Experimentation under Uninsurable Idiosyncratic Risk: An Application to Entrepreneurial Survival[*]

Jianjun Miao[†] and Neng Wang[‡]

May 28, 2007

Abstract

We propose an analytically tractable continuous-time model of experimentation in which a risk-averse entrepreneur cannot fully diversify the idiosyncratic risk from his business investment. He makes consumption/savings and business exit decisions jointly, while learning about the unknown quality of the project over time. Using the closed-form solutions, we show that (i) the entrepreneur may stay in business even though the project's net present value (NPV) is negative; (ii) entrepreneurial risk aversion erodes option value and lowers private project value so that a sufficiently risk-averse entrepreneur may exit even when the NPV is positive; (iii) a more risk-averse or a more pessimistic entrepreneur exits earlier; and (iv) the model can generate a positive relation between wealth and entrepreneurial survival duration from undiversifiable idiosyncratic risk without liquidity constraints.

JEL Classification: D81, D83, D91, E21, L26

Keywords: bandit problems, real options, learning, private firm value, survival, precautionary savings, incomplete markets

1   Introduction

We analyze a two-armed bandit problem in which one arm is safe and offers a deterministic payoff, and the other arm is risky and has unknown quality. If the risky arm is good, then it generates a payoff higher than the safe arm after an exponentially distributed random time. If it is bad, then it generates a payoff lower than the safe arm forever. The decision maker is risk averse and chooses between these two arms, recognizing the dynamic nature of this decision problem. In addition, the decision maker chooses his intertemporal consumption/savings decision, and cannot fully diversify the idiosyncratic risk of the risky arm.

One application of this type of bandit problems is entrepreneurship. Consider an example in which a risk-averse entrepreneur decides whether to stay in his risky business or to quit and accept a safe job. If the entrepreneur stays in his risky business, then he receives profits from an investment with unknown quality. If the investment project is of low quality, then it represents a business failure and thus generates smaller present values than the safe project. However, if this project is of high quality, then it generates larger payoffs after the arrival of profitable growth opportunities. The arrival of these opportunities may be modeled as a Poisson process.

Bandit problems have been analyzed extensively under risk neutrality in the literature.[1] To the best of our knowledge, we are the first to analyze the impact of the decision maker's risk aversion on his experimentation strategy. The effect of risk aversion on experimentation is potentially important when the entrepreneur cannot fully diversify the idiosyncratic risk of the risky project for reasons such as moral hazard and information asymmetries. As a result, the entrepreneur's intertemporal consumption/savings decisions become intertwined with his experimentation strategy. Ample empirical evidence documents that an entrepreneur's investment opportunities have substantial undiversifiable idiosyncratic risks.[2] The entrepreneur's well-being depends heavily on the outcome of his investments. Moreover, the quality of his investment opportunities is often unknown to the entrepreneur before starting the project. By experimenting with his investment opportunities, the entrepreneur learns about the quality of

---

[1]See the early related contribution by Jovanovic (1979, 1982). See Bergemann and Valimaki (2006) for a survey and references cited therein.

[2]See Gentry and Hubbard (2004), Heaton and Lucas (2000), and Moskowitz and Vissing-Jorgensen (2002), among others.

these opportunities over time. Unlike models based on risk neutrality as in the literature, the risk from experimentation cannot be fully diversified and hence demands a risk premium from the entrepreneur's perspective. Hence, entrepreneurs may exhibit different choice behavior due to the undiversifiable idiosyncratic project risk. That is, we expect that entrepreneurs' risk attitudes play an important role in determining their interdependent consumption/savings and investment/disinvestment decisions.

In this paper, we use entrepreneurial investment as a primary application of our model. While entrepreneurial activities have other important dimensions, such as how much to invest, and how to finance investment, for simplicity, we focus on the exit-timing aspect of entrepreneurial activities. We extend the standard two-armed bandit models by using a utility-maximization framework to analyze the effects of the entrepreneur's risk aversion and uninsurable idiosyncratic risks on his learning and exit decisions.

We show that learning has option value. That is, exit is not a now-or-never decision. The entrepreneur may stay in business for a while and continue to experiment with his risky investment because he waits for new information to come in, hoping the quality of the risky project is high. We also show that risk aversion erodes the option value of learning when the entrepreneur cannot fully diversify the idiosyncratic risk from his investment. The net present value (NPV) rule often recommended by practitioners and economics and business textbooks can go wrong when applied to analyzing the entrepreneurial exit decisions. To elaborate on this point, we start with the constant absolute risk aversion (CARA) utility specification. We show that when the entrepreneur's degree of risk aversion is not very high (including risk neutral entrepreneur as a special case), he continues to stay in business even though the NPV of the risky project is negative. This result may provide a potential explanation for Hamilton's (2000) empirical evidence that entrepreneurial projects do not seem to generate enough NPV. We also show that a more risk-averse entrepreneur exits earlier because the nontraded risky project is less valuable to a more risk-averse entrepreneur. When the entrepreneur is sufficiently risk averse, he may exit at a time even when the NPV of the risky project is positive.

Finally, we study the impact of wealth effect on business exit decisions. We use a power utility function which features constant relative risk aversion (CRRA). Our model generates a

positive relation between entrepreneurial wealth and survival duration without liquidity constraints.[3] The intuition behind this result is as follows: First, higher wealth makes the risk-averse entrepreneur more prone to take risks. This point has been made by Cressy (2000) in a static model. Second, in our dynamic model, the entrepreneur has preferences for intertemporal consumption smoothing. Higher wealth gives the entrepreneur more buffer to guard against future income fluctuation. Thus, entrepreneurs with higher wealth survive longer in business.

While our analysis borrows insights from the standard real options approach to investment,[4] our model differs from this approach along several dimensions. First, the standard real options approach to investment assumes that all idiosyncratic risk is diversifiable or investors are risk neutral. Thus, the decision maker's risk aversion does not play any direct role in conventional real options models (after the usual risk premium correction). Miao and Wang (2007) is an exception to the literature. As they analyze optimal investment timing decision when the entrepreneur cannot fully diversify the idiosyncratic risks (also see Henderson (2005) and Hugonnier and Morellec (2005)). Unlike Miao and Wang (2007), here we analyze an exit (disinvestment) problem when the entrepreneur learns about the quality of his investment opportunity. Second, in the standard real options approach, the investment cost or the outside option is exogenous. In our model, the outside option is endogenous in the sense that its value is given by the optimal life-time utility obtained from the alternative safe job and hence is wealth dependent. Finally, there is no learning in the standard real options approach, while in our model the entrepreneur learns about the quality of the investment opportunity over time and makes the exit decision based on the updated belief about this quality.

Our model studies a single entrepreneur's decision problem and does not involve any strategic experimentation issues.[5] Unlike most continuous-time experimentation models, which assume that uncertainty is generated by a Brownian motion, our model assumes that uncertainty about the project quality is resolved upon the arrival of a Poisson shock, as in Keller et al. (2005) and other related papers cited therein. Unlike these papers, which assume risk neu-

---

[3]Evans and Jovanovic (1989) Holtz-Eakin et al. (1994) argue that liquidity constraints are important for entrepreneurial entry and exit decisions.

[4]See Brennan and Schwartz (1985), McDonald and Siegel (1986), and Dixit and Pindyck (1994).

[5]See Bolton and Harris (1999), Grenadier (1999), and references cited therein for models of strategic experimentation.

trality, in our model the decision maker is risk averse and also makes consumption-savings decisions. Despite this added complexity, we are still able to derive closed-form solutions for two commonly used utility specifications (CARA and CRRA).

The remainder of the paper is organized as follows. Section 2 sets up the model and the decision problem. Section 3 analyzes the model with the CARA utility specification. Section 4 studies the wealth effect on entrepreneurial exit decisions by using the CRRA utility specification. Section 5 concludes. Proofs are relegated to an appendix.

2   The Model

In this section, we first set up the model. We then describe the entrepreneur's learning process. Finally, we formulate his decision problem.

2.1   Setup

Consider an infinite horizon continuous time model. An entrepreneur derives utility from a consumption process $\{c(t) : t \geq 0\}$ according to the following expected utility function:

$$E\left[\int_0^\infty e^{-\rho t} u(c_t)\, dt\right],\tag{1}$$

where $\rho > 0$ is the rate of time preference, and $u(c)$ is concave and strictly increasing. Since the discount rate does not play an important role in our analysis, we assume that ρ is equal to the constant interest rate r. The entrepreneur starts with initial wealth x_0. He makes consumption/saving decisions over time by borrowing or lending at the constant risk-free interest rate r. He also decides whether to continue in business or to quit and accept a safe job.

If the entrepreneur accepts the safe job, he obtains a constant flow $z > 0$ of income per period. If he stays in business, he receives income from a risky investment. The quality of the risky investment could be high or low. The entrepreneur does not know about this quality. The entrepreneur may only learn about the quality of this risky investment by taking (experimenting with) this risky investment. Any time the entrepreneur leaves this risky investment, he no longer receives information about the quality of this investment. Let $0 < p_0 < 1$ denote his prior belief that the risky investment is of low quality. If the risky investment is indeed of

4

low quality, the entrepreneur will receive a perpetual income stream at a constant flow rate $y_1 > 0$. In order to have a non-trivial trade-off for the entrepreneur, assume that the payoff from the low quality investment is strictly dominated by the income from the safe job, in that $y_1 < z$. If the investment is of high quality, then the entrepreneur will receive a jump of his income from y_1 to a permanently higher level y_2 at the stochastic time. The arrival time for the permanent jump from y_1 to y_2 occurs randomly with probability $\lambda \Delta t$, over a small time interval Δt, if the risky investment project is of high quality. The average time for the entrepreneur to receive an income jump (from the high quality risky investment project) is $1/\lambda$, because of the exponential distribution for the stochastic arrival time. Given the stochastic arrival nature of the permanent raise for the high-quality investment, the quality of the risky investment may never be fully discovered for sure in any finite time.

While the safe job pays more than the income from the low quality investment $(y_1 < z)$, the entrepreneur may receive a higher income level in the future if the quality of the risky investment turns out to be high. Thus, the entrepreneur may have incentives to continue in business and experiment with the risky investment. Since the entrepreneur does not know the quality of the risky investment, he learns about it by updating his prior using the Bayes Rule given the information about the past investment outcomes. The entrepreneur trades off the benefit of discovering more information about the quality of the risky investment with the cost of receiving a lower income level $y_1 < z$ today. When the entrepreneur's belief about the quality of the risky project is pessimistic enough, he quits from his business and accepts the safe job.

We assume that the entrepreneur is not well diversified and has limited access to the capital market. In order to focus on the effect of undiversified investment risk, we assume that the entrepreneur can only trade a risk-free asset to partially insure himself against his project risk. Specifically, let $\{x(t) : t \geq 0\}$ denote the wealth process and $\{y(t) : t \geq 0\}$ denote the risky income process. Then, the entrepreneur's wealth process $\{x(t) : t \geq 0\}$ satisfies

$$dx_t = (rx_t + y_t - c_t) \, dt. \tag{2}$$

The entrepreneur's objective is to maximize his lifetime utility function (1), subject to his wealth accumulation dynamics (2), his business opportunity sets (including the safe job and

the risky investments), and certain regularity conditions to rule out Ponzi schemes.[6]

2.2 Learning

We now describe the entrepreneur's learning problem. Let $p(t)$ denote the entrepreneur's posterior belief that the quality of the risky investment is low at time t, given that the entrepreneur has received no jump in income until time t. Over a small time interval Δt, the entrepreneur's income jumps to y_2 permanently with probability $\lambda \Delta t$, conditional on the investment being of high quality. The Bayes rule implies that the posterior probability $p(t + \Delta t)$ is given by

$$p(t + \Delta t) = \frac{p(t)}{p(t) + (1 - p(t))(1 - \lambda \Delta t)}. \tag{3}$$

Taking the limit as Δt goes to 0 gives the following differential equation[7]

$$dp(t) = \lambda p(t)(1 - p(t))dt, \tag{4}$$

with initial belief $p(0) = p_0$.

The Bayesian updating process (4) implies the following properties: (i) the posterior belief $p(t)$ is increasing in time t; (ii) the change of belief $\dot{p}(t)$ is symmetric in $p(t)$ and $(1 - p(t))$; and (iii) the speed of learning increases with the arrival rate λ. The intuition behind property (i) is as follows. The longer the entrepreneur takes the risky investment and does not receive any income jump, the more likely the investment is of low quality. In the limit $(t \to \infty)$, belief $p(t)$ will converge to 1 and the entrepreneur will learn the truth. Property (ii) reflects the fact that the quality of the risky investment has to be either high or low. Thus, that rate of increase in posterior belief $p(t)$ is equal to the absolute value of the rate of decrease in the posterior belief $(1 - p(t))$ that investment is of low quality. Property (iii) states that the entrepreneur learns the investment quality faster when the arrival of income jump is faster.

---

[6]For the CARA utility, we impose the transversality condition given in the appendix. For the CRRA utility, we use the borrowing constraint $x_t \geq -y_1/r$. It turns out that this constraint is never binding at optimum.

[7]Note that

$$\frac{dp(t)}{dt} = \lim_{\Delta t \to 0} \frac{p(t + \Delta t) - p(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{p(t)}{\Delta t} \frac{\lambda(1 - p(t))\Delta t}{1 - \lambda(1 - p(t))\Delta t} = \lambda p(t)(1 - p(t)).$$

Solving (4) gives the following explicit formula for the posterior belief $p(t)$:[8]

$$p(t) = \left[1 + \left(\frac{1}{p_0} - 1 \right) e^{-\lambda t} \right]^{-1}, \tag{5}$$

provided that the entrepreneur's income remains at y_1 up to time t. Of course, at any time when the entrepreneur's income jumps to y_2, the investment is revealed to be of high quality with probability 1. Figure 1 plots the dynamics of the posterior belief $p(t)$ over time t. Differentiating the belief's dynamic evolution equation (4) gives

$$p''(t) = \lambda p'(t)(1 - 2p(t)) \gtrless 0 \quad \text{if} \quad p(t) \lessgtr 1/2. \tag{6}$$

¿From the preceding, we conclude that the posterior belief $p(t)$ is convex in time t, for $p \leq 1/2$ and concave in time t, for $p \geq 1/2$. When the entrepreneur believes that the risky investment is more likely to be of high quality ($p(t) < 1/2$), the convexity of $p(\cdot)$ implies that the speed of learning, $\dot{p}(t)$, is increasing over time. Intuitively, learning reveals information, which is against the prior that the risky investment is more likely to be of high quality ($p \leq 1/2$). On the other hand, when the entrepreneur believes that it is more likely that the risky investment is of low quality ($p > 1/2$), then the incremental learning, which provides more signal confirming his prior, is less valuable. Thus, the speed of learning, $\dot{p}(t)$, is decreasing over time.

[Insert Figure 1 Here]

2.3 Decision problem

We solve the entrepreneur's decision problem by dividing it into two sub-problems. First, we solve the entrepreneur's deterministic consumption/saving problem when he receives a constant level of income forever. Second, we solve the entrepreneur's decision problem in which he needs to learn about the quality of the risky investment and to choose his optimal consumption-saving plan accordingly.

We start with the first problem. Let $V(x; w)$ denote the corresponding value function given that the entrepreneur's current wealth level is x and his income level is constant over time and

---

[8]See the appendix for derivations.

equal to w.[9] Then $V(x; w)$ satisfies

$$V(x; w) = \max_c \int_0^\infty e^{-rt} u(c_t) \, dt \tag{7}$$

subject to

$$dx_t = (rx_t + w - c_t) \, dt, \quad x_0 = x, \tag{8}$$

and the no Ponzi games condition. By a standard argument, $V(x; w)$ satisfies the following Bellman equation:

$$rV(x; w) = \max_c \; (rx + w - c) \, V_x(x; w) + u(c). \tag{9}$$

Since $\rho = r$, we can easily show that the entrepreneur's optimal consumption is equal to $rx+w$, the sum of interest income and constant wage income, which are constant over time. Therefore, his wealth remains constant at x at all times. Consequently, the value function is given by

$$V(x; w) = \frac{1}{r} u(rx + w). \tag{10}$$

Next, we turn to the entrepreneur's decision problem when he stays in business. Intuitively, the entrepreneur will continue to experiment with the risky investment, provided that there is sufficiently high benefits from potentially discovering the risky investment to be high quality. Namely, it is optimal for the entrepreneur to experiment with the risky investment, provided that the posterior belief about the risky investment to be low quality p_t is sufficiently low. This intuition suggests that the posterior belief p_t is an additional state variable for the entrepreneur's decision making. Let $W(x, p)$ be the value function when the current wealth level and belief are given by x and p, respectively. Let τ be the first time when the entrepreneur stops experimenting with the risky investment and accepts the safe job. The principle of optimality implies that $W(x, p)$ satisfies

$$W(x, p) = \max_{c, \, \tau} E \left[\int_0^\tau e^{-rt} u(c_t) \, dt + e^{-r\tau} V(x_\tau; z) \right], \tag{11}$$

subject to the belief updating process (4) and

$$dx_t = (rx_t + y_t - c_t) \, dt \, , \quad x_0 = x, \; p_0 = p. \tag{12}$$

---

[9]Notice that here income w is not a state variable.

8

We proceed heuristically to derive the optimality conditions. First, when the entrepreneur continues to stay in business, it follows from the principle of optimality that the value function $W(x,p)$ satisfies the Bellman equation

$$W(x,p) = \max_c E\left[\int_0^{\Delta t} e^{-rt} u(c_t)\,dt + e^{-r\Delta t} W(x_{\Delta t}, p_{\Delta t})\right], \tag{13}$$

for a short time Δt. We apply Ito's Lemma and let Δt goes to zero to obtain

$$rW(x,p) = \max_c\ u(c) + \mathcal{D}W(x,p), \tag{14}$$

where we define

$$\mathcal{D}W(x,p) = (rx + y_1 - c)\,W_x(x,p) + \lambda(1-p)\left(V(x;y_2) - W(x,p)\right) + \lambda p(1-p)W_p(x,p). \tag{15}$$

The intuition behind equation (14) is as follows. The entrepreneur optimally stays in business and chooses his consumption process c such that the flow measure of his value function $rW(x,p)$ on the left side of (14) is equal to the right side of (14), which is given by the sum of the instantaneous utility payoff $u(c)$ and the instantaneous expected changes of the value function given by (15). These changes consist of three terms on the right side of equation (15). The first term gives the marginal increase of the value function from saving of $(rx + y_1 - c)$ units when the entrepreneur does not discover that his investment is high quality. The second term gives the expected increase in the value function if the risky investment is revealed to be high quality. Note that $\lambda\Delta t\,(1-p)$ measure the conditional likelihood that risky investment will be revealed to be high quality over a small time interval Δt and $(V(x;y_2) - W(x,p))$ is the change in the value function associated with the discovery of the risky investment to be high quality. The last term measures the impact of rational learning about the quality of the risky investment on the entrepreneur's value function. When the entrepreneur updates his belief, he changes his expectation about the future and hence his value function.

Second, when the entrepreneur stops experimenting and accepts the safe job, the following value-matching condition is satisfied

$$W(x,p) = V(x;z). \tag{16}$$

This equation implicitly determines a boundary $\bar{p}(x)$, which states that belief p is a function of wealth x. It is intuitive that if the entrepreneur attaches more probability to the low quality investment, then the value function will be smaller. That is, $W(x,p)$ is decreasing in p. Thus, for any given wealth level x_t, if the posterior belief $p_t \leq \bar{p}(x_t)$, then experimenting with the risky investment for some time is optimal. On the other hand, if the posterior belief $p_t \geq \bar{p}(x_t)$, then abandoning the risky investment and accepting the safe investment is optimal.

Finally, for the boundary $\bar{p}(x)$ to be optimal, the following smooth-pasting conditions must also be satisfied (see, for example, Krylov (1980) and Dumas (1991)):

$$W_x(x,\bar{p}(x)) = V_x(x;z), \tag{17}$$
$$W_p(x,\bar{p}(x)) = 0. \tag{18}$$

Intuitively, the above two smooth pasting conditions capture the optimal tradeoffs for the entrepreneur when he chooses the boundaries between experimenting with the risky business and exiting to take the safe job.

From a mathematical point of view, the above decision problem is a two-dimensional combined control and stopping problem, which is generally hard to solve. This is due to the fact that the boundaries for the partial differential equation (14) are also parts of the solutions, (not inputs) of the problem. In order to obtain a more detailed characterization of the underlying economics, we next analyze the optimization/learning problem by considering two widely used utility specifications in the next two sections.

3 Survival without wealth effect: CARA utility

In this section, we assume that the entrepreneur's preferences are represented by CARA utility function $u(c) = -e^{-\gamma c}/\gamma$, where $\gamma > 0$ is the constant absolute risk aversion parameter. The parameter γ also measures the entrepreneur's precautionary motive (Kimball (1990)). It is well known that the CARA utility specification is analytically convenient for a variety of applications in economics due to its lack of wealth effect.[10] We adopt this utility function as a first step

---

[10]See Caballero (1991) and Wang (2004) for applications of CARA utility to incomplete markets consumption-saving models.

towards the understanding of the effect of learning on the consumption-saving and investment decisions for a risk averse (and precautionary) entrepreneur.

3.1 Solution

We first solve the deterministic consumption/saving problem in which the entrepreneur receives a stream of constant level w of income. Note that the entrepreneur's consumption is constant over time and given by $(rx + w)$. Using our earlier analysis and equation (10), we may conclude that the value function is given by

$$V(x; w) = -\frac{1}{\gamma r} \exp\left[-\gamma (rx + w)\right]. \tag{19}$$

We now turn to the entrepreneur's decision problem when he stays in business. We take advantage of the CARA utility's lack of wealth effect and (19) to conjecture that the value function takes the following form:

$$W(x, p) = -\frac{1}{\gamma r} \exp\left[-\gamma r (x + f(p))\right], \tag{20}$$

where $f(p)$ is a smooth function to be determined. Given the functional forms in (19) and (20) for the CARA utility specification, we note that the wealth state variable x is cancelled out in the value matching condition (16) and the smooth-pasting conditions (17). Thus, the problem is reduced to a one dimensional one and the belief boundary $\bar{p}(x)$ does not depend on the wealth level x. Consequently, we simply use $\bar{p}$ to denote the constant belief threshold.

We present the solution in the following proposition.

Proposition 1 *Let $u(c) = -e^{-\gamma c}/\gamma$. Suppose*

$$0 < r\delta_1 < \frac{\lambda}{\gamma}\left(1 - e^{-\gamma\delta_2}\right), \tag{21}$$

where $\delta_1 = z - y_1$ and $\delta_2 = y_2 - z$. Then the value function $W(x, p)$ is given by (20) and the optimal consumption rule is given by

$$c(x, p) = r(x + f(p)), \text{ for } p < \bar{p}, \tag{22}$$

where $f(p)$ satisfies the differential equation

$$rf(p) = y_1 + \lambda(1 - p)\frac{1}{\gamma r}\left[1 - e^{-\gamma(y_2 - rf(p))}\right] + \lambda p(1 - p)f'(p), \tag{23}$$

11

subject to $f(\bar{p}) = z/r$. Finally, the belief threshold $\bar{p}$ is given by

$$\bar{p} = 1 - \frac{\gamma r \delta_1}{\lambda \left(1 - e^{-\gamma \delta_2}\right)}. \tag{24}$$

We first discuss the assumption of this proposition. The assumption given in (21) ensures that the belief threshold $\bar{p}$ lies in $(0, 1)$. If $z \leq y_1$ or $\delta_1 \leq 0$, then the risky investment dominates the safe investment for every possible realization of the risky investment payoff. Therefore, the entrepreneur will always stay with the risky investment. On the other hand, if $\gamma r \delta_1 \geq \lambda \left(1 - e^{-\gamma \delta_2}\right)$, then the entrepreneur will always stay with the safe investment. This situation happens when any one of the following conditions is satisfied: (i) the entrepreneur is sufficiently risk averse, i.e., γ is high enough; (ii) the cost of experimenting with the risky investment, represented by δ_1, is sufficiently high; or (iii) the benefit from experimenting with the risky investment, represented by λ and δ_2, is sufficiently small.

We next turn to the consumption rule. Equation (22) implies that the entrepreneur consumes the annuity value of the sum of financial wealth x and certainty equivalent nonfinancial wealth $f(p)$. Equation (23) reveals that the certainty equivalent nonfinancial wealth $f(p)$ comes from three sources. The first is y_1, the income if the investment quality is permanently low, which is reflected by the first term on the right side of (23). The second is the increase in "certainty equivalent" wealth if the investment quality is revealed to be high over the next instant. Since the probability of a jump of income from y_1 to y_2 is $\lambda (1 - p) \Delta t$ over the time interval Δt, the entrepreneur faces income fluctuation. Consequently, he has a precautionary motive to save part of his wealth to buffer against income fluctuation. The second term on the right side of (23) captures precautionary savings. As expected, precautionary savings depend on the entrepreneur's risk attitude γ. Finally, the third source of nonfinancial wealth is due to learning, which is reflected by the last term on the right side of (23). Note that this component is negative because $f(p)$ is decreasing in belief p. Intuitively, belief is deteriorating over time (after controlling for the upside (captured by the second term in (23))). Hence, learning by the rational entrepreneur on average brings bad news, relative to myopic ones who simply ignore the dynamics of belief updating. Because the entrepreneur gradually updates his prior over the investment quality, he changes his valuation of investment payoffs over time.

3.2 Beliefs and private firm value

To further understand the determination of the belief threshold, we first analyze the valuation of the private firm. Since the entrepreneur is subject to the uninsurable idiosyncratic investment risk and the investment project cannot be traded in the market, we cannot use the standard valuation method under complete markets to price the private firm. We adopt the certainty equivalent approach in the literature on the pricing of non-traded assets.[11]

We define private firm value as the lowest level of the compensating differential needed in order to make entrepreneur willing to give up his risky investment opportunity and taking the safe job. Let q denote this private firm value. Then by definition, we have

$$W(x, p) = V(x + q; z). \tag{25}$$

Given the explicit functional forms of the value functions in (19) and (20), we can show that the private firm value is given by $q = f(p) - z/r$.

Figure 2 plots the function $f(p)$. This figure reveals the following: First, $f(p)$ is a decreasing function. This result is intuitive. When the entrepreneur becomes more pessimistic about the quality of the risky investment, his valuation of the firm should be lower. Second, at the belief threshold $\bar{p}$, certainty equivalent wealth $f(p)$ is equal to the present value of income from the safe job, z/r, and therefore the private firm value q is zero. That is, at $\bar{p}$ the entrepreneur is indifferent between stay and exit. This fact follows from the value matching condition (16). Third, at $\bar{p}$ the curve $f(p)$ is tangent to the line z/r. This fact follows from the smooth-pasting conditions (17)-(18). Finally, for the posterior belief $p < \bar{p}$, the private firm value $q = f(p) - z/r$ is always positive. Thus, the entrepreneur will stay in business for $p < \bar{p}$. When the entrepreneur is pessimistic enough such that p reaches the threshold value $\bar{p}$, the entrepreneur's private valuation of the firm is zero that he is willing to exit and accept the safe job.

[Insert Figure 2 Here]

---

[11]See Carpenter (1998), Detemple and Sundaresan (1999), Hall and Murphy (2000), Kahl, Liu and Longstaff (2003), among others, on nontraded asset valuation such as employee stock options.

We can offer another interpretation for $\bar{p}$ by rewriting (24) as follows:

$$(z - y_1) e^{-\gamma(rx+z)} = (1 - \bar{p}) \lambda \left[V(x; y_2) - V(x; z) \right], \tag{26}$$

where $V(x; w)$ is the value function for CARA utility and is given in (19). This equation reveals that the entrepreneur will stop experimenting until the cost of experimenting is equal to the associated benefit. The cost is the income loss $(z - y_1)$ measured in terms of units of marginal utility $e^{-\gamma(rx+z)}$. The benefit occurs if the entrepreneur discovers that the quality of the project is high with probability $(1 - \bar{p})$ and if there is an income jump with arrival rate λ. When this situation happens, the entrepreneur obtains a perpetual increase of income, and hence he obtains benefit measured by the difference in the value function: $[V(x; y_2) - V(x; z)]$.

3.3 Survival duration

From the preceding analysis, we know that the entrepreneur exits from his business when his belief about the investment quality is pessimistic enough. More specifically, when the belief process $p(t)$ crosses the threshold value $\bar{p}$ given in (24), the entrepreneur exits. Using equation (5), we can derive the entrepreneur's survival duration,

$$T = \frac{1}{\lambda} \ln \left(\frac{1/p_0 - 1}{1/\bar{p} - 1} \right), \text{ for } p_0 < \bar{p}. \tag{27}$$

The above equation measures the time that the entrepreneur will wait (provided that he does not discover that the project is high up until time T.) Note that duration T in (27) gives the maximum time that the entrepreneur will experiment with the risky project. This equation implies that the entrepreneur stays in business longer if the initial prior p_0 is smaller or if the belief threshold $\bar{p}$ is larger. This result is intuitive. A smaller value of the initial prior p_0 implies that the entrepreneur is more optimistic about the investment project. He will then believe that the chance of getting a high quality investment is larger. Thus, he will experiment with the risky investment longer. A larger value of the belief threshold $\bar{p}$ implies that the degree of pessimism triggering exit is higher. Thus, it will take a longer time for the entrepreneur to reach this trigger value. That is, the entrepreneur will stay in business longer.

Since the belief threshold $\bar{p}$ is important for survival duration, we conduct further comparative statics analysis regarding $\bar{p}$.

Corollary 1 *Given the assumptions in Proposition 1, the belief threshold $\bar{p}$ increases with λ and δ_2, and decreases with γ and δ_1.*

It is natural to expect that the option value of learning about the quality of the risky investment is higher when the arrival rate λ of the high quality is higher, or the cost of learning δ_1 is smaller, or the benefit of learning δ_2 is larger, *ceteris paribus*. When the option value of learning is higher, the belief threshold $\bar{p}$ is higher. For fixed λ, this result implies that the entrepreneur stays in business longer for a higher value of δ_2 or a lower value of δ_1. Although a higher value of λ also raises $\bar{p}$, there is another opposite effect. That is, a higher value of λ makes the entrepreneur learn faster, as seen from the duration survival function (27). Thus, the overall effect of an increase in λ on the survival duration is ambiguous.

Turn to the effect of risk aversion. A larger coefficient of absolute risk aversion γ induces a stronger precautionary motive for the entrepreneur. Thus, an increase in γ raises precautionary savings and lowers the certainty equivalent nonfinancial wealth or the entrepreneur's valuation of private equity $f(p)$. As shown in Figure 3, this fact implies that a larger γ lowers the belief threshold $\bar{p}$ at which the entrepreneur is willing to stop learning. Thus, a more risk averse entrepreneur exits earlier. In particular, when γ is large enough, $\bar{p}$ goes to zero. This result implies that when the entrepreneur is sufficiently risk averse, he will exit immediately and accept the safe job. This is consistent with the conventional wisdom that entrepreneurs are more likely to be risk takers, ceteris paribus.

[Insert Figure 3 Here]

3.4 NPV rule, risk aversion, and option value of learning

When thinking about whether to stay in business or exit, people often apply the following net present value (NPV) rule: First, calculate the present value of the expected stream of income generated from the entrepreneurial activities. Second, calculate the present value of the expected income generated from the safe job. Finally, determine whether the difference between the two–the NPV–is greater than zero. If it is, then stay in business; otherwise, exit and take the safe job. Using this method, Hamilton (2000) finds that the NPV for many

entrepreneurs is significantly less than zero. This raises the question as to why entrepreneurs still want to stay in business.

We now use our model to shed light on this question. We can compute the present value of the expected stream of income generated from the risky investment as follows:

$$PV(p) \equiv p \int_0^\infty e^{-rt} y_1 dt + (1-p) E \left[\int_0^{T_a} e^{-rt} y_1 dt + \int_{T_a}^\infty e^{-rt} y_2 dt \right], \tag{28}$$

where T_a is the stochastic arrival time of the permanent income jump at Poisson intensity λ, and the expectation is taken with respect to the exponential distribution for the arrival time T_a. Solving the above equation gives

$$PV(p) \equiv p \frac{y_1}{r} + (1-p) \frac{y_1 + \lambda y_2 / r}{r + \lambda}, \tag{29}$$

where p is the probability that the quality of the investment is low. Note that the above present value calculation (28)-(29) ignores the fact that the entrepreneur's belief will be updated over time. In macroeconomics, this assumption is sometimes invoked as an approximation for an otherwise complicated Bayesian dynamic optimization problem. This assumption is dubbed "anticipated utility" following Kreps (1998) and Sargent (1993, 1998).[12] While the entrepreneur's belief p in (29) is correct, the entrepreneur ignores that his belief is changing over time by holding his belief p unchanged, as if he is myopic. This is the same as in these anticipated utility models.

Thus, according to the NPV rule (28) as in the anticipated utility analysis of Kreps and Sargent, there is a value of belief threshold p^* above which the entrepreneur is so pessimistic that he prefers to exit. This value is defined by the equation

$$PV(p^*) = \frac{z}{r}. \tag{30}$$

Solving this equation gives

$$p^* = 1 - \frac{\delta_1}{\delta_1 + \delta_2} \left(1 + \frac{r}{\lambda} \right). \tag{31}$$

To see how this NPV rule (as in anticipated utility models) can go wrong, we first consider the risk neutral case.

---

[12]See Cogley and Sargent (2005) for a recent quantitative analysis on the approximation of anticipated utility model for Bayesian decision making problems.

Corollary 2 *Let the entrepreneur be risk neutral. Suppose $r\delta_1 < \lambda\delta_2$. Then the belief threshold is given by*

$$\bar{p} = 1 - \frac{r\delta_1}{\lambda\delta_2}, \tag{32}$$

and the certainty equivalent wealth is given by

$$f(p) = PV(p) + \left(\frac{z}{r} - PV(\bar{p})\right) \frac{p}{\bar{p}} \left(\frac{\Omega(p)}{\Omega(\bar{p})}\right)^{\frac{r}{\lambda}} \tag{33}$$

where $\Omega(p) \equiv p/(1-p)$.

To interpret equation (32), we rewrite it as

$$\delta_1 = \lambda(1-\bar{p})\frac{\delta_2}{r}. \tag{34}$$

Thus, for the risk-neutral entrepreneur, at the belief threshold $\bar{p}$, the income loss from experimentation, δ_1, is equal to the present value of the income gain from experimentation, $\lambda(1-\bar{p})\delta_2/r$.

Given the assumption $r\delta_1 < \lambda\delta_2$, we can easily show that $p^* < \bar{p}$ given in (32). That is, the rational entrepreneur exits too early relative to the entrepreneur who applies the NPV rule as in anticipated utility analysis. Moreover, we can show that $PV(\bar{p}) < z/r$. That is, the entrepreneur may continue to stay in business even if the NPV is negative. The reason for this behavior is that there is an option value of learning. The option value of learning is given by the second term in equation (33). Equation (33) shows that the certainty equivalent wealth is equal to the present value of the investment payoffs plus the option value of learning. Note that the option value of learning is positive since $PV(\bar{p}) < z/r$.

We next consider the case with risk aversion. Corollary 1 shows that the belief threshold $\bar{p}$ decreases with risk aversion γ. The intuition is that an increase in the degree of risk aversion lowers certainty equivalent wealth $f(p)$ and erodes the option value of learning, as illustrated in Figure 3. To compare with the NPV rule, Figure 4 plots $\bar{p}$ as a function of γ. It shows that there is a critical value γ^* of the risk aversion parameter such that $\bar{p} = p^*$. When $\gamma < \gamma^*$, we have $\bar{p} > p^*$, implying that the entrepreneur may stay in business even though the NPV is negative. By contrast, when $\gamma > \gamma^*$, we have $\bar{p} < p^*$, implying that the entrepreneur exits too

early even though the NPV is positive. This result seems surprising. The intuition is simple. When the degree of risk aversion is sufficiently large, the entrepreneur's precautionary savings motive dominates and hence substantially erodes the option value of learning.

From the preceding discussion, we conclude that for the risk averse entrepreneur, the NPV rule can be wrong for two reasons: (i) it ignores the option value, and (ii) it does not capture risk aversion and the undiversifiable idiosyncratic risks. Therefore, the NPV rule may imply either too early or too late exit, depending on the degree of risk aversion and the value of learning.

[Insert Figure 4 Here]

4 Wealth effect without liquidity constraint: CRRA utility

So far, we have analyzed the effect of learning on consumption/saving and exit decisions for the CARA utility specification. The upside of the CARA utility specification is that we are able to derive analytical solutions so that we can conduct an intuitive analysis. The downside of this specification is that we abstract away from the wealth effect. To analyze this effect, we now consider the CRRA utility specification, $u(c) = c^{1-\alpha}/(1-\alpha)$, where $\alpha > 0$ is the coefficient of constant relative risk aversion. The log utility corresponds to $\alpha = 1$.

First, consider the case with constant income w. Since $\rho = r$, we may immediately conclude that the value function given in (10) is given by

$$V(x;w) = \frac{1}{r} \frac{(rx+w)^{1-\alpha}}{1-\alpha}, \ x > -w/r. \tag{35}$$

Plugging this expression into the HJB equation (14), we obtain a partial differential equation for $W(x,p)$. Unlike the CARA utility specification analyzed in the previous section, there is no closed-form expression for $W(x,p)$. However, we can still characterize the belief boundary.

Proposition 2 *Let $u(c) = c^{1-\alpha}/(1-\alpha)$, $\alpha > 0$ and $\alpha =\neq 1$, and $x > -y_1/r$. Suppose*

$$0 < r(z-y_1) < \lambda \frac{(rx+z)}{(1-\alpha)} \left[\left(\frac{rx+y_2}{rx+z} \right)^{1-\alpha} - 1 \right]. \tag{36}$$

Then the belief boundary $\bar{p}(x)$ is given by[13]

$$\bar{p}(x) = 1 - \frac{r(z - y_1)(1 - \alpha)}{\lambda(rx + z)\left[\left(\frac{rx + y_2}{rx + z}\right)^{1-\alpha} - 1\right]}. \tag{37}$$

Moreover, $\bar{p}'(x) > 0$, and $\bar{p}(x)$ decreases in α for any given $x > -y_1/r$.

Much intuition for the case with the CARA utility carries over for case with the CRRA utility. First, similar to (21), assumption (36) ensures that the belief boundary $\bar{p}(x)$ lies in $(0,1)$. To understand this boundary, we rewrite (37) as

$$(z - y_1)(rx + z)^{-\alpha} = (1 - \bar{p}(x))\lambda(V(x; y_2) - V(x; y_1)). \tag{38}$$

This equation admits a similar interpretation to that for (26). So we do not repeat it here. Second, Proposition 2 also shows that the belief boundary decreases with the risk aversion parameter α. That is, a more risk averse entrepreneur is more likely to exit. A similar result is obtained in the case of CARA utility. The intuition is also similar. Since for the CRRA utility precautionary motive increases with the risk aversion parameter α, an increase in α raises precautionary savings. This increase in turn lowers the entrepreneur's valuation of private equity.

It is important to emphasize that unlike the CARA utility specification, the belief boundary depends on wealth. Furthermore, this boundary increases with wealth. This result has an important implication. That is, a wealthier entrepreneur stays in business longer. This result seems not obvious since an increase in wealth also raises the outside option value or the life-time utility $V(x; z)$ achieved from the safe job. Thus, this increase also raises the entrepreneur's incentive to exit. However, a higher wealth level makes the entrepreneur less risk averse as in the static model (Cressy (2000)). Moreover, a higher wealth level provides more buffer for the entrepreneur to guard against the intertemporal investment risk. These static and dynamic effects provide the entrepreneur with incentives to stay in business. Proposition 2 implies that the latter two effects dominate.

---

[13]For log utility, we take limit as α goes to 1 to obtain

$$\bar{p}(x) = 1 - \frac{r(z - y_1)}{\lambda(rx + z)\ln\left(\frac{rx + y_2}{rx + z}\right)}.$$

[Insert Figure 5 Here]

Figure 5 plots the belief boundary $\bar{p}(x)$ for $\alpha = 2$ and 5. The belief boundary partitions the state space for (x, p) into two regions. For state variables in the region below the belief boundary, the entrepreneur stays in business. When the state variables move into the region above the belief boundary for the first time, the entrepreneur exits and accepts the safe job. Figure 5 also illustrates that an increase in α lowers the belief boundary.

5   Conclusion

Entrepreneurs often do not know the qualities of their investment opportunities. They learn about them by experimenting with the projects. Moreover, in the process of experimentation, entrepreneurs are often exposed to the project's idiosyncratic risks, which cannot be fully hedged. Motivated by these two important frictions in entrepreneurial settings, we develop a model of experimentation/learning where the decision maker is subject to uninsurable idiosyncratic risk.

We show that a naive NPV calculation ignores the option value of learning. Hence, an entrepreneur may stay in business even though the NPV of his business investment opportunity is negative. In addition, risk aversion and beliefs are important for entrepreneurial survival. In particular, a more risk averse or a more pessimistic entrepreneur exits earlier. When the entrepreneur is sufficiently risk averse, he may exit even though the NPV is positive. This result reflects the fact that risk aversion erodes the option value. We also show that wealth is positively related to survival duration even though there is no liquidity constraint for utility specification that captures wealth effect (such as CRRA). Our model thus provide an alternative explanation (other than liquidity constraints) for the empirically documented positive relationship between entrepreneurial liquid wealth and investment/experimentation threshold.

We have intentionally chosen our model setup in a parsimonious way in order to deliver the economic insights into the effect of learning on entrepreneurial survival when markets are not complete. Parsimonious modeling comes with some limitations. First, we do not model entry decision in order to focus on entrepreneurial survival. Second, our model cannot quantitatively

address the private equity premium puzzle (Moskowitz and Vissing-Jorgensen (2002)). To understand the size of empirically observed private equity premium, one may need a general equilibrium model in which individuals invest in both public and private equity. Finally, in order to derive intuitive and analytical results, we assume a stylized income process. It would be interesting to consider more realistic income processes and calibrate the model so that one can confront the model with data. We leave these issues for future research.

Appendix

A Proofs

Derivation of Equation (5): A simple way to solve for the posterior belief $p(t)$ is as follows. Multiplying a power function of $p(t)$, say $Ap(t)^{A-1}$, on both sides of (4) gives

$$Ap(t)^{A-1}p'(t) = A\lambda p(t)^A - A\lambda p(t)^{A+1}. \tag{A.1}$$

Define $q(t) = p(t)^A$. Then, we obtain

$$\dot{q}(t) - A\lambda q(t) = -\lambda A\left[q(t)\right]^{(A+1)/A}. \tag{A.2}$$

Choosing $A = -1$ yields

$$\dot{q}(t) + \lambda q(t) = \lambda. \tag{A.3}$$

Solving gives

$$q(t) = q(0)e^{-\lambda t} + \left(1 - e^{-\lambda t}\right). \tag{A.4}$$

Inverting $q(t) = p(t)^{-1}$ gives equation (5). Q.E.D.

Proof of Proposition 1: By the first-order condition $u'(c) = W_x(x, p)$ and the CARA utility specification, we have

$$u\left(c\right) = \frac{-W_x\left(x, p\right)}{\gamma} \tag{A.5}$$

at the optimum. Using the conjectured functional form given in (20) yields

$$W_x(x, p) = \exp\left[-\gamma r\left(x + f\left(p\right)\right)\right] = -\gamma r W\left(x, p\right), \tag{A.6}$$

$$W_p(x, p) = f'\left(p\right)\exp\left[-\gamma r\left(x + f\left(p\right)\right)\right] = f'(p)W_x\left(x, p\right) \tag{A.7}$$

Using equations (19) and (20) yields

$$V\left(x; y_2\right) = -\frac{W_x\left(x, p\right)}{\gamma r}e^{-\gamma(y_2 - rf(p))}. \tag{A.8}$$

Plugging the above equations (A.5)-(A.8) into (14) gives

$$\frac{-W_x}{\gamma} = \frac{-W_x}{\gamma} + \left(y_1 - rf(p)\right)W_x + \lambda\left(1 - p\right)\frac{W_x}{\gamma r}\left[1 - e^{-\gamma(y_2 - rf(p))}\right] + \lambda p(1 - p)W_x f'(p). \tag{A.9}$$

Dropping W_x gives

$$0 = y_1 - rf(p) + \frac{\lambda(1-p)}{\gamma r}\left[1 - \exp\left(-\gamma\left(y_2 - rf(p)\right)\right)\right] + \lambda p(1-p)f'(p). \tag{A.10}$$

For $0 < p < 1$, we may write

$$f'(p) = \frac{1}{p}\left[\frac{rf(p) - y_1}{\lambda(1-p)} - \frac{1}{\gamma r}\left(1 - e^{-\gamma(y_2 - rf(p))}\right)\right]. \tag{A.11}$$

Given the conjectured form of the value function (20), the value-matching condition (16) and the smooth pasting-conditions (17)-(18) become

$$f(\bar{p}) = \frac{z}{r} \tag{A.12}$$

$$f'(\bar{p}) = 0. \tag{A.13}$$

Using $\delta_1 = z - y_1$ and $\delta_2 = y_2 - z$, and plugging (A.12)-(A.13) into (A.11) give

$$0 = \frac{\delta_1}{\lambda(1-\bar{p})} - \frac{1}{\gamma r}\left(1 - e^{-\gamma\delta_2}\right). \tag{A.14}$$

Re-arranging yields (24). To ensure that the posterior belief threshold to lie within the interesting region $(0, 1)$, we assume (21).[14]   Q.E.D.

Proof of Corollary 1:   The following are straightforward comparative statics results:

$$\frac{\partial \bar{p}}{\partial \lambda} = \frac{\gamma r \delta_1}{\lambda^2\left(1 - e^{-\gamma\delta_2}\right)} = \frac{1-\bar{p}}{\lambda} > 0, \tag{A.15}$$

$$\frac{\partial \bar{p}}{\partial \delta_1} = -\frac{\gamma r}{\lambda\left(1 - e^{-\gamma\delta_2}\right)} < 0, \tag{A.16}$$

$$\frac{\partial \bar{p}}{\partial \delta_2} = \frac{\gamma^2 r \delta_1}{\lambda\left(1 - e^{-\gamma\delta_2}\right)^2}e^{-\gamma\delta_2} > 0. \tag{A.17}$$

We turn to the effect of risk aversion on the posterior threshold. We have

$$\frac{\partial \bar{p}}{\partial \gamma} = -\frac{r\delta_1}{\lambda\left(1 - e^{-\gamma\delta_2}\right)} + \frac{\gamma r\delta_1\delta_2}{\lambda\left(1 - e^{-\gamma\delta_2}\right)^2}e^{-\gamma\delta_2}$$

$$= \frac{r\delta_1}{\lambda\left(1 - e^{-\gamma\delta_2}\right)^2}\left(\gamma\delta_2 e^{-\gamma\delta_2} + e^{-\gamma\delta_2} - 1\right). \tag{A.18}$$

To show $\partial\bar{p}/\partial\gamma < 0$, we only need to show the function

$$h(x) \equiv xe^{-x} + e^{-x} - 1 < 0 \text{ for all } x > 0. \tag{A.19}$$

This is true since $h(0) = 0$ and $h'(x) = -xe^{-x} < 0$ for $x > 0$.   Q.E.D.

---

[14]The transversality condition $\lim_{t\to\infty} E\left[e^{-rt}W\left(x_t, p_t\right)\right] = 0$ must also be satisfied.

Proof of Corollary 2: Taking limit in (23) as γ goes to zero, we obtain the differential equation for $f(p)$ under risk neutrality,

$$(r + \lambda(1-p)) f(p) = y_1 + \lambda(1-p)\frac{y_2}{r} + \lambda p(1-p) f'(p).\tag{A.20}$$

It can be easily verified the function given in (33) is the solution to the above equation. Finally using conditions (A.12)-(A.13), we obtain (32). Q.E.D.

Proof of Proposition 2: Given the CRRA utility specification, the first-order condition with respect to consumption gives

$$c = (W_x(x,p))^{-1/\alpha}.\tag{A.21}$$

Thus, at the optimum,

$$u(c) = \frac{1}{1-\alpha} (W_x(x,p))^{-(1-\alpha)/\alpha}.\tag{A.22}$$

Substituting the above two equations into the Bellman equation (14) and evaluating at the belief boundary $(x,\bar{p}(x))$, we obtain

$$
\begin{aligned}
rW(x,\bar{p}(x)) &= \frac{1}{1-\alpha} (W_x(x,\bar{p}(x)))^{-(1-\alpha)/\alpha} \\
&\quad + \left(rx + y_1 - (W_x(x,\bar{p}(x)))^{-1/\alpha}\right) W_x(x,\bar{p}(x)) \\
&\quad + \lambda(1-\bar{p}(x))(V(x;y_2) - W(x,\bar{p}(x))) + \lambda p(1-\bar{p}(x))W_p(x,\bar{p}(x))
\end{aligned}\tag{A.23}
$$

Substituting the value-matching and smooth-pasting conditions (16)-(18) into the preceding equation, we obtain

$$
\begin{aligned}
rV(x;z) &= \frac{1}{1-\alpha} (V_x(x;z))^{-(1-\alpha)/\alpha} + \left(rx + y_1 - (V_x(x;z))^{-1/\alpha}\right) V_x(x;z) \\
&\quad + \lambda(1-\bar{p}(x))(V(x;y_2) - V(x;z)) \\
&= \frac{\alpha}{1-\alpha} (V_x(x;z))^{-(1-\alpha)/\alpha} + (rx + y_1) V_x(x;z) + \lambda(1-\bar{p}(x))(V(x;y_2) - V(x;z)).
\end{aligned}\tag{A.24}
$$

Plugging the explicit expression for the value function $V(x;w)$ given in (35) for $w = z$ and y_2 into the preceding equation and simplifying yield equation (37).

We now show that $\bar{p}'(x) > 0$. First, we rewrite (37) as

$$\left[\lambda(1-\bar{p}(x))\left(x+\frac{z}{r}\right) - (\alpha-1)(z-y_1)\right]\left(x+\frac{z}{r}\right)^{-\alpha} = \lambda(1-\bar{p}(x))\left(x+\frac{y_2}{r}\right)^{1-\alpha}.\tag{A.25}$$

24

Apply the Implicit Function Theorem to (A.25) to derive

$$\lambda \bar{p}'(x) \left[\left(x + \frac{z}{r} \right)^{1-\alpha} - \left(x + \frac{y_2}{r} \right)^{1-\alpha} \right] \tag{A.26}$$

$$= \lambda (1 - \bar{p}(x)) \left(x + \frac{z}{r} \right)^{-\alpha} - \alpha \left[\lambda (1 - \bar{p}(x)) \left(x + \frac{z}{r} \right) - (\alpha - 1)(z - y_1) \right] \left(x + \frac{z}{r} \right)^{-\alpha - 1}$$

$$- \lambda (1 - \bar{p}(x)) (1 - \alpha) \left(x + \frac{y_2}{r} \right)^{-\alpha}.$$

Substituting (A.25) into the second term on the right side yields

$$\lambda \bar{p}'(x) \left[\left(x + \frac{z}{r} \right)^{1-\alpha} - \left(x + \frac{y_2}{r} \right)^{1-\alpha} \right] \tag{A.27}$$

$$= \lambda (1 - \bar{p}(x)) \left[\left(x + \frac{z}{r} \right)^{-\alpha} - \alpha \left(x + \frac{z}{r} \right)^{-1} \left(x + \frac{y_2}{r} \right)^{1-\alpha} - (1 - \alpha) \left(x + \frac{y_2}{r} \right)^{-\alpha} \right].$$

We will show the expressions in the two square brackets have the same sign by considering two cases.

Case I. $0 < \alpha < 1$. Then since $z < y_2$

$$\left(x + \frac{z}{r} \right)^{1-\alpha} - \left(x + \frac{y_2}{r} \right)^{1-\alpha} < 0. \tag{A.28}$$

To show $\bar{p}'(x) > 0$, we only need to show

$$\left(x + \frac{z}{r} \right)^{-\alpha} - \alpha \left(x + \frac{z}{r} \right)^{-1} \left(x + \frac{y_2}{r} \right)^{1-\alpha} - (1 - \alpha) \left(x + \frac{y_2}{r} \right)^{-\alpha} < 0. \tag{A.29}$$

Or

$$\left(\frac{x + \frac{y_2}{r}}{x + \frac{z}{r}} \right)^{\alpha} - 1 < \alpha \left(\frac{x + \frac{y_2}{r}}{x + \frac{z}{r}} - 1 \right). \tag{A.30}$$

Let

$$k = \frac{x + \frac{y_2}{r}}{x + \frac{z}{r}}. \tag{A.31}$$

Then, by assumption, $k > 1$. By the Mean Value Theorem and the fact that $\alpha \in (0, 1)$,

$$k^{\alpha} - 1 = \alpha k_0^{\alpha - 1} (k - 1) < \alpha (k - 1), \tag{A.32}$$

where $k_0 \in [1, k]$. The desired result follows.

Case II. $\alpha > 1$. Then

$$\left(x + \frac{z}{r} \right)^{1-\alpha} - \left(x + \frac{y_2}{r} \right)^{1-\alpha} > 0. \tag{A.33}$$

25

To show $\bar{p}'(x) > 0$, we only need to show

$$\left(x + \frac{z}{r}\right)^{-\alpha} - \alpha\left(x + \frac{z}{r}\right)^{-1}\left(x + \frac{y_2}{r}\right)^{1-\alpha} - (1-\alpha)\left(x + \frac{y_2}{r}\right)^{-\alpha} > 0. \qquad (A.34)$$

Or,

$$\left(\frac{x + \frac{y_2}{r}}{x + \frac{z}{r}}\right)^{\alpha} - 1 > \alpha\left(\frac{x + \frac{y_2}{r}}{x + \frac{z}{r}} - 1\right). \qquad (A.35)$$

By the Mean Value Theorem and the fact that $\alpha > 1$,

$$k^{\alpha} - 1 = \alpha k_1^{\alpha-1}(k-1) > \alpha(k-1), \qquad (A.36)$$

where $k_1 \in [1, k]$. The desired result follows.

Finally, to prove $\bar{p}(x)$ decreases with α for any given x, we only need to prove the expression

$$\frac{1}{1-\alpha}\left[\left(\frac{rx + y_2}{rx + z}\right)^{1-\alpha} - 1\right] \qquad (A.37)$$

decreases with α. Let $b = (rx + y_2)/(rx + z)$. By assumption $b > 1$. We now compute

$$\frac{\partial\left(\frac{b^{1-\alpha}-1}{1-\alpha}\right)}{\partial\alpha} = \frac{(b - b\ln b + \alpha b\ln b - b^{\alpha})}{(1-\alpha)^2 b^{\alpha}}. \qquad (A.38)$$

Define the function

$$g(\alpha) = b - b\ln b + \alpha b\ln b - b^{\alpha}. \qquad (A.39)$$

It is sufficient to prove $g(\alpha) < 0$. We observe that $g(\alpha)$ is concave since $g''(\alpha) = -(\ln b)^2 b^{\alpha} < 0$. Consequently, we solve the first-order condition $g'(\alpha) = b\ln b - (\ln b) b^{\alpha} = 0$ to conclude that $g(\alpha)$ is maximized at the value $\alpha = 1$. Since $g(1) = 0$, we conclude that $g(\alpha) < 0$ for all $\alpha > 0$ and $\alpha \neq 1$. Q.E.D.

References

Bergemann, D. and Valimaki, 2006, Bandit Problems, forthcoming in Steven Durlauf and Larry Blume, eds., The New Palgrave Dictionary of Economics, 2nd ed. Macmillan Press.

Brennan, M. J., and E. S. Schwartz, 1985, Evaluating Natural Resource Investments, *Journal of Business*, 58, 135-57.

Bolton, P. and C. Harris, 1999, Strategic Experimentation, *Econometrica* 67, 349-374.

Caballero, R. J., 1991, Earnings uncertainty and aggregate wealth accumulation, *American Economic Review* 81(4), 859-71.

Cogley, T., and T. J. Sargent, 2005, Anticipated Utility and Rational Expectations as Approximations of Bayesian Decision Making, working paper, New York University

Cressy, R., 2000, Credit Rationing or Entrepreneurial Risk Aversion? An Alternative Explanation for the Evans and Jovanovic Finding, *Economics Letters* 66, 235-240.

Dixit, A. and R. Pindyck, 1994, *Investment Under Uncertainty*, Princeton University Press, Princeton, NJ.

Dumas, B., 1991, Super contact and related optimality conditions, *Journal of Economic Dynamics and Control* 4, 675-685.

Evans, D. and B. Jovanovic, 1989, An Estimated Model of Entrepreneurial Choice Under Liquidity Constraints, *Journal of Political Economy* 97, 808-827.

Gentry, W., Hubbard, R., 2004. Entrepreneurship and household saving. Working paper. Columbia Business School, New York.

Grenadier, S., 1999, Information Revelation Through Option Exercise, *Review of Financial Studies* 12, 95-130.

Hamilton, B. H., 2000, Does Entrepreneurship Pay? An Empirical Analysis of the Returns to Self-Employment, *Journal of Political Economy* 108, 604-631.

Henderson, V., 2005, Valuing the Option to Invest in an Incomplete Market, working paper, Princeton University.

Hugonnier, J. and E. Morellec, 2005, Investment under Uncertainty and Incomplete Markets, working paper, University of Lausanne.

Holtz-Eaking, D., Joulfaian, D. and H. Rosen, 1994, Sticking it Out: Entrepreneurial Survival and Liquidity Constraints, *Journal of Political Economy* 102, 53-75.

Jovanovic, B., 1979, Job matching and the theory of turnover, *Journal of Political Economy* 87, 972-990.

Jovanovic, B., 1982, Selection and the Evolution of Industry, *Econometrica* 50, 649-670.

Keller, G., S. Rady, and M. Cripps, 2005, Strategic Experimentation with Exponential Bandits, *Econometrica* 73, 39-68.

Kihlstrom, R. and J. Laffont, 1979, A General Equilibrium Entrepreneurial Theory of Firm Formation Based on Risk Aversion, *Journal of Political Economy* 87, 719-749.

Kimball, M. S., 1990, Precautionary Saving in the Small and in the Large, *Econometrica* 58, 53-73.

Kreps, D., 1998, Anticipated utility and dynamic choice, 1997 Schwartz Lecture, in *Frontiers of Research in Economic Theory*, Edited by D. Jacobs, E. Kalai, and M. Kamien, Cambridge University Press, Cambridge, UK.

Krylov, N. V., 1980, *Controlled Diffusion Processes*, Springer, Berlin.

McDonald, R. and D. Siegel, 1986, The value of waiting to invest, *Quarterly Journal of Economics* 101, 707-728.

Miao, J. and N. Wang, 2007, Investment, Consumption, and Hedging under Incomplete Markets, forthcoming, *Journal of Financial Economics*.

Moskowitz, T. and A. Vissing-Jorgensen, 2002, The Returns to Entrepreneurial Investment: A Private Equity Premium Puzzle? *American Economic Review* 92, 745-778.

Sargent, T. J., 1993, *Bounded Rationality in Macroeconomics*, Clarendon Press: Oxford.

Sargent, T. J., 1999, The Conquest of American Inflation, Princeton University Press: Princeton.

Wang, N., 2004, Precautionary saving with partially observed income, *Journal of Monetary Economics*, 51(8), 1645-1681.
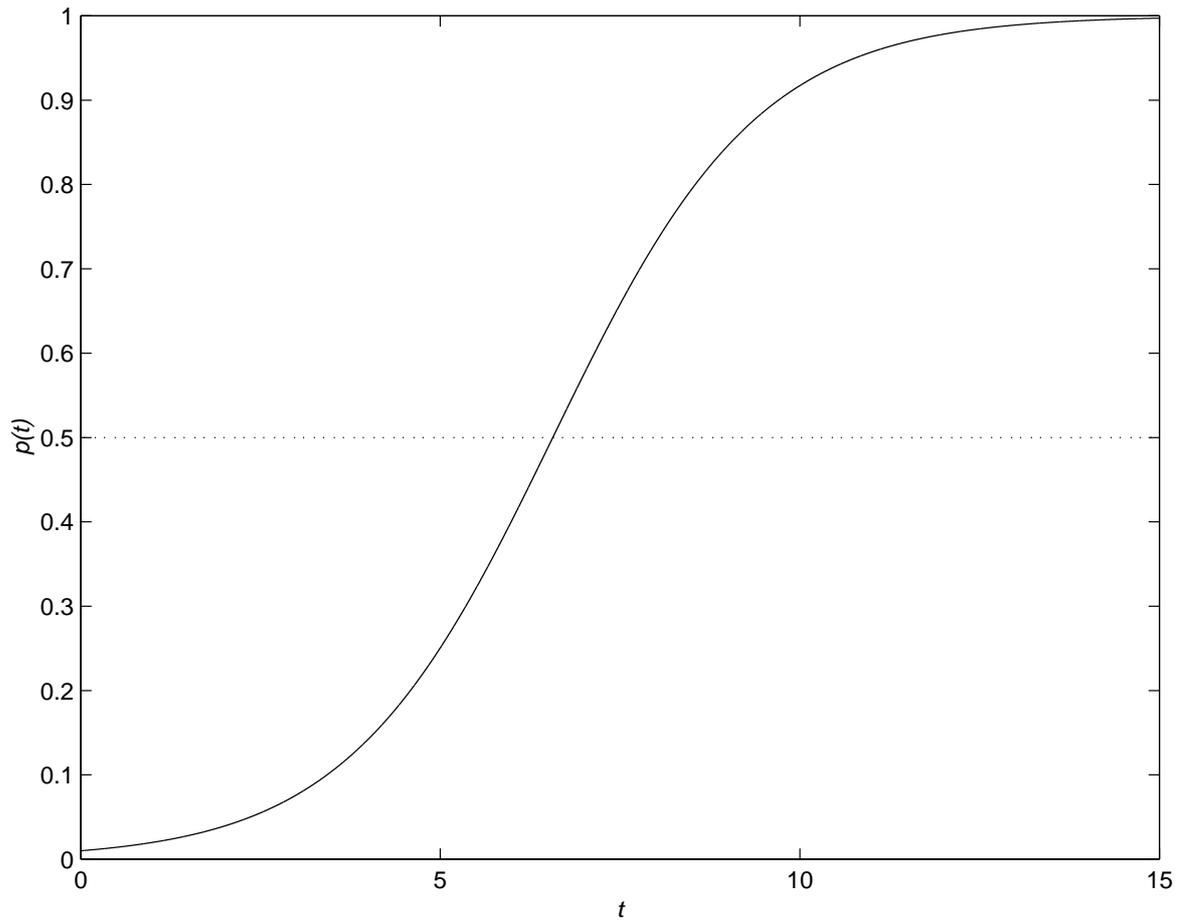
Figure 1: **Posterior beliefs** $p(t)$. The parameter values are set as follows: $p_0 = 0.01$ and $\lambda = 0.7$.
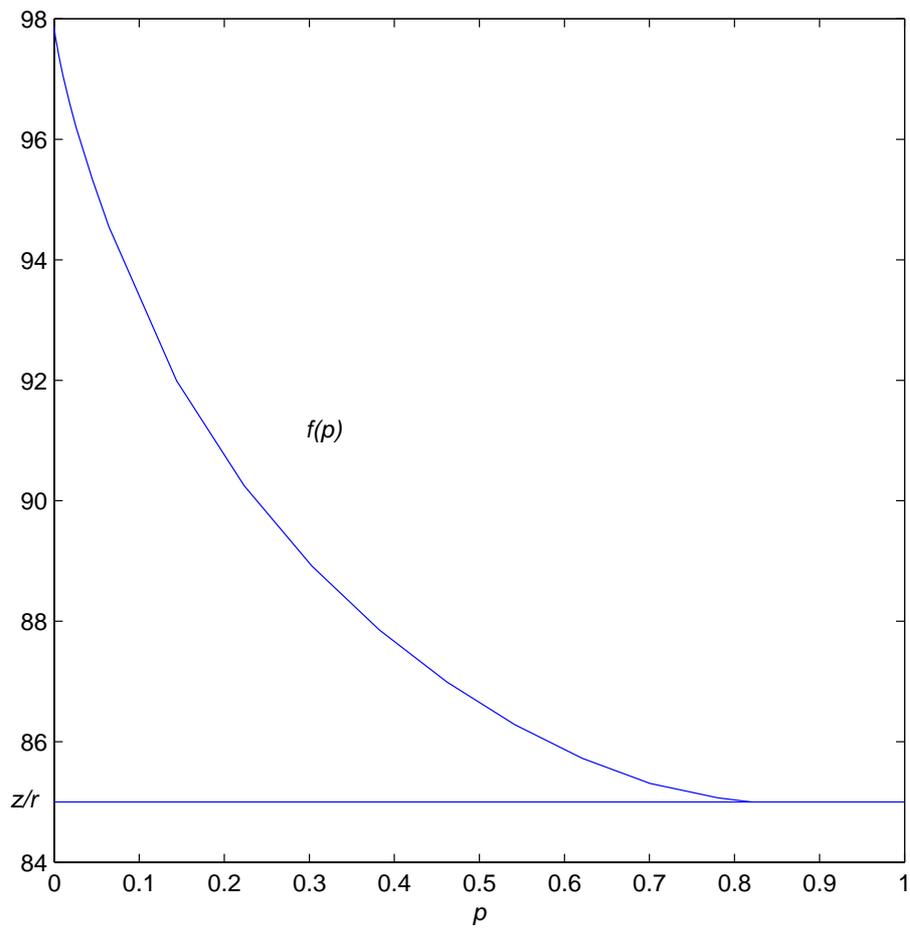
Figure 2: **Private firm value** $f(p)$**.** The parameter values are set as follows: $y_1 = 1$, $y_2 = 2$, $z = 1.7$, $\gamma = 5$, $r = 2\%$, and $\lambda = 0.5$.
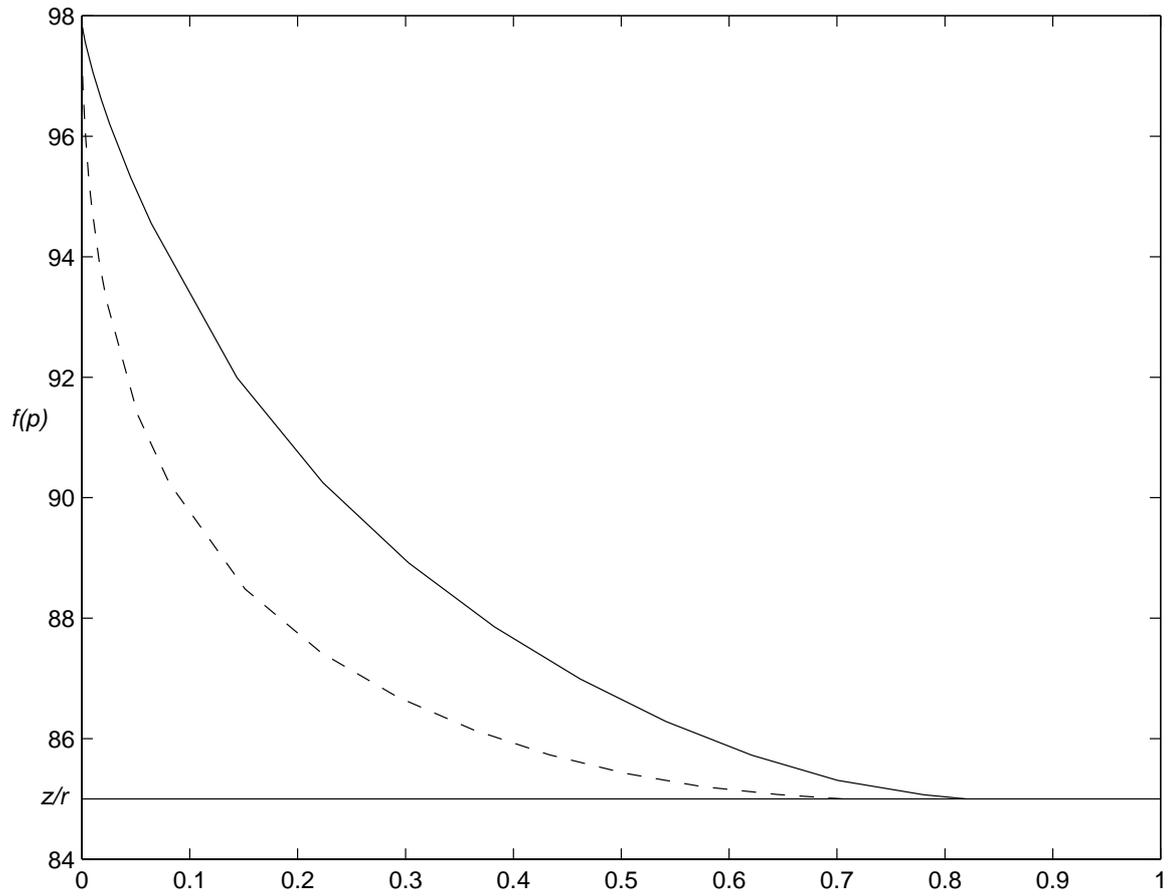
Figure 3: **The effect of** γ **on** $f(p)$**.** The solid curve plots $f(p)$ for $\gamma = 5$. The dashed curve plots $f(p)$ for $\gamma = 10$. The parameter values are set as follows: $y_1 = 1$, $y_2 = 2$, $z = 1.7$, $r = 2\%$, and $\lambda = 0.5$.
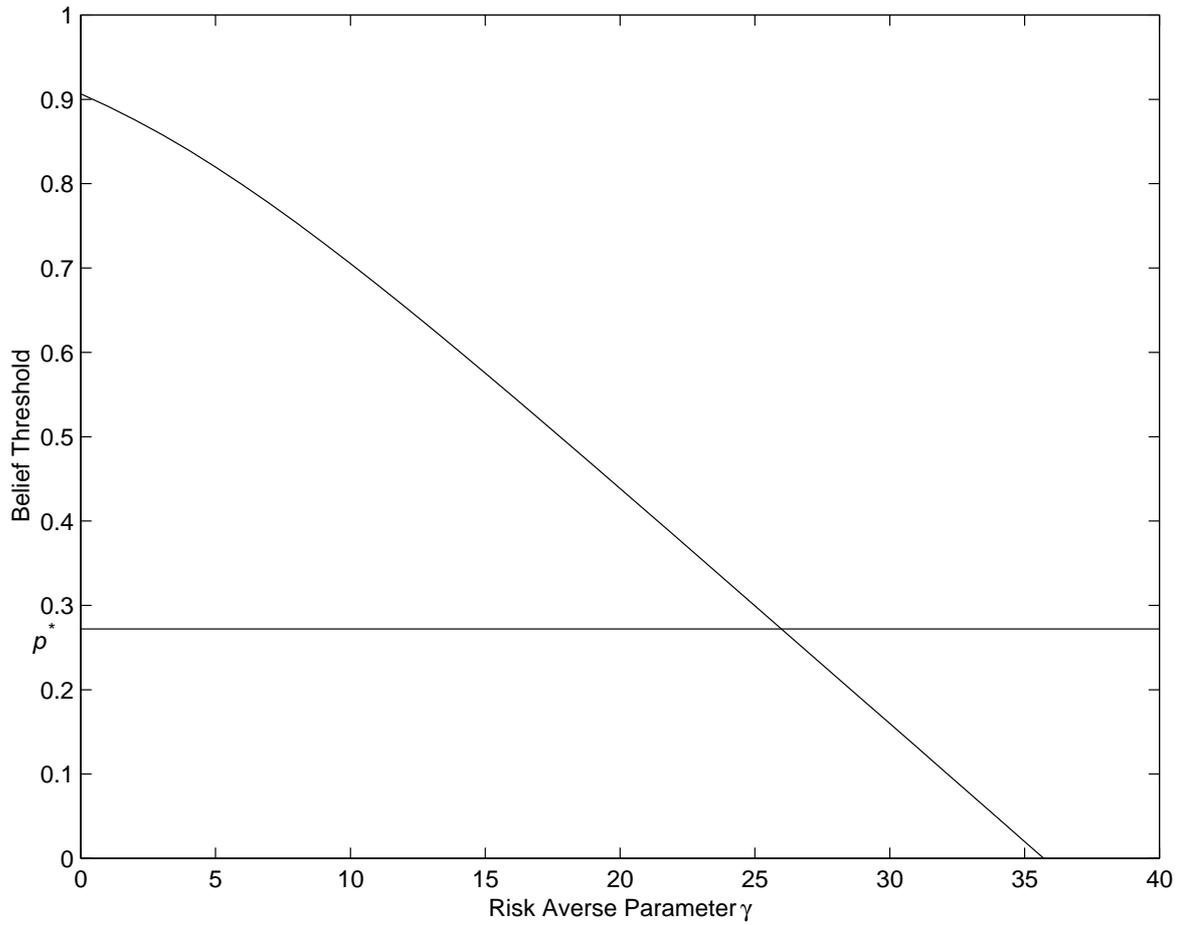
Figure 4: **The belief threshold $\bar{p}$ as a function of γ.** The parameter values are set as follows: $y_1 = 1$, $y_2 = 2$, $z = 1.7$, $r = 2\%$, and $\lambda = 0.5$.
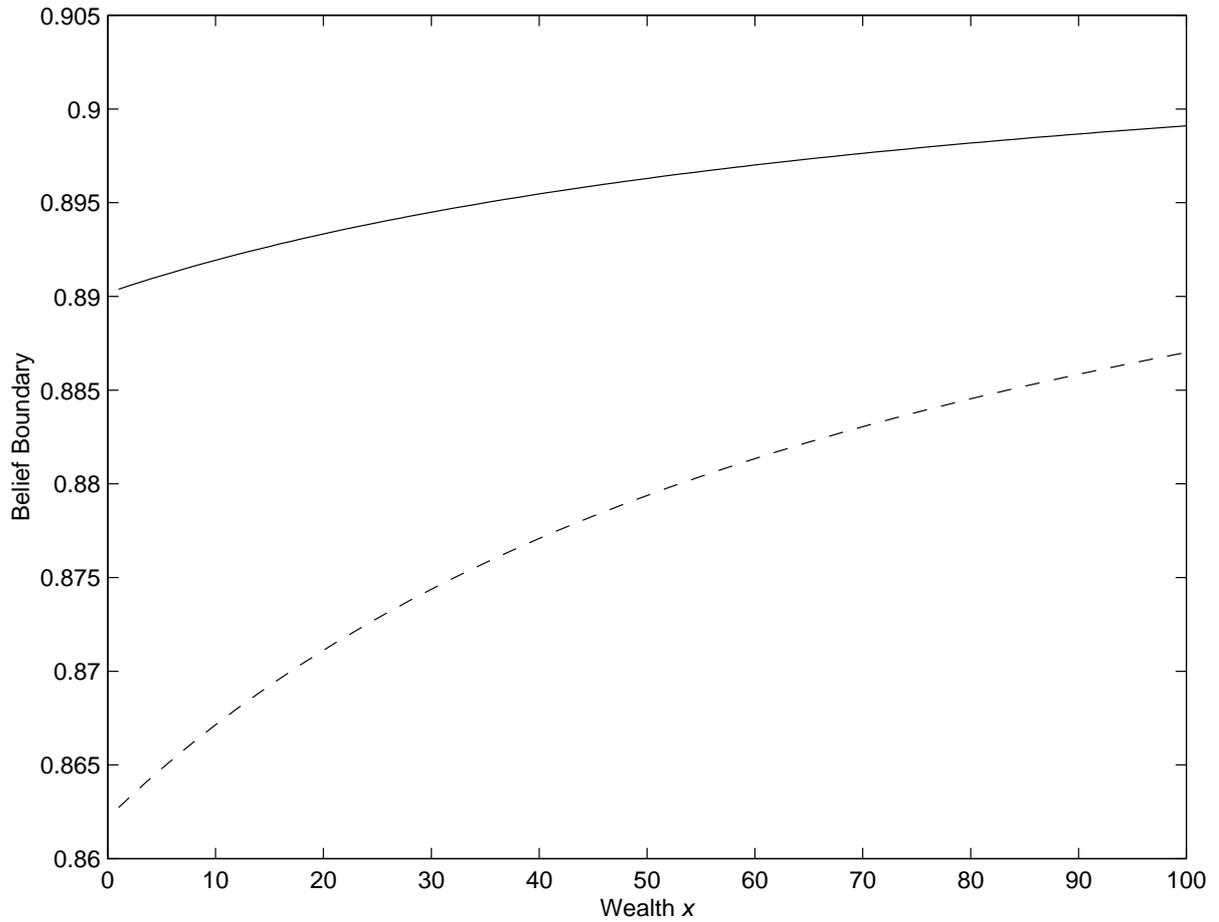
Figure 5: **The belief boundary $\bar{p}(x)$ as a function of x.** The solid curve is for $\alpha = 2$, and the dashed curve is for $\alpha = 5$. The parameter values are set as follows: $y_1 = 1$, $y_2 = 2$, $z = 1.7$, $r = 2\%$, and $\lambda = 0.5$.