewritten as

$$0 = \hat{L}(J_w v) - r J_w v - v(\hat{L} J_w). \quad (\text{A.10})$$

To see this, note that $\hat{L}v = \beta_v$ because v is not an explicit function of w so that $v_w = 0$, etc., and note that $\partial J_w / \partial t = 0$ since J is not an explicit function of t . Substitute $\hat{L} J_w$ from (A.9) into (A.10) to get

$$\hat{L}(J_w v) - \rho(J_w v) = 0. \quad (\text{A.11})$$

By Theorem 5.2 of Friedman (1975, p. 147) the solution to (A.11) is

$$J_w(t)F(t) = E_t[J_w(T)\zeta(T)e^{-\rho(T-t)}]. \quad (\text{A.12})$$

From the first-order condition (A.2) we know that $J_w(t) = u_1(c(t))$ which when substituted into (A.12) gives (A.8). ■

Continuous-payoff contracts with arbitrary boundary conditions are evaluated according to:

Theorem A.2. Let $f(Q(t), X(t), t, T) = f(t)$ denote the time t price written into a continuous-payoff contract with boundary condition at T ,

$$f(Q(T), X(T), T, T) = \zeta(Q(T), X(T)). \quad (\text{A.13})$$

Then

$$f(t) = E_t \left[\frac{u_1(c(T))}{u_1(c(t))} \zeta \exp \left(\int_t^T (r(s) - \rho) ds \right) \right]. \quad (\text{A.14})$$

Proof. The proof is exactly parallel to Theorem A.1 using eq. (A.6) instead of (A.5). Eq. (A.6) can be rewritten as

$$0 = \hat{L}(J_w f) - f(\hat{L}J_w),$$

or using (A.9) to eliminate $\hat{L}J_w$,

$$0 = \hat{L}(J_w f) - J_w f(\rho - r). \quad (\text{A.15})$$

Again by Theorem 5.2 of Friedman (1975) the solutions to (A.15) is

$$J_w(t)f(t) = E_t \left[J_w(T) \zeta(T) \exp \left(\int_t^T (r(s) - \rho) ds \right) \right],$$

which is equivalent to (A.14) since $J_w(t) = u_1(c(t))$. ■

Theorem 1 of the text is simply Theorem A.1 with $\zeta(T) = x_T P_T$. The futures price, eq. (31), is found from Theorem A.2 using the boundary condition $\zeta(T) = P_T(T)$. The spot prices $P_i(t) = u_i(c(t))/u_1(c(t))$ are found from (A.3) and are the usual marginal rates of substitution.

We need one final result for use in section 5. Apply Theorem 5.2 of Friedman (1975, p. 147) to find the solution of (A.9),

$$1 = E_t \left[\frac{u_1(c(T))}{u_1(c(t))} \exp \left\{ \int_t^T (r(s) - \rho) ds \right\} \right]. \quad (\text{A.16})$$

Eq. (A.16) says that the time t present value of the proceeds of investing one unit of numeraire good in rolling over instantaneous riskless bonds from time t until time T is, of course, one unit of numeraire good.

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