A Double-Exponential Fast Gauss Transform Algorithm for Pricing Discrete Path-Dependent Options

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This paper develops algorithms for the pricing of discretely sampled barrier, lookback, and hindsight options and discretely exercisable American options. Under the Black-Scholes framework, the pricing of these options can be reduced to evaluation of a series of convolutions of the Gaussian distribution and a known function. We compute these convolutions efficiently using the double-exponential integration formula and the fast Gauss transform. The resulting algorithms have computational complexity of $O(nN)$, where the number of monitoring/exercise dates is n and the number of sample points at each date is N, and our results show the error decreases exponentially with N. We also extend the approach and provide results for Merton's lognormal jump-diffusion model.

1. Introduction

There are many traded options whose payoff depends on the maximum, the minimum, or the average of the underlying asset price during the whole or part of the life of the option. An example of such path-dependent options is the barrier option, which has a payoff equal to that of the European option except that the option is nullified or activated if the underlying asset price reaches a barrier during the life of the option. Other examples include the lookback option, whose payoff depends on the difference between the asset price at maturity and the maximum or the minimum of the asset price; and the hindsight option, whose payoff depends on the difference between the maximum or the minimum and a fixed strike price. American options are also sometimes called quasi path-dependent options because the decision of early exercise is made based on the path of the asset price.

Many of these options have closed-form solutions or analytical approximation formulas under the Black-Scholes (Black and Scholes 1973) model. See Goldman et al. (1979) and Conze and Viswanathan (1991) for lookback options and Rubinstein and Reiner (2000) and Rich (1995) for barrier options. These formulas are based on the assumption that the extremum or the average is taken over the entire continuous path of the underlying asset, or early exercise is possible at any moment during the life of the option. However, for many traded path-dependent options,

the extremum or the average is taken over a finite set of time points called monitoring dates. Similarly, for some American-style options, early exercise is only allowed on a finite set of time points called exercise dates, and such options are known as Bermudan options. For these discrete options, it has long been recognized that the formulas developed under the assumption of continuous monitoring/exercise give only a poor approximation. If the number of monitoring dates is not too small, corrections to these formulas proposed by Broadie et al. (1997, 1999) work well and give satisfactory results for the standard barrier and lookback options. However, when the number of monitoring dates is small or if one needs to price a more exotic form of path-dependent options, such as the lookback option with American features, one has to resort to numerical methods.

In principle, the price of these discrete options can be calculated by simple forward or backward recursion over time. For example, in the case of the discrete knockout option, which is a barrier option that is nullified if the underlying asset price reaches a barrier at one of the monitoring dates, one needs a probability density function (pdf) $P_i(S_i)$ at each date i such that $P_i(S_i)dS_i$ represents the probability that the option is still alive at time t_i and the asset price at t_i is between S_i and $S_i + dS_i$. Of course, all pricing computations are done with risk-neutral pdfs (see Duffie 2001 for more detail). Strictly speaking, it would

be more appropriate to call $P_i(S_i)$s "subprobability" density functions (consistent with a risk-neutral distribution) because they do not integrate to one. However, hereafter we simply call them pdfs for brevity. The pdf $P_i(S_i)$ can be calculated by multiplying $P_{i-1}(S_{i-1})$ with the transition pdf (tpdf) $p(S_i \mid S_{i-1})$, integrating over S_{i-1} and setting to zero any probability outside the barrier. In the case of a Bermudan option, one has to compute the continuation value at time t_{i-1}. This is calculated by multiplying the option value at time t_i with the tpdf $p(S_i \mid S_{i-1})$ and integrating over S_i. In both cases, the key computation in one step of the recursion is the integration of a product of a known function and the tpdf. In particular, under the Black-Scholes model, the tpdf becomes a Gaussian probability density function by adopting the log asset price as a variable and the above integration reduces to a convolution of a given function and the Gaussian density.

Many numerical methods for option pricing are based on this idea. For instance, binomial and trinomial methods (see Cox et al. 1979, Boyle 1988) correspond to dividing the time between the two monitoring/exercise dates into smaller time steps so that the width of the tpdf for one time step is small enough that the convolution can be approximated by a sum of two or three terms. However, such an approach needs many time steps even when the number of monitoring/exercise dates is small. Even then the generic binomial and trinomial algorithms require special modifications to price barrier and lookback options (see Ahn et al. 1999, Boyle and Lau 1994, Cheuk and Vorst 1996, Ritchken 1995). Finite-difference approaches to this problem are explored in Boyle and Tian (1998, 1999) and Zvan et al. (2000). Numerical methods for pricing discrete barrier and lookback options under alternative (i.e., non Black-Scholes) processes are considered in Boyle and Tian (1998, 1999), Davydov and Linetsky (2001), and Duan et al. (2003).

An efficient approach in the Black-Scholes model in the case of discretely monitored/exercisable options is to compute the convolution by numerical integration without using intermediate time steps. The convolution method proposed by Reiner (2000) adopts this latter approach. The method uses equally spaced grid points to discretize the log asset price at each date and computes the convolution by the trapezoidal rule or a quadrature based on cubic fitting. The convolution is computed via the fast Fourier transform (FFT), so when the number of points at each date is N, the computational work for one convolution is $O(N \log N)$, which is far less than the work of $O(N^2)$ that would be required for direct evaluation. However, the convergence of this method is not very fast. The error due to numerical integration decreases only as N^{-c}, where c is a constant and $c = 2$ for the trapezoidal rule. This means that to attain d digits accuracy, one needs $O(e^{d/c})$ sample points. This is because for the integral appearing in the convolution, either the integration region is a half-infinite line (in the case of barrier and lookback options) or the integrand has

a discontinuity in its derivative (in the case of Bermudan options). The trapezoidal rule and other lower-order integration rules are known to be inefficient for either of the cases.

Another development along this direction is the tridiagonal probability algorithm for barrier and hindsight options proposed by Tse et al. (2001). The method uses Gaussian quadrature, which can attain far more accuracy than the trapezoidal rule with the same number of sample points, and can compute the option price to a specified accuracy. The main disadvantage of this method is that the sample points of the Gaussian quadrature are not equally spaced and the FFT can no longer be used to compute the convolution. This results in $O(N^2)$ computational work for each time step. Gaussian quadrature is also used in the approach of Sullivan (2000).

In this paper, we propose a new algorithm for pricing discrete path-dependent and quasi path-dependent options. Our algorithm is also based on the idea of computing the convolution by numerical integration, but attains faster convergence and less computational work for each time step at the same time by adopting two techniques from numerical analysis, namely, the double-exponential (DE) integration formula (Takahashi and Mori 1974, Mori 1985) and the fast Gauss transform (FGT) (Greengard and Strain 1991). The DE formula computes an integral over a finite interval or a half-infinite line by converting it to an integral over the entire real axis. Its convergence is very fast and the error decreases faster than any negative power of N. Typically, the error decreases as e^{-cN}, where c is some constant, which means that one needs only $O(d)$ sample points to attain d digits accuracy. However, as in the case of the Gaussian quadrature, its sample points are not equally spaced and the FFT cannot be used. We therefore use the FGT, which is a fast algorithm to compute the convolution of a given function and a Gaussian in $O(N)$ work. In addition to being asymptotically faster than the FFT, it has a marked advantage that the sample points need not be equally spaced. By combining these two techniques, we can construct a fast and accurate algorithm for the pricing of barrier, lookback, hindsight, and Bermudan options.

In this paper, we assume that the underlying asset follows geometric Brownian motion (Black-Scholes model) or geometric Brownian motion with lognormal jumps (Merton model). However, it is in principle possible to extend our algorithm to other asset price models if the analytical expression of the transition probability density function is given. See, for example, Broadie and Yamamoto (2003) for an idea for applying our approach to Kou's jump-diffusion model (2002). It is also possible to extend the algorithm to options on more than one asset by using the multidimensional version of the DE formula and the FGT. Our algorithm can also be used to price true American options by letting the time interval Δt between monitoring dates approach zero. However, in this case, the fast convergence property of our algorithm is masked by the large

monitoring date error of $O(\Delta t)$. Therefore, the advantage of our algorithm is fully exploited when pricing purely discrete options.

In the next section, we first consider barrier options and show how the pricing computation can be reduced to a series of convolutions of a function with a Gaussian density. We also introduce the basic ideas of the double-exponential integration formula and the FGT and formulate our algorithm by combining them. Extensions of the method to Merton's model and to barrier options with Bermudan features are also discussed. Section 3 deals with the application of our method to lookback and hindsight options, along with extensions. The application to Bermudan options is investigated in §4. Numerical results that demonstrate the effectiveness of our method are shown at the end of each section. Section 5 contains concluding remarks.

2. Barrier Options

2.1. The Pricing Problem

In this section, we set up a framework for pricing a European barrier option with discrete monitoring dates. Let us consider a Black-Scholes economy with a dividend-paying asset S_t and a money-market account B_t with a constant risk-free interest rate r. Then, under the risk-neutral probability measure, S_t follows the stochastic differential equation

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t, \qquad (1)$$

where q is a constant dividend yield and W_t is a standard Brownian motion process. Also, the value of the money-market account is given by

$$B_t = \exp(rt). \qquad (2)$$

We now consider a time horizon $[0, T]$ and $n + 1$ discrete time points $t_i = i\Delta t$ ($i = 0, 1, \ldots, n$), where $\Delta t = T/n$, and denote S_{t_i} by S_i. The discrete down-and-out call option with maturity T, monitoring dates $\{t_i\}_{i=1}^{n-1}$, barrier H, and strike K is defined as an option whose payoff at time T is $(S_n - K)^+$ if $S_i > H$ for all i ($1 \leqslant i \leqslant n - 1$) and zero otherwise. Here $(x)^+$ denotes $\max(x, 0)$. Other types of barrier options such as the down-and-in call and the up-and-out put are defined similarly, and the numerical method we develop in this section can be applied in a similar way, so we limit ourselves to the down-and-out call in this section.

To formulate the option-pricing problem, we introduce a set of (risk-neutral) probability density functions $\{P_i(S_i)\}_{i=1}^{n}$ such that $P_i(S) \, dS$ represents the probability that $S_j > H$ for $1 \leqslant \forall j \leqslant i$ and $S \leqslant S_i \leqslant S + dS$. Then, the price Q_0^{DOC} of the discrete down-and-out call option at time 0 can be written as follows:

$$Q_0^{DOC}(S_0, K, H) = e^{-rT} \int_K^\infty P_n(S_n)(S_n - K) \, dS_n. \qquad (3)$$

Let $p(S_i \mid S_{i-1})$ denote the transition probability density of the asset price. Then, from the definition, $\{P_i(S_i)\}_{i=1}^{n}$ satisfy the following recursion formula:

$$P_1(S_1) = \begin{cases} p(S_1 \mid S_0) & \text{if } S_1 > H, \\ 0 & \text{otherwise,} \end{cases} \qquad (4)$$

$$P_i(S_i) = \begin{cases} \displaystyle\int_H^\infty p(S_i \mid S_{i-1})P_{i-1}(S_{i-1}) \, dS_{i-1} \\ \qquad \text{if } S_i > H, \\ 0 \quad \text{otherwise} \end{cases} \qquad (5)$$

$$(i = 2, \ldots, n).$$

We can therefore start from $P_1(S_1)$ defined by Equation (4), compute $P_2(S_2), \ldots, P_n(S_n)$ by the forward recursion formula (5) and, finally, compute the option price by Equation (3).

For our purposes, it is more convenient to change variables and work with

$$x_t = \ln(S_t) - \left(r - q - \tfrac{1}{2}\sigma^2\right)t. \qquad (6)$$

Then, x_t evolves according to the stochastic differential equation

$$dx_t = \sigma \, dW_t, \qquad (7)$$

and the transition probability density function is seen to be

$$p(x_i \mid x_{i-1}) = p^G(x_i - x_{i-1})$$
$$\equiv \frac{1}{\sqrt{2\pi\Delta t}\,\sigma} \exp\left\{-\frac{(x_i - x_{i-1})^2}{2\sigma^2\Delta t}\right\}, \qquad (8)$$

which is a Gaussian density. The option-pricing formula (3) and the recursion formulas (4) and (5) become

$$Q_0^{DOC}(S_0, K, H)$$
$$= e^{-rT} \int_k^\infty P_n(x_n)\left[\exp\left\{x_n + \left(r - q - \tfrac{1}{2}\sigma^2\right)T\right\} - K\right] dx_n, \qquad (9)$$

and

$$P_1(x_1) = \begin{cases} p^G(x_1 - \ln S_0) & \text{if } x_1 > h_1, \\ 0 & \text{otherwise,} \end{cases} \qquad (10)$$

$$P_i(x_i) = \begin{cases} \displaystyle\int_{i-1}^\infty p^G(x_i - x_{i-1})P_{i-1}(x_{i-1}) \, dx_{i-1} \\ \qquad \text{if } x_i > h_i, \\ 0 \quad \text{otherwise,} \end{cases} \qquad (11)$$

$$(i = 2, \ldots, n)$$

respectively. Here,

$$k = \ln K - \left(r - q - \tfrac{1}{2}\sigma^2\right)T \quad \text{and}$$
$$h_i = \ln H - \left(r - q - \tfrac{1}{2}\sigma^2\right)i\Delta t. \qquad (12)$$

Thus, we have established that the price of the down-and-call price can be computed by a series of convolutions of $P_i(x)$ and the Gaussian density.

2.2. The Double-Exponential Formula

Note that $P_i(x)$ $(i = 1, \ldots, n-1)$ is analytical over the region $[h_i, \infty)$ by construction, so the integral in Equation (11) is an integral of an analytical function over a half-infinite line. The double-exponential formula proposed by Takahashi and Mori (1974), Mori (1985), and Ooura and Mori (1991) is particularly efficient for computing such an integral, and it is known that when the number of sample points N is increased, its discretization error decreases faster than any negative power of N.

To derive the formula heuristically, we consider an integral of a function $g(u)$ over the entire real axis. We assume that $g(u)$ is analytical in this region, and the function value and all of its derivatives approach zero quickly when $x \to \pm\infty$. We then truncate the infinite integral $\int_{-\infty}^{+\infty} g(u)\,du$ at a lower limit u^- and an upper limit u^+. These limits are chosen so that $g(u)$ and its higher-order derivatives can be regarded as zero outside the limits (for more rigorous discussion, see the complex functional analysis-based approach in Takahashi and Mori 1974, Mori 1985, Sugihara 1997). We then determine step size h, choose integers $N^+ > 0$ and $N^- < 0$ so that $(N^- - 1)h < u^- \leqslant N^- h$ and $N^+ h \leqslant u^+ < (N^+ + 1)h$, and apply the trapezoidal rule with step size h to the integral $\int_{N^- h}^{N^+ h} g(u)\,du$. Let the value of the integral approximated in this way be I_h. Then, because we can neglect the truncation error, we can evaluate the error in I_h by the Euler-Maclaurin formula (Evans 1993, Krommer and Ueberhuber 1998) as follows:

$$
I_h - \int_{-\infty}^{+\infty} g(u)\,du
$$
$$
\simeq \left\{ \sum_{j=N^-}^{N^+} g(jh) - \tfrac{1}{2}\{g(N^- h) + g(N^+ h)\} \right\} h - \int_{N^- h}^{N^+ h} g(u)\,du
$$
$$
= \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} h^{2k-1} \{ g^{(2k-1)}(N^+ h) - g^{(2k-1)}(N^- h) \} + E_m, \quad (13)
$$

where B_k are the Bernoulli numbers with $B_3 = B_5 = \cdots = 0$ and $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $g^{(n)}(u)$ is the nth derivative of $g(u)$, and

$$
E_m = (N^+ h - N^- h) \frac{B_{2m+2} h^{2m+2}}{(2m+2)!} g^{(2m+2)}(\xi) \quad (14)
$$

with $N^- h < \xi < N^+ h$. Equation (13) holds for any fixed positive integer m. Recalling that higher-order derivatives of $g(u)$ can be regarded as zero at $u = N^- h$ and $u = N^+ h$, we know that the error can be approximated by E_m, which is $O(h^{2m+2})$. Because m is arbitrary, we can say that the error of the trapezoidal rule decreases faster than any power of h in this case.

The DE formula exploits this fact by converting an integral over a half-infinite region into one over the entire real axis. Let the integral we want to compute be

$$
I = \int_{c}^{\infty} f(x)\,dx. \quad (15)
$$

We introduce the following change of variables, the double-exponential transformation

$$
x = c + \exp\left(\frac{\pi}{2} \sinh u \right), \quad (16)
$$

which leads to a new expression for the integral

$$
I = \int_{-\infty}^{\infty} f\left(c + \exp\left(\frac{\pi}{2} \sinh u \right) \right) \exp\left(\frac{\pi}{2} \sinh u \right) \frac{\pi}{2} \cosh u\,du. \quad (17)
$$

We can now apply the trapezoidal rule with step size h to this integral, obtaining the approximation

$$
I_h = h \sum_{j=-\infty}^{\infty} f\left(c + \exp\left(\frac{\pi}{2} \sinh(jh) \right) \right)
$$
$$
\cdot \exp\left(\frac{\pi}{2} \sinh(jh) \right) \frac{\pi}{2} \cosh(jh). \quad (18)
$$

It is known that if the original integral (15) converges, the integrand in Equation (17) decreases at least as fast as the double-exponential function $\exp(-\exp(|u|))$ as $u \to \pm\infty$, and the sum (18) can be truncated at a modest value of $|jh|$ without affecting the computed result. To see this, note first that the integrand in Equation (15) has to decrease at least faster than $1/x$ as $x \to \infty$ for the infinite integral to exist. Therefore, we assume that $f(x) \sim x^{-1-\alpha}$ as $x \to \infty$, where $\alpha > 0$, and put this into the integrand of Equation (17). Then, the integrand becomes

$$
\left(c + \exp\left(\frac{\pi}{2} \sinh u \right) \right)^{-1-\alpha} \exp\left(\frac{\pi}{2} \sinh u \right) \frac{\pi}{2} \cosh u
$$
$$
\sim \left(\exp\left(\frac{\pi}{2} \sinh u \right) \right)^{-\alpha} \frac{\pi}{2} \cosh u
$$
$$
\sim \exp\left(-\frac{\pi\alpha}{4} \exp u \right) \frac{\pi}{4} \exp u
$$
$$
= \frac{\pi}{4} \exp\left(u - \frac{\pi\alpha}{4} \exp u \right). \quad (19)
$$

When, for example, $\alpha \sim 1$, this function becomes less than 10^{-16} at $u = 4$, and the infinite sum of Equation (18) can be safely truncated at $|jh| \sim 4$ if double precision arithmetic is used. Thus, the number of the sample points is $N = O(h^{-1})$, and we can conclude that the discretization error decreases faster than any negative power of N.

Takahashi and Mori (1974) show that the double-exponential transformation (16) is optimal for a wide class of functions in the sense that it can achieve the maximum accuracy for a given number of points (Takahashi and Mori 1974, Mori 1985, Sugihara 1997). However, convergence faster than any negative power of N is attained as long as (a) the transformation is analytical, (b) it maps the original integration region onto the entire real axis, and (c) the integrand in Equation (17) approaches zero fast enough

when $x \to \pm\infty$. Hence, it is sometimes more appropriate to modify the transformation by taking into account special properties of the original integrand $f(x)$. We will do this for the convolution (11) in §2.4.

It is important to note that the integrand in Equation (11) is analytical for asset price models other than the Black-Scholes model. For example, the integrand for Merton's lognormal jump-diffusion model (Merton 1992) is also analytical because the jump size is lognormally distributed and the transition probability density $p(x_i \mid x_{i-1})$ can be represented as a sum of Gaussians. Also, the tpdf in Kou's jump-diffusion model (2002) is analytical because it can be represented as a sum of products of an exponential function and the Hh function, both of which are analytical. This indicates that the DE method has potential application to these models. We illustrate an example of this in §2.5.

2.3. The Fast Gauss Transform

To compute the convolution, we first rewrite the DE formula by defining the sample points a_j and w_j as follows:

$$I_h^N = \sum_{j=N^-}^{N^+} f(a_j) w_j, \quad (20)$$

$$a_j^i = h_i + \exp\left(\frac{\pi}{2}\sinh(jh)\right), \quad (21)$$

$$w_j = h\exp\left(\frac{\pi}{2}\sinh(jh)\right)\frac{\pi}{2}\cosh(jh), \quad (22)$$

where N^- and N^+ are determined so that $N^+h \sim -N^-h \sim 4$ and the total number of sample points is $N = N^+ - N^- + 1$. Using this formula, we can approximate Equation (11) by

$$P_i(a_j^i) = \sum_{j'=N^-}^{N^+} p^G(a_j^i - a_{j'}^{i-1})P_{i-1}(a_{j'}^{i-1})w_{j'}$$
$$(j = N^-, \ldots, N^+). \quad (23)$$

Note that we do not need sample points in the region $x_i < h_i$ because $p_i(x_i)$ is always zero there. Apparently, Equation (23) requires $O(N^2)$ computation for each time step. Moreover, the FFT cannot be used here to reduce the computational work because the sample points $\{a_j^i\}$ and $\{a_{j'}^{i-1}\}$ are not equally spaced. We therefore use the FGT, which is a fast algorithm introduced by Greengard and Strain (1991) and Greengard and Sun (1998) to compute the discrete convolution of a given function with a Gaussian function in $O(N)$ work.

Suppose that we want to calculate the sums

$$G(x_j) = \sum_{k=1}^{N} q_k \exp\left\{-\frac{(x_j-y_k)^2}{\delta}\right\}, \quad j=1,2,\ldots,M. \quad (24)$$

To compute the multiple sums efficiently, the FGT uses the following expansion of the Gaussian:

$$e^{-(x_j-y_k)^2/\delta} = \sum_{\beta=0}^{\infty}\sum_{\alpha=0}^{\infty}\frac{1}{\beta!}\frac{1}{\alpha!}\left(\frac{y_k-y_0}{\sqrt{\delta}}\right)^\alpha$$
$$\cdot h_{\alpha+\beta}\left(\frac{x_0-y_0}{\sqrt{\delta}}\right)\left(\frac{x_j-x_0}{\sqrt{\delta}}\right)^\beta. \quad (25)$$

This formula can be shown easily by using the expansion of the Gaussian in terms of Hermite functions $h_\alpha(x)$:

$$e^{-(x-y)^2} = \sum_{\alpha=0}^{\infty}\frac{y^\alpha}{\alpha!}h_\alpha(x), \quad (26)$$

$$h_\alpha(x) = (-1)^\alpha\left(\frac{d}{dx}\right)^\alpha e^{-x^2}, \quad (27)$$

and by further expanding the Hermite function in the right-hand side of Equation (26). It is known that this expansion converges very quickly and the double infinite sum over α and β can be truncated at a reasonably small integer, $\alpha = \beta = \alpha_{\max}$. It is known that $\alpha_{\max} = 8$ is sufficient to achieve a relative error of 10^{-8} when $|(y_k-y_0)/\sqrt{\delta}| < 1/2$ and $|(x_j-x_0)/\sqrt{\delta}| < 1/2$ (Greengard and Strain 1991).

Now we consider a special case where all the target points $\{x_j\}$ are in an interval with center x_0 and length $\sqrt{\delta}$ and all the source points $\{y_k\}$ are in another interval with center y_0 and length $\sqrt{\delta}$. Then, the expansion (25) converges quickly by truncating the sums over α and β at $\alpha = \beta = \alpha_{\max}$. By substituting this into Equation (24), we can approximate $G(x_j)$ as

$$G(x_j) \cong \sum_{k=1}^{N} q_k \sum_{\beta=0}^{\alpha_{\max}}\sum_{\alpha=0}^{\alpha_{\max}}\frac{1}{\beta!}\frac{1}{\alpha!}\left(\frac{y_k-y_0}{\sqrt{\delta}}\right)^\alpha$$
$$\cdot h_{\alpha+\beta}\left(\frac{x_0-y_0}{\sqrt{\delta}}\right)\left(\frac{x_j-x_0}{\sqrt{\delta}}\right)^\beta$$
$$= \sum_{\beta=0}^{\alpha_{\max}}\frac{1}{\beta!}\left(\frac{x_j-x_0}{\sqrt{\delta}}\right)^\beta$$
$$\cdot\left\{\sum_{\alpha=0}^{\alpha_{\max}}h_{\alpha+\beta}\left(\frac{x_0-y_0}{\sqrt{\delta}}\right)\left\{\frac{1}{\alpha!}\sum_{k=1}^{N}q_j\left(\frac{y_k-y_0}{\sqrt{\delta}}\right)^\alpha\right\}\right\}. \quad (28)$$

This expression shows that the computation of $G(x_j)$ can be divided into three steps:

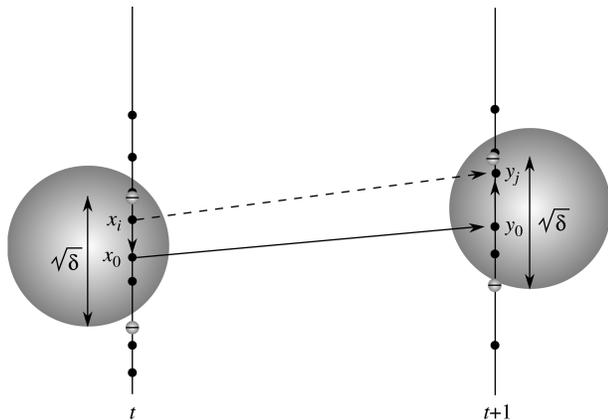*Step* 1. Compute

$$A_\alpha \equiv \frac{1}{\alpha!}\sum_{k=1}^{N}q_k\left(\frac{y_k-y_0}{\sqrt{\delta}}\right)^\alpha \quad \text{for } \alpha = 0,\ldots,\alpha_{\max}.$$

Step 2. Compute

$$B_\beta \equiv \sum_{\alpha=0}^{\alpha_{\max}}A_\alpha h_{\alpha+\beta}\left(\frac{x_0-y_0}{\sqrt{\delta}}\right) \quad \text{for } \beta = 0,\ldots,\alpha_{\max}.$$

Figure 1.   Illustration of the FGT algorithm.

Note. The source points x_i and target points y_j lie in intervals of length $\sqrt{\delta}$ centered at x_0 and y_0, respectively.

Step 3. Compute

$$G(x_j) = \sum_{\beta=0}^{\alpha_{\max}} B_\beta \frac{1}{\beta!} \left(\frac{x_j - x_0}{\sqrt{\delta}} \right)^\beta \quad \text{for } j = 1, \ldots, M.$$

When $\alpha_{\max}$ is fixed, Steps 1 and 3 require $O(N)$ and $O(M)$ computational effort, respectively, while Step 2 can be done in a constant time that does not depend either on N or M. See Figure 1 for an illustration of the FGT algorithm.

In the general case, we divide the space into intervals of length $\sqrt{\delta}$ and apply the above method to each of the possible pairs of a source interval and a target interval. Let K and J denote the source interval and the target interval, respectively, and y_K and x_J denote their centers. The algorithm can be written as follows:

Step 1. Compute

$$A_{\alpha, K} \equiv \frac{1}{\alpha!} \sum_{y_k \in K} q_k \left(\frac{y_k - y_K}{\sqrt{\delta}} \right)^\alpha \quad \text{for } \alpha = 0, \ldots, \alpha_{\max}$$

and for each source interval K.

Step 2. Compute

$$B_{\beta, J} \equiv \sum_K \sum_{\alpha=0}^{\alpha_{\max}} A_{\alpha, K} h_{\alpha+\beta} \left(\frac{x_J - y_K}{\sqrt{\delta}} \right) \quad \text{for } \beta = 0, \ldots, \alpha_{\max}$$

and for each target interval J.

Step 3. Compute

$$G(x_j) = \sum_{\beta=0}^{\alpha_{\max}} B_{\beta, J} \frac{1}{\beta!} \left(\frac{x_j - x_J}{\sqrt{\delta}} \right)^\beta \quad \text{for } j = 1, \ldots, M.$$

Here J is the target interval x_j belongs to.

Because each x_j and y_k belong to only one interval, the total work for Step 1 and Step 3 is still $O(N)$ and $O(M)$, respectively, while Step 2 needs work proportional to $O(N_{\text{int}}^2)$, where N_{int} is the number of intervals.

By applying the fast Gauss transform with source points $\{a_{j'}^{i-1}\}$, target points $\{a_j^i\}$, and

$$q_k = P_{i-1}(a_k^{i-1})w_k, \tag{29}$$

$$\delta = 2\sigma^2 \Delta t, \tag{30}$$

we can compute the discrete convolution (23) in $O(N)$ work.

Note that we gave only the basic idea of the FGT, and in the actual algorithm several techniques to further speed up the computation are employed. For the details of these techniques, as well as for the error analysis and extensions to higher dimensions, consult Greengard and Strain (1991), Greengard and Sun (1998), and Baxter and Roussos (2002).

We conclude this subsection by noting that there are other approaches to computing the discrete convolution (23) with computational effort less than $O(N^2)$. One possibility is to use nonuniform FFTs, or variants of the FFT for unequally spaced grids Dutt and Rokhlin (1993) and Ware (1998). One of the disadvantages of this approach is that it needs $O(N \log N)$ work when the number of grid points is N, as opposed to $O(N)$ work required by the FGT. In addition, from our preliminary experiments, nonuniform FFTs were seen to be about 10 times slower than FFTs for equally spaced grids. We conclude that it is more efficient to use problem-specific convolution methods such as the FGT when they are available. However, convolution based on nonuniform FFTs has a marked advantage that it can deal with a much wider class of transition probability density functions. We are exploring the possibility of fast pricing methods based on this approach.

2.4. The DE-FGT Method for Barrier Options

The method described in the previous subsection gives a straightforward approach to computing the price of discrete down-and-out options using the combination of the DE formula and the FGT. However, we can construct a more efficient method by taking into account the properties of the integrand in Equation (11) and by slightly modifying the transformation (16).

As $x_{i-1} \to \infty$, the integrand in Equation (11) approaches zero faster than the Gaussian, because $p^G(x_i - x_{i-1})$ is Gaussian and $P_{i-1}(x_{i-1})$ also approaches zero. If we apply the ordinary double-exponential transformation to this integral, we have another factor that approaches zero double exponentially. Although this gives rise to an advantage that the integrand vanishes extremely fast and the integration region can be truncated at a smaller upper bound, it also causes a disadvantage that the integrand changes very rapidly and a finer mesh is necessary in the integration region. According to our numerical experiments, this disadvantage exceeds the advantage.

Intuitively, it seems more appropriate to construct a transformation that works as the DE transformation near the lower end of the integral and approaches an identity

transformation as $x_{i-1} \to \infty$, because the original integrand $p^G(x_i - x_{i-1})P_{i-1}(x_{i-1})$ vanishes rapidly enough. We therefore adopt the following transformation:

$$x = \ln\left\{ e^c + \exp\left(\frac{\pi}{2}(1 + u - e^{-u})\right)\right\}. \tag{31}$$

This is obtained by replacing the e^u in the sinh function with $1 + u$ and taking the logarithm. It has the property that it approaches the DE transformation $x = c + e^{-c}\exp(-(\pi/2)\exp(-u))$ when $u \to -\infty$ and approaches a linear transformation $x = (\pi/2)u$ when $u \to \infty$. The original integral (15) is converted into

$$I = \int_{-\infty}^{\infty} f\left(\ln\left\{e^c + \exp\left(\frac{\pi}{2}(1 + u - e^{-u})\right)\right\}\right)$$
$$\cdot \frac{\exp((\pi/2)(1 + u - e^{-u}))(\pi/2)(1 + e^{-u})}{e^c + \exp((\pi/2)(1 + u - e^{-u}))}\, du. \tag{32}$$

Applying the trapezoidal rule with step size h to this integral, we can find the sample points and the weights of the modified DE formula as follows:

$$a_j^i = \ln\left\{e^c + \exp\left(\frac{\pi}{2}(1 + jh - e^{-jh})\right)\right\} \quad \text{and} \tag{33}$$

$$w_j = h\frac{\exp((\pi/2)(1 + jh - e^{-jh}))(\pi/2)(1 + e^{-jh})}{e^c + \exp((\pi/2)(1 + jh - e^{-jh}))}. \tag{34}$$

By putting these into Equation (23) and computing the discrete convolution using the FGT, we can construct a method for pricing the discrete down-and-out options efficiently. We name this the DE-FGT method.

Our numerical experiments show that numerical integration based on the transformation (31) requires only half as many sample points as that based on (16) to attain the same level of accuracy. We therefore use this transformation in the subsequent sections whenever an integral over a half-infinite line is necessary.

2.5. Extensions of the Basic Algorithm

2.5.1. Time-Varying Barriers and Double Barriers. So far we have assumed, for simplicity, that the barrier level H is constant through time and the monitoring dates are equally spaced between $t = 0$ and $t = T$. However, as can be inferred from the derivation of the algorithm, there is no difficulty in dealing with the case of time-varying barriers or nonequally spaced monitoring dates. We can also extend the algorithm to a double-barrier option with a lower barrier H_L and an upper barrier H_H. In this case, the integral appearing in convolution (11) becomes one over a finite interval $[h_L, h_H]$, and one can use the finite-interval version of the double-exponential transformation

$$x = \frac{h_H + h_L}{2} + \frac{h_H - h_L}{2}\tanh\left(\frac{\pi}{2}\sinh u\right) \tag{35}$$

instead of (31).

2.5.2. Computation of Delta and Gamma. Our method can also be used to compute the option delta and gamma. To see this, differentiate both sides of Equations (10) and (11) with respect to the initial stock price S_0. Then, we know that $\partial P_i/\partial S_0$ satisfies the same recurrence as P_i itself and can therefore be computed with our DE-FGT method. After obtaining $\partial P_n/\partial S_0$, we can differentiate Equation (3) with respect to S_0, put $\partial P_n/\partial S_0$ in the right-hand side, and compute the option delta $\partial Q_0^{DOC}/\partial S_0$. The option gamma can be computed in the same manner.

2.5.3. Application to Merton's Model. We can extend our method to deal with the lognormal jump-diffusion model introduced by Merton (1992). In this model, the asset price follows the equation

$$S_i = S_{i-1}\exp\left\{(r - q - \tfrac{1}{2}\sigma^2 - \nu\lambda)\Delta t + \sigma\sqrt{\Delta t}z_0 \right.$$
$$\left. + \sum_{l=1}^{N_t^P(\Delta t)}(\delta z_l + \gamma - \tfrac{1}{2}\delta^2)\right\}, \tag{36}$$

where Δt is the time interval between t_{i-1} and t_i, $N_t^P(\Delta t)$ is the number of jumps during this interval, which follows a Poisson process with intensity λ, and z_l $(l = 0, 1, \ldots)$ are independent and follow the standard normal distribution $N(0, 1)$. The constants γ and δ determine the mean and the standard deviation of the jumps, respectively, and

$$\nu = e^\gamma - 1. \tag{37}$$

In this model, the market becomes incomplete due to the existence of jumps, and the standard argument for option pricing based on the replicating portfolio no longer holds. Merton (1992) derives an option-pricing formula under the assumption that jump risk is diversifiable so that risk-neutral pricing is appropriate. Bates (1996) and Naik and Lee (1990) derive option-pricing formulas in representative agent general equilibrium models. The form of their pricing equations are identical to the Merton formula, but with altered parameter values that account for the market price of jump risk. The pricing problems in these models are therefore equivalent from a computational viewpoint: One simply substitutes the appropriate "risk-adjusted" parameters into the risk-neutral pricing formula.

We can apply the change of variable

$$x_t = \ln(S_t) - (r - q - \tfrac{1}{2}\sigma^2 - \nu\lambda)t \tag{38}$$

to Equation (36) and obtain an equation for x_i:

$$x_i = x_{i-1} + \sigma\sqrt{\Delta t}z_0 + \sum_{l=1}^{N_t^P(\Delta t)}(\delta z_l + \gamma - \tfrac{1}{2}\delta^2). \tag{39}$$

Note that the Poisson probability can be written as

$$\Pr(N_t^P(\Delta t) = n) = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^n}{n!}, \tag{40}$$

and when the number of jumps is n, $x_i - x_{i-1}$ follows a Gaussian distribution with the variance and mean given by

$$\sigma_n^2 = \sigma^2 \Delta t + n\delta^2 \quad \text{and} \tag{41}$$

$$\mu_n = n\left(\gamma - \tfrac{1}{2}\delta^2\right), \tag{42}$$

respectively. We can then write the tpdf of x_i as

$$p(x_i \mid x_{i-1}) = p^M(x_i - x_{i-1})$$
$$\equiv \sum_{n=0}^{\infty} e^{-\lambda \Delta t} \frac{(\lambda \Delta t)^n}{n!} \frac{1}{\sqrt{2\pi}\sigma_n}$$
$$\cdot \exp\left\{-\frac{(x_i - x_{i-1} - \mu_n)^2}{2\sigma_n^2}\right\}. \tag{43}$$

As can be easily seen, the probability density

$$p^M(x_i - x_{i-1})$$

has the following expansion, which corresponds to Equation (25) in the Gaussian case (Broadie and Yamamoto 2003):

$$p^M(x_i - x_{i-1}) = \sum_{\beta=0}^{\alpha_{max}} \sum_{\alpha=0}^{\alpha_{max}} \frac{1}{\beta!} \frac{1}{\alpha!} \left(\frac{x_i - x''}{\sqrt{2}\sigma}\right)^\alpha$$
$$\cdot \left\{\sum_{n=0}^{\infty} e^{-\lambda \Delta t} \frac{(\lambda \Delta t)^n}{n!} \frac{1}{\sqrt{2\pi}\sigma_n} \left(\frac{\sigma}{\sigma_n}\right)^{\alpha+\beta}\right.$$
$$\left. \cdot h_{\alpha+\beta}\left(\frac{x' - x'' + \mu_n}{\sqrt{2}\sigma_n}\right)\right\} \left(\frac{x_{i-1} - x'}{\sqrt{2}\sigma}\right)^\beta, \tag{44}$$

where x' and x'' are the centers of intervals of length $\sqrt{2}\sigma$ containing x_{i-1} and x_i, respectively. Comparing this expression with Equation (25), we know that we can construct an algorithm similar to the FGT by replacing the Hermite function with a weighted sum of shifted and scaled Hermite functions. Specifically, we have only to replace the formula to compute B_β (see §2.3) with the following:

$$B_\beta = \sum_{\beta=0}^{\alpha_{max}} A_\alpha \left\{\sum_{n=1}^{N_{jump}} e^{-\lambda \Delta t} \frac{(\lambda t)^n}{n!} \frac{1}{\sqrt{2\pi}\sigma_n} \left(\frac{\sigma}{\sigma_n}\right)^{\alpha+\beta}\right.$$
$$\left. \cdot h_{\alpha+\beta}\left(\frac{x' - x'' + \mu_n}{\sqrt{2}\sigma_n}\right)\right\}, \tag{45}$$

where we have truncated the sum over the number of jumps at N_{jump}. This algorithm enables us to compute the convolution of $p^M(x)$ and a given function almost as easily as in the Gaussian case. We refer the reader to Broadie and Yamamoto (2003) for more detailed derivation of the algorithm and implementation issues.

2.5.4. Barrier Options with Bermudan Features. It is also easy to extend our DE-FGT method to the pricing of barrier options with Bermudan features. To do this, we use the DE-FGT method for Bermudan options, which we will

introduce in §4 as a basis. In this method, we use backward recursion and compute the continuation value at time t_{i-1} as a convolution of the option price $Q_i(x_i)$ at time t_i and the transition probability density function. Because $Q_i(x_i)$ has a discontinuity in the derivatives at the optimal exercise boundary $x_i = x_c$, we divide the integration region into two parts, namely $(-\infty, x_c]$ and $[x_c, \infty)$, and apply the DE formula to each of them.

To price barrier options such as down-and-out options with a Bermudan feature using this algorithm, we only have to replace the lower integral region with $[h_i, x_c]$, where h_i is the barrier level given by Equation (12), and use the DE formula (35) for a finite interval.

2.6. Numerical Results

We implemented the DE-FGT method for European down-and-out call options under the Black-Scholes model and Merton's model and studied its speed and accuracy. All of the experiments, including those in the later sections, were done on a 266 MHz Pentium II PC with Red Hat Linux using gnu C++ compiler, unless otherwise noted.

2.6.1. Down-and-Out Call Option Under the Black-Scholes Model. We show results for European down-and-out call options under the Black-Scholes model in Figure 2. The parameters are $S_0 = K = 100$, $T = 0.2$, $r = 0.1$, $q = 0$, and $\sigma = 0.3$. We varied the barrier level from $H = 91$ to $H = 99$ in increments of 2 and set the number of monitoring dates to $n = 5$, 25, or 50. These are the values used in

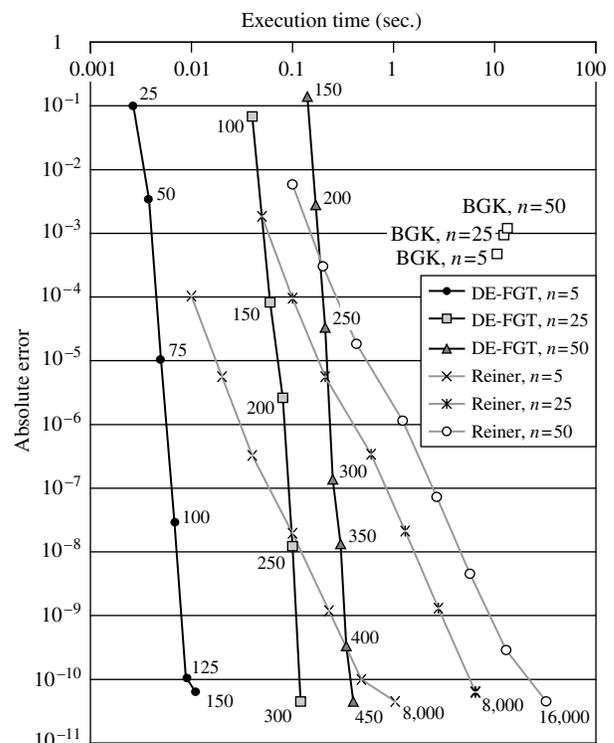**Figure 2.** European down-and-out call option price under the Black-Scholes model.

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Table 1. European down-and-out call prices under the Black-Scholes model.

	$n = 5$	$n = 25$	$n = 50$	$n = \infty$
$H = 91$	6.1872900302	6.0320261243	5.9770686565	5.8077713405
$H = 93$	5.9997553594	5.6875323983	5.5843399451	5.2768140270
$H = 95$	5.6711051343	5.0814151587	4.9067890354	4.3975025272
$H = 97$	5.1672453684	4.1158152250	3.8339777052	3.0595631676
$H = 99$	4.4891724312	2.8124392982	2.3363868958	1.1707931057

Broadie et al. (1997), Reiner (2000), and Tse et al. (2001). As reference prices against which to compute the errors, we used the values shown in Table 1, which were obtained by Reiner's convolution method (2000) with the cubic fitting quadrature and $N = 32{,}000$ grid points for each date. These values agree with the results of Tse et al. (2001) to at least 10 digits after the decimal point for $n = 5$ and $n = 25$. For comparison, we also show the prices of the continuously monitored down-and-out call options (denoted as $n = \infty$) computed using the analytical formula (Kwok 1998).

In Figure 2, the vertical axis and the horizontal axis represent the error in the calculated option price and the computational time, respectively, both in log scale. The errors are root mean square errors of five options with different barrier levels, and the time is for computing one option price. In the computation of Equation (32), we truncated the integral at the lower bound $u_{\min} = -4.0$ and the upper bound $u_{\max} = \ln H + 5\sigma\sqrt{T}$ and used N sample points to approximate the integral. The value of N is also shown in the graph. For comparison, we also plotted the error and the computational time of Reiner's method with up to 16,000 points and Broadie et al.'s (BGK) (1999) trinomial lattice method with 5,000 time steps. For the former method, we truncated the integral of Equation (11) at the upper bound of $10\sigma\sqrt{T}$. For the latter method, the computation was done on a 2.0 GHz Xeon PC, and their computation times were multiplied by 4, for the Xeon machine was found to be about four times faster than the Pentium II machine.

As can be seen from the graph, our method converges much faster than Reiner's method and attains the same level of accuracy. Compared with the trinomial lattice method, our method gives much more accurate prices in much shorter time. The error of our method decreases almost exponentially with N, because as N is incremented by a constant, the position of the corresponding point in the graph moves downward by a constant distance. Finally, our method is extremely fast, because even the option with 50 monitoring dates can be priced within 0.5 CPU second to an accuracy of 10^{-10}.

As an example of down-and-out call options with longer times to maturity and larger number of monitoring dates, we show the results for one-year ($T = 1.0$) options with 252 monitoring dates in Figure 3. Other parameters are the same as those for options mentioned above. The reference prices computed by Reiner's method with 64,000 points are

shown in Table 2. In this case, the computation was done on a 2.0 GHz Xeon PC.

It can be seen from the graph that the convergence behavior of both Reiner's method and our DE-FGT method is the same as that for options with a smaller number of monitoring dates, and the DE-FGT method is superior when higher accuracy is needed.

2.6.2. Up-and-Out Call Option Under the Black-Scholes Model. Up-and-out call options, which are nullified when the underlying asset price reaches a barrier level from below, are known to present difficulties for most numerical pricing procedures due to the discontinuous nature of their payoff. However, our DE-FGT method, as well as Reiner's convolution method, does not suffer from this at all. In fact, all we need to do is to replace the half-infinite integration interval $[k, \infty)$ in Equation (9) with a finite interval $[k, h_n]$ and use the DE formula based on Equation (35).

Figure 3. European down-and-out call option price under the Black-Scholes model ($T = 1.0$, 252 monitoring dates).
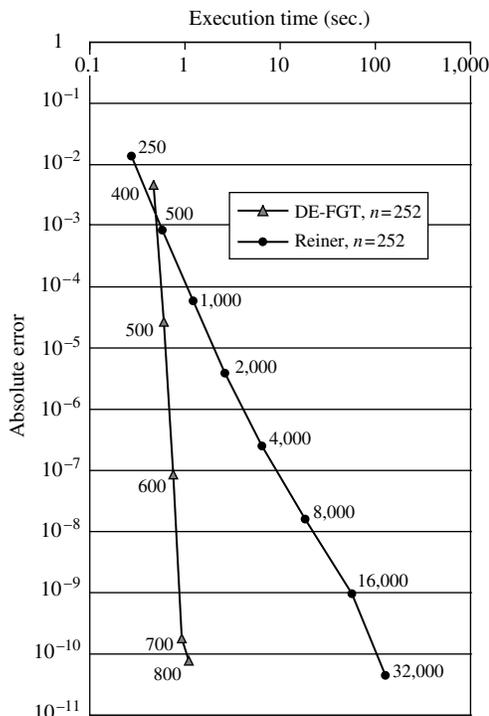
Table 2. European down-and-out call prices under the Black-Scholes model ($T = 1.0$, 252 monitoring dates).

	$n = 252$
$H = 91$	11.3121524522
$H = 93$	9.7292574722
$H = 95$	7.8438846454
$H = 97$	5.6306538930
$H = 99$	3.1673854834

Results for up-and-call options with $S_0 = K = 100$, $T = 0.2$, $r = 0.1$, $q = 0$, and $\sigma = 0.3$ are illustrated in Figure 4. Here, the barrier level is varied from $H = 121$ to $H = 129$ and the number of monitoring dates was set to $n = 5, 25$, or 50. The reference values computed by Reiner's method are shown in Table 3. In this case, the computation was also done on a 2.0 GHz Xeon PC. It can easily be seen from the graph that both our method and Reiner's method can compute the prices with the same speed and accuracy as for the down-and-out call options.

2.6.3. Down-and-Out Call Option Under Merton's Model. We also computed European down-and-out call option prices under Merton's model. The parameter values are the same as for the down-and-out call options under the Black-Scholes model with $T = 0.2$, and for the jump parameters, we used $\lambda = 2.0$, $\gamma = 0$, and $\delta = 0.3$. The

Figure 4. European up-and-out call option price under the Black-Scholes model.
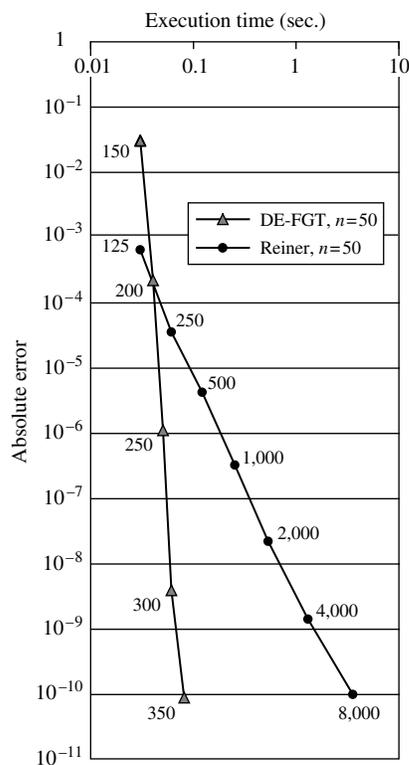

Table 3. European up-and-out call prices under the Black-Scholes model.

	$n = 50$
$H = 121$	2.9102779978
$H = 123$	3.3933815021
$H = 125$	3.8446456577
$H = 127$	4.2550087291
$H = 129$	4.6196180375

reference prices computed by the convolution method with $N = 64,000$ are shown in Table 4.

Figure 5 shows the convergence of the DE-FGT method and the convolution method. In this case, we extended the upper bound of integral Equation (11) for the convolution method to $30\sigma\sqrt{T}$ to attain the same level of accuracy as in the Black-Scholes case. For the DE-FGT method, we used the same value of $u_{\min}$ as in the Black-Scholes case and increased $u_{\max}$ to $\ln H + 5\sigma\sqrt{T}$. Again, it is clear from the graph that the error of our method decreases exponentially with N.

3. Lookback Options

3.1. Reduction to a One-Dimensional Problem

Here we formulate the problem of pricing discrete lookback put options using the notation of §2.1. A lookback put option on an asset is the right to sell the asset at maturity at the highest asset price between initial and maturity dates. Lookback call options are defined similarly and can be priced in a completely parallel manner, so here we deal with only put options. The asset price is monitored on a finite number of dates for discrete lookback options. For details, see Goldman et al. (1979) or Broadie et al. (1999).

Let us consider a discrete lookback option with maturity T and monitoring dates $\{t_i\}_{i=1}^{n-1}$. Also, denote the asset price on these monitoring dates by $\{S_i\}_{i=1}^{n-1}$ and let

$$M_i = \max_{1 \leqslant k \leqslant i} S_k. \tag{46}$$

Then, the payoff of this option at maturity can be written as $M_n - S_n$. Under the risk-neutral probability measure Q, the price of the option at time 0 is given by

$$Q_0^{LP}(S_0) = e^{-rT} E_0[M_n - S_n], \tag{47}$$

Table 4. European down-and-out call option prices under Merton's model.

	$n = 5$	$n = 25$	$n = 50$
$H = 91$	8.6304893283	8.2843010923	8.1796345791
$H = 93$	8.2883832522	7.7161307812	7.5470008678
$H = 95$	7.7707276025	6.8204546460	6.5607004413
$H = 97$	7.0559324990	5.4877084298	5.0916199042
$H = 99$	6.1639697190	3.7626493142	3.1078183986

Figure 5. European down-and-out call option price under Merton's model.
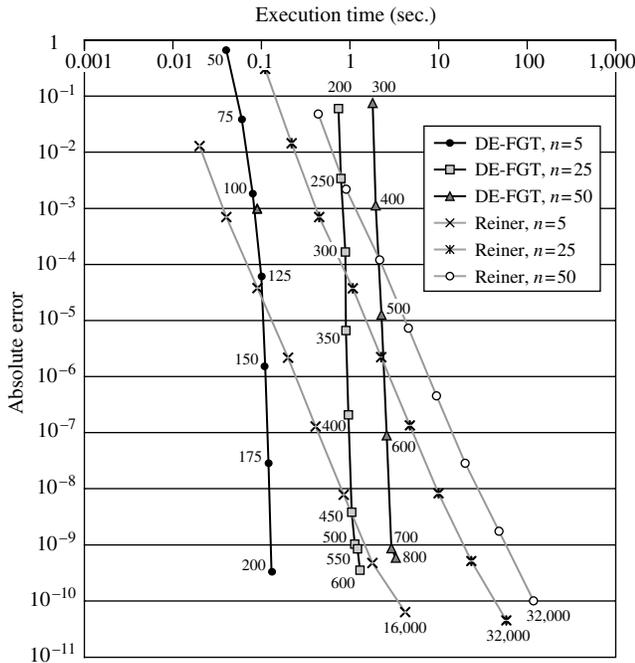
where $E_t[\cdot]$ is the conditional expectation operator under Q given information up to time t.

Equation (47) involves two random variables S_n and M_n at time T, so we need the joint distribution function of them to compute the expectation value. To reduce the dimensionality of the problem, we apply a change of measure (see Babbs 2000, Andreasen 1998) and rewrite Equation (47) as

$$Q_0^{LP}(S_0) = e^{-qT} S_0 E_0' \left[\frac{M_n}{S_n} - 1 \right], \tag{48}$$

where $E_t'[\cdot]$ denotes the conditional expectation operator under a new measure Q' defined by

$$dQ' = \frac{S_n}{S_t e^{(r-q)(T-t)}} dQ. \tag{49}$$

Under this measure,

$$W_t' = W_t - \sigma t \tag{50}$$

becomes a Brownian motion and the stochastic differential equation (1) becomes

$$dS_t = (r - q + \sigma^2) S_t \, dt + \sigma S_t \, dW_t'. \tag{51}$$

By further introducing the log stock prices

$$s_i = \ln(S_i/S_0) \quad \text{and} \tag{52}$$

$$m_i = \ln(M_i/S_0), \tag{53}$$

we can finally write the option price as

$$Q_0^{LP}(S_0) = e^{-qT} S_0 E_0'[e^{m_n - s_n} - 1]. \tag{54}$$

This shows that it is sufficient to find the distribution of $m_n - s_n$ to compute the option price.

To compute Equation (54), we consider the distribution of $m_i - s_i$ $(i = 0, 1, \ldots, n)$. Apparently, the distribution function is zero when $m_i - s_i < 0$ and there is a finite probability mass at $m_i - s_i = 0$. We therefore represent the distribution of $m_i - s_i$ by two quantities, namely, a scalar c_i, which represents the probability that $m_i - s_i = 0$, and a function $g(x)$ $(x > 0)$, which represents the probability density in the region $m_i - s_i > 0$. Note that the pdf of $m_i - s_i$ can be formally written as $c_i \delta(x) + g_i(x)$, where $\delta(x)$ is the Dirac delta function. At time 0, we have

$$c_0 = 1 \quad \text{and} \tag{55}$$

$$g_0(x) = 0 \tag{56}$$

by definition. To compute c_i and $g_i(x)$ given c_{i-1} and $g_{i-1}(x)$, we use the identity

$$m_i - s_i = \max(0, m_{i-1} - s_i)$$
$$= \max(0, (m_{i-1} - s_{i-1}) + (s_{i-1} - s_i)). \tag{57}$$

Because $s_{i-1} - s_i$ is an increment of the Brownian motion between time t_{i-1} and t_i and is independent of $m_{i-1} - s_{i-1}$, Equation (57) shows that the pdf of $m_i - s_i$ is obtained by computing the convolution of the pdf of $m_{i-1} - s_{i-1}$ with that of $s_{i-1} - s_i$ and collecting all the probability mass corresponding to $m_i - s_i < 0$ to the point $m_i - s_i = 0$. In summary, we have the following recursion formula:

$$\bar{g}_i(x) = \int_{-\infty}^{\infty} \{c_{i-1}\delta(y) + g_{i-1}(y)\} f(x - y) \, dy$$
$$= c_{i-1} f(x) + \int_0^{\infty} g_{i-1}(y) f(x - y) \, dy, \tag{58}$$

$$c_i = \int_{-\infty}^0 \bar{g}_i(x), \tag{59}$$

$$g_i(x) = \bar{g}_i(x) \quad (x > 0), \tag{60}$$

where $f(x)$ is the probability density function of $s_{i-1} - s_i$ given by

$$f(x) = \frac{1}{\sqrt{2\pi \Delta t}\sigma} \exp\left\{ -\frac{\left(x + (r - q + \frac{1}{2}\sigma^2)\Delta t\right)^2}{2\sigma^2 \Delta t} \right\}$$
$$= p^G \left(x + (r - q + \frac{1}{2}\sigma^2)\Delta t\right). \tag{61}$$

Equations (55), (56), (58), (59), and (60), along with Equation (54), provide us with a means of computing the lookback option price by a series of convolutions and integrations.

3.2. Application of the DE-FGT Method

The application of the DE formula and the FGT is now straightforward. We first rewrite the integral (58) as

$$\bar{g}_i(x) = c_{i-1}f(x) + \int_{-b}^{\infty} g_{i-1}(y'+b)p^G(x-y')\,dy', \quad (62)$$

where

$$b = \left(r - q + \tfrac{1}{2}\sigma^2\right)\Delta t. \quad (63)$$

This convolution has the same form as that in Equation (11), and the method described in the previous section can be applied. After calculating $g_i(x)$, we can use the DE formula again to compute c_i by Equation (59).

3.3. Extensions of the Basic Algorithm

3.3.1. Pricing of Hindsight Options.
Using the notation we have defined in §3.1, the discrete hindsight call option with strike K is defined as an option whose payoff at T is $(M_n - K)^+$. To compute the price $Q_0^{HS}(S_0, K)$ of this option, we use the following relationship between the hindsight calls and the lookback puts (see, e.g., Broadie et al. 1999):

$$Q_0^{HS}(S_0, K) = Q_0^{LP}(S_0, K) + S_0 - e^{-rT}K, \quad (64)$$

where $Q_0^{LP}(S_0, M)$ is the price of a generalized lookback option for which the historical maximum asset price at time 0 is regarded to be a given value M instead of S_0. We can compute the price of such an option easily by changing the distribution of $m_0 - s_0$ from $\delta(x)$ (as given in Equations (55) and (56)) to $\delta(x - \ln(M/S_0))$. Then, by putting the result into Equation (64), we obtain the hindsight option price.

3.3.2. Application to Merton's Model.
We next consider the pricing of the lookback option under Merton's model (1995). Andreasen (1998) shows that under the probability measure Q' defined by Equation (49), S_i follows a new equation:

$$S_i = S_{i-1}\exp\left\{\left(r - q + \tfrac{1}{2}\sigma^2 - \nu\lambda\right)\Delta t + \sigma\sqrt{\Delta t}z_0'\right.$$

$$\left. + \sum_{l=1}^{N_t'^P(\Delta t)}\left(\delta z_l' + \gamma + \tfrac{1}{2}\delta^2\right)\right\}, \quad (65)$$

where $N_t'^P(\Delta t)$ follows a Q' Poisson process with intensity

$$\lambda' = \lambda e^{\gamma} \quad (66)$$

and z_l' $(l = 0, 1, \ldots)$ are independent and follow the standard normal distribution $N(0, 1)$ under Q'. If we define the log stock price by Equation (52), we can show that $s_{i-1} - s_i$ follows the distribution

$$f(x) = \sum_{n=0}^{\infty} e^{-\lambda'\Delta t}\frac{(\lambda'\Delta t)^n}{n!}\frac{1}{\sqrt{2\pi}\sigma_n}\exp\left\{-\frac{(x+\mu_n)^2}{2\sigma_n^2}\right\}, \quad (67)$$

where

$$\sigma_n^2 = \sigma^2\Delta t + n\delta^2 \quad \text{and} \quad (68)$$

$$\mu_n = \left(r - q + \tfrac{1}{2}\sigma^2 - \nu\lambda\right)\Delta t + n\left(\gamma + \tfrac{1}{2}\delta^2\right). \quad (69)$$

Then, we have only to replace the function $f(x)$ given by Equation (61) with that given by Equation (67) in the computation of Equation (58). The resulting convolution can be computed efficiently by the DE-FGT method explained in §2.5.3.

3.4. Numerical Results

3.4.1. Hindsight Call Option Under the Black-Scholes Model.
Figure 6 shows the results of our DE-FGT method applied to hindsight call options under the Black-Scholes model. The parameters are $S_0 = K = 100$, $T = 0.5$, $r = 0.1$, $q = 0$, and $\sigma = 0.3$. The number of monitoring dates are $n = 5$, 25, or 50. As reference prices, we used the values given in Tse et al. (2001), which were computed using the tridiagonal probability algorithm and are claimed to be accurate up to 10 decimal places. These values, along with the computational time for the tridiagonal algorithm, are shown in Table 5. For comparison, we also included the results obtained with Broadie et al.'s (1999) trinomial lattice method with 5,000 time steps.

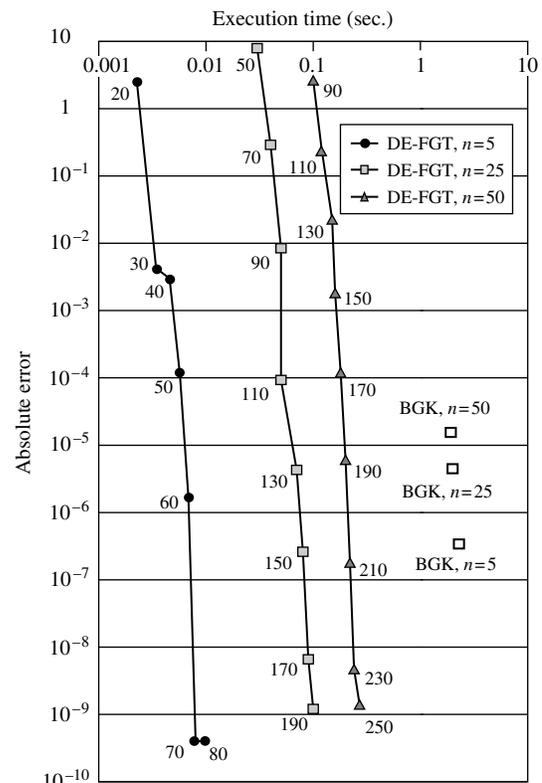**Figure 6.** Hindsight call option price under the Black-Scholes model.

Table 5.   Hindsight call option prices under the Black-Scholes model.

	$n = 5$	$n = 25$	$n = 50$
Price	14.9413046399	17.6028684623	18.3264598300
Time (tridiagonal)	0.66s	20.55s	442.48s

Here we again see that our method is extremely fast and attains an accuracy of 10^{-10} within a fraction of one second even for the case of $n = 50$. The lattice method gives reasonably accurate results, but requires much longer computational time. It is not easy to compare the performance of our method with that of the tridiagonal algorithm, because the implementation in Tse et al. (2001) is based on MATLAB, while ours is based on C++. However, we can point out that while the computational time of the tridiagonal probability algorithm increases drastically with the number n of monitoring dates, our computational time seems to increase only as $O(n\sqrt{n})$ when the accuracy of the result is fixed. This is natural, because the width of the transition probability density function $p(x_i \mid x_{i-1})$ is proportional to $\sqrt{\Delta t} = \sqrt{T/n}$, and therefore we need the step size proportional to $\sqrt{T/n}$ to attain the same level of accuracy. Hence, the computational work for each convolution is $O(\sqrt{n})$ and the total work is $O(n\sqrt{n})$.

4. Bermudan Options

4.1. The DE-FGT Method for Bermudan Options

The rational price of Bermudan options can be calculated as a discounted expectation value (under the risk-neutral measure) of the payoff under the optimal (adapted) exercise strategy, that is,

$$Q_0(S_0) = \sup_{\tau} e^{-r\tau} E_0[h_\tau(S_\tau)], \tag{70}$$

where $h_t(S_t)$ is the payoff from exercise at time t and τ is a stopping time that takes a value on the discrete set $\{t_i\}_{i=1}^n$. It is well known that $Q_0(S_0)$ can be computed by the backward recursion

$$Q_i(S_i) = \max\{h_i(S_i), C_i(S_i)\}, \tag{71}$$

$$C_i(S_i) = e^{-r\Delta t} E_i[Q_{i+1}(S_{i+1})], \tag{72}$$

where Q_i and C_i are the option value and the continuation value at time t_i, respectively, and $E_i[\cdot]$ is the expectation value operator given information up to time t_i. Equation (72) can be written more explicitly using the transition probability density function $p(S_{i+1} \mid S_i)$:

$$C_i(S_i) = e^{-r\Delta t} \int_0^\infty p(S_{i+1} \mid S_i)$$
$$\cdot \max\{h_{i+1}(S_{i+1}), C_{i+1}(S_{i+1})\} \, dS_{i+1}. \tag{73}$$

From the numerical point of view, Equation (73) is difficult to evaluate accurately because the integrand contains the max operator and its higher-order derivatives are discontinuous. For the standard Bermudan option,

$$h_i(S_i) = (S_i - K)^+, \tag{74}$$

and there is a value S_i^c called the optimal exercise price for each exercise date i ($i = 0, 1, \ldots, n$) such that $h_i(S_i) > C_i(S_i)$ for $S > S_i^c$ and $h_i(S_i) \leqslant C_i(S_i)$ for $S \leqslant S_i^c$. Hence, the higher-order derivatives of the integrand are discontinuous at $S_i = S_i^c$. This is the main reason why the convergence of the high-order multinomial methods (Alford and Webber 2001, Broadie and Yamamoto 2003) is slower for Bermudan options than for European options. The DE formula applied directly to Equation (73) will not work well either, for the efficiency of the DE formula hinges on the assumption that the integrand is analytical.

We therefore choose to locate the optimal exercise price at each exercise date by the bisection method, divide the integration region into two at the price, and then apply the DE formula for each of the subregions. We move to the log asset price defined by Equation (6) and denote the log optimal exercise price at t_i by x_i^c. Then, Equation (73) can be rewritten as

$$C_i(x_i) = e^{-r\Delta t} \left\{ \int_{-\infty}^{x_{i+1}^c} p(x_{i+1} \mid x_i) h_{i+1}(x_{i+1}) \, dx_{i+1} \right.$$
$$\left. + \int_{x_{i+1}^c}^\infty p(x_{i+1} \mid x_i) C_{i+1}(x_{i+1}) \, dx_{i+1} \right\}. \tag{75}$$

Both of the integrals appearing in Equation (75) become convolutions under the Black-Scholes or Merton's model and can be computed efficiently by the DE-FGT methods we have described in §§2.4 and 2.5.3, respectively.

To be more specific, assume that we know the value of x_{i+1}^c, that $\{a_j^{i+1}\}$ and $\{b_j^{i+1}\}$ are the sample points of the DE formula in the regions $[x_{i+1}^c, \infty]$ and $[-\infty, x_{i+1}^c]$, respectively, and that we have the the values of $h_{i+1}(x_{i+1})$ at points $\{a_j^{i+1}\}$ and the values of $C_{i+1}(x_{i+1})$ at points $\{b_j^{i+1}\}$. Clearly, these assumptions hold for $i = n - 1$. Then, one time step of our method proceeds as follows:

(1) Determine a lower bound x_L^c and an upper bound x_H^c for x_i^c. Then, using the fact that the value of $C_i(x_i)$ for an arbitrary point x_i can be computed from the given values of $h_{i+1}(x_{i+1})$ and $C_{i+1}(x_{i+1})$ through Equation (75) and the DE formula, compute the log optimal exercise price x_i^c by the bisection method.

(2) Generate the sample points $\{a_{j'}^i\}$ of the DE formula in the region $[x_i^c, \infty]$. Also, generate sample points $\{b_{j'}^i\}$ of the DE formula in the region $[-\infty, x_i^c]$.

(3) Compute the values of $C_i(x_i)$ at $\{a_{j'}^i\}$ and $\{b_{j'}^i\}$ by the DE-FGT method.

It may appear that the computation of x_i^c by the bisection method requires extra work and lowers the efficiency of this method. However, we can reduce this work by also

using the FGT for Step 1 above. Let m and N denote the number of points used in the bisection and the number of sample points used in the DE formula, respectively. If direct evaluation is used in the bisection to compute $C_i(x_i)$, the computational work is $O(mN)$. Instead, we can perform Steps 1 and 2 of the FGT (see §2.3) before the bisection, using the sample points $\{a_j^{i+1}\}$ and the values of $h_{i+1}(x_{i+1})$ there. Note that in these two steps, no information on the target points is necessary. We then have the coefficients $B_{\beta, J}$. Similarly, we perform Steps 1 and 2 using the sample points $\{b_j^{i+1}\}$ and the values of $C_{i+1}(x_{i+1})$ there, and obtain another set of coefficients $B'_{\beta, J}$. Once we have these coefficients, we can evaluate $C_i(x_i)$ for each x_i in $O(1)$ work, using Step 3 of the FGT. Hence, the work for the bisection can be reduced to $O(m)$. Note that the computation of $B_{\beta, J}$ and $B'_{\beta, J}$ was originally done in Step 3 of the above algorithm, so they incur no extra work.

It is also possible to reduce m by a judicious choice of the lower bound x_L^c and the upper bound x_H^c. For example, because the optimal exercise boundary of the Bermudan call option is a monotonically decreasing function of t, x_{i+1}^c gives a lower bound on x_i^c. An upper bound can easily be obtained from the boundary for an infinite maturity American call option.

The DE-FGT method we proposed in this subsection has computational work of $O(n(m+N))$ when the number of exercise dates is n. It can also be expected that the computational error decreases faster than any negative power of N because both of the integrands appearing in Equation (75) are analytical. We will confirm this in the next subsection.

4.2. Numerical Results

4.2.1. Bermudan Call Options Under the Black-Scholes Model.
We show results for Bermudan call options under the Black-Scholes model in Figure 7. The parameters are $S_0 = 100$, $T = 0.5$, $r = 0.03$, $q = 0.07$, $\sigma = 0.2$, and $n = 10$. The strike price K was varied from 90 to 110 in increments of 5. The reference values computed using the multinomial FGT method (Broadie and Yamamoto 2003) are shown in Table 6. The multinomial-FGT method is a variant of the multinomial method, which has a large number of branches and which uses the FGT to speed up the computation of the continuation values. The number of branches used is $2b + 1$, where $b = 409,600$.

In Figure 7, we show the results of three numerical methods: the binomial method, the multinomial-FGT method, and our DE-FGT method. The number of time steps for the binomial method, the value of b for the multinomial-FGT method, and the number of sample points N for the

Figure 7. Bermudan call option price under the Black-Scholes model.
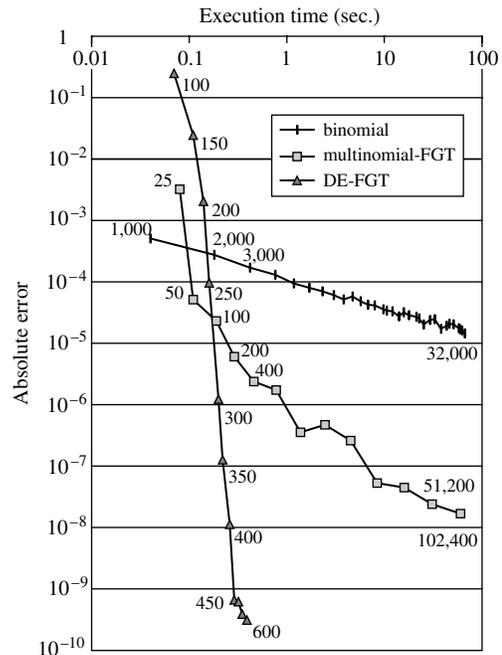
DE-FGT method are also shown in the graph. It can be seen from the graph that the binomial method is acceptable if low accuracies are required, but if higher accuracies are required, the multinomial-FGT method is much faster. Our DE-FGT method is even faster and shows exponential decrease of the absolute error, attaining an accuracy of 10^{-10} within one CPU second.

4.2.2. Bermudan Put Options Under Merton's Model.
Finally, we show pricing results for Bermudan put options under Merton's jump-diffusion model. The parameters are $S_0 = 40$, $K = 30$ to 50 (in increments of 5), $T = 1.0$, $r = 0.08$, $q = 0$, $\sigma = \sqrt{0.05}$, $\lambda = 5.0$, $\gamma = 0$, and $\delta = \sqrt{0.05}$. We show the reference values computed using a modified version of the multinomial-FGT method (referred to as FGT II in Broadie and Yamamoto 2003) with $b = 102,400$ in Table 7.

Here we compare the performance of four numerical methods: Amin's algorithm (1993), FGT I, FGT II (Broadie and Yamamoto 2003), and our DE-FGT method. FGT I is a variant of Amin's algorithm that uses the FGT to reduce the work of computing the continuation values at each time step from $O(N^2)$ to $O(N)$. FGT II is a multinomial-FGT

Table 6. Bermudan call option price under the Black-Scholes model.

	$K = 90$	$K = 95$	$K = 100$	$K = 105$	$K = 110$
Price	10.73001013	7.32288562	4.75727741	2.94105489	1.73255637

Table 7. Put option prices under Merton's model.

|	$K = 30$	$K = 35$	$K = 40$	$K = 45$	$K = 50$
Price	2.70636133	4.58094453	6.99527475	9.90671277	13.25586096

method that takes the effect of jumps into account. See Broadie and Yamamoto (2003) for more details about these two methods. The convergence results of the four methods are shown in Figure 8. The numbers in the graph represent the number of grid points or sample points used at each time step. Again, the two FGT-based methods, FGT I and FGT II, converge much faster than Amin's algorithm, but the DE-FGT method shows exponential convergence and is even faster.

5. Conclusion

In this paper, we have shown that under the Black-Scholes framework, the price of many path-dependent and quasi path-dependent options—such as the barrier, lookback, hindsight, and Bermudan options—can be computed by a series of convolutions of the Gaussian distribution and a known function. Using this fact, we proposed a new pricing algorithm, the DE-FGT method, that computes this convolution by a combination of the double-exponential integration formula and the FGT.

The computational work of our method is $O(nN)$ when the number of monitoring/exercise dates is n and the number of sample points at each date is N. Theoretically, the errors of our method decrease faster than any negative power of N, and our experiments on the above four options show an exponential decrease of the errors. We also extend the method to Merton's lognormal jump-diffusion model
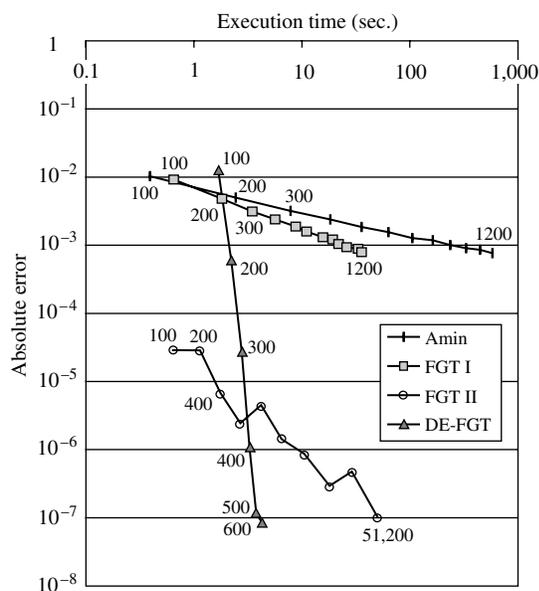
and obtain the same order of computational work and convergence properties as in the Black-Scholes case.

Acknowledgments

References

Ahn, D. H., S. Figlewski, B. Gao. 1999. Pricing discrete barrier options with an adaptive mesh model. *J. Derivatives* **6**(4) 33–44.

Alford, J., N. Webber. 2001. Very high order lattice methods for one factor models. Working paper, Cass Business School, London, UK.

Amin, K. 1993. Jump diffusion option valuation in discrete time. *J. Finance* **48**(5) 1833–1863.

Andreasen, J. 1998. The pricing of discretely sampled Asian and lookback options: A change of numeraire approach. *J. Computational Finance* **2**(1) 5–30.

Babbs, S. 2000. Binomial valuation of lookback options. *J. Econom. Dynamics Control* **24**(11–12) 1499–1525.

Bates, D. 1996. Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche mark options. *Rev. Financial Stud.* **9** 69–107.

Baxter, B., G. Roussos. 2002. A new error estimate of the fast Gauss transform. *SIAM J. Sci. Comput.* **24**(1) 257–259.

Black, F., M. Scholes. 1973. The pricing of options and corporate liabilities. *J. Political Econom.* **81** 637–654.

Boyle, P. 1988. A lattice framework for option pricing with two state variables. *J. Financial Quant. Anal.* **23** 1–12.

Boyle, P., S. H. Lau. 1994. Bumping up against the barrier with the binomial method. *J. Derivatives* **1**(4) 6–14.

Boyle, P., Y. S. Tian. 1998. An explicit finite difference approach to the pricing of barrier options. *Appl. Math. Finance* **5**(1) 17–43.

Boyle, P., Y. S. Tian. 1999. Pricing lookback and barrier options under the CEV process. *J. Financial Quant. Anal.* **34**(2) 241–264.

Broadie, M., Y. Yamamoto. 2003. Application of the fast Gauss transform to option pricing. *Management Sci.* **49**(8) 1071–1088.

Broadie, M., P. Glasserman, S. G. Kou. 1997. A continuity correction for discrete barrier options. *Math. Finance* **7**(4) 325–349.

Broadie, M., P. Glasserman, S. G. Kou. 1999. Connecting discrete and continuous path-dependent options. *Finance Stochastics* **3**(1) 55–82.

Cheuk, T., T. Vorst. 1996. Complex barrier options. *J. Derivatives* **4**(1) 8–22.

Conze, A., R. Viswanathan. 1991. Path dependent options: The case of lookback options. *J. Financial Econom.* **7** 229–263.

Cox, J., S. Ross, M. Rubinstein. 1979. Option pricing: A simplified approach. *J. Financial Econom.* **7** 229–263.

Davydov, D., V. Linetsky. 2001. The valuation and hedging of barrier and lookback options under the CEV process. *Management Sci.* **47** 949–965.

Duan, J.-C., E. Dudley, G. Gauthier, J.-G. Simonato. 2003. Pricing discretely monitored barrier options by a Markov chain. *J. Derivatives* **10**(4) 9–31.

Figure 8. Bermudan put option price under the Merton jump-diffusion model.

Duffie, D. 2001. *Dynamic Asset Pricing Theory*, 3rd ed. Princeton University Press, Princeton, NJ.

Dutt, A., V. Rokhlin. 1993. Fast Fourier transforms for nonequispaced data. *SIAM J. Sci. Comput.* **14**(6) 1368–1393.

Evans, G. 1993. *Practical Numerical Integration*. John Wiley and Sons, New York.

Goldman, M., H. Sosin, M. A. Gatto. 1979. Path-dependent options: Buy at the low, sell at the high. *J. Finance* **34** 1111–1128.

Greengard, L., J. Strain. 1991. The fast Gauss transform. *SIAM J. Sci. Statist. Comput.* **12**(1) 79–94.

Greengard, L., X. Sun. 1998. A new version of the fast Gauss transform. *Documenta Mathematica*, extra volume **ICM**(III) 575–584.

Kou, S. 2002. A jump diffusion model for option pricing. *Management Sci.* **48**(8) 1086–1101.

Krommer, A., C. Ueberhuber. 1998. *Computational Integration*. SIAM, Philadelphia, PA.

Kwok, Y. K. 1998. *Mathematical Models of Financial Derivatives*. Springer-Verlag, Singapore.

Merton, R. 1992. *Continuous-Time Finance*. Basil Blackwell, London, UK.

Mori, M. 1985. Quadrature formulas obtained by variable transformation and the DE-rule. *J. Computational Appl. Math.* **12 & 13** 119–130.

Naik, V., M. Lee. 1990. General equilibrium pricing of options on the market portfolio with discontinuous returns. *Rev. Financial Stud.* **3**(4) 493–521.

Ooura, T., M. Mori. 1991. The double exponential formula for oscillatory functions over the half infinite interval. *J. Computational Appl. Math.* **38** 353–360.

Reiner, E. 2000. Convolution methods for exotic options. Presentation at Columbia University, New York.

Rich, D. R. 1994. The mathematical foundations of barrier option pricing theory. *Adv. Futures Options Res.* **7** 267–371.

Ritchken, P. 1995. On pricing barrier options. *J. Derivatives* **3**(2) 19–28.

Rubinstein, M., E. Reiner. 2000. Breaking down the barriers. *RISK* **8**(September) 28–35.

Sugihara, M. 1997. Optimality of the double exponential formula—Functional analysis approach. *Numerische Mathematik* **75** 379–395.

Sullivan, M. A. 2000. Pricing discretely monitored barrier options. *J. Computational Finance* **3**(4) 35–52.

Takahashi, H., M. Mori. 1974. Double exponential formulas for numerical integration. *Publ. RIMS Kyoto Univ.* **9** 721–741.

Tse, W. M., L. K. Li, K. W. Ng. 2001. Pricing discrete barrier and hindsight options with the tridiagonal probability algorithm. *Management Sci.* **47**(3) 383–393.

Ware, A. F. 1998. Fast approximate Fourier transforms for irregularly spaced data. *SIAM Rev.* **40**(4) 838–856.

Zvan, R., K. Vetzal, P. Forsyth. 2000. PDE methods for pricing barrier options. *J. Econom. Dynam. Control* **24** 1563–1590.