

# A Method for Staffing Large Call Centers Based on Stochastic Fluid Models

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We consider a call center model with  $m$  input flows and  $r$  pools of agents; the  $m$ -vector  $\lambda$  of instantaneous arrival rates is allowed to be time dependent and to vary stochastically. Seeking to optimize the trade-off between personnel costs and abandonment penalties, we develop and illustrate a practical method for sizing the  $r$  agent pools. Using stochastic fluid models, this method reduces the staffing problem to a multidimensional newsvendor problem, which can be solved numerically by a combination of linear programming and Monte Carlo simulation. Numerical examples are presented, and in all cases the pool sizes derived by means of the proposed method are very close to optimal.

*Key words:* capacity sizing; call centers; fluid analysis; multidimensional newsvendor; nonstationarity; queueing; random environment; stochastic programming

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## 1. Introduction

From an operations research (OR) perspective, the two central problems of telephone call center management are (a) the assignment of agents to work schedules, which we call the *staff scheduling problem*, and (b) the dynamic routing of calls to agents given system status, which we call the *dynamic routing problem*. This paper is directly concerned with the first of those problems, and because the two problems are inextricably linked, it is indirectly concerned with the second one as well.

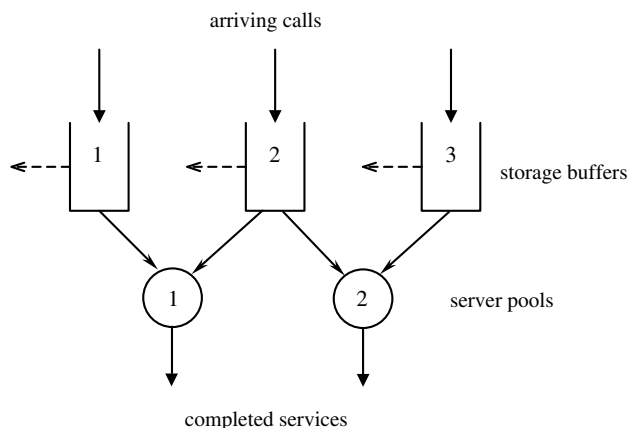
As usual in OR studies, we view a call center as a queueing system, frequently referring to callers as “customers” and to call center agents as “servers.” The general model that we adopt has  $m$  customer classes and  $r$  server pools. Server pool  $k$  consists of  $b_k$  interchangeable agents ( $k = 1, \dots, r$ ), whose capabilities will be described shortly. Customers of the various classes arrive randomly over time, and those who cannot be served immediately wait in a (possibly virtual) infinite-capacity buffer that is dedicated to their specific class. An example with  $m = 3$  customer classes and  $r = 2$  server pools is portrayed schematically

in Figure 1; buffers are represented by open-ended rectangles and server pools by circles. An important assumption of our model is that customers of any given class will abandon their calls if forced to wait too long before commencement of service; abandoned calls are represented by the horizontal dotted arrows emanating from the storage buffers in Figure 1. Our assumptions regarding speed of abandonment will be explained later.

The servers in a given pool may be cross-trained to handle customers of several different classes, and by the same token, several pools may be able to handle a given customer class. For the example portrayed in Figure 1, each server pool can handle two of the three customer classes; customers of Class 2 can be served by either pool, but each of the other two classes can be served by just one pool. In general, we allow the service time distribution of a customer to depend on both the customer’s class and on the pool from which the server comes.

The dynamic routing problem referred to earlier is the following. First, when a customer arrives and one or more idle servers can handle that customer’s class,

**Figure 1** A Schematic Model of a Call Center with Three Customer Classes and Two Agent Pools



the system manager must choose between routing the customer immediately to one of them versus putting the customer into buffer storage for later disposition. If the customer is routed immediately, there may be a further choice regarding the server pool to which it will be routed. Second, each time a server completes the processing of a customer, and there exist waiting customers of one or more classes that the server can handle, then the system manager must choose between routing one of those customers to the server immediately versus idling the server in anticipation of future arrivals. These resource allocation decisions are conditioned on system status information at the time of the choice, including the number of customers waiting in the various buffers and the number of servers that are idle in the various pools.

Generally speaking, the system manager wants to route calls to servers that can handle them most efficiently, but must also keep in mind the full spectrum of work to be done and the relative advantage of the different server pools in doing different kinds of work. Of course, it is skills acquired through training and experience that determine which classes of calls a given server pool can handle, and how efficiently it can handle them. Thus, the problem laid out in the previous paragraph is often referred to as one of *skills-based routing*; see Gans et al. (2003, §5.1) for further discussion of skills-based routing.

In describing a call center and the associated dynamic routing problem, we have suppressed virtually all physical detail. A recent survey paper by

Gans et al. (2003) explains some of the technological reality that lies behind a standard queueing model of the kind employed here, and further provides a good account of the various problems involved in call center management. As those authors emphasize, capacity management is a matter of *hierarchical* decision making: Decisions about hiring, training, and retention determine personnel levels over relatively long time spans; those personnel levels constrain staff scheduling decisions that fix pool sizes over intermediate time spans; and then pool sizes constrain dynamic routing decisions that are made and revised over short time spans.

Ignoring the first or highest level of that hierarchy, we shall address the following, somewhat stylized version of the second-level problem in the body of this paper, explaining afterward how the analysis can be extended to recognize more of the fine structure that characterizes real-world staff scheduling. First, the decision variables in our formulation are the pool sizes  $b_1, \dots, b_r$  identified earlier, which we treat as continuous variables. The treatment of pool sizes as continuous variables reflects our primary focus on *large* call centers; see §3 for further discussion.

Second, in our formulation of the staff scheduling problem, a system manager must determine in advance the capacity vector  $b = (b_1, \dots, b_r)$  to be employed during a specified planning period; by assumption, that decision cannot be revised as actual demand is observed during the period. (In a typical application, a day would be broken into several such planning periods.) Third, we express service-level concerns in our formulation by attaching a penalty of  $p_i$  dollars to each class  $i$  customer that abandons his or her call; as we shall explain later (see §6), the formulation can easily be extended to further incorporate a linear waiting cost for each customer class. Finally, given the personnel cost  $c_k$  associated with employing one server in pool  $k$  for the duration of the planning period ( $k = 1, \dots, r$ ), our objective is to minimize the sum of personnel costs plus expected total abandonment cost.

The crucial task in addressing this problem is to estimate *best achievable performance* with a given capacity vector  $b$ , by which we mean the smallest expected abandonment cost that can be achieved over the course of the planning period with the specified

capacity vector. Our proposed method for call center staffing is based on a linear programming estimate of best achievable performance, an estimate that appears crude but has nonetheless proved surprisingly accurate in a realistic parameter regime (see §3).

Roughly speaking, our method ignores all uncertainty and all variability in the call center environment except for that associated with average arrival rates, or average demand rates. We employ a very general model of call center demand in which the  $m$ -vector  $\lambda$  of average arrival rates for the various customer classes (expressed in units like calls per minute) is both temporally and stochastically variable; that is, we view  $\lambda$  itself as a stochastic process. As Gans et al. (2003) acknowledge in §4.4 of their survey paper, such a view is realistic, although most published papers on both call center staffing and dynamic routing treat average arrival rates as known and constant over the relevant planning period. (Moreover, to the best of our knowledge, all commercial software products that support those functions are based on similar modeling assumptions.)

To repeat, we take the view that temporal and stochastic variability of average demand rates, over a time span that is appropriate for staff scheduling purposes, are not only significant, but actually dominant. We treat all other sources of variability as essentially negligible by comparison. (To capture that perspective mathematically, we adopt a stochastic fluid model in which arrivals and departures both appear as continuous fluid flows.)

The previous paragraphs suggest that our method for call center staffing stands in contrast to other methods that have been proposed in the literature. That statement is somewhat deceptive, however, because there *is* no literature on staffing methods with multiple pools and multiple customer classes, except for the fragmentary results reported in §5 of Gans et al. (2003). The correct statement is that when one specializes our method to the simple case of one customer class and one server pool, which is essentially the only setting considered in the current literature, it is clearly distinguished from earlier work. A striking virtue of our method is that it applies to the general multiclass-and-multipool problem, which Gans et al. (2003) characterize in §5 of their survey paper as far beyond the reach of current theory.

### Existing Modeling Approaches and Related Work.

As indicated above, studies of call center staffing have focused on the case of a single pool of homogeneous agents. Basic queueing models, in particular, the Erlang-C formula for the  $M/M/N$  queueing model (see Gans et al. 2003), provide the main mathematical analysis tool in that setting. A widely used rule-of-thumb that emerges from the Erlang-C formula is the *square-root safety staffing rule*, cf. Kolesar and Green (1998), which recommends a server pool size of the form  $N = R + \beta\sqrt{R}$ , where  $R$  is the nominal incoming load measured in Erlangs. This relationship apparently dates back to early work of Erlang (1924), see Erlang (1948) and Gans et al. (2003, §4.1.1) for further discussion and references, and Halfin and Whitt (1981) for a more rigorous justification using diffusion limits. A recent asymptotic analysis of the staffing problem in the context of the  $M/M/N$  single-class/single-pool call center model was carried out by Borst et al. (2004). They refine the square-root rule by optimizing over  $\beta$  to balance queueing and staffing costs. Garnett et al. (2002) extend the square-root staffing principle to account for abandonments, while Jennings et al. (1996) adjust this formalism to account for nonstationary demand using infinite-server approximations. All of these results pertain to the single-class/single-pool Markovian queueing model. In the context of temporally varying demand, heuristics such as *pointwise stationary approximations* (see Green and Kolesar 1991) are often used. Fluid limits, which take a macroscopic view of the system dynamics, provide a more rigorous analysis framework for nonstationary queueing systems; see Mandelbaum et al. (1998) for a treatment of Markovian service networks that are inspired by call center models, and Changa et al. (2003), which studies a web-service system with transient overload and uncertain demand using stochastic fluid models. Effects of uncertainty and nonstationarity are discussed in Chen and Henderson (2001), and Atlason et al. (2004) uses simulation and cutting plane methods to optimize costs subject to service-level constraints.

**Planning Periods and Objective Function.** Two important elements of our problem formulation deserve further emphasis. With regard to the critical notion of “planning period,” we implicitly assume

that one or both of the following conditions prevail: (a) the staffing decision for any given planning period must be finalized well before the period actually begins, so there is significant uncertainty at the time of the decision about the average arrival rates that will apply; (b) planning periods cannot be made so short (that is, staff levels cannot be changed so often) that temporal variation in average arrival rates within a planning period is negligible. Thus, in the typical situation that we envision, a call center will be either clearly overstaffed or clearly understaffed most of the time, even with optimal decision making, because the system manager lacks the ability to finely tune capacity in response to observed demand; one might say that our modeling assumptions constitute a reduced-expectations view of system management. Based on conversations with industry professionals, we believe this view is consistent with practical reality, but we have no hard evidence to support that belief.

The system manager's objective in our formulation is to minimize the sum of staffing costs and expected abandonment penalties for the various customer classes. That is not the standard view of optimal staffing in the call center industry. Call center managers are much more comfortable with planning based on percentiles of the waiting time distribution. For example, a commonly stated goal is to minimize staffing costs subject to the constraint that 80% of calls should be answered in 10 seconds or less; with two customer classes, it might be specified that 90% of Class 1 customers wait less than 10 seconds, and 80% of Class 2 wait less than 20 seconds. At least on the surface, it is much more difficult to minimize staffing costs subject to an array of such performance constraints than to minimize the objective function that we propose. In our formulation, abandonment penalties serve as surrogates for the system manager's performance goals. Our hope and belief is that by iteratively adjusting the abandonment penalties, one can eventually derive a policy that makes a reasonable trade-off between staffing costs and customer service, and between service goals for the different classes. To put this in another way, we proceed under the assumption that performance goals can be adequately "dualized" by a suitable choice of abandonment penalties.

**The Remainder of the Paper.** In §2 we describe our method, emphasizing data inputs and computational mechanics rather than model assumptions. Readers will find that our specification of the method is less than fully detailed. In a sense, we reduce one problem to another, without saying exactly how the latter is to be solved, but the missing elements can reasonably be characterized as discretionary technical details. In §3 we provide supporting logic for the proposed method, including a description of the parameter regime in which we expect it to work well. In the course of that discussion, two mathematical conjectures regarding limit theory are advanced in rough form, but we make no attempt to justify our method in a rigorous mathematical sense, or even to specify with complete precision just what our model of a call center is.

Section 4 is devoted to the simple case with one customer class and one server pool, its goals being to give readers a clearer understanding of our method's essential character and to make connections with existing literature. In §5 we present a family of closely related numerical examples that all have the following crucial and relatively rare property: There exists an obvious "dominant strategy" for dynamic routing, and so one can use brute-force simulation to determine a nearly optimal staffing plan for each example, then compare its total cost against that achieved by our method. (One cannot make such a comparison for an arbitrary example because one does not know the optimal dynamic routing policy given pool sizes.) In all cases considered, the vector of pool sizes determined by our method is nearly optimal, and our seemingly crude estimate of best achievable performance is very accurate as well. Section 6 contains some concluding remarks, including refinements of our method (described in broad outline) to account for aspects of real-world staff scheduling that are ignored in the body of the paper. That section also identifies several obvious directions for further research.

## 2. Description of the Proposed Staffing Method

To describe server capabilities in our call center model, we shall use the previously established notion of processing "activities" as in Harrison and Lopez

(1999); see also Harrison (2002) for a broader discussion of this concept and its role in stochastic systems theory. A total of  $n$  processing activities are available to the system manager in our general call center model; each corresponds to agents from one particular pool serving customers of one particular class. (Thus, the total number of activities is  $n = 4$  for the system portrayed in Figure 1.) For each activity  $j = 1, \dots, n$  we denote by  $i(j)$  the customer class being served, by  $k(j)$  the server pool involved, and by  $\mu_j$  the associated mean service rate (that is, the reciprocal of the mean of the associated service time distribution).

Let  $R$  and  $A$  be an  $m \times n$  matrix and an  $r \times n$  matrix, respectively, defined as follows: For each  $j = 1, \dots, n$  set  $R_{ij} = \mu_j$  if  $i = i(j)$  and  $R_{ij} = 0$  otherwise, and set  $A_{kj} = 1$  if  $k = k(j)$  and  $A_{kj} = 0$  otherwise. Thus, one interprets  $R$  as an input-output matrix, precisely as in Harrison and Lopez (1999). Its  $(i, j)$ th element specifies the average rate at which activity  $j$  removes class  $i$  customers from the system. Also,  $A$  is a capacity consumption matrix as in Harrison and Lopez (1999). Its  $(k, j)$ th element is 1 if activity  $j$  draws on the capacity of server pool  $k$  and is 0 otherwise. In addition to the matrices  $R$  and  $A$ , our method for call center staffing requires as data the vector  $p = (p_1, \dots, p_m)$  of penalty rates, and the vector  $c = (c_1, \dots, c_r)$  of personnel costs (see §1). The only other input required is a probability distribution  $F$  on  $\mathbb{R}_+^m$  that is associated with the demand process; this will be explained in the paragraphs that follow.

In the following discussion of demand modeling, time  $t = 0$  represents the start of the planning period and time  $t = T$  is its end. For the sake of concreteness we shall speak initially in terms of a doubly stochastic Poisson model of demand, which means the following. There is given a stochastic process  $\Lambda = (\Lambda(t): 0 \leq t \leq T)$  taking values in  $\mathbb{R}_+^m$ , and given that  $\Lambda(t) = \lambda = (\lambda_1, \dots, \lambda_m)$ , the conditional distribution of arrivals in customer classes  $1, \dots, m$  immediately after time  $t$  is that of independent Poisson processes with average arrival rates  $\lambda_1, \dots, \lambda_m$  respectively ( $0 \leq t < T$ ). It is the distribution of the stochastic process  $\Lambda$  with which we shall be concerned.

For each  $\lambda \in \mathbb{R}_+^m$  and  $b \in \mathbb{R}_+^r$  let us denote by  $\pi^*(\lambda, b)$  the optimal objective value of the following linear program (LP): choose an  $n$ -vector  $x$  to

$$\text{minimize } \pi = p \cdot (\lambda - Rx) \quad (1)$$

subject to

$$Rx \leq \lambda, \quad Ax \leq b, \quad \text{and } x \geq 0. \quad (2)$$

This LP problem represents what might be called a *local fluid version* of the system manager's dynamic scheduling problem. Attaching to  $\lambda$  and  $b$  the same meanings as before, one interprets  $x$  as the number of servers dedicated to activity  $j$  in the immediate future ( $j = 1, \dots, n$ );  $Rx$  is the vector of output rates from the various customer classes generated by that program of activities, and  $Ax$  is a vector whose components show how many servers in the various pools are occupied (as opposed to idle). Additional interpretation and motivation of the LP problem (1)–(2) will be provided in the next section.

The initial specification of our proposed method for call center staffing, to be simplified shortly, is the following. Choose the capacity vector  $b$  to

$$\text{minimize } c \cdot b + \mathbb{E} \left\{ \int_0^T \pi^*(\Lambda(t), b) dt \right\}, \quad (3)$$

where  $\mathbb{E}\{\cdot\}$  denotes expected value over possible realizations of the stochastic process  $\Lambda$ . Again, the reasoning that supports this recommendation is delayed until the next section. However, because the initial term  $c \cdot b$  in the objective function (3) represents the total personnel cost associated with capacity vector  $b$ , readers may have already inferred that the second term in (3) is the LP-based estimate of best achievable performance that was referred to in §1. This is indeed the case.

To recast the optimization problem (3) in a standard form, let us define the cumulative distribution function

$$F(\lambda) := \frac{1}{T} \int_0^T \mathbb{P}\{\Lambda(t) \leq \lambda\} dt \quad \text{for } \lambda \in \mathbb{R}_+^m. \quad (4)$$

One interprets  $F(\lambda)$  as the expected fraction of time (within the planning period under study) during which  $\Lambda(\cdot) \leq \lambda$ . It is now an elementary exercise to prove that (3) is equivalent to the following (if  $\Lambda$  is a finite-valued process, this is just a matter of definition, and then one can use monotone, finite-valued approximations to establish the general equivalence):

$$\text{minimize } c \cdot b + T \int_{\mathbb{R}_+^m} \pi^*(\lambda, b) dF(\lambda) =: \phi(b). \quad (5)$$

In the literature of stochastic programming, this kind of problem is called a *two-stage LP with recourse*. At the first stage a system manager chooses the capacity vector  $b$  and incurs cost  $c \cdot b$ ; then a random demand vector  $\lambda$  with distribution  $F$  is observed, and given that observation, the system manager chooses at the second stage a vector  $x$  of activity levels that solve the linear programming problem (1)–(2). The particular kind of two-stage problem embodied in (5) is sometimes called a *multidimensional newsvendor problem*; see, e.g., the recent survey by Van Mieghem (2003).

With regard to numerical solution techniques, two distinct computational approaches appear in the literature of stochastic programming, cf. Birge and Louveaux (1997). First, various exact methods can be used when the distribution  $F$  concentrates its mass on a relatively small number of points. Second, approximate methods based on Monte Carlo simulation can be used in the general case. To elaborate on the latter approach in our particular context, observe that  $\pi^*(\lambda, \cdot)$  is convex for each fixed  $\lambda$  (this is a standard result in linear programming theory), which directly implies the following.

**PROPOSITION 1.** *The minimand  $\phi$  in (5) is a convex function on  $\mathbb{R}_+^r$ .*

Given that our problem is one of convex optimization, the gradient-descent method can be used for its numerical solution, with Monte Carlo simulation providing the means to estimate  $\nabla\phi(b)$  for each trial value of  $b$ ; see Shapiro (2003) for a recent survey of such methods. This approach will be applied to a small-scale example in §5, but it looks to be practical for problems of the size encountered in real call center applications.

As stated earlier, the central feature of our call center model is the assumption of a *random demand environment*, meaning that the vector of average or expected arrival rates is itself viewed as a stochastic process  $\Lambda$ . We have thus far spoken in terms of doubly stochastic Poisson arrivals, but the Poisson assumption has not actually been used. The method that we propose for call center staffing does not depend on the fine stochastic structure of customer arrival processes given  $\Lambda$  (it would make no difference, for example, if arrival streams were correlated

given  $\Lambda$ ) because we treat routine stochastic variability given  $\Lambda$  as insignificant compared to variations in  $\Lambda$  itself.

Thus far, nothing has been said about how to estimate the distribution  $F$ , which summarizes all that is relevant about demand variability for purposes of our method, from operational data in a given call center environment. That could be the subject of a paper by itself, and we cannot claim to have even thought through the various alternatives systematically as yet. However, one relatively simple and widely applicable approach will be described in §6.

### 3. Supporting Logic

To motivate the staffing method described in §2, we need to justify the second term in (3) as a reasonable estimate of best achievable performance for a given capacity vector  $b$ . First, more must be said about the abandonment mechanism in our call center model. We assume there exist *abandonment rates*  $\gamma_1, \dots, \gamma_m > 0$  such that, when there are  $q_i$  customers waiting for service in the class  $i$  buffer at time  $t$ , the expected number of class  $i$  abandonments in the interval  $(t, t + h)$  is approximately  $(\gamma_i q_i)h$  for small  $h > 0$  ( $i = 1, \dots, m$ ). This may be viewed as a consequence of the following more detailed assumption, which is standard in call center modeling, see Garnett et al. (2002), Harrison and Zeevi (2004), and Gans et al. (2003). There is associated with each class  $i$  caller an exponentially distributed random variable  $\tau$  that has mean  $1/\gamma_i$ , and the customer will abandon the call when his or her waiting time (exclusive of service time) reaches a total of  $\tau$  time units.

To justify the performance estimate embodied in (3), let us first consider a scenario where the vector  $\lambda$  of average arrival rates is known and constant (that is, not time-varying) and the time horizon for the dynamic routing problem is infinite. Assuming that the arrival rates  $\lambda_i$  are large enough to make large pool sizes (on the order of tens, say) economically desirable, the law of large numbers provides a rough justification for the following *fluid model* of dynamic routing, cf. Maglaras (2000). First, the total number of class  $i$  customers at any given time  $t$  is modeled as a continuous variable  $z_i(t) \geq 0$ , and the state of the system at time  $t$  is represented by the vector  $z(t) = [z_1(t), \dots, z_m(t)]$ . Second,

having observed  $z(t)$ , the system manager chooses a control vector  $x(t) = [x_1(t), \dots, x_n(t)]$  that satisfies the constraints

$$Ax(t) \leq b, \quad Bx(t) \leq z(t), \quad \text{and} \quad x(t) \geq 0, \quad (6)$$

where  $B$  is an  $m \times n$  matrix with  $B_{ij} = 1$  if  $i = i(j)$  and  $B_{ij} = 0$  otherwise. One interprets  $x_j(t)$  as the number of servers devoted to activity  $j$  at time  $t$ ; recall that  $k(j)$  is the pool to which such servers belong, and  $i(j)$  is the class of customers that they serve when engaged in activity  $j$ . The first set of constraints in (6) requires that, for each pool  $k = 1, \dots, r$ , the total number of servers from pool  $k$  that are assigned to various activities is no larger than the number  $b_k$  that exist. The second set of constraints requires that, for each customer class  $i = 1, \dots, m$ , the total number of servers assigned to processing class  $i$  customers is no larger than the number of such customers present in the system.

Let us define the  $m$ -vector  $q(t) = z(t) - Bx(t)$ , so that  $q_i(t)$  represents the number of class  $i$  customers waiting at time  $t$ , excluding the ones being served, as above. Then  $y_i(t) := \gamma_i q_i(t)$  is the instantaneous departure rate from buffer  $i$  at time  $t$  due to abandonment. That is, setting  $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_m)$ , we define an  $m$ -vector

$$y(t) = \Gamma[z(t) - Bx(t)], \quad t \geq 0, \quad (7)$$

and then the dynamic evolution of our fluid control problem is governed by the following system of ordinary differential equations:

$$\dot{z}(t) = \lambda - Rx(t) - y(t), \quad t \geq 0. \quad (8)$$

Also, the instantaneous cost rate at time  $t$  is

$$\pi(t) := \sum_{i=1}^m p_i \gamma_i q_i(t) = p \cdot y(t), \quad t \geq 0. \quad (9)$$

To avoid technical distractions, we shall restrict attention from the outset to controls  $x(\cdot)$  for which

$$\bar{x} := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t) dt \quad (10)$$

exists. Because all the abandonment rates  $\gamma_i$  are strictly positive by assumption, it is easy to prove that

$z(\cdot)$  is bounded. Thus, integrating (10) over the interval  $[0, T]$ , dividing both sides by  $T$  and letting  $T \rightarrow \infty$ , one has

$$\frac{1}{T} \int_0^T y(t) dt \rightarrow (\lambda - R\bar{x}) =: \bar{y} \quad \text{as } T \rightarrow \infty, \quad (11)$$

and the long-run average cost rate  $\bar{\pi}$  that we are striving to minimize in our fluid control problem is given by

$$\bar{\pi} := p \cdot \bar{y} = p \cdot (\lambda - R\bar{x}). \quad (12)$$

Let  $\pi^*(\lambda, b)$  be the optimal objective value of the linear programming problem (1)–(2) as in §2. Because  $y(t) \geq 0$  for all  $t \geq 0$ , and hence  $\bar{y} \geq 0$ , we have from (11) that  $R\bar{x} \leq \lambda$ , and (6) implies  $A\bar{x} \leq b$  and  $\bar{x} \geq 0$ . That is,  $\bar{x}$  is a feasible solution for the LP problem (1)–(2). Thus, under any admissible control  $x(\cdot)$  one has

$$\bar{\pi} \geq \pi^*(\lambda, b). \quad (13)$$

Fixing an optimal solution  $x^*$  of the LP problem (1)–(2), it is easy to construct from  $x^*$  an admissible control that achieves the lower bound in (13). In verbal terms, one simple way to accomplish this is the following. First, for each  $j = 1, \dots, n$  we permanently dedicate  $x_j^*$  servers from pool  $k(j)$  to activity  $j$ , which means that those servers are permanently dedicated to processing class  $i(j)$  customers. Any servers who are *not* permanently dedicated in that first step remain permanently idle. Second, at each time  $t$  we match or assign as many customers as possible to servers that have been dedicated to that class, and it is immaterial just how that matching is done.

To express this mathematically, let us focus specifically on customer Class 1. Let the  $n$  activities be numbered in such a way that activities  $j = 1, \dots, \ell$  are the only ones that have  $i(j) = 1$  and  $x_j^* > 0$ . That is, activities  $1, \dots, \ell$  are precisely the ones to which servers are permanently dedicated for the processing of Class 1 customers. Let

$$f_j(z_1) := [z_1 - (x_1^* + \dots + x_{j-1}^*)]^+ \wedge x_j^* \quad \text{for } j = 1, \dots, \ell \quad \text{and } z_1 \geq 0, \quad (14)$$

with  $f_j(\cdot) = 0$  for all  $j \in \{\ell + 1, \dots, n\}$  such that  $i(j) = 1$ . Now we take

$$x_j(t) = f_j(z_1(t)), \quad t \geq 0, \quad (15)$$

for all  $j \in \{1, \dots, n\}$  such that  $i(j) = 1$ . In words, this means the following. Of the  $z_1(t)$  Class 1 customers who are present at time  $t$ , we first allocate as many as possible to the  $x_1^*$  servers from pool  $k(1)$  who are dedicated to Activity 1, then allocate as many as possible to the  $x_2^*$  servers from pool  $k(2)$  who are dedicated to Activity 2, and so on through the allocation of Class 1 customers to servers from pool  $k(\ell)$  who are dedicated to activity  $\ell$ ; if not all Class 1 customers have been allocated to a server at that point, the remainder wait in buffer storage. The assignment of customers first to servers from pool  $k(1)$ ,  $\dots$ , and last to those from pool  $k(\ell)$  is arbitrary. What matters for the argument below is that we assign or match as many customers as possible to dedicated servers.

It is easy to verify that, under the control strategy just described, the general relationship (8) specializes to give

$$\dot{z}_1(t) = g_1(z_1(t)), \quad t \geq 0, \quad (16)$$

where  $g_1(\cdot)$  is piecewise linear and strictly decreasing on  $[0, \infty)$  with the following properties:

$$g_1(0) = \lambda_1 > 0, \quad (17)$$

$$g_1(z_1^*) = 0, \quad \text{where } z_1^* = (x_1^* + \dots + x_\ell^*) + \frac{1}{\gamma_1}(\lambda - Rx^*), \quad (18)$$

$$g_1(z_1) = -\gamma_1(z_1 - z_1^*) \quad \text{for } z_1 \geq z_1^*. \quad (19)$$

It follows that  $z_1(t) \rightarrow z_1^*$  as  $t \rightarrow \infty$ ; in fact,  $z_1(t) \uparrow z_1^*$  if  $z_1(0) \leq z_1^*$  and  $z_1(t) \downarrow z_1^*$  if  $z_1(0) \geq z_1^*$ . For each  $j = 1, \dots, \ell$  we have  $f_j(z_1^*) = x_j^*$ , implying that  $x_j(t) \rightarrow x_j^*$  as  $t \rightarrow \infty$ , and thus  $\bar{x}_j = x_j^*$ .

Repeating the construction and argument in identical fashion for other customer classes, one obtains  $\bar{x}_j = x_j^*$  for all  $j = 1, \dots, n$ , and hence  $\bar{\pi} = (\lambda - Rx^*) = \pi^*(\lambda, b)$  by (11) and (12). In words,  $\pi^*(\lambda, b)$  represents the best achievable performance (that is, the smallest achievable long-run average cost rate) in our fluid approximation to the steady-state dynamic scheduling problem with  $\lambda$  known and constant.

To further develop our justification for the objective function in (3), consider a scenario where sample paths of the stochastic process  $\Lambda(\cdot)$  are constant over one-hour intervals within the planning period, but the value of  $\Lambda(\cdot)$  that will obtain over each such interval is unknown at the time when pool sizes must be

decided. Let us further suppose that the average service rates  $\mu_1, \dots, \mu_n$  and average abandonment rates  $\gamma_1, \dots, \gamma_m$  are all about 60 per hour, which means that the average service time for each customer class is something close to one minute, and so is the average time that a customer will wait before abandoning a call. In this situation, our dynamic routing problem evolves on a much faster time scale than does the vector of average demand rates. Given the value  $\lambda$  taken on by  $\Lambda(\cdot)$  at the beginning of a one-hour interval, and assuming as before that total flow rates are large enough to justify a fluid approximation, one is led to approximate the minimum average cost rate achievable over the interval by the steady-state performance estimate  $\pi^*(\lambda, b)$ .

Generalizing that piecewise-constant demand scenario in the obvious way, we assume that average service rates  $\mu_j$  and average abandonment rates  $\gamma_j$  are large relative to the time scale on which demand changes occur. Thus, one can reasonably employ a steady-state approximation for the lowest cost rate that is achievable in the dynamic scheduling problem given the value of  $\Lambda(\cdot)$  that pertains at any given time. Further, assuming that call volumes are adequate to justify a fluid approximation for the problem of steady-state performance estimation given that  $\Lambda(\cdot) = \lambda$ , one arrives at (3).

There are two standard means of bolstering user confidence in approximations like ours. The first is to analyze numerical examples; that course will be followed in §§4 and 5. Second, one may strive to prove that the proposed approximation is in some sense “asymptotically optimal” in a limiting parameter regime. We shall not attempt such an analysis here, but it may be worthwhile to at least articulate in a concrete manner the form of such a result. (For further details and a rigorous derivation see Bassamboo et al. 2004.) Starting with a single call center model of the kind described in §1, let us denote by  $b^*$  an optimal solution of the minimization problem (3), and let  $f: \mathbb{R}_+ \mapsto \mathbb{R}_+$  be any nondecreasing and super-linear function (that is,  $\kappa^{-1}f(\kappa) \rightarrow \infty$  as  $\kappa \rightarrow \infty$ ). Now consider a cognate model defined as follows. First, all service and abandonment processes are accelerated by a uniform factor of  $\kappa > 1$ . Second, the original cost vector  $c$  is replaced by  $\kappa c$ , which means that the effective cost of capacity, expressed in terms of potential



services per time unit, remains constant. Finally, all arrival processes are accelerated by a factor of  $f(\kappa)$ , meaning that  $\Lambda(\cdot)$  is replaced by  $f(\kappa)\Lambda(\cdot)$ .

It is easy to verify that in the cognate model, the vector of pool sizes recommended by our method is  $\kappa^{-1}f(\kappa)b^*$ . That is, the original capacity choice  $b^*$  is scaled up by a factor of  $f(\kappa)$  to reflect the acceleration of arrival processes, but then scaled down by a factor of  $\kappa$  to reflect the acceleration of service processes. In Bassamboo et al. (2004) it is shown that this choice is asymptotically optimal in the following sense: both the expected total cost (over the planning period being analyzed) with our proposed vector of pool sizes, and the *minimum* expected total cost achievable with any choice, are asymptotic to  $f(\kappa)\phi(b^*)$  as  $\kappa \rightarrow \infty$ , where  $\phi(\cdot)$  is defined for the original model via (8) and (9). This statement corresponds to what is called *fluid-scale asymptotic optimality* in the literature of applied probability, cf. Maglaras (2000) and Meyn (1997).

#### 4. A Special Case: Homogeneous Customers and Agents

Let us consider now the special case with a single customer class ( $m = 1$ ) and a single agent pool ( $r = 1$ ). There is then a single processing activity ( $n = 1$ ), and we denote by  $\mu > 0$  the associated average service rate. Recalling that  $T$  denotes the length of planning horizon, we assume that

$$c < T\rho\mu. \quad (20)$$

That is, the cost to employ one server for the length of the planning period is less than the expected total abandonment cost that the server can prevent if continuously busy. (If this inequality did not hold, the optimal pool size would be  $b^* = 0$ .)

In the current context, one can solve the LP problem (1)–(2) by inspection; the optimal solution is  $x^* = \lambda \wedge (b\mu)$ , and the optimal objective value is

$$\pi^*(\lambda, b) = p(\lambda - x^*) = p(\lambda - b\mu)^+. \quad (21)$$

Of course,  $F$  is now a probability distribution on  $[0, \infty)$ , and using (21) one can express the objective function  $\phi$  in our optimization problem (5) as

$$\begin{aligned} \phi(b) &= cb + T\rho \int_0^\infty (\lambda - b\mu)^+ dF(\lambda) \\ &= cb + T\rho \int_{b\mu}^\infty (1 - F(\lambda)) d\lambda. \end{aligned} \quad (22)$$

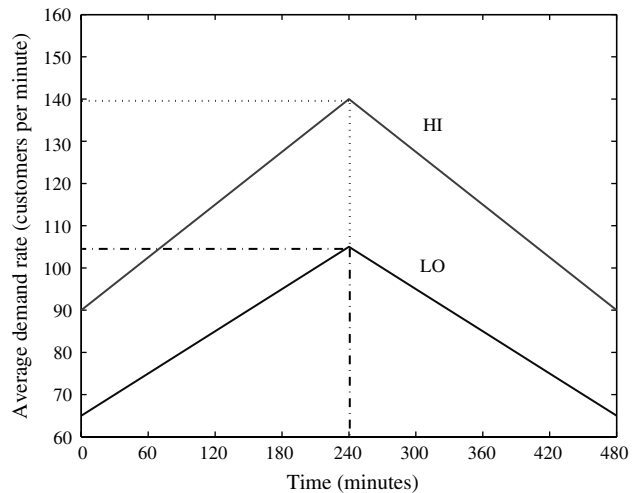
Assuming for simplicity that  $F$  is a continuous (atomless) distribution, one can differentiate (22) with respect to  $b$  and set the derivative equal to zero to obtain the following characterization of the optimal pool size  $b^*$ :

$$F(b^*\mu) = 1 - \frac{c}{T\rho\mu}. \quad (23)$$

Minimization of the objective function (22) is the classical *newsvendor problem* of operations research: If one could know the demand  $\lambda$  before choosing a capacity  $b$ , the best choice would be  $b = \lambda/\mu$ , but lacking such clairvoyance, one must choose  $b$  so as to optimize the trade-off between (linear) *overage costs* and (linear) *underage costs*; the choice which optimizes that trade-off is the *critical fractile solution* (23).

To illustrate various features of our proposed staffing method in the simple context described above, consider the artificial but illuminating demand scenario portrayed in Figure 2. In this example, the planning period is a 480-minute day (that is,  $T = 480$ ) and each day's demand is either HI or LO with equal probability; the system manager does not know which case pertains when the pool size must be set. In each of those cases, the average arrival rate grows and then dissipates during the course of a day according to the deterministic pattern portrayed in Figure 2. Thus, the demand distribution function  $F$ , defined in (4), for this

Figure 2 Illustrative Demand Pattern



example is

$$F(\lambda) = \begin{cases} \frac{1}{2} \frac{\lambda - 65}{40} & \text{for } 0 \leq \lambda \leq 90 \\ \frac{1}{2} \left( \frac{\lambda - 65}{40} + \frac{\lambda - 90}{50} \right) & \text{for } 90 \leq \lambda \leq 105 \\ \frac{1}{2} \left( 1 + \frac{\lambda - 90}{50} \right) & \text{for } 105 \leq \lambda \leq 140, \end{cases}$$

and the overall average arrival rate is

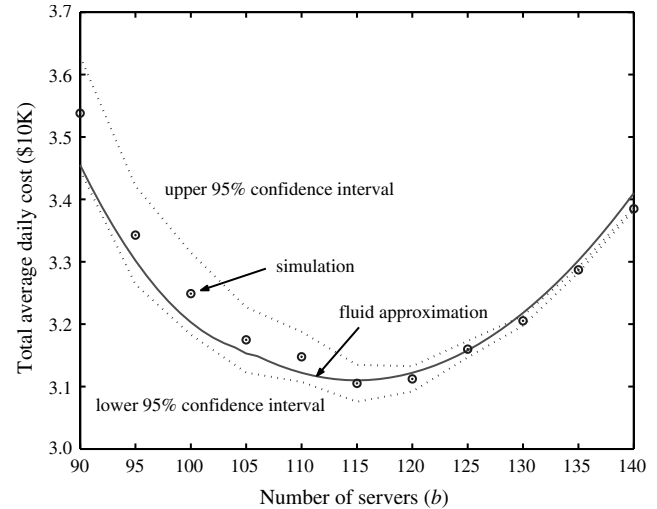
$$\bar{\lambda} = \int_0^{\infty} \lambda dF(\lambda) = 100. \quad (24)$$

The cost of employing one server for one day is taken to be  $c = \$240$ , and the abandonment penalty is  $p = \$2$  per customer. Finally, assuming the mean service rate to be  $\mu = 1$  customers per minute, the critical fractile formula (23) gives  $F(b^*) = 0.75$ , which means that for this example our staffing method dictates a pool size of  $b^* = 115$ .

To simplify the analysis of the example, we make the following assumptions. First, both service times and interabandonment times are exponentially distributed, with parameters  $\mu = 1$  and  $\gamma = 0.5$ , respectively. Thus, the average service time is one minute, and the average time that a customer will wait in the queue before abandoning is two minutes. Second, arrivals occur according to a nonhomogeneous Poisson process that has an intensity parameter  $\Lambda(\cdot)$  that evolves according to the pattern portrayed in Figure 2. Of course, no dynamic routing decisions are to be made in this simple system: Servers process customers on a first-in-first-out basis, and there is no motivation to interrupt a service once it has begun, even if one assumes that is possible.

For each of the trial values  $b = 90, \dots, 140$ , we simulated system performance over 1,000 statistically independent days, and recorded the average cost per day for each  $b$  value. Those average cost values, along with their upper and lower 95% confidence intervals, are plotted in Figure 3, where the example under discussion is identified as our “variable demand scenario,” to distinguish it from another case considered below. Also plotted in Figure 3 is the estimate of the average daily cost (as a function of  $b$ ) derived from our fluid approximation (22). Three important conclusions are to be drawn from

Figure 3 Total Average Daily Cost as a Function of the Number of Servers: Variable Demand Scenario



Note. Because of high variability in the simulation results, the plot depicts the 95% confidence interval (dotted lines).

Figure 3. First, for each  $b$  value in the range considered, our fluid-based performance estimate is accurate to within about 1%. Second, the pool size of  $b^* = 115$ , recommended by our fluid-based method, is essentially identical to the optimal staffing level obtained in the simulation study. To facilitate future discussion, the first two rows of Table 1 summarize pool size recommendations and associated daily cost estimates for our variable demand scenario, first based on the simulation study, then based on our fluid approximation. The final conclusion to be drawn from Figure 3 is that system performance is relatively insensitive to the pool size  $b$  in the neighborhood of the minimizer, according to both our simulation results and the fluid approximation. Finally, statistical variability in the simulation decreases as the pool size increases, as seen in the “shrinking” confidence intervals in the figure, because more customers are served immediately upon arrival and queueing effects decrease.

It is interesting to contrast the example just discussed with a “constant demand scenario” where  $\Lambda(\cdot)$  is identically equal to the overall average value,  $\bar{\lambda} = 100$ , which was identified earlier in (24). Figure 4 presents both the fluid-based performance estimate derived from (22) for that case, and the corresponding simulation results, again based on a sample of 1,000 independent days for each  $b$  value. (In this case the

**Table 1 Staffing Levels and System Average Daily Costs for Variable and Constant Demand Scenarios**

	Optimal pool size	Personnel cost at optimum	Average abandonment cost at optimum	Average total cost at optimum
Variable demand scenario simulation results	116	\$27,840	\$3,220	\$31,060
Variable demand scenario fluid approximation	115	\$27,600	\$3,500	\$31,100
Constant demand scenario simulation results	105	\$25,200	\$1439	\$26,639
Constant demand scenario fluid approximation	100	\$24,000	0	\$24,000

*Note.* The table depicts staffing levels and cost estimates computed by the proposed fluid approximation versus optimal results obtained via simulation.

95% confidence intervals are so close to the simulation estimates of average daily cost, relative to the scale of the graph, that we have omitted them.) The third and fourth rows of Table 1 summarize pool size recommendations and associated daily cost estimates for the constant demand scenario, first based on the simulation study and then based on our fluid approximation. Of course, the staffing level recommended by our method is trivially  $b^* = 100$ .

For choices of  $b$  close to the optimal value, our fluid-based approximation of average daily cost is almost 10% lower in the constant-demand scenario than our simulation estimate. (Recall that the error is about 1% in the variable-demand scenario.) On the

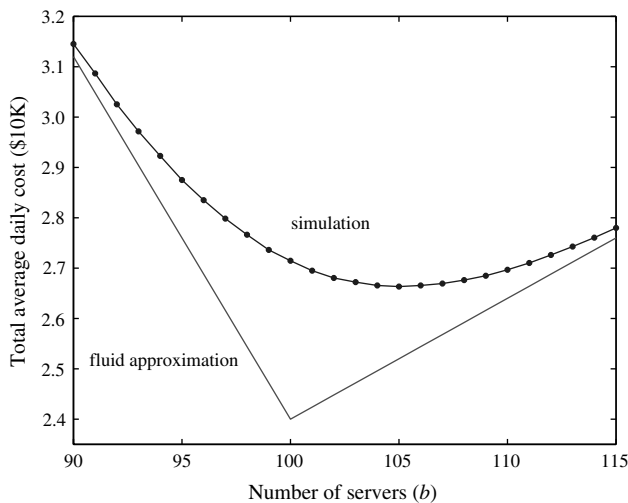
other hand, the naïve staffing proposal derived by our method in the constant-demand scenario is still quite good, producing an average daily cost that is only 1%–2% greater than the lowest achievable value according to our simulation study. Of course, the most striking aspect of the figures presented in Table 1 is that best achievable performance is much worse in the variable demand scenario: Both average daily cost and the prescribed pool size are about 10% higher with variable demand than with constant demand.

For the constant-demand scenario, we see in Figure 4 that the fluid-based estimate of average daily cost is considerably “steeper” in the vicinity of its minimizer than the simulation-based estimate. For the variable-demand scenario, both estimates are much “flatter” in the neighborhood of their respective minimizers (see Figure 3). Formula (22) for the fluid-based performance estimate helps one to understand this phenomenon. Its derivative is

$$\phi'(b) = c - Tp\mu(1 - F(b\mu)).$$

(The existence of this derivative is implied by the distribution  $F$  having no point masses.) Thus,  $\phi'(\cdot)$  increases monotonically from  $-(Tp\mu - c)$  to  $+c$ , and the rate of increase depends on how “spread out” the distribution  $F$  is. That is, according to our fluid approximation, as the demand distribution  $F$  becomes more “spread out,” a larger range of staffing levels  $b$  achieves near-optimal performance. The results depicted in Table 1 and Figures 3 and 4 are consistent with this observation. In particular, note that average daily cost in the variable demand case would be quite

**Figure 4 Total Average Daily Cost as a Function of the Number of Servers: Constant Demand Scenario**



close to optimal even if the fluid-optimal staffing level were not as close to the true optimum as it turned out to be.

### 5. Numerical Examples

Having considered the special case with a single customer class, homogenous agents and a simple stylized demand pattern, we now turn to a sequence of examples that illustrate the performance of the proposed fluid-based staffing method in more complicated scenarios. In particular, we consider system models that have two customer classes and an operation that involves dynamic routing decisions. These systems will first be analyzed with another stylized demand pattern, and subsequently a more realistic demand scenario is introduced and analyzed.

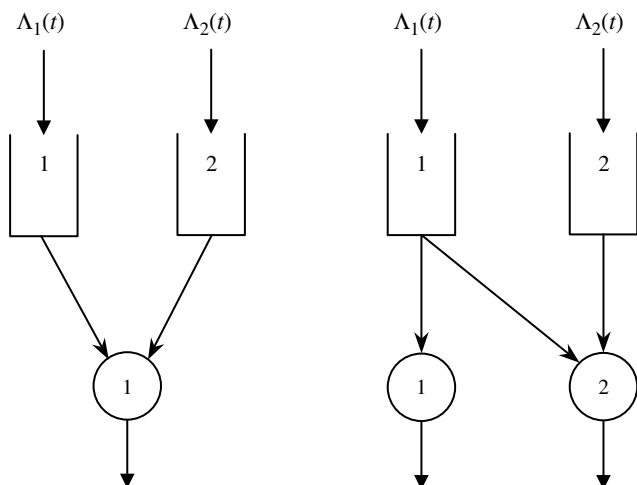
The system models depicted in Figure 5 both have two customer classes ( $m = 2$ ) that are served by a single agent pool ( $r = 1$ ) and two agent pools ( $r = 2$ ), respectively. Callers of Classes 1 and 2 arrive according to nonhomogenous Poisson processes with stochastic intensities  $\Lambda_1(t)$  and  $\Lambda_2(t)$ . There are  $b_k$  servers in pool  $k$  ( $k = 1, 2$ ), and the possible server-to-customer matchings are depicted in the figure. In the terminology of §2, the single-pool model has two processing activities (one for each customer class), and the two-pool model has three processing activities. To simplify the dynamic scheduling decisions

that are involved in allocating servers to incoming and waiting calls, we assume that all service times are exponentially distributed with rate  $\mu_j = 1$  customer per minute ( $j = 1, 2$  for the single-pool system, and  $j = 1, 2, 3$  for the other). Moreover, in the interest of simplicity, we assume that services can be interrupted at any time and later resumed from the point of preemption without incurring a penalty. Customers of class  $i$  who are waiting in queue abandon at rate  $\gamma_i = 0.5$  defections per minute, i.e., the interabandonment times are exponentially distributed with mean  $1/\gamma_i = 2$  minutes ( $i = 1, 2$ ) and are independent of the arrival and service time processes. The abandonment penalties are  $p_1 = \$1$  and  $p_2 = \$2$  per abandonment. With these assumptions and input data, it is always best to give Class 2 priority when a server allocation decision must be made. The length of the planning horizon is taken to be  $T = 480$  minutes, and the cost of employing a server for a working day is  $c_1 = \$240$  in the single-pool system, and  $c_1 = \$160$  and  $c_2 = \$240$  in the two-pool system. These costs reflect the fact that flexible (cross-trained) agents are paid more than those who can only process a single customer class.

#### 5.1. Analysis of a Stylized Demand Scenario

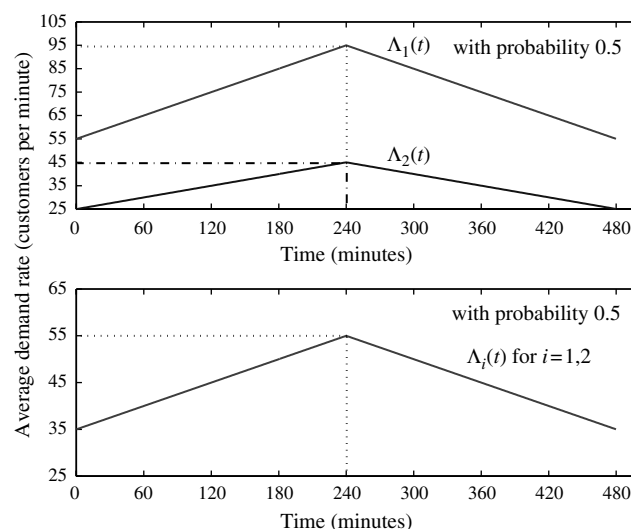
We consider the stylized demand pattern depicted in Figure 6, for which the demand distribution func-

Figure 5 Schematic Models of a Call Center with Two Customer Classes



Note. The left model has a single agent pool and the right model has two agent pools.

Figure 6 A Two-Dimensional Stylized Demand Pattern



**Table 2** Staffing Levels and Average Daily Costs for the Two System Models

	Optimal pool sizes	Personnel cost at optimum	Average abandonment cost at optimum	Average total cost at optimum
Single-pool system simulation results	94	\$22,560	\$5,001	\$27,561
Single-pool system fluid approximation	93	\$22,320	\$5,000	\$27,320
Two-pool system simulation results	(53, 54)	\$21,440	\$2,499	\$23,939
Two-pool system fluid approximation	(54, 52)	\$21,120	\$2,654	\$23,774

*Note.* The table depicts staffing levels and cost estimates computed by the proposed fluid approximation versus optimal results obtained via simulation. (The 95% confidence intervals for the total cost derived via simulation are roughly  $\pm 330$  for the first system, and  $\pm 250$  for the second, respectively.)

tion  $F$ , defined in (4), distributes mass 1/2 uniformly on each of the two line segments

$$\{(\lambda_1, \lambda_2) : \lambda_1 = 2\lambda_2 + 5 = x, 55 \leq x \leq 95\} \quad \text{and}$$

$$\{(\lambda_1, \lambda_2) : \lambda_1 = 2\lambda_2 = x, 35 \leq x \leq 55\}.$$

For both of the system models portrayed in Figure 5, the fluid approximation produces very accurate estimates of system performance, as well as pool sizes that are on par with the optimal values derived via simulation. Table 2 contrasts the staffing level, personnel costs, and abandonment costs derived using our fluid approximation and via simulation (based on 1,000 statistically independent replications of the planning period). Using our proposed fluid approximation, the optimal staffing decision is obtained by solving a variant of the newsvendor problem that has piecewise linear underage cost; in the system with two server pools there are two design variables resulting in a two-dimensional newsvendor problem;

see Van Mieghem (2003) for examples and further discussion of multidimensional newsvendor problems. The simulation results reported in Table 3 for our two-pool model show that the objective function is quite “flat” in the vicinity of the optimal staffing levels. A similar insensitivity was observed in the single-pool model.

## 5.2. A More Realistic Demand Scenario

We now restrict attention to the two-class/two-pool system and consider a more realistic demand scenario. Let  $Z = (Z_n : n \in \mathbb{Z})$  be an i.i.d. sequence of  $\mathbb{R}^2$ -valued random variables, which are normally distributed with zero mean and covariance matrix

$$\Sigma = \begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix},$$

where  $q \in [0, 1)$ . Fix  $r \in [0, 1)$  and let  $X = (X_n : n \in \mathbb{Z})$  be given by the recursion

$$X_{n+1} = \alpha(1-r)e + rX_n + Z_{n+1},$$

**Table 3** Simulated Average Total Costs for the Two-Class/Two-Pool System for Various Staffing Levels

Pool 1	Pool 2						
	49	50	51	52	53	54	55
51	24,171.9	24,349.5	24,220.5	24,048.6	24,054.5	24,295.2	24,318.9
52	24,244.8	24,094.3	24,125.4	24,285.7	24,117.8	24,325.9	23,982.6
53	24,387.0	24,089.2	24,051.8	24,060.5	24,161.3	<b>23,938.7<sup>†</sup></b>	23,977.7
54	24,194.8	24,392.9	24,162.8	24,172.1	24,280.2	24,102.0	24,203.3
55	24,112.9	24,209.5	24,226.7	24,220.5	24,085.3	24,190.9	24,135.0
56	24,241.9	24,151.2	24,085.4	24,115.4	24,261.0	24,288.5	24,320.5
57	24,337.3	24,372.3	24,293.1	24,335.7	24,123.0	24,244.4	24,419.9

*Note.* The 95% confidence intervals are  $\pm 250$ . The optimal cost is denoted with a <sup>†</sup>.

**Table 4** Average Daily Total Cost for the Two-Class/Two-Pool System with Correlated and Nonstationary Demand as a Function of Staffing Levels

Pool 1	Pool 2				
	50	55	58	<b>60</b>	65
45	28,220.2 [27,531.7]	27,865.8 [27,451.3]	27,300.6 [27,226.4]	27,501.2 [27,311.8]	27,493.4 [27,236.3]
50	27,903.6 [27,342.3]	27,395.2 [27,182.8]	27,316.1 [27,132.2]	27,751.7 [27,050.2]	27,438.5 [27,253.5]
<b>52</b>	27,996.3 [27,313.4]	27,299.6 [27,211.1]	27,276.8 <sup>†</sup> [27,054.3]	27,390.4 [ <b>27,039.6</b> ]	27,434.2 [27,231.4]
55	27,847.3 [27,274.8]	27,362.6 [27,166.4]	27,577.4 [27,112.5]	27,471.4 [27,111.0]	27,602.3 [27,445.3]
60	27,683.8 [27,244.4]	27,526.2 [27,223.2]	27,355.0 [27,264.1]	27,683.6 [27,288.2]	27,835.5 [27,446.0]

*Note.* Shown in the body of the table are the simulation estimates of average daily total cost, followed by the fluid estimates in brackets. Fluid-optimal staffing levels and the associated fluid approximation to the total cost are given in boldface, and the optimal simulated cost is denoted with a †. The 95% confidence intervals for simulated costs are roughly  $\pm 400$ .

where  $e = [1, 1]^T$ . Then  $X$  is a stationary autoregressive process in  $\mathbb{R}^2$  with a marginal distribution that is bivariate normal with mean  $[\alpha, \alpha]^T$  and covariance matrix  $\Sigma_X = (1 - r^2)^{-1}\Sigma$ . If we consider the first coordinate of  $X$  to be the instantaneous arrival rate of Class 1 calls over a certain interval of time, and the second coordinate as the Class 2 arrival rate, we have a demand process that exhibits both interclass correlation, the extent of which is controlled by the value of  $q$ , and temporal correlation, the extent of which is controlled by the value of  $r$ . In real-world systems the call volumes from different classes typically increase and decrease together, and temporal correlations are most often positive. With this in mind, we set  $q = 0.5$  and  $r = 0.5$  so that these correlations are positive and moderate in value, and set  $\alpha = 3$  so that  $X$  takes on nonnegative values with very high probability. The demand scenarios we consider in the next two examples are related to the stylized arrival rate patterns depicted earlier in Figure 6. In preparation, let us partition the planning period (recall that its length is  $T = 480$  minutes) into 18 subintervals of length  $\Delta = T/18$  minutes, and denote by  $a_i(j)$  the expected average arrival rate for class  $i$  customers over the  $j$ th subinterval when using the stylized demand model portrayed in Figure 6 ( $i = 1, 2$  and  $j = 1, \dots, 18$ ). Also, let  $a_i$  denote the expected average arrival rate for class  $i$  customers over the entire planning period in the same demand model.

**EXAMPLE 1: A NONSTATIONARY DEMAND SCENARIO.** Consider a model where  $\Lambda(\cdot)$  is constant over each of the 18 subintervals described immediately above. More specifically, let  $(X_1, \dots, X_{18})$  be consecutive observations in the stationary autoregressive sequence  $X$  defined earlier, and suppose that  $\Lambda_i(t) =$

$a_i(j) \cdot (X_{ij}/3)$  for all  $t$  in the  $j$ th subinterval ( $i = 1, 2$  and  $j = 1, \dots, 18$ ). Thus, because  $\mathbb{E}X_{ij} = 3$  for each  $i$  and  $j$ , the expected number of arrivals into each class is the same over each subinterval as in the stylized demand model portrayed in Figure 6. Table 4 shows the average daily costs with various pool size combinations, contrasting values predicted by our proposed fluid approximation with those calculated using simulation (averaged over 1,000 statistically independent replications of the planning period).

**EXAMPLE 2: A STATIONARY DEMAND SCENARIO.** Again let the planning period be partitioned into 18 equal-sized intervals, but now set  $\Lambda_i(\cdot) = a_i \cdot (X_{ij}/3)$  for all  $t$  in the  $j$ th subinterval ( $i = 1, 2$  and  $j = 1, \dots, 18$ ). Thus, the expected number of arrivals over the entire day is the same for each input flow as in the nonstationary example immediately above. Table 5 depicts the average daily costs, contrasting values predicted by our proposed fluid approximation with those calculated using simulation (averaged over 1,000 statistically independent replications of the planning period).

**Discussion.** An inspection of Tables 4 and 5 reveals three noteworthy features that were also present in the previous examples that focused on a more stylized demand model. First, the cost surface is relatively “flat” in the region of the optimal staffing vector because of the uncertainty and variability in demand. Second, the fluid approximation results in near-optimal staffing-level decisions and accurately predicts system costs. To elaborate on the results presented in Table 4, for example, the simulated cost of lost calls due to abandonments at the fluid-optimal staffing level (52, 60) is \$4,800, while the

**Table 5** Average Daily Total Cost for the Two-Class/Two-Pool System with Correlated and Stationary Demand as a Function of Staffing Levels

Pool 1	Pool 2				
	50	55	<b>58</b>	60	65
45	27,432.3 [26,931.1]	26,913.9 [26,751.3]	26,600.7 [26,545.4]	26,742.9 [26,341.8]	26,472.8 [26,037.8]
50	27,277.3 [26,412.4]	26,524.2 [26,223.8]	26,351.7 [26,213.4]	26,358.9 [26,134.4]	26,543.6 [26,163.5]
<b>53</b>	26,880.9 [26,411.2]	26,621.6 [26,148.7]	26,214.9 <sup>†</sup> [ <b>26,118.6</b> ]	26,411.6 [26,134.6]	26,539.5 [26,331.3]
55	26,946.8 [26,314.0]	26,466.1 [26,194.0]	26,320.5 [26,165.4]	226,799.4 [26,195.3]	26,539.5 [26,238.7]
60	26,740.1 [27,244.4]	26,409.0 [26,226.1]	26,389.9 [26,241.5]	26,759.1 [26,382.9]	27,069.8 [26,459.0]

*Note.* Shown in the body of the table are the simulation estimates of average daily total cost, followed by the fluid estimates in brackets. Fluid-optimal staffing levels and the associated fluid approximation to the total cost are given in boldface, and the optimal simulated cost is denoted with a †. The 95% confidence intervals for simulated costs are roughly  $\pm 370$ .

fluid approximation predicts this cost to be \$4,511. Third, as Table 6 below indicates, the cost of “lost business” due to abandonments is roughly 15% of the total cost, a consequence of the fact that the system spends a nonnegligible fraction of the day in an overloaded mode where the incoming rate of work exceeds capacity. (In contrast, if a system operates close to “heavy-traffic” yet slightly underloaded, high service efficiency and arbitrarily small abandonment probabilities can be achieved; see, e.g., Garnett et al. 2002.) Finally, as is also evident from Table 6, the nonstationary demand scenario results in a larger overall cost, due to an increase in “lost business” relative to the stationary demand scenario, as one would have anticipated.

### 5.3. General Comments on Computational Procedures

The simulation results in all the examples covered in this section were obtained by generating an appropri-

ate nonhomogenous Poisson process. (The simulation of the latter is straightforward using the thinning procedure described in Ross 1997.) For the two examples discussed in §5.2, the autoregressive process was generated first, then rescaled appropriately by the average rates, and subsequently the nonhomogenous Poisson process was simulated using those rates. In terms of optimization procedures, the stylized examples considered in §5.1 are simple enough that one can calculate the fluid-model optimal staffing levels  $b^*$  by hand, but for the more complex examples in §5.2 the fluid approximation objective function was optimized using gradient descent as follows. The server allocation LP given in (1)–(2) is two-dimensional and can therefore be solved graphically. Given a staffing vector  $(b_1, b_2)$ , the LP shadow prices can be found in a straightforward manner, and it is not difficult to see that the aforementioned shadow prices are constant over four regions, which partition  $\mathbb{R}^2$ . (For further details see the recent sur-

**Table 6** Staffing Levels and Average Daily Costs for the Two Demand Scenarios

	Optimal pool sizes	Personnel cost at optimum	Average abandonment cost at optimum	Average total cost at optimum
Example 1 (nonstationary demand): simulation results	(52, 58)	\$22,720	\$4,511	\$27,271
Example 1 (nonstationary demand): fluid approximation	(52, 60)	\$22,240	\$4,800	\$27,040
Example 2 (stationary demand): simulation results	(53, 58)	\$22,400	\$3,815	\$26,215
Example 2 (stationary demand): fluid approximation	(53, 58)	\$22,400	\$3,719	\$26,119

*Note.* The table depicts staffing levels and cost estimates computed by the proposed fluid approximation versus optimal results obtained via simulation. The 95% confidence intervals for the total cost derived via simulation are roughly  $\pm 410$  for the first example, and  $\pm 370$  for the second example, respectively.

vey by Van Mieghem 2003, which includes various examples of multidimensional newsvendor problems of this sort.) Thus, starting from any initial guess of the staffing vector, the shadow prices can be generated using the arrival rate over each time interval  $[(j-1)\Delta, j\Delta)$  and each region in  $\mathbb{R}^2$ . Using simulation and averaging over a number of such days give an estimate of the expected gradient of the objective function and thus the direction of descent. (Implicit in this is an interchange argument that can be justified under mild assumptions; cf. Van Mieghem 2003 for further details.) Computing the total average daily cost of the system using the proposed fluid approximation was carried out by generating the arrival rate vector for the whole day, then solving the LP over each time interval where the arrival rate is constant. For further discussion of solution methods for such LP problems with recourse see Birge and Louveaux (1997) and the recent survey by Shapiro (2003), which discusses various Monte Carlo simulation-based optimization approaches.

## 6. Discussion and Concluding Remarks

We have adopted a formulation in which the only congestion-related costs are abandonment penalties. However, our method extends readily to the situation where, in addition to abandonment penalties, the system manager incurs a cost of  $h_i > 0$  for each unit of time that a class  $i$  customer spends waiting in the queue. (The term “linear holding cost” is commonly used to describe this added model element.) By modifying appropriately the supporting logic described in §2, readers can verify that one arrives at the same staffing algorithm as before, except that the abandonment penalty  $p_i$  that appears in the objective function (1) of our linear program is replaced by  $p_i + h_i/\gamma_i$  ( $i = 1, \dots, m$ ) in the case with linear holding costs.

In this paper, we have taken as given the basic system configuration, including the number of server pools, the number of customer classes, the possible matchings of the latter to the former, and the average service rates embodied in the input-output matrix  $R$ . We have developed a fluid approximation that allows one to estimate the best achievable system performance with a given vector of pool sizes, and used

that estimate to optimize pool sizes given the system configuration. However, one can obviously extend this analytical method to evaluate and contrast competing design configurations that propose different processing activities and agent pool structures. For example, one can compare the two-class/two-pool system model depicted in Figure 5 with an alternative that has two *dedicated* agent pools, each serving only a single designated class. In this manner, it is possible to characterize the value of incorporating cross-trained agents into a given system, as well as determine the extent to which such cross-training is necessary to achieve target operating costs.

At least in principle, our method can be modified to consider the workforce management considerations described by Gans et al. (2003) in §3.2 of their paper. When one considers the added structure that captures shift schedules, etc., one confronts a large set of discrete staffing alternatives, rather than the continuum of choices assumed here. It is then possible to use our proposed stochastic fluid approximation to estimate costs under each of those discrete alternatives. Of course, this would require linking our performance evaluation method to an appropriate discrete optimization technique. (An example of simulation used for purposes of performance evaluation in conjunction with an integer program that is used to set staffing levels is found in Epelman et al. 2004.)

To apply our method in a given real-world context, the main task is to estimate the demand distribution  $F$  defined by (4). A reasonable procedure for doing this, which accords well with the estimation methods commonly used in call center practice, is the following. First, divide the planning period into  $L$  intervals or “time buckets” of fixed length  $\Delta$ , treating the process  $\Lambda$  as piecewise constant over such intervals. (In practice it is common to take  $\Delta =$  one-half hour.) For each realization of the arrival process over the planning period, and each time bucket  $\ell = 1, \dots, L$ , one can calculate an  $m$ -vector  $\xi_\ell$  of average arrival rates by taking the total number of arrivals in each class  $i$  and dividing by  $\Delta$ , and then  $\xi := (\xi_1, \dots, \xi_L)$  describes one realization of the process  $\Lambda$ . The empirical distribution of the finite sequence  $\xi$ , compiled from repeated observations of the arrival process over the complete planning period, then provides the estimate of  $F$  for



purposes of our method. That is, drawing at random from the distribution  $F$  amounts to choosing at random among the observed realizations of  $\xi$ , and then choosing at random a time bucket  $\ell \in \{1, \dots, L\}$ .

Obviously, the staffing method that we have proposed is only valid if one can devise a dynamic routing policy whose performance with regard to expected abandonment costs approaches the LP-based estimate of best achievable performance embodied in (3). In the interest of brevity we have not attempted a systematic discussion of dynamic routing in this paper. However, a promising general approach to dynamic routing is embodied in the “supporting logic” described in §3. Roughly speaking, the idea is to solve the linear programming problem (1)–(2) in real time, based on a current estimate of the arrival rate vector  $\lambda$ , then allocate servers to customer classes over a moderate time span based on the solution obtained, repeating this procedure at the end of that time span. The asymptotic optimality of such an approach (in the parameter regime identified in §3), as well as variants of this basic idea that seem promising for practical implementation, are the subject of Bassamboo et al. (2004).

In this paper we have only emphasized uncertainty about the call center’s demand environment. In a similar fashion, call center managers are typically uncertain about actual capacity (in the sense of average potential processing rates) given their staffing choices, because of factors such as absenteeism, heterogeneity among agents who nominally belong to the same skill category, learning effects, and a host of other factors. That is, call center managers are uncertain about their own internal capabilities, quite apart from the statistical variation in service times. It seems plausible that our method of accounting for large-scale demand uncertainty could be extended or modified to account for uncertainty regarding internal capabilities, but no serious effort has been expended in that direction to date.

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