Hedging with Futures in an Intertemporal Portfolio Context

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I. INTRODUCTION

The traditional hedging model (THM) posits investors with undiversified portfolios, each consisting of a cash position with a definite maturity and one or more futures.[1] The identity of the cash position is not a question in the THM. For the farmer, it is the value of his crop at harvest time; for the institutional investor, it is the value of a future foreign-currency cash flow. The main problem posed in the futures market literature to date is to determine the optimal hedge, defined as the quantity of futures that either minimizes the variance of the cash-cum-futures position or that maximizes its expected utility.[2] A variance-minimizing hedge generally appears as a component of an expected-utility maximizing strategy.

The purpose of this paper is to explore the intertemporal structure of the optimal hedging decision when, in addition to a single cash position, investors can also hold (a portfolio of) freely shortable, traded assets. Among the major implications of this extension are that futures generally will not be used exclusively for hedging purposes; that the existence of a futures contract that is perfectly correlated with the price of the nontraded position will not generally be sufficient for a perfect, zero-variance hedge; and that the conditions for optimal hedges to be preference-free and, therefore, implementable, will rarely be met in practice.

Demands for assets for hedging purposes in an intertemporal portfolio choice framework were first identified by Merton (1971 and 1973). What gen-

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[1]We mean nothing invidious by this characterization. The term "traditional" does not mean obsolete and is used here simply as a short-hand device. The literature on hedging using the THM is long and distinguished. It contains the classics in the field: Working (1953 and 1962), Johnson (1960), Telser (1955 and 1958) and Stein (1961). Recently, with the growth of organized futures markets, the topic has been fruitfully revisited: In one-period models by Ederington (1979), Stoll (1979), Rolfo (1980), Anderson and Danthine (1981), and Benninga, Eldor, and Zilcha (1985); and in intertemporal models, following Merton (1971 and 1973), and Breeden (1979 and 1984), by Ho (1984), and Stulz (1984).

[2]Variance-minimizing hedging rules generally will not maximize expected utility, a point clearly recognized by Ederington (1979) and Anderson and Danthine (1981) and those who came after them. See also Section 6, below.

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erates hedging behavior in Merton's model is the possibility of shifts in the opportunity set. More specifically, investors demand financial assets for hedging purposes when the future consumption of each depends on economic state variable in addition to the level of his wealth. State variables are best described as information processes that cause the future means and variances of traded asset returns and, therefore, the opportunity set, to be random: they may include the asset prices themselves. The presence of state variables in the specification of traded-asset return-diffusions is tantamount to the assumption of an imperfection, in the form of incomplete information about the future local moments of asset-return distributions. The only kind of hedging that arises in Merton's model or in that of Breeden (1979 and 1984) is of this information-based variety.

To obtain hedging in the THM sense, that is, the use of one or more assets and futures to hedge a pre-designated position, three further requirements must be met. The position cannot be continuously revisable; it must be correlated with existing assets; and it must appear as the source of an identifiable future cash flow that affects the investor's welfare. Merton and Breeden did not find any THM hedging demands because their investors did not take positions in state variables directly. However, Mayers (1972) obtained a (variance-minimizing) hedging term in a one-period portfolio-choice model due to his introduction of nontraded assets that affect end-of-period consumption and that are correlated with traded assets. A puzzle in the literature, which we resolve below, is why, when Ho (1984) replaced the freely traded assets in Merton's model with a nontraded position, he did not obtain the intertemporal counterpart of Mayers' minimum-variance hedge while Stultz (1984), working in a very similar setting, did.

As a device for generating traditional hedging decisions, nonmarketability is sufficient and convenient. The nontradedness paradigm may indeed capture empirically relevant aspects of reality. The liabilities that pension managers are locked into may legitimately be viewed as non-traded. So may other financial assets such as unregistered stock or privately-placed corporate and municipal securities. Empirically, however, observed hedging behavior is often directed at positions that are not permanently nonmarketable, like large managed portfolios that are subject to short selling constraints but that can be liquidated at a cost. Even farmers may add to or sell off part of their holdings before the harvest season. What generates hedging in these latter cases is high liquidation costs and costly restrictions on short-selling that produce discontinuities in trading for some assets. The imperfections that give rise to what might be called temporary nontradedness are unduly hard to model in any detail. For simplicity we therefore proxy their effects in what follows by allowing some positions to be completely nontradeable.

The structure of the paper is as follows. Section II sets forth our assumptions. Section III demonstrates the equivalence between alternative formulations of the investor's expected-utility maximization program, presents the optimal demands for traded assets and futures, and decomposes these demands into speculative and hedging components. Section IV establishes that futures will be preferred to traded assets as hedging instruments only under conditions where the hedging demand for traded assets disappears. Section V explores the feasibility of perfect, zero-variance hedges. Section VI investigates implementable hedges and shows that they will rarely also be optimal for general utility structures. Conclusions and final remarks appear in Section VII.

II. ASSUMPTIONS

Assumption 1. The investor is endowed at time-0 with the right to an uncertain quantity, $\tilde{Q}_T$, of a specific commodity or asset that will be received at time T. Costless information about this delivery is gathered continuously and allows the agent to revise his beliefs. The continuous information process, $\{Q_t\}$ satisfies the stochastic differential equation:

$$dQ_t = \mu_q(Q, t)dt + \sigma(Q, t)dz_q(t), \tag{1}$$

where $\{z_q(t); 0 \leqq t \leqq T\}$ is a standard Brownian motion process (BMP). The process $\{Q_t\}$ may be interpreted as a weather report that causes a farmer to update his forecast of his field's yield at harvest-time. Alternatively, take the case of a common stock, traded on a foreign exchange and quoted in a foreign currency, that the investor either cannot or does not want to liquidate or sell short before time T; $\tilde{Q}_T$ then represents the random foreign-currency value of the stock at time T while $\{Q_t\}$ is the current foreign-currency price of the stock.

Assumption 2, (Nontradedness). The initial endowment is fixed and cannot be revised or altered between times 0 to T, even parsimoniously.

Assumption 3. The spot price of the nontraded position is specified exogenously and satisfies the stochastic differential equation:

$$dP_t = \mu_p(P, t) + \sigma_p(P, t)dz_p(t), \tag{2}$$

where $\{z_p(t); 0 \leqq t \leqq T\}$ is a standard BMP and the instantaneous correlation between the price and quantity processes is given by: $\rho_{p,q}dt = dz_p \cdot dz_q$.

Assumption 4. The investor has free and unrestricted access to a financial market in which a number, A, of financial assets are traded continuously. The $A \times 1$ vector of prices, P_a, of these assets satisfies the stochastic differential equation:[3]

$$dP_{a,t} = I_a[\mu_a dt + \sigma_a dz_a(t)] \tag{3}$$

where I_a is an $A \times A$ diagonal matrix whose diagonal is the vector of prices; μ_a represents the constant vector of mean rates of return on these assets and

[3]Notice that while the price dynamics of the information processes, P and Q, are written in terms of their levels, the dynamics of traded-assets' prices are specified in terms of their percentage changes. For the ith traded asset: $dP_{ai}/P_{ai} = \mu_{ai}dt + \sigma_{ai}dz_{ai}(t)$, where the moments are at most deterministic functions of time.

$V_{aa} \equiv \sigma_a \sigma_a'$ is the constant instantaneous variance-covariance matrix. The ith component, z_{a_i} of the vector Wiener process, z_a, has instantaneous correlations, $\rho_{a_i,p}$ and $\rho_{a_i,q}$, with the processes z_p and z_q, respectively.

Assumption 5. An instantaneously riskless bond is available. Its rate of return, r, is constant over the period $[O, T]$. The investor may borrow or lend freely at this rate.

Assumption 6. The investor also has free and unrestricted access to a futures market. One of the futures contracts is written on the nontraded position. It promises delivery at time T of a fixed amount of Q at a futures price fixed at time 0. This contract is marked to market continuously leaving its net value equal to zero. The settlement price of the contract, F_t, is assumed to satisfy:

$$dF_t = \mu_f(P, t)dt + \sigma_f(P, t)dz_f(t), \tag{4}$$

where the BMP, $\{z_f(t); 0 \leqq t \leqq T\}$ has correlations $\rho_{f,p}$ and $\rho_{f,q}$ with the processes z_p and z_q that in general are not perfect due, possibly, to random convenience yields.[4] When the futures contract is priced by arbitrage and the convenience yield is proportional to the spot price ($c(P, t) = cP$), $F_t = P_t e^{(r-c)(T-t)}$ and $\rho_{f,p} = 1$. When, in addition, the contract is assumed, following Breeden (1984), to mature instantaneously, $F_t = P_t$.

In what follows, we contemplate at several points the introduction of a separate futures contract that is perfectly correlated with Q.[5] In the case of a foreign stock, for instance, where Q represents its foreign currency price, the contract would promise the delivery of one share at time T, at the futures price fixed at time-0. Perfect correlation is obtained if the variations in the settlement price of this contract are spanned by the variations in the foreign currency price and in the riskless (constant interest rate) instantaneous foreign bond.

Assumption 7. The investor chooses a consumption stream $\{c_t\}$, a trading strategy in futures $\{x_t\}$ and in traded assets $\{w_t W_t\}$ adapted to his information, so as to maximize his expected lifetime utility. For simplicity, we shall suppose that the life of the investor corresponds to the delivery interval $[O, T]$. During his lifetime, the investor instantaneously consumes an amount of the good c_t and trades in the asset market so as to secure future consumption. At maturity, he consumes his terminal wealth consisting of the cash amount generated by his past trading strategies and of the proceeds of his nontraded posi-

[4]For instance, if the convenience yield satisfies the stochastic differential equation $dc = \mu_c(P, t)dt + \sigma_p(P, t)dz_c$ where $\{z_c(t); 0 \leqq t \leqq T\}$ is a BMP distinct from the process $\{z_p(t); 0 \leqq t \leqq T\}$, an arbitrage argument involving only the commodity, the futures contract and the riskless bond cannot be implemented. In this instance the futures settlement price is imperfectly correlated with the commodity price. Obviously the assumption that the correlation between the futures and spot price is imperfect and the representation of the futures settlement price process assumed in Equation (4) raise questions of compatibility with a full-fledged equilibrium model.

[5]Such a contract would be written much like index contracts are now. At maturity they promise delivery of an amount of cash equal to some pre-set multiple times Q. For a farmer, for example, to view the contract as perfectly correlated with his yield, it would have to be written on quantity of his specific harvest.

tion $P_T Q_T$. The instantaneous utility function $u(c_t, t)$ and the terminal utility function $B(W_T + P_T Q_T, T)$ are assumed to satisfy the usual assumptions.

Assumption 8. In the course of his life, it is quite possible that the investor may adopt strategies that will deplete his cash resources or that will generate a positive probability of obtaining a negative cash-flow at the maturity date T. For the purpose of this paper, we shall not deal explicitly with this bankruptcy problem and its consequences in terms of decision rules. We will instead suppose that the structure of the economy produces a positive cash flow with certainty at date T along the optimal path.

III. THE OPTIMAL HEDGING DECISION

This section investigates the investor's program for optimally hedging his nontraded position in the presence of traded assets and futures. A main result is that two apparently competing specifications of the intertemporal budget constraint are, in fact, equivalent and lead to the same optimal decision. This proposition serves to identify the minimum-variance component of the optimal hedge as an intertemporal demand in the sense of Merton (1971). Following Assumption 7, the investor's program is:

$$\underset{\{c_t,(wW_t),x_t\}}{\text{Maximize}}\; E\left[\int_0^T u(c_t, t) + B(Y_T, T)\right] \tag{5}$$

where W_T represents the cash position at the target date; $P_T Q_T$ is the unique cash flow generated by the nontraded position at time T and total wealth at maturity is given by their sum: $Y_T = W_T + P_T Q_T$.

To solve this intertemporal consumption-investment-hedging problem, the investor must maximize Equation (5) subject to a dynamic budget constraint. Two different formulations of this constraint are possible. The first is a constraint on the cash-balance:

$$dW = w'W\frac{dP_a}{P_a} + xdF + [(1 - w'l)rW - c]dt \tag{6}$$

where $w'l = \Sigma_{i=1}^{A} w_i$ is the sum of the portfolio weights of traded assets. Alternatively, the constraint may be expressed in terms of total wealth at time t; $Y_t = W_t + P_t Q_t$:

$$dY = d(PQ) + w'W\frac{dP_a}{P_a} + xdF + [(1 - w'l)rW - c]dt \tag{7}$$

Except for the inclusion of the cash flows from traded assets, Equation (6) is the form used by Ho (1984), and Equation (7) is the one employed by Stulz (1984). The difference between the two formulations is that Equation (7) contains $d(PQ)_t$ while Equation (6) does not. This apparent difference is illusory.

In the cash balance constraint formulation, Equation (6), the investor chooses a trading strategy in traded assets and futures so as to guarantee optimal consumption and to control the level of the cash balance position. Although there are no intermediate cash flows associated with the nontraded position, its presence causes the investor to modify his intertemporal trading. The boundary condition, which stipulates that the value function at time-T be equal to the terminal utility of the cash balance plus the cash flow generated by the nontraded position, is what produces this effect.

In the wealth constraint formulation, Equation (7), P_tQ_t is the present value of P_TQ_T. As the position is nontraded, $d(PQ)_t$ does not represent a cash flow and does not have a direct control variable associated with it. Consequently, $d(PQ)_t$ operates in Equation (7) as a pure information process. The investor using this formulation of the constraint in effect trades in assets and futures to adjust the cash balance, using information on the current value of the nontraded position, so as to maximize his expected lifetime utility. What may not be obvious is that the trading strategy of an investor who uses the cash balance formulation of the constraint is identical. A straightforward generalization of Adler and Detemple (1986) yields,

Proposition 1. The optimal trading strategies for futures and traded assets and the investor's expected utility based on the cash balance constraint, Equation (6), and those based on the wealth constraint Equation (7) are the same.

What drives the proof is the fact that the information conveyed by the processes $\{Y_t, P_t, Q_t\}$ and the processes $\{W_t, P_t, Q_t\}$ is identical, as wealth and cash balances are linked by the additive relationship, $Y_t = W_t + P_tQ_t$. Conservation of information suggests that the solution not be modified by moving from one formulation to the other. Consequently, the optimal hedges obtained by Ho and Stulz should have been, despite any apparent difference, the same.

The proof of Proposition 1 in the appendix also enables us to write out the optimal vector asset demands for traded assets and futures as:

$$\begin{pmatrix} wW \\ x \end{pmatrix} = -\frac{J_y}{J_{yy}} V^{-1} \begin{pmatrix} \mu_a - rl \\ \mu_f \end{pmatrix} - V^{-1} \left\{ \begin{pmatrix} V_{a,p} \\ V_{f,p} \end{pmatrix} \left(\frac{J_{py}}{J_{yy}} + Q \right) + \begin{pmatrix} V_{a,q} \\ V_{f,q} \end{pmatrix} \left(\frac{J_{qy}}{J_{yy}} + P \right) \right\} \tag{8}$$

where, in addition to notation already introduced, J is the indirect utility or value function and subscripts denote its partial derivatives; $V_{a,p}$ and $V_{a,q}$ are respectively the $(A \times 1)$ vectors of the covariances, $\sigma_{a_i,p}$ and $\sigma_{a_i,q}$, between the traded assets and the price or quantity of the nontraded position; $V_{f,p}$ and $V_{f,q}$ are the (scalar) covariances between the futures price and, respectively, the spot price and the quantity; and V is the $(A + 1) \times (A + 1)$ variance-covariance matrix with the (single) futures contract appearing in the $(A + 1)st$ position.

Equation (8) decomposes the demand for assets and futures into two parts, a locally speculative demand in the first term on the right hand side (RHS) and a hedging demand in the second. To see this interpretation, notice first that the speculative demand consists of the mean-variance-efficient portfolio, weighted by the investor's risk tolerance, that solves the problem:[6]

$$\underset{\{wW,x\}}{\text{Max}}\ E(dW) \text{ subject to } \text{Var}(dW) = \text{a constant}$$

As in Anderson and Danthine (1981), the THM result, reached first by Working (1953) and McKinnon (1967) and repeated more recently by Rolfo (1980) and by Benninga, Eldor and Zilcha (1985), that the sign of μ_f automatically determines the sign of the weight of the futures position in the speculative demand portfolio, breaks down in our model. Whether the investor is long or short in futures depends, in general, not only on the expected change in the futures price but also, via the inverse of the variance-covariance matrix, on its correlations with every other traded asset. Finally, we note that the speculative demand would be the only demand were there no nontraded position (or other exogenous state variables). This observation illustrates the point, implied also in Merton (1973), that investors will not hedge at all when all assets can freely be sold short and there are no shifts in the opportunity set.[7]

[6]As pointed out also in Stulz (1984), the problem is:

$$\underset{\{wW,x\}}{\text{Max}}\ [\mu_{pq} + w'W(\mu_a - rl) + x'\mu_f] + \lambda[k - (W^2 w'V_{aa}w + x'V_{ff}x + 2Ww'V_{a,f}x)].$$

The first order conditions are:

$$\mu_a - rl = \lambda(V_{aa}wW + V_{a,f}x)$$

$$\mu_f = \lambda(V_{f,a}wW + V_{ff}x)$$

which, after manipulation, provides up to a scalar multiple:

$$\begin{pmatrix} wW \\ x \end{pmatrix} = \begin{bmatrix} V_{aa} & V_{af} \\ V_{fa} & V_{ff} \end{bmatrix}^{-1} \begin{pmatrix} \mu_a - rl \\ \mu_f \end{pmatrix}$$

This portfolio is also called a speculative demand because it depends on the direction in which securities' and futures' prices are expected to move.

[7]Notice that as an interim step in obtaining the mean-variance efficient portfolio in footnote 6 one could have written the demand as:

$$wW^m = [V_{aa} - \beta' V_{ff}\beta]^{-1}[(\mu_a - rl) - \beta'\mu_f]$$

$$x^m = V_{ff}^{-1}(\mu_f - V_{f,a}wW)$$

where $\beta' = V_{a,f}V_{ff}^{-1}$ = the matrix of regression coefficients of a on f, and $V_{aa} - \beta' V_{ff}\beta$ = the matrix of the residuals of these regressions. Following Sercu (1980), the equation for wW^m can conceivably be interpreted as a demand for traded assets "hedged by futures" and x^m is then the sum of the demand for futures to hedge traded assets plus a demand for futures as part of the efficient portfolio. The point to stress, however, is that the notion of hedging traded assets with futures is completely arbitrary. It is an artifact, due purely to an arbitrary partitioning of the variance-covariance matrix: other partitions would produce different "hedging" demands. It is meaningless to define hedging in the absence of state variables that shift the opportunity set.

The hedging expression, the second term on the RHS of Equation (8), represents (twice) the portfolio that maximizes the covariance between the controllable cash balance component of wealth and the relative change in the marginal utility of total wealth; that is, it is (two times) the portfolio that solves the unconstrained program:[8]

$$\underset{\{wW,x\}}{\text{Max}}\ \text{Cov}\left[dW, \frac{dJ_y}{J_y}(Y, P, Q, t)\right]$$

It is useful to decompose the hedging term one stage further. Denoting the hedge portfolio by h, we obtain from Equation (8):

$$\begin{pmatrix} wW \\ x \end{pmatrix}^h = -V^{-1}\left\{\begin{pmatrix} V_{a,p} \\ V_{f,p} \end{pmatrix}\frac{J_{py}}{J_{yy}} + \begin{pmatrix} V_{a,q} \\ V_{f,q} \end{pmatrix}\frac{J_{qy}}{J_{yy}}\right\} - V^{-1}\left\{\begin{pmatrix} V_{a,p} \\ V_{f,p} \end{pmatrix}Q + \begin{pmatrix} V_{a,p} \\ V_{f,q} \end{pmatrix}P\right\} \tag{9}$$

The two preference weighted portfolios in the first term on the RHS of Equation (9) replicate the Merton-Breeden dynamic hedges in this case with two state variables. The second term on the RHS of Equation (9) is the one that did not emerge in their setting as it is due to the introduction of a nontraded position into Equation (5). Since the stochastic part of $d(PQ)$ equals the stochastic

[8] The program is $\underset{\{wW,x\}}{\text{Max}}\ \text{Cov}[dW, (dJ_y/J_{yy})(Y, P, Q, t)]$, where

$$dJ_y = \pounds J_y(\cdot) + J_{yy}[wW\sigma_a dz_a + x\sigma_f dz_f + Q\sigma_p dz_p + P\sigma_q dz_q] + J_{yp}\sigma_p dz_p + J_{yq}\sigma_q dz_q$$

and where $\pounds$ is the instantaneous mean operator. This program is equivalent to

$$\underset{\{wW,x\}}{\text{Max}}\ \frac{J_{yy}}{J_y}\left\{(w'W, x')V\begin{pmatrix} wW \\ x \end{pmatrix} + (w'W, x')\left[\begin{pmatrix} V_{a,p} \\ V_{f,p} \end{pmatrix}Q + \begin{pmatrix} V_{a,q} \\ V_{f,q} \end{pmatrix}P\right]\right\}$$
$$+ (w'W, x')\left[\begin{pmatrix} V_{a,p} \\ V_{f,p} \end{pmatrix}\frac{J_{py}}{J_y} + \begin{pmatrix} V_{a,q} \\ V_{f,q} \end{pmatrix}\frac{J_{yq}}{J_y}\right]$$

for which the first order conditions are:

$$2V\begin{pmatrix} wW \\ x \end{pmatrix} + \begin{pmatrix} V_{a,p} \\ V_{f,p} \end{pmatrix}\left(\frac{J_{py}}{J_{yy}} + Q\right) + \begin{pmatrix} V_{a,q} \\ V_{f,q} \end{pmatrix}\left(\frac{J_{qy}}{J_{yy}} + P\right) = 0$$

$$\Rightarrow \begin{pmatrix} wW \\ x \end{pmatrix} = -\frac{1}{2}\left[V^{-1}\begin{pmatrix} V_{a,p} \\ V_{f,p} \end{pmatrix}\left(\frac{J_{py}}{J_{yy}} + Q\right) + V^{-1}\begin{pmatrix} V_{a,q} \\ V_{f,q} \end{pmatrix}\left(\frac{J_{qy}}{J_{yy}} + P\right)\right]$$

part of $QdP + PdQ$, it is immediate that this term represents the portfolio that solves the program:[9]

$$\underset{\{wW,x\}}{\text{Min}}\ \text{Var}(dY) = (w'W, x)\begin{bmatrix} V_{aa} & V_{af} \\ V_{fa} & V_{ff} \end{bmatrix}\begin{pmatrix} wW \\ x \end{pmatrix} + 2(w'W, x)\begin{pmatrix} V_{a,pq} \\ V_{f,pq} \end{pmatrix} + \text{Var}[d(PQ)], \quad (10)$$

where $V_{i,pq}$ = covariance of asset i with the value of the nontraded position.

In what follows, we shall refer to this portfolio as the minimum-variance hedge. As the sequence of decompositions indicates, it is a component of the intertemporal demand for assets and futures, for hedging shifts in the opportunity set. As distinct from all the other components of the optimal demand, it is preference-free and is, therefore, potentially implementable by regression analysis. Finally, it is the one term that distinguishes hedging in the sense of the THM from the more general problem of determining the optimal intertemporal demand for assets.

IV. ON THE EXCLUSIVE USE OF FUTURES FOR HEDGING

This section addresses the question of whether futures will be preferred to other traded assets as hedging instruments when transactions costs are zero. An immediate implication of the construction of the minimum-variance hedge is that, in general, neither will be preferred and both will be used together. To see this, we write out the first order conditions for Equation (10), denoting the minimum-variance hedge portfolio by v, as follows:

$$\begin{aligned} (wW)^v &= -V_{aa}^{-1}V_{af}x^v - V_{aa}^{-1}V_{a,pq} \\ x^v &= -V_{ff}^{-1}V_{fa}(wW)^v - V_{ff}^{-1}V_{f,pq} \end{aligned} \quad (11)$$

[9] $\underset{\{wW,x\}}{\text{Min}}\ (\text{Var}[d(PQ)] + W^2w'V_{aa}w + x'V_{ff}x + 2Ww'V_{a,pq} + 2x'V_{f,pq} + 2Ww'V_{a,f}x)$ provides the first order conditions:

$$V_{aa}wW + V_{af}x + V_{a,pq} = 0$$

$$V_{fa}wW + V_{ff}x + V_{f,pq} = 0, \qquad \text{so that}$$

$$\begin{pmatrix} wW \\ x \end{pmatrix}^v = -V^{-1}\begin{pmatrix} V_{a,pq} \\ V_{f,pq} \end{pmatrix} = -V^{-1}\left[\begin{pmatrix} V_{a,p} \\ V_{f,p} \end{pmatrix}Q + \begin{pmatrix} V_{a,q} \\ V_{f,q} \end{pmatrix}P\right]$$

The second term in each line of Equation (11) reveals that the nontraded position will be hedged by both traded assets and futures. The first terms in each line reveal further that the variance-minimizing demands for traded assets and futures each respectively depends on the other. It follows that futures will be the preferred hedging instrument under conditions that set the variance-minimizing demand for traded assets, $(wW)^v$, equal to zero. These conditions are summarized in the following sequence of linked propositions.

Proposition 2. If there exists for a given state variable a futures price that is perfectly correlated with it, then: (*i*) only that futures contract will be used for hedging it; and (*ii*) a portfolio of the future and other assets will, in general, be used to hedge other state variables.

Proposition 2 implies directly that, where a futures is perfectly correlated with the underlying asset, it and it alone will be used to hedge the underlying asset. The proposition generalizes readily. When there are many state variables and futures contracts, each state variable will be hedged exclusively by the one futures that is perfectly correlated with it.[10]

In general, however, the perfect correlation requirements of Proposition 2 may be violated. Random convenience yields can produce imperfect correlations between futures prices and the associated spot prices. In addition, there may exist no futures contracts at all for such variables as the weather in various locations, foreign GNPs and interest rates or, for that matter, for the quantity component of the nontraded position in our model. Under what conditions will futures be preferred over traded assets for hedging in these more general circumstances? In the presence of many futures and state variables (or nontraded positions), the answer has two parts. The first is:

Proposition 3. The hedging demand for traded assets will disappear and only futures will be used for hedging if the return on each traded asset is independent of each futures price and of each state variable, including the spot prices and quantities of each nontraded position.

Proposition 3 incorporates the possibility, discussed by Anderson and Danthine (1981), that a given nontraded position will be cross-hedged by more than one future in a world with no perfect correlations. Moreover, it generalizes their result. The hedge portfolio will include also traded assets unless the latter are independent of both the futures and the nontraded position. Finally, Proposition 3 leads naturally to the statement of the very strong conditions to be met if the routine, one-position, one-futures hedge of the THM is to arise in this case.

[10]See Breeden (1984) for the complete markets case with instantaneously maturing futures that are perfectly correlated with each and every state variable. Due to these perfect correlations and with spanning, the demand for traded securities to hedge state variables disappears. Under more general conditions where correlation is imperfect, however, the link that Breeden forges between risk aversion and the J_{y}/J_{yy} terms and his subsequent result, that the logarithmic utility separates "long" hedgers from "reverse" hedgers, both break down. Logarithmic utility does not produce myopia in this instance when the underlying cash position is nontraded, as we show in Adler and Detemple (1986).

Proposition 4. Each nontraded position will be hedged exclusively by the one, imperfectly correlated futures contract written on it if each nontraded position is correlated with only one futures price (independently of all other security prices) and each such futures price is independent of all other futures and asset prices.

To sum up this section, futures can be said to be preferred over other assets as hedging instruments in the absence of transaction-cost differentials only under conditions where the hedging-demand for traded assets disappears. The notion of futures being better for the purpose of hedging when traded assets also appear in the hedging demands is vacuous. The two sets of conditions in which futures only are used for hedging involve a trade-off. Propositions 3 and 4 allow futures to be imperfectly correlated with (the components of) the nontraded position but then require these two sources of uncertainty to be orthogonal to all other traded assets. Proposition 2 is more restrictive in that it requires perfect correlation between a futures contract and each component of the nontraded position, but is less restrictive in that it allows all sources of uncertainty to be correlated.

In practice, transaction cost differences may account for what casual empiricism suggests is a market preference for hedging with futures. To short a stock, for example, one initially borrows and sells it; puts the proceeds after paying a commission in escrow at zero interest; and simultaneously posts a 50%, interest-earning margin. Subsequently, with maintenance margins set at 30%, margin calls are satisfied by cash payments. When shorting a future, there are no initial proceeds on which interest income is lost. In other respects, the cost structure is similar: Commissions must be paid; margin of between 2% and 6% can be posted with T-bills; and margin calls are paid in cash. The main cost advantages of futures are the lower margins and the avoidance of the lost interest on short-sale proceeds, and these may be substantial. However, if transactions costs distort pricing in equilibrium in such a way as to remove the advantage of futures, investors in general will use all traded assets for hedging purposes except under circumstances like those described above.

V. THE FEASIBILITY OF ZERO-VARIANCE HEDGES

We maintain the setting of Section 3, with many traded assets, one nontraded position and no state variables other than P and Q. The question in this section is: When will minimum-variance hedges also be zero-variance, or what have been called perfect, hedges in the presence of quantity uncertainty? As anticipated, it is enough to have access to futures whose prices are perfectly correlated with P and Q, respectively.[11] This is the immediate implication of the following proposition.

[11] The possibility of perfect insurance in continuous time models where futures contracts based on each of the sources of uncertainty and perfectly correlated with these sources are available is discussed in Ho (1984). In the BEZ (1985) one-period model a perfect hedge is possible, but the hedge ratio for the exchange rate depends on the hedge ratio for the quantity. This dependence arises because the multiplicative uncertainty faced in the single-period setting cannot be linearized as in a continuous time model.

Proposition 5. A perfect hedge, which reduces the variance of changes in total wealth to zero, exists if there are futures that are perfectly correlated with changes in P and changes in Q, respectively.

It is of interest to note that the additive combination of the two futures contracts described in the proposition provides the investor with perfect insurance against the product of two variables, PQ. This follows from the fact that, in continuous-time models with diffusion processes, the local uncertainty faced is linear-additive. That is, the stochastic part of the change in the product PQ is a linear combination of the stochastic parts of the change in P and of the change in Q. As long as the investor can trade continuously, he can then locally offset each of these sources of uncertainty by the selection of an appropriate trading strategy in the two futures, thereby achieving a perfect hedge.

It is perhaps notable that Proposition 5 is unaffected by the presence of traded assets, other than the two assumed futures, that may be partially correlated with the nontraded position. This is a reflection of Proposition 2 above, which establishes that the demand for traded assets for hedging any state variable that has a perfectly correlated future written on it will be zero. As a result, the vector zero-variance demand for the two futures may be written simply as:

$$\begin{pmatrix} x \\ y \end{pmatrix}^{v=0} = -\begin{pmatrix} Q\sigma_p/\sigma_f \\ P\sigma_q/\sigma_y \end{pmatrix} \tag{12}$$

where y denotes the position in the futures that is perfectly correlated with dQ and σ_y is the standard deviation of its price, while σ_p, σ_q and σ_f are the standard deviations, respectively, of the spot price, the quantity and the price of the future that is perfectly correlated with dP.

Of course, futures contracts on the quantities of nontraded positions are, with rare exceptions, not available. The usual situation is that of having to hedge a position where, at best, there is a contract perfectly correlated with the spot price. The following proposition summarizes all the circumstances, mentioned at various points in the literature, in which a zero-variance hedge is available in this case.[12]

Proposition 6. When there is one futures contract whose price changes are perfectly correlated with dP and no other asset or futures that is perfectly correlated with dQ, a perfect hedge exists either:

(*i*) if quantity is deterministic ($\sigma_q = 0$), where the zero-variance hedge is given by $x^{v=0} = -Q\sigma_p/\sigma_f$; or

(*ii*) if quantity is random but is perfectly instantaneously correlated with the price ($\rho_{pq} = \pm 1$), where the hedge is given by $x^{v=0} = -(Q\sigma_p \pm P\sigma_q)/\sigma_f$.

[12]See, especially, Anderson and Danthine (1981) and Ho (1984).

This is perhaps the appropriate point to compare the zero-variance hedges of Proposition 6 with the "delta" hedges of the futures and options literature. A delta hedge is defined as the number of futures (or options) required to insure perfectly a unitary position in the underlying commodity or asset, where perfect insurance means that the insured position has zero variance and returns the riskless rate. Delta hedge ratios are derived assuming that $F = F(P, t)$, that is, that the futures price (and the convenience yield) is a function of at most the current spot price and time. A delta hedge then involves going short F_p^{-1} futures contracts for each unit of the commodity purchased.

To obtain the relationship between delta and zero-variance hedges, notice that when $F = F(P, t)$, $\sigma_f = F_p \sigma_p$. Consequently, the hedge ratio in case (*i*) of Proposition 6 may be written as: $(x^{v=0}/Q) = -F_p^{-1}$. When quantity is nonrandom, the zero-variance hedge is a delta hedge. In case (*ii*), where quantity is random but perfectly correlated with price, the zero-variance hedge ratio becomes: $(x^{v=0}/Q) = -F_p{}^{-1}[1 \pm (P\sigma_q/Q\sigma_p)]$. In this case, the zero-variance hedge is proportional but not equal to the delta hedge, where the proportionality constant depends on the ratio of the normalized standard deviations, σ_q/Q and σ_p/P. In general, a minimum-variance hedge that is not perfect is neither equal to nor proportional to a delta hedge.

To close this section, we explore the conditions for the zero-variance hedge ratio to be constant, that is, not to require revision, over time. Clearly, this is impossible in case (*ii*) of Proposition 6, where quantity may fluctuate, or under Proposition 5, where both P and Q may vary. In case (*i*) of Proposition 6, the hedge ratio will at least be deterministic. When the futures is priced by arbitrage, as in Black (1976), Brennan and Schwartz (1985), and Richard and Sundaresan (1981), and in the absence of a convenience yield, $F_t = P_t e^{r(T-t)}$, and the perfect-hedge ratio, $F_p^{-1} = e^{-r(T-t)}$, is at most a function of time to maturity: it is constant for Breeden-type futures that mature instantaneously.

VI. IMPLEMENTABLE HEDGES

For our final purpose, we define as implementable those hedges that are preference-free and that depend (for all preference structures) only on the measurable characteristics of the opportunity set. Under what conditions, we then ask, will preference-free hedges be optimal, that is, expected-utility maximizing, for arbitrary preferences? The answer to this question is of considerable practical interest.

The only preference-free hedge in our framework is the minimum-variance hedge. To see this, we need only rewrite Equation (8), following Equation (9), as:

$$\begin{pmatrix} wW \\ x \end{pmatrix} = -\frac{J_y}{J_{yy}} V^{-1} \begin{pmatrix} \mu_a - rl \\ \mu_f \end{pmatrix} - V^{-1} \left[\begin{pmatrix} V_{a,p} \\ V_{f,p} \end{pmatrix} \frac{J_{py}}{J_{yy}} + \begin{pmatrix} V_{a,q} \\ V_{f,q} \end{pmatrix} \frac{J_{qy}}{J_{yy}} \right] - V^{-1} \begin{pmatrix} V_{a,pq} \\ V_{f,pq} \end{pmatrix} \tag{13}$$

Clearly, the only preference-free term is the last. There is widespread recognition throughout the literature that the minimum-variance hedge is potentially nonoptimal (even for quadratic preferences). What is perhaps less well-known is the precise set of conditions under which it will be optimal.

Restated, the question can now be asked in two ways. First, what are the conditions that set both of the first two, preference-laden terms on the RHS of Equation (13) equal to zero? This is the subject of Proposition 7, below. Alternatively, can we determine conditions under which: Only futures appear in the hedge portfolio; only traded assets feature in the speculative portfolio; and the middle, Merton-Breeden hedging term on the RHS of Equation (13) drops out? This is the subject of Proposition 8. Both are proved in the appendix.

For simplicity, we maintain the following assumptions throughout this section: (a) there is one traded asset, one nontraded position, and a single futures contract; (b) there is no quantity uncertainty; (c) the futures contract promises delivery of one unit of the nontraded position and is perfectly correlated with it (i.e., $\rho_{pf} = 1$); and (d) the traded asset is distinct from the futures (i.e., $\rho_{fa} \neq \pm 1$). Under these assumptions, we have the following sufficient conditions.

Proposition 7. Under assumptions (1)–(4) above, the optimal demands for traded assets and futures consist only of the minimum-variance demands if:

(*i*) $\mu_f = 0$; the futures price is a pure martingale, and

(*ii*) $\mu_a = r$; the risky traded asset is dominated by the riskless asset for all risk-averse investors.

Proposition 7 merits further discussion. The condition, $\rho_{af} \neq \pm 1$, in principle permits both the traded asset and the futures to appear in the optimal demand. However, following Proposition 2 above, only the futures contract will actually be used for hedging as it is perfectly correlated with the spot price. Together, conditions (*i*) and (*ii*) reduce the speculative, mean-variance demand to zero. And, as the appendix demonstrates, they are sufficient also to set the cross-partial derivative of the value function, $J_{py} = 0$. Thus, the first two terms on the RHS of Equation (13) become equal to zero and the optimal demands, denoted by an asterisk, are given in this case by:

$$\begin{pmatrix} wW^* \\ x^* \end{pmatrix} = -\begin{pmatrix} 0 \\ Q\sigma_p/\sigma_f \end{pmatrix} \tag{14}$$

Notably, the traded asset, which is dominated in the mean-variance sense, also serves no useful hedging purpose. The hedging demand for futures, as before, can be computed as a regression coefficient, as $Q\sigma_p/\sigma_f = Q\sigma_{fp}/\sigma_f^2$ when $\rho_{fp} = 1$. Following Proposition 6, the hedge is perfect.

Notice further that conditions (*i*) and (*ii*) of Proposition 7 would be met were assets priced in the capital markets as if aggregate risk-tolerance were infinite, that is, risk-neutrally. These conditions are therefore inconsistent with any single-agent model of capital market equilibrium where the representative individual is risk-averse. They are at best consistent with a multiple-agent equilibrium in which risk-averters are dominated at the level of pricing by the presence of at least one risk-neutral agent who would end up holding all assets in positive net supply. There is sufficient evidence of risk-premia in the mean returns on capital assets in general to suggest that the two conditions of Proposition 7 are violated empirically.[13]

The question then remains: Can the minimum-variance hedging demand for futures be optimal under less restrictive circumstances? In the next proposition, the traded asset's expected returns can differ from the riskless rate: it is, therefore, not dominated. Proposition 8 identifies the condition under which there is a separation of functions: only the traded asset is held for speculative purposes; only the futures is used for hedging; and the hedging demand is implementable.

Proposition 8. Under the assumptions preceding Proposition 7 and with $\mu_a \neq r$, the optimal speculative demand contains only the traded asset and the optimal hedging demand is a minimum-variance portfolio containing only the future if: $\mu_f = (\mu_a - r)\sigma_{fa}/\sigma_a^2$.

In this case, the optimal demands for traded assets and futures are given, as the proof of the proposition in the appendix implies, by:

$$\begin{pmatrix} wW^* \\ x^* \end{pmatrix} = -\frac{J_y}{J_{yy}} \begin{pmatrix} (\mu_a - r)/\sigma_a^2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ Q\sigma_p/\sigma_f \end{pmatrix} \tag{15}$$

where, by virtue of Proposition 6, the hedge again is perfect, i.e., zero-variance.

The intuition behind Proposition 8 is straightforward. A risk-averse investor who owns a nontraded position will, in the absence of a perfect hedging instrument, diversify his portfolio in a distorted way, so as to offset the imbalance due to that position. In the limiting case where there exists a future whose returns are perfectly correlated with those of the nontraded asset, the investor first hedges by shorting the future in a manner that exactly cancels the presence of the nontraded position. He then proceeds to choose an efficient portfolio from among the remaining traded assets (or, in this case, that consists of the traded asset). Three-fund separation is achieved. The optimal portfolio consists of combinations of the riskless asset, the risky traded asset and the hedge portfolio. The condition that the future be priced so that its excess expected returns are linearly related to those of the traded asset, that is, that

[13]Here we are taking exception to the thrust of the argument in Benninga, Eldor, and Zilcha (1984) who assume "unbiased capital markets," i.e., risk neutrality, as an empirically verified proposition. Their paper also assumes "regressivity," that is, that the spot price is a linear function of the futures price plus some error term of the form: $\tilde{P}_t = a + b\tilde{F}_t + \tilde{e}_t$. This formulation is hard to motivate on the basis of existing theory. Linearity holds under arbitrage pricing but with no error. When the futures price depends on the spot price plus additional (imperfectly hedgable) state variables, there is no prior reason for a linear relationship to materialize.

$\mu_f = (\mu_a - r)\sigma_{fa}/\sigma_a^2$, is what guarantees that the future will not be held for mean-variance efficiency purposes.[14]

What is perhaps less obvious but true nonetheless is that the same pricing condition (combined with the same perfect hedging opportunity) also removes the preference-dependent component of the hedging demand for futures. Intuitively, at the point where the nontraded position is hedged perfectly and the pricing condition is met, changes in wealth are independent of the futures price and, therefore, of the nontraded position, so that $J_{py} = 0$.[15]

The discussion of Proposition 8 reveals also its fragility. Following Proposition 2 above, its desirable, function-separating properties break down in the presence of quantity uncertainty that cannot separately be covered by a perfectly correlated futures contract. More generally, one would expect that incomplete hedging opportunities would render the pricing condition of the proposition inconsistent with a capital market equilibrium with nontraded assets. In short, the conditions for implementable, variance-minimizing hedges also to be optimal are likely to be breached empirically.

VII. CONCLUDING REMARKS

Hedging in the sense of the THM does not occur in perfect financial markets. For a demand to appear, for some assets to hedge other, predesignated assets,

[14]To see this last point more precisely, note that the speculative component of the demand for the asset and the future, denoted by m, is:

$$\begin{pmatrix} wW \\ x \end{pmatrix}^m = -\frac{J_y}{J_{yy}} V^{-1} \begin{pmatrix} \mu_a - r \\ \mu_f \end{pmatrix}$$

In this two-by-two case, we also have

$$V^{-1} = D^{-1} \begin{pmatrix} \sigma_f^2 & -\sigma_{af} \\ -\sigma_{fa} & \sigma_a^2 \end{pmatrix}, \qquad \text{where} \quad D = \sigma_a^2\sigma_f^2(1 - \rho_{af}^2).$$

Substituting, the mean variance demand becomes:

$$\begin{pmatrix} wW \\ x \end{pmatrix}^m = -\frac{J_y}{J_{yy}} D^{-1} \begin{bmatrix} (\mu_a - r)\sigma_f^2 - \mu_f\sigma_{af} \\ -(\mu_a - r)\sigma_{fa} + \mu_f\sigma_a^2 \end{bmatrix} = -\frac{J_y}{J_{yy}} \begin{bmatrix} (\mu_a - r)/\sigma_a^2 \\ 0 \end{bmatrix}$$

since the condition of Proposition 8 implies that the bottom term in the square bracket equals zero and, rewritten as $\mu_f/\sigma_f = \rho_{af}(\mu_a - r)/\sigma_a$, that the top term is given by $(\mu_a - r)\sigma_f^2 - (\sigma_{af}^2/\sigma_a^2)(\mu_a - r) = (\mu_a - r)\sigma_f^2(1 - \rho_{af}^2)$.

[15]Note the proof in the appendix. Another way of seeing this is to write out the wealth budget constraint as:

$$dY = \{(Q\mu_p + x\mu_f - rPQ)dt + Q\sigma_p dz_p + x\sigma_f dz_p\} + \{(wW(\mu_a - r) + rY)dt + wW\sigma_a dz_a\}$$

where the first bracketed term is zero by arbitrage. What remains is the second bracketed expression, changes in which can be associated with the total returns on the mean-variance portfolio of traded assets. When futures are priced by the condition that $\mu_f = (\mu_a - r)\sigma_{fa}/\sigma_a^2$, these returns are independent of the moments of the futures price and, therefore, of the spot price, as footnote 13 shows. Consequently, dY is similarly independent of P.

some imperfection that causes trading discontinuities in the pre-identified assets is required.[16] Taking some positions as completely nontraded most closely conforms to the fixed-cash-position assumption of the THM. When perfect hedges are infeasible, investors will generally hedge their nontraded positions with portfolios that contain both traded assets and futures.

The only implementable hedges are minimum-variance hedges. They can be estimated by ordinary least squares (OLS) techniques provided that the regression coefficients are intertemporally constant. If the regression coefficients depend on exogenous state variables, OLS procedures at best provide an approximation and more complex statistical techniques are required. In general, minimum-variance hedges must be rebalanced continuously. Only if there is no quantity uncertainty and a perfect hedge is possible can the hedge ratio be constant. In any event, minimum or zero-variance hedges will seldom be optimal in the expected-utility-maximizing sense.

The main limitation of the model on which these results are based is, as was implied in the introduction, its assumption of complete nonmarketability for the positions that investors choose to hedge. By virtue of this assumption, this paper, along with most of the hedging literature, considers only the problem of hedging a given exposure. Over time, of course, each exposure, that is, the size of each temporarily nontraded position, can in principle be varied. Were this position instantaneously and costlessly modifiable in any direction, the hedging model would reduce once more to the general portfolio model. Only shifts in the opportunity set would be left to motivate hedging behavior as the direct hedging term, which reflects THM-type hedging, would disappear. A full theory of hedging therefore lies between the two extremes: Of the complete nontradedness of target positions that underlies the THM on the one hand, and pure portfolio theory with freely variable spot positions on the other.

A notable line of contributions, including Stein (1961), Stoll (1979), Rolfo (1980), Anderson and Danthine (1981), and Benninga, Eldor, and Zilcha (1985), all in a one-period framework; and Ho (1984) in an intertemporal model, have explored the interaction between hedging and an initial, once-off production decision. Their analyses are most clearly applicable to circumstances like those of a farmer who, once his field is cultivated and planted, may be unable thereafter to increase his acreage or liquidate part of it. However, this approach cannot capture the essence of hedging behavior in financial markets where exposures themselves (i.e., production decisions) are periodically revisable. What is required is a theory of the optimal choice of spot positions that can be adjusted, perhaps sluggishly, over time. Explicitly modelling this problem will undoubtedly require the introduction of transactions costs, short-selling constraints and stopping times. While this task is formidable, hedging theory will remain incomplete until the work is done.

[16]The assumption of nontradedness of an asset or position, however, also raises the issue of its compatibility with a continuous process for that asset, i.e., the existence of a market with continuous trading. This apparent conflict is resolved as follows. First a market with continuous trading may exist, but access to the market may be restricted for various reasons: divisibility problems, minimum transaction size requirements, discrimination among classes of investors, and so on. Second, investors may trade at discrete points in time in some markets, but the arrival process of orders to the market may still produce continuous trading. In both cases a continuous price process at the aggregate level coexists with barriers to continuous trading at the individual level.

Appendix

Proof of Proposition 7

Under the maintained assumptions (1)-(4) in the text, the demand function are:

$$\begin{pmatrix} wW \\ x \end{pmatrix} = -\frac{J_y}{J_{yy}} \begin{pmatrix} \sigma_a^2 & \sigma_{af} \\ \sigma_{fa} & \sigma_f^2 \end{pmatrix}^{-1} \begin{pmatrix} \mu_a - r \\ \mu_f \end{pmatrix} - \begin{pmatrix} 0 \\ \sigma_p/\sigma_f \end{pmatrix} \frac{J_{py}}{J_{yy}} - \begin{pmatrix} 0 \\ Q\sigma_p/\sigma_f \end{pmatrix}$$

and substituting in the Bellman equation leads to,

$$\begin{aligned} -J_t = {} & J_y[Q\mu_p + rW - c] + J_p\mu_p + 1/2 J_{pp}\sigma_p^2 + u(c, t) \\ & + 1/2 J_{yy}\sigma_p^2 Q^2 + J_{yp}\sigma_p^2 Q - 1/2\Bigg[\frac{J_y^2}{J_{yy}}(\mu_a - r, \mu_f)V^{-1}\begin{pmatrix} \mu_a - r \\ \mu_f \end{pmatrix} \\ & + 2J_y(\mu_a - r, \mu_f)V^{-1}\begin{pmatrix} \sigma_{ap} \\ \sigma_{fp} \end{pmatrix}\frac{J_{py}}{J_{yy}} + J_{yp}(\sigma_{pa}, \sigma_{pf})V^{-1}\begin{pmatrix} \sigma_{ap} \\ \sigma_{fp} \end{pmatrix}\frac{J_{py}}{J_{yy}} \\ & + J_{yy}Q^2(\sigma_{pa}, \sigma_{pf})V^{-1}\begin{pmatrix} \sigma_{ap} \\ \sigma_{fp} \end{pmatrix} + 2J_y(\mu_a - r, \mu_f)V^{-1}\begin{pmatrix} \sigma_{ap} \\ \sigma_{fp} \end{pmatrix} Q \\ & + 2J_{yp}(\sigma_{pa}, \sigma_{pf})V^{-1}\begin{pmatrix} \sigma_{ap} \\ \sigma_{fp} \end{pmatrix}\Bigg] \end{aligned}$$

Further noticing that:

$$\sigma_p^2 - (\sigma_{pa}, \sigma_{pf})V^{-1}\begin{pmatrix} \sigma_{ap} \\ \sigma_{fp} \end{pmatrix} = 0$$

$$(\mu_a - r, \mu_f)V^{-1}\begin{pmatrix} \sigma_{ap} \\ \sigma_{fp} \end{pmatrix} = (\sigma_p/\sigma_f)\mu_f$$

and that by arbitrage:

$$\mu_p - (\sigma_p/\sigma_f)\mu_f = rP$$

we obtain:

$$-J_t = J_y(rY - c) + J_p\mu_p + 1/2 J_{pp}\sigma_p^2 + u(c, t)$$

$$- 1/2 \frac{J_y^2}{J_{yy}} (\mu_a - r, \mu_f) V^{-1} \begin{pmatrix} \mu_a - r \\ \mu_f \end{pmatrix} - J_y(\sigma_p/\sigma_f)\mu_f \frac{J_{py}}{J_{yy}}$$

$$- 1/2 J_{yp}(\sigma_{pa}, \sigma_{pf}) V^{-1} \begin{pmatrix} \sigma_{ap} \\ \sigma_{fp} \end{pmatrix} \frac{J_{py}}{J_{yy}}$$

Now the mean variance demands are zero if:

(C1)
$$\begin{cases} \sigma_f^2(\mu_a - r) - \sigma_{fa}\mu_f = 0 \\ -\sigma_{fa}(\mu_a - r) + \sigma_a^2\mu_f = 0 \end{cases}$$

and the value function J is of the separable form $J[Y, P, t] = \psi(Y, t) + \phi(P, t)$ (so that $J_{yp} = 0$) if:

$$(\mu_a - r, \mu_f) V^{-1} \begin{pmatrix} \mu_a - r \\ \mu_f \end{pmatrix} \text{ is independent of } P.$$

and there exist two functions ψ and ϕ satisfying

(C2)
$$-\psi_t = \psi_y(rY - c) + u(c, t) - 1/2 \frac{\psi_y^2}{\psi_{yy}} (\mu_a - r, \mu_f) V^{-1} \begin{pmatrix} \mu_a - r \\ \mu_f \end{pmatrix}$$

$$-\phi_t = \phi_p\mu_p + 1/2\phi_{pp}\sigma_p^2$$

$$\phi(P, T) = 0; \qquad \psi(Y, T) = B[Y, T] \quad \text{where} \quad c = u_c^{-1}(\psi_y).$$

Now if conditions (*i*) and (*ii*) of the proposition are satisfied, we immediately get (C1) and the first condition of (C2) holding. Thus, the result is obtained if in addition there exists a solution ψ and ϕ of the system:

$$-\psi_t = \psi_y(rY - c) + u(c, t)$$

$$-\phi_t = \phi_p \mu_p + 1/2 \phi_{pp} \sigma_p^2$$

$$\phi(P, T) = 0; \qquad \psi(Y, T) = B[Y, T]$$

$$\text{where} \quad c = u_c^{-1}(\psi_y)$$

In the statement of the proposition we omit these existence conditions. That is, we implicitly suppose that there exists a space of utility functions, U, for which these conditions are met.

Proof of Proposition 8

Following the proof of proposition 7, we need the conditions:

(C′1) $$-\sigma_{fa}(\mu_a - r) + \sigma_a^2 \mu_f = 0$$

and (C2) to be satisfied. Under the condition of the proposition, (C′1) is trivially satisfied and the first condition of (C2), namely:

$$[(\mu_a - r)^2 \sigma_f^2 - 2\sigma_{af}(\mu_a - r)\mu_f + \mu_f^2 \sigma_a^2][\sigma_a^2 \sigma_f^2 (1 - \rho_{af}^2)]^{-1} \quad \text{independent of } P$$

becomes

$$[(\mu_a - r)^2 / \sigma_a^2] \quad \text{independent of } P$$

This condition is clearly satisfied. Assuming away the existence problem mentioned earlier completes this proof of Proposition 8.

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