Integrability of Demand in Incomplete Markets: Kreps-Porteus-Selden Preferences

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Abstract

We provide necessary and sufficient conditions such that consumption and asset demands in an incomplete market setting can be rationalized by Kreps-Porteus-Selden preferences and provide a means for recovering the underlying unique representations of risk and time preferences. The incompleteness of asset markets introduces two serious problems in attempting to use the classic Slutsky symmetry and negative semidefiniteness properties employed in certainty demand analysis. First, contingent claim prices are not unique and second, they do not vary independently. Non-uniqueness is a key obstacle to rationalizing conditional asset demand by a representation of risk preferences. Non-independence precludes proving the existences of an overall Kreps-Porteus-Selden representation defined over consumption and contingent claims and hence the existence of a representation of time preferences. Mechanisms are provided for overcoming both obstacles.

KEYWORDS. Kreps-Porteus-Selden preferences, expected utility, contingent claims, integrability, incomplete markets.

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1 Introduction

The classic integrability problem in demand theory asks (i) what conditions guarantee that demand functions can be rationalized by a well-behaved utility function and if such a utility exists, (ii) when is it unique and (iii) how can it be recovered? This problem is of intrinsic theoretical interest and has obvious practical applications: While preferences are not observable, an individual's demand is, at least in principle. And one is naturally interested in conditions under which there exist preferences that generate or rationalize given demand. One would also like to recover the underlying preferences in order to make statements for example about the welfare effects of economic policy. Building on Samuelson (1947), Hurwicz and Uzawa (1971) and Mas-Colell (1978) give complete answers to questions (i) - (iii) for the case of demand under certainty. Surprisingly, almost no work has been done seeking to extend their results to the case of uncertainty where financial markets are incomplete.

In this paper we argue that when financial markets are incomplete, it is generally impossible to extend the Hurwicz and Uzawa (1971) solution to the most direct case, a utility for assets. But what if preferences are defined over contingent claims? Here one encounters the immediate difficulty that when the number of states exceeds the number of assets, contingent claim prices become indeterminate. However, if one makes the *a priori* reasonable assumption that when asset prices change state probabilities can also change, it becomes possible to determine a unique vector of supporting contingent claim prices from asset demand. Then introducing mild assumptions on asset demand, one can apply the conditions in Hurwicz and Uzawa (1971) and Mas-Colell (1978) to a space of extended contingent claim demand and prove the existence of a utility function that rationalizes demand, show that it is unique for the class of preferences considered and recover the utility from the given asset demand.

More specifically, we consider the classic two period consumption-portfolio problem, where period 1 consumption is certain and period 2 consumption is risky. In the first period the consumer chooses a level of period 1 consumption and a portfolio of financial assets, where the market for assets is incomplete. We assume that consumers have preferences of the form axiomatized by Kreps and Porteus (1978) and Selden (1978) which include two period Expected Utility preferences as a special case. These Kreps-Porteus-Selden preferences are fully characterized by a representation of **time preferences** defined over certain periods 1 and 2 consumption and **conditional risk preferences**, where the latter are parameterized by period one consumption and are defined over risky period 2 consumption. This separation of time and risk preferences is well known and has been widely used in the analysis of saving behavior and asset pricing.[1]

Consider the case where the demand for assets is observable as a function of asset prices, income and state probabilities. In Theorem 1, assuming the existence of a unique set of contingent claim prices consistent with the no-arbitrage condition on assets, we derive necessary and sufficient conditions for rationalizing asset demand (conditional on period one consumption) by a unique, twice continuously differentiable, strictly increasing, strictly concave state independent Expected Utility function and give a means for recovering the specific utility. The conditions on demand include satisfaction of the classic Slutsky symmetry and negative semidefiniteness properties as well as a restriction on the specific functional form for conditional asset demand.[2] The latter restriction can be viewed as a concrete test for the existence of conditional risk preferences that are representable by an Expected Utility function.[3] One complication that arises in the context of Theorem 1 is that demand may not be unique due to the non-independence of contingent claim prices. Verification of the Slutsky properties would then not be possible. This difficulty is overcome by the Theorem 1 assumption of a unique contingent claim price vector which facilitates introduction of an extended contingent claim demand. Theorem 2 shows that this assumption will indeed hold if conditional demand is globally invertible and twice continuously differentiable.

Theorem 3 gives conditions such that there exists a unique representation of time preferences defined over periods 1 and 2 certain consumption and provides a means for recovering the utility. In order to prove the existence of the representation of time preferences, we introduce a new restriction on unconditional demands. This property, referred to as Certainty Regularity, ensures that the representation

[1] For dynamic extensions (i.e., more than two periods) of these preferences such as the widely used Epstein and Zin (1989) model, it is not possible in general to achieve a complete separation of time and risk preferences (over consumption). See Epstein, Farhi and Strzalecki (2014, p. 2687).

[2] It will be seen that the conditions we derive do not include the global Lipschitzian property which is assumed for the analogous integrability result of Hurwicz and Uzawa (1971) and Mas-Colell (1978). See Remark 2 below.

[3] For the case of asset demand with incomplete markets, the question of identifying the rationalizing utility, utilizing the approach of Mas-Colell (1977), has been addressed in a single period setting (e.g., Green, Lau and Polemarchakis 1979 and Dybvig and Polemarchakis 1981) and in a two period setting (e.g., Polemarchakis and Selden 1984). Each of these applications assumes not only the existence of a rationalizing utility but also that it takes the state independent Expected Utility form. However for the case of incomplete markets, there is no known test to verify whether asset demand was generated by the maximization of an Expected Utility function as assumed.

of time preferences is probability independent. To see why such a restriction is needed, first note that in a single period contingent claim setting Kubler, Selden and Wei (2014) show that when probabilities are allowed to vary, risk preferences can be represented by an Expected Utility function with NM indices that vary (by more than a positive affine transformation) in response to the change in probabilities.[4] Experimental support for the possibility of probability dependent NM indices has recently been provided by Polisson, et al. (2016) in the form of revealed preference tests based on asset demands. If one extends these Expected Utility analyses to the two period case considered in this paper, the presence of probability dependent NM indices will necessarily imply that the corresponding representation of time preferences inherit probability dependence as well. This possibility of probability dependent conditional risk and time preferences is not considered in Kreps-Porteus-Selden preferences. Indeed the integrability result we obtain in this paper for Expected Utility conditional risk preferences rules out the possibility of probability dependent NM indices and an induced probability dependence for time preferences. However for general two period Kreps-Porteus-Selden preferences where risk and time preferences are independent, it is possible to have probability independent conditional risk preferences and probability dependent time preferences. We provide a simple example illustrating this point. The Certainty Regularity property ensures that the representation of time preferences is probability independent as required by the standard Kreps-Porteus-Selden utility.

Together Theorems 1 - 3 extend the integrability results of Hurwicz and Uzawa (1971) and Mas-Colell (1978) to the consumption-portfolio problem, where consumption and asset demands are generated using the conditional risk and time preference building blocks defining Kreps-Porteus-Selden preferences. To illustrate the application of our key results, we include a comprehensive example (Examples 6, 7 and 10) in which given demands are shown to satisfy the necessary and sufficient conditions for the existence of both a representation of conditional risk preferences and a representation of time preferences. We also identify the two utilities. The resulting Kreps-Porteus-Selden utility is quite simple and takes a familiar form despite the fact that the demand functions for period one consumption and assets are quite complex. In incomplete markets these demands typically provide little or no hint as to whether they are rationalizable and, if they are, what form the generating utility might take.

The desire to separately identify risk and time preferences from given consump-

[4] Preference axioms are introduced in Kubler, Selden and Wei (2016) which rule out this dependence.

tion and asset demands is a clear motivation for why we have chosen to focus on the consumption-portfolio optimization rather than just the portfolio problem. With regard to potential applications of the theoretical results in this paper, recent laboratory experimental work investigating the separate roles of risk and time preferences would seem quite complementary. Numerous studies have been conducted in this area (see, for example, Andreoni and Sprenger 2012, 2015, Wölbert and Riedl 2013, Cheung 2015, Epper and Fehr-Duda 2015 and Miao and Zhong 2015). Because in lab settings of this type, an experimenter could naturally vary both prices and state probabilities, they would seem to offer a potentially attractive environment in which to test whether (i) consumption and asset demands can indeed be rationalized by Kreps-Porteus-Selden preferences and (ii) the representations of conditional risk and time preferences are independent.

The rest of the paper is organized as follows. In the next section, we introduce the setup and define notation. In Section 3, we provide several examples illustrating a number of specific obstacles in an incomplete market setting to directly solving the integrability problem for a utility over assets rather than contingent claims. In Section 4, we consider integrability for the case where conditional risk preferences are representable by Expected Utility. Section 5 provides necessary and sufficient conditions for the existence of a utility representing time preferences over certain periods 1 and 2 consumption and a means for identifying the utility. Section 6 provides concluding comments. Proofs and supporting calculations are given respectively in Appendices A and B.

2 Preliminaries

In this section, we first describe the consumption-portfolio setting and then review the structure and properties of Kreps-Porteus-Selden preferences. One of the motivations for assuming these preferences is to be able to identify, based on consumption and asset demands, the specific underlying risk and time preferences. To achieve this, it will prove useful to utilize a two stage process for solving the consumption-portfolio optimization.

At the beginning of period 1, the consumer chooses a level of certain first period consumption c_1 and a set of asset holdings, where the returns on the latter fund consumption in period 2. The asset market can be incomplete with $J \geq 2$ independent assets and S states, where $J \leq S$. Denote the payoff for asset j ($j \in \{1, ..., J\}$) in state s ($s \in \{1, ..., S\}$) by $\xi_{js} \geq 0$, where for each j, there exists at least one $s \in \{1, ..., S\}$ such that $\xi_{js} > 0$.[5] The quantities of assets and

[5] This standard assumption rules out contingent claims being negative.

contingent claims are denoted, respectively, by z_j and c_{2s}, with $\mathbf{z}$ and $\mathbf{c}_2$ being the corresponding vectors. Random period 2 consumption can thus be expressed as

$$c_{2s} = \sum_{j=1}^{J} z_j \xi_{js} \quad (s = 1, ..., S) . \tag{1}$$

The prices of period 1 consumption, asset z_j and contingent claim c_{2s} are given by, respectively, p_1, q_j and p_{2s}. The vector of state probabilities is denoted $\boldsymbol{\pi} \in \mathbb{R}^S_{++}$. Both asset prices and state probabilities are allowed to vary. We assume throughout that the payoffs of the J assets across states, $(\xi_{j1}, \ldots, \xi_{jS})$, are linearly independent for all $j = 1, \ldots, J$. Asset prices preclude arbitrage in that there are $p_{2s} > 0$, $s = 1, \ldots, S$ such that

$$q_j = \sum_{s=1}^{S} \xi_{js} p_{2s} \quad (j = 1, 2, , ..., J) . \tag{2}$$

The consumer's preferences over the consumption vectors $(c_1, c_{21}, ..., c_{2S})$ can be represented in two equivalent ways. First following Kreps and Porteus (1978), the utility can take the form

$$\mathcal{U}(c_1, \sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})). \tag{3}$$

The expression $\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})$ is the standard single period state independent Expected Utility representation over risky period 2 consumption, where V_{c_1} is the NM (von Neuman-Morgenstern) index conditional on period 1 consumption.[6] If V_{c_1} takes the form

$$V_{c_1}(\cdot) = a(c_1) V(\cdot) + b(c_1), \tag{4}$$

where $a(c_1) > 0$ and $b(c_1)$ are functions of c_1 and V is independent of c_1, it will be said to exhibit RPI (risk preference independence). Otherwise, it will be said to exhibit RPD (risk preference dependence). In general, the representation (3) is clearly not linear in probabilities and diverges from the classic two period Expected Utility

$$\sum_{s=1}^{S} \pi_s W(c_1, c_{2s}). \tag{5}$$

The index $\mathcal{U}$ in (3) can be viewed as a utility over period 1 consumption and period 2 Expected Utility.

[6] One familiar example of such a dependence is the internal habit formation formulation in Constantinides (1990).

The second representation, due to Selden (1978), is given by

$$U(c,\widehat{c}_2)=U\left(c_1,V_{c_1}^{-1}\left(\sum_{s=1}^{S}\pi_s V_{c_1}(c_{2s})\right)\right), \tag{6}$$

where V_{c_1} is strictly increasing in c_{2s} and continuous in c_1 and c_{2s}. The second argument of U in (6) is the period 2 certainty equivalent associated with $(c_{21},...,c_{2S})$

$$\widehat{c}_2=V_{c_1}^{-1}\left(\sum_{s=1}^{S}\pi_s V_{c_1}(c_{2s})\right). \tag{7}$$

Clearly for the case of an RPD conditional NM index, $\widehat{c}_2$ will depend not only on $(c_{21},...,c_{2S})$ but also on c_1. Because this representation is defined over certain period 1 consumption and period 2 certainty equivalent consumption and the index U is defined up to an increasing transformation, it is referred to as OCE (ordinal certainty equivalent) utility. The representation is fully defined by the indices $(U,\{V_{c_1}\})$.

Remark 1 *It is clear that an analogous representation could be based on risk preferences being represented by a different utility such as RDU (rank dependent utility).*[7] *Certainty equivalent period 2 consumption would then be based on the non-Expected Utility representation and U could continue to be used to represent certainty time preferences. However in this paper, as in Selden (1978), it will be assumed that OCE preferences are necessarily characterized by conditional risk preferences being represented by an Expected Utility function. Thus if period one consumption and asset demand fail the integrability tests developed in this paper, it is fully possible that they might be representable by a different combination of conditional risk and time preferences.*

Clearly (6) and (3) are equivalent if one defines

$$\mathcal{U}_{c_1}(\cdot)=U_{c_1}\circ V_{c_1}^{-1}(\cdot). \tag{8}$$

Given our goal of separating risk preferences defined over risky period two consumption and time preferences over certain consumption pairs, the OCE representation (6) is more natural than (3) where $\mathcal{U}$ is defined over consumption-utility pairs. It is also more intuitive when utilizing the two stage optimization process described below.

The OCE representation includes the two period Expected Utility (5) as a special case. To see this, suppose

$$U_{c_1}(c_2)=T\circ\left(a(c_1)V_{c_1}(c_2)+b(c_1)\right), \tag{9}$$

[7] See Wakker (2010, Chapter 5).

where $T' > 0$ and $a(c_1) > 0$ and $b(c_1)$ are arbitrary functions of c_1. Then

$$U(c_1, \widehat{c}_2) = T\left(\sum_{s=1}^{S} \pi_s V(c_1, c_{2s})\right), \tag{10}$$

where $V(c_1, c_2)$ and $W(c_1, c_2)$ in (5) are equivalent up to a positive affine transformation. In the case where U is additively separable and risk preferences are RPI,

$$U(c_1, c_2) = u_1(c_1) + u(c_2) \qquad \text{and} \qquad V_{c_1}(c_2) = u(c_2). \tag{11}$$

Then $U(c_1, \widehat{c}_2)$ is ordinally equivalent to the additively separable two period Expected Utility function

$$\begin{aligned} U(c, \widehat{c}_2) &= u_1(c_1) + u\left(V^{-1} \sum_{s=1}^{S} \pi_s V(c_{2s})\right) = u_1(c_1) + \sum_{s=1}^{S} \pi_s u(c_{2s}) \\ &= \sum_{s=1}^{S} \pi_s \left(u_1(c_1) + u(c_{2s})\right). \end{aligned} \tag{12}$$

Given the dual contingent claim structure assumed, the consumer's consumption-portfolio optimization problem is given by

$$\max_{c_1, \mathbf{c}_2, \mathbf{z}} U(c_1, \widehat{c}_2) \tag{13}$$

$$S.T. \;\; \widehat{c}_2(\mathbf{c}_2; \boldsymbol{\pi}) = V_{c_1}^{-1}\left(\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})\right), \tag{14}$$

$$c_{2s} = \sum_{j=1}^{J} \xi_{js} z_j \quad \text{and} \quad p_1 c_1 + \sum_{j=1}^{J} q_j z_j = I. \tag{15}$$

It should be noted that the standard assumptions that U is strictly quasiconcave and V_{c_1} is strictly concave for each c_1 do not imply that the first order conditions for the problem (13) - (15) are sufficient for a unique maximum. This can be ensured by requiring

$$U\left(c_1, V_{c_1}^{-1} \sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})\right) \tag{16}$$

to be strictly quasiconcave in $(c_1, c_{21}, ..., c_{2s})$. A sufficient condition for this to be the case is that U_{c_1} is more concave than V_{c_1}, i.e.,[8]

$$-\frac{U''_{c_1}}{U'_{c_1}} \geq -\frac{V''_{c_1}}{V'_{c_1}}. \tag{17}$$

[8]Kimball and Weil (2009) utilize the OCE model in their analysis of the precautionary savings effect and they discuss condition (17) for the special case where U is additively separable and V is independent of c_1 (see Kimball and Weil 2009, p. 248).

(Once the utilities $(U, \{V_{c_1}\})$ have been identified based on Theorems 1 - 3, the condition (17) can be verified.)

The solution to (13) - (15) can be expressed as the period one consumption $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and asset demand function $\mathbf{z}(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$. The open sets of period one consumption price, no-arbitrage asset prices, probabilities and incomes are denoted respectively by $\mathcal{P} \subset \mathbb{R}_{++}$, $\mathcal{Q} \subset \mathbb{R}^J_{++}$, $\Pi \subset \Delta^{S-1}_{++} = \{\boldsymbol{\pi} \in \mathbb{R}^S_{++} | \sum_{s=1}^S \pi_s = 1\}$ and $\mathcal{I} \subset \mathbb{R}_{++}$.[9] It will be understood that when we write $\pi_1, ..., \pi_S$, one can always replace π_S by $1 - \sum_{s=1}^{S-1} \pi_s$. Throughout this paper, we assume that one is given the functions $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and $\mathbf{z}(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and the question is whether they were generated as the result of the optimization (13) - (15) and hence are rationalizable by OCE preferences. Consistent with the above simplex normalization of probabilities, corresponding to any change in π_s $(s \neq S)$ it will be understood that π_S will have a compensating change. Given this convention, $\partial c_1/\partial \pi_s$ and $\partial \mathbf{z}/\partial \pi_s$ are defined for $s = 1, ..., S-1$.

Throughout the paper, we assume that there exists a risk free asset or an effectively risk free asset. The latter is a portfolio constructed from other assets, which has the same payoff in each state. The necessary and sufficient condition for the existence of an effectively risk free asset is

$$rank \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1S} \\ \vdots & \vdots & \cdots & \vdots \\ \xi_{J1} & \xi_{J2} & \cdots & \xi_{JS} \\ 1 & 1 & \cdots & 1 \end{pmatrix} = J. \tag{18}$$

The optimization (13) - (15) can be decomposed into a two stage problem.[10] First conditional on a given value of c_1, one solves the single period problem

$$\max_{\mathbf{c}_2, \mathbf{z}} \sum_{s=1}^S \pi_s V_{c_1}(c_{2s}) \quad S.T. \sum_{j=1}^J q_j z_j = \sum_{s=1}^S p_{2s} c_{2s} = I - p_1 c_1 = I_2, \tag{19}$$

where I_2 denotes period 2 residual income. The solution to (19) is referred to as the conditional asset demand function and denoted by $\mathbf{z}(\mathbf{q}, \boldsymbol{\pi}, I_2 | \, c_1)$. Then the second stage optimization

$$\max_{c_1, \mathbf{c}_2} U(c_1, \widehat{c}_2) = \max_{c_1} U(c_1, \widehat{c}_2 (\mathbf{z}(\, \mathbf{q}, \boldsymbol{\pi}, I_2 | \, c_1)\,)) \tag{20}$$

[9] In the consumption-portfolio setting with incomplete markets, income may not be defined over the full positive orthant. For example, in the demand function (52) in Example 6 below, if $I \to 0$, then $c_1 < 0$ which violates the positivity requirement for consumption.

[10] A necessary and sufficient condition for being able to perform this decomposition is that the conditions in Lemma 1 below are satisfied.

is solved. The resulting optimal period 1 consumption demand $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ can be substituted into $\mathbf{z}(\mathbf{q}, \boldsymbol{\pi}, I_2 | \; c_1)$ yielding the unconditional asset demand $\mathbf{z}(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$. It will prove convenient to base our demand test for and identification of conditional risk preferences, corresponding to $\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})$, on the solution to the conditional asset optimization problem (19). Then for the existence and identification of time preferences represented by $U(c_1, c_2)$, we utilize the solution to (20).

As noted above, we allow state probabilities to vary. This is different from the traditional Arrow-Debreu setting, where probabilities are assumed to be given and fixed. This key difference enables us to define an implicit relationship between probabilities and asset prices from the given asset demand functions. As a result, we can identify a unique set of contingent claim prices and an extended conditional contingent claim demand system which can be utilized in testing whether the asset demand was generated by solving the optimization (19), conditional on each value of period 1 consumption.

3 Obstacles to Rationalizing Demand by a Utility for Assets

The most natural and direct way to solve the integrability problem would be to prove the existence of a rationalizing utility defined over period one consumption and assets rather than period one consumption and contingent claims. However there are a number of serious difficulties with this approach. First, the Hurwicz and Uzawa (1971) existence result in the certainty case makes the perfectly reasonable assumption that consumption is positive. Clearly in an asset setting, this is too strong as it rules out short-selling. It is not obvious whether their argument can be extended to a setting with negative asset holdings. Second, even if one could prove the existence of an increasing and quasiconcave utility over assets, there is no guarantee that this would imply the existence of an increasing and quasiconcave utility over consumption. This difficulty is illustrated by the following simple example based on the first stage optimization problem (19). The Slutsky symmetry and negative semi-definiteness properties necessary for the existence of a utility over assets are satisfied, but there is no utility over contingent claims which is increasing and quasiconcave.

Example 1 *Assume three states with a risk free asset and a risky asset, where* *the payoffs are given by*

$$\xi_{11} = 1,\ \xi_{12} = 1,\ \xi_{13} = 1,\ \xi_{21} = 2,\ \xi_{22} = 0,\ \xi_{23} = \frac{1}{2}. \tag{21}$$

The conditional asset demand is given by

$$z_1(\mathbf{q}) = \frac{I_2}{2q_1} \qquad \text{and} \qquad z_2(\mathbf{q}) = \frac{I_2}{2q_2}. \tag{22}$$

Slutsky symmetry and negative semidefiniteness are satisfied where the former holds automatically since there are only two assets. Applying the Hurwicz and Uzawa (1971) recovery process yields the familiar Cobb-Douglas form defined over assets, $V(z_1, z_2) = z_1 z_2$. *This in turn, corresponds to a utility function over the contingent claim domain* $\left\{ (c_{21}, c_{22}, c_{23}) \in \mathbb{R}^3_{++} \middle| \ c_{23} = \frac{c_{21}}{4} + \frac{3c_{22}}{4} \right\}$ *which is given by*

$$u(c_{21}, c_{22}, c_{23}) = c_{22}\frac{c_{21} - c_{22}}{2}. \tag{23}$$

By the Tietze extension theorem (see e.g., Hazewinkel 2001), this can be extended to a continuous utility function over the entire contingent claim space. However, it is easy to see that the utility is not everywhere monotone in the contingent claim c_{22}.

A third obstacle relates to the actual recovery of the utility for assets using the Hurwicz and Uzawa (1971) process assuming existence is somehow known. The following example illustrates that even for an extremely simple form of utility over contingent claims, in incomplete markets the corresponding asset demand functions, (27) and (28), are far too complicated to be able to integrate back and determine the generating utility.

Example 2 *Assume three states with a risk free asset and a risky asset, where the payoffs are given by*

$$\xi_{11} = 1,\ \xi_{12} = 1,\ \xi_{13} = 1,\ \xi_{21} = 2,\ \xi_{22} = 0,\ \xi_{23} = \frac{1}{2}. \tag{24}$$

Risk preferences over contingent claims conditional on a given c_1 *are represented by the log additive Expected Utility*

$$\pi_1 \ln c_{21} + \pi_2 \ln c_{22} + \pi_3 \ln c_{23}, \tag{25}$$

which can be rewritten as the following utility over assets

$$\pi_1 \ln (z_1 + 2z_2) + \pi_2 \ln z_1 + \pi_3 \ln \left(z_1 + \frac{z_2}{2} \right). \tag{26}$$

Maximizing (26) subject to the budget constraint $q_1 z_1 + q_2 z_2 = I_2$, yields the following very complicated conditional asset demand functions

$$z_1 = \frac{2(\pi_1 + 2\pi_2 + \pi_3) q_1 - (\pi_1 + 5\pi_2 + 4\pi_3) q_2 - \sqrt{A}}{2(\pi_1 + \pi_2 + \pi_3)(q_1 - 2q_2)(2q_1 - q_2)} I_2 \tag{27}$$

and

$$z_2 = \frac{2(\pi_1 + \pi_3) q_1^2 - (9\pi_1 + 5\pi_2 + 6\pi_3) q_1 q_2 + 4(\pi_1 + \pi_2 + \pi_3) q_2^2 + \sqrt{A}}{2(\pi_1 + \pi_2 + \pi_3) q_2 (q_1 - 2q_2)(2q_1 - q_2)} I_2, \tag{28}$$

where

$$A = \pi_1^2 (2q_1 - q_2)^2 + 2\pi_1 (2q_1 - q_2)((3\pi_2 - 4\pi_3) q_2 + 2\pi_3 q_1) + ((3\pi_2 + 4\pi_3) q_2 - 2\pi_3 q_1)^2 . \tag{29}$$

The fourth obstacle also relates to the recovery of a generating (conditional) utility defined over assets. In the recovery process developed in Green, Lau and Polemarchakis (1979) and Dybvig and Polemarchakis (1981), it is assumed that the given asset demand is generated by the maximization of Expected Utility. The next example shows that if this assumption turns out to be incorrect and one nevertheless utilizes their recovery process, it is possible to recover a completely different utility for assets from the one which actually generates the demand.

Example 3 *Assume three states with a risk free asset paying off* 1 *in each state and a risky asset paying off ξ_{2s}. Conditional risk preferences are represented by*

$$\sum_{s=1}^{3} \pi_s \ln(z_1 + (\xi_{2s} + 1) z_2), \tag{30}$$

which is defined over assets and is linear in the probabilities but is clearly not an Expected Utility function over contingent claims. Also, the utility is not strictly increasing in contingent claims. Then following the identification process in Dybvig and Polemarchakis (1981), one is led to conclude that the agent's conditional risk preferences over assets are represented by

$$\sum_{s=1}^{3} \pi_s \ln(z_1 + \xi_{2s} z_2), \tag{31}$$

which is not ordinally equivalent to the representation in eqn. (30). Given the risky asset payoffs ξ_{2s} $(s = 1, 2, 3)$, it is clear that (31) is an Expected Utility representation defined over contingent claims whereas (30) is not. To avoid this problem, in the rest of the paper, we always give tests for the assumed utility forms before employing our identification process. (Supporting material is given in Appendix B.1.)

4 Integrability and Identification of Conditional Asset Demands

To verify that period 1 consumption and asset demand were generated by OCE preferences, we first show that the implied conditional asset demand is a solution to the first stage maximization (19) based on an Expected Utility representation of risk preferences. In the first subsection, necessary and sufficient conditions are given for this to be the case in an incomplete market setting. Moreover, we provide a means for recovering the set of conditional NM indices $\{V_{c_1}\}$. One of the conditions in our integrability result assumes that there exists a unique set of contingent claim prices. The second subsection provides conditions such that this assumption is validated and a means for deriving the unique contingent claim price vector consistent with the Expected Utility representation. Examples 6 and 7 provide a comprehensive, concrete illustration where the conditions for the existence of a conditional Expected Utility are satisfied, a unique contingent claim price vector is derived and the conditional NM indices are recovered.

4.1 Conditional Risk Preferences

Varian (1983) showed that in revealed preference analyses, the Afriat inequalities for asset demand can be used to test whether demand and price observations are consistent with the maximization of an Expected Utility representation in an incomplete market setting. Green and Srivastava (1986) had the insight in their revealed preference analysis involving assets and contingent claims to simply require that there exist contingent claim prices which are consistent with the no-arbitrage relation and satisfy the standard Afriat inequalities for contingent claim demands. We apply this insight in our derivation below of necessary and sufficient conditions for asset demand to be rationalizable by an Expected Utility representation of conditional risk preferences.

The question of existence of a conditional Expected Utility can only be answered in terms of restrictions on conditional asset demand.[11] However it is reasonable to suppose that one is given unconditional demands for period 1 consumption and assets. So we first need to ensure that it is possible to derive a unique

[11] If the unconditional demands can be rationalized by a twice continuously differentiable, strictly increasing and strictly quasiconcave utility function, then one can always consider the two stage optimization and hence unique twice continuously differentiable conditional asset demand exists. But if the utility defined over period 1 consumption and assets is not differentiable, then it is possible that conditional demand is not unique, as shown in Example 4 below, and a unique rationalization taking the Expected Utility form may not exist.

twice continuously differentiable conditional asset demand from the unconditional demands. However as the next two examples show, respectively, multiple conditional asset demand functions corresponding to the first stage optimization (19) may exist and no conditional asset demand may be derivable from the unconditional demands.

Example 4 *Assume that*

$$c_1 = \frac{I}{p_1 + q_1 + q_2}, \; z_1 = \frac{\pi_1 (q_1 + q_2) I}{(\pi_1 + \pi_2)(p_1 + q_1 + q_2) q_1} \quad \text{and } z_2 = \frac{\pi_2 (q_1 + q_2) I}{(\pi_1 + \pi_2)(p_1 + q_1 + q_2) q_2}. \tag{32}$$

It can be verified that the Slutsky matrix associated with (c_1, z_1, z_2) *is symmetric and negative semidefinite. Noting that*

$$I_2 = I - p_1 c_1 = \frac{(q_1 + q_2) I}{p_1 + q_1 + q_2} \tag{33}$$

and substituting the above I_2 *and* c_1*, respectively, into the formula for* z_1*, two different expressions can be derived for conditional demand* $z_1(\mathbf{q}, \boldsymbol{\pi}, I_2 | \; c_1)$

$$z_1(\mathbf{q}, \boldsymbol{\pi}, I_2 | \; c_1) = \frac{\pi_1 I_2}{(\pi_1 + \pi_2) q_1} \quad \text{and} \quad z_1(\mathbf{q}, \boldsymbol{\pi}, I_2 | \; c_1) = \frac{\pi_1 (q_1 + q_2) c_1}{(\pi_1 + \pi_2) q_1}. \tag{34}$$

Thus in this case, a unique conditional asset demand function cannot be identified as the solution to the first stage optimization and the given unconditional demands cannot be uniquely rationalized by representations of conditional risk and time preferences.

Example 5 *Assume that*

$$c_1 = \frac{I}{p_1 + q_1 + q_2}, \quad z_1 = \frac{\pi_1 (q_1 + q_2) I^2}{(p_1 + q_1 + q_2) q_1^2} \quad \text{and} \quad z_2 = \frac{(q_1 - \pi_1 I)(q_1 + q_2) I}{(p_1 + q_1 + q_2) q_1 q_2}. \tag{35}$$

It can be verified that the Slutsky matrix associated with (c_1, z_1, z_2) *is not symmetric. But we still have*

$$I_2 = I - p_1 c_1 = \frac{(q_1 + q_2) I}{p_1 + q_1 + q_2}. \tag{36}$$

Since c_1 *and* I_2 *are not independent, one cannot solve for* (p_1, I) *as a function of* (c_1, I_2) *from the following two equations*[12]

$$c_1 = \frac{I}{p_1 + q_1 + q_2} \quad \text{and} \quad I_2 = \frac{(q_1 + q_2) I}{p_1 + q_1 + q_2}. \tag{37}$$

Therefore it is not possible to derive the conditional demand $z_i(\mathbf{q}, \boldsymbol{\pi}, I_2 | \; c_1)$ $(i = 1, 2)$ *from the above unconditional demand.*

[12] The expressions for c_1 and I_2 are the same as they were in Example 4. However in that example even though (p_1, I) cannot be solved for as a function of (c_1, I_2), an infinite number of conditional demand functions can be obtained due to the specific forms of z_1 and z_2.

The following lemma provides a sufficient condition for the existence of unique twice continuously differentiable conditional asset demand functions. It will prove useful to denote the Jacobian matrix of derivatives of the vector function (c_1, I_2) with respect to (p_1, I) as

$$J_c = \frac{\partial (c_1, I_2)}{\partial (p_1, I)}. \tag{38}$$

The nonsingularity of the Jacobian matrix (38) ensures that (p_1, I) can be uniquely expressed as functions of $(c_1, I_2, \mathbf{q}, \boldsymbol{\pi})$.[13] Substituting these functions into the unconditional demand $(z_1, ..., z_J)$, one obtains the conditional demand. This process is illustrated in Example 6 below.

Lemma 1 *For given twice continuously differentiable demands* $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ *and* $\mathbf{z}(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$, *if (i) the vector map* $(c_1, I_2)(p_1, I)$ *is proper*[14] *and (ii)* $\forall (p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$, $\det J_c \neq 0$, *then* $\forall (p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$, *there exists unique twice continuously differentiable conditional asset demand*

$$z_i(\mathbf{q}, \boldsymbol{\pi}, I_2 |\ c_1) = z_i(p_1, \mathbf{q}, \boldsymbol{\pi}, I) \qquad (i = 1, ..., J). \tag{39}$$

Condition (ii) ensures the local existence of conditional demand and condition (i) guarantees that conditional demand exists globally. If one starts with the U and $\{V_{c_1}\}$ defining OCE utility being, respectively, strictly increasing and strictly quasiconcave and strictly increasing and strictly concave, then it is always possible to express the consumption-portfolio optimization in two stages and there will always be a unique conditional asset demand. However, suppose one begins with unconditional demands and $\det J_c = 0$ as in Examples 4 and 5. Then there does not exist a unique twice continuously differentiable conditional asset demand derivable from the unconditional demands and the two stage optimization process cannot be employed.

The key to applying the ideas of Varian and Green and Srivastava to our setting is to postulate the existence of a function, $\mathbf{p}_2 : \mathcal{Q} \times \Pi \times \mathcal{I} \rightarrow \mathbb{R}^S_{++}$ that maps asset prices to contingent claim prices. This allows us to define the conditional contingent claim demand for all $(\mathbf{q}, \boldsymbol{\pi}, I_2) \in \mathcal{Q} \times \Pi \times \mathcal{I}$ as

$$c_{2s}(\mathbf{p}_2(\mathbf{q}, \boldsymbol{\pi}, I_2), \boldsymbol{\pi}, I_2 |\ c_1) = \sum_{j=1}^{J} z_j(\mathbf{q}, \boldsymbol{\pi}, I_2 |\ c_1)\xi_{js}. \tag{40}$$

[13] The reason for including $\mathbf{q}$ and $\boldsymbol{\pi}$ as arguments in the inverse demand functions is to ensure that $\mathbf{q}$ and $\boldsymbol{\pi}$ will enter into the unconditional demand $(z_1, ..., z_J)$ as parameters.

[14] A function between topological spaces is called proper if inverse images of compact subsets are compact.

It will also prove convenient to introduce the "constrained asset demand range" $\mathcal{A}$ which corresponds to the set of conditional asset demands that maximize utility and do not violate the no bankruptcy constraint, i.e.,[15]

$$\mathcal{A}=\left\{ (z_1,...,z_J)\in\mathbb{R}^J \middle| \left(\sum_{j=1}^{J}\xi_{j1}z_j,...,\sum_{j=1}^{J}\xi_{jS}z_j\right)\in\mathbb{R}^S_{++},\ (\mathbf{q},\boldsymbol{\pi},I_2)\in\mathcal{Q}\times\Pi\times\mathcal{I}\right\}. \tag{41}$$

Also define the "constrained contingent claim demand range" $\mathcal{C}$ which corresponds to the set of conditional contingent claim demands associated with $\mathcal{A}$ and will be the domain of the utility in Theorem 1.

$$\mathcal{C}=\left\{(c_{21},...,c_{2S})\in\mathbb{R}^S_{++} \;\middle|\; \begin{array}{c}(c_{21},...,c_{2S})=\left(\sum_{j=1}^{J}\xi_{j1}z_j,...,\sum_{j=1}^{J}\xi_{jS}z_j\right),\\ (\mathbf{q},\boldsymbol{\pi},I_2)\in\mathcal{Q}\times\Pi\times\mathcal{I}\end{array}\right\}. \tag{42}$$

Then we have the following theorem.

Theorem 1 *Assume $S>2$, and conditions (i) and (ii) in Lemma 1 are satisfied. Then the unique twice continuously differentiable conditional asset demand $\mathbf{z}(\mathbf{q},\boldsymbol{\pi},I_2|\ c_1)$ over $\mathcal{A}$ can be rationalized by a unique state independent Expected Utility function defined over $\mathcal{C}$*

$$\sum_{s=1}^{S}\pi_s V_{c_1}\left(\sum_{j=1}^{J}\xi_{js}z_j\right)=\sum_{s=1}^{S}\pi_s V_{c_1}(c_{2s}), \tag{43}$$

where V_{c_1} is twice continuously differentiable, strictly increasing and strictly concave if and only if

(i) *there exists a twice continuously differentiable price function $\mathbf{p}_2$ satisfying*

$$q_j=\sum_{s=1}^{S}p_{2s}(\mathbf{q},\boldsymbol{\pi},I_2)\,\xi_{js}; \tag{44}$$

(ii) *there exists a twice continuously differentiable function $f_{c_1}:\mathbb{R}^2_{++}\to\mathbb{R}_{++}$ such that for all $(\mathbf{q},\boldsymbol{\pi},I_2)\in\mathcal{Q}\times\Pi\times\mathcal{I}$ the contingent claim demand (40) derived from asset demand, satisfies $c_{2s}=f_{c_1}(c_{21},k_s)$, where k_s is defined by*

$$k_s=_{def}\frac{\pi_s p_{21}}{\pi_1 p_{2s}}, \tag{45}$$

and $f_{c_1}(c_{21},k_s)$ is strictly increasing in k_s and $f_{c_1}(x,1)=x$ for all x; and

[15] The assumption that each $\sum_{j=1}^{J}\xi_{js}z_j\in\mathbb{R}_{++}$ is consistent with Mas-Colell (1978, p. 122).

(iii) *the extended contingent claim demand* $\bar{c}_{2s}(p_{21},...,p_{2S},I_2)$ $(s=1,...,S)$ *derived from the set of equations*

$$c_{2s} = f_{c_1}(c_{21},k_s) \quad (s=2,...,S) \tag{46}$$

and

$$I_2 = \sum_{s=1}^{S} p_{2s}c_{2s} \tag{47}$$

is twice continuously differentiable and satisfies Slutsky symmetry and negative semidefiniteness for all $(p_{21},...,p_{2S},I_2) \in \mathbb{R}^S_{++} \times \mathcal{I}$*, where the Slutsky matrix is defined by* $(\sigma_{ij})_{S\times S}$ *and*

$$\sigma_{ij} = \frac{\partial \bar{c}_{2i}}{\partial p_{2j}} + \bar{c}_{2j}\frac{\partial \bar{c}_{2i}}{\partial I_2}. \tag{48}$$

The NM index V_{c_1} *is uniquely identified up to a positive affine transformation over* $\mathcal{C}$ *by the following relation*

$$f_{c_1}(c_{21},k_s) = V'^{-1}_{c_1}\left(\frac{V'_{c_1}(c_{21})}{k_s}\right). \tag{49}$$

It should be emphasized that the extended contingent claim demand in Theorem 1(iii) is derived from the conditional contingent claim demand. As a result, whereas the extended demand functions are uniquely defined, this is not true for the unconditional contingent claim demand functions, since the latter depend on c_1, which does not have a unique expression due to the interdependence of contingent claim prices (see the discussion of eqns. (87) and (88) in Section 5 below).

The utility function obtained in Theorem 1 is unique but only in the subspace of $\mathbb{R}^S_{++}$ spanned by conditional asset demand. As Example 8 at the end of Section 4.2 demonstrates, the conditional asset demand may have been generated by a non-Expected Utility as well as the Expected Utility function identified using the differential equation (49) in Theorem 1. Introduction of the extended contingent claim demand in condition (iii) of the theorem enables us, once a unique contingent claim price vector has been derived, to extend the rationalizing utility to the full space $\mathbb{R}^S_{++}$. As a result, we can then apply the Slutsky symmetry and negative semidefiniteness tests. However, this in no way enables us to overcome the fact that the full space is not spanned and non-Expected Utility functions which generate the same demand may exist.

Remark 2 *Theorem 1 is similar to the classic integrability result of Hurwicz and Uzawa (1971) in that the resulting utility is local in the sense of being defined* *only over the range of demand and not over the full choice space. Also both results assume that demand satisfies Slutsky symmetry and negative semidefiniteness. However, they differ in a number of ways. First, Theorem 1 does not assume the global Lipschitz condition of Hurwicz and Uzawa (1971), which specifies that for every positive $a' < a''$, there exists a positive $K_{a',a''}$ such that for* $s = 1, ..., S$,

$$\left| \frac{\partial \overline{c}_{2s} \left(p_1, \mathbf{p}_2, I \right)}{\partial I} \right| < K_{a',a''} \tag{50}$$

for any I and $a' < p_1 < a''$, $a' < p_{2s} < a''$. As pointed out in a private communication from Phil Reny, this condition, which must hold even for incomes of zero, is actually not required for the Hurwicz and Uzawa (1971) result. It is enough if it holds only for incomes and prices between a' and a''. This weaker restriction is satisfied, for example, when demand is continuously differentiable on the strictly positive orthant of prices and incomes. Second, our conditions are necessary and sufficient whereas those of Hurwicz and Uzawa (1971) are only sufficient. This difference is due to our rationalizing utility being strictly increasing and strictly quasiconcave instead of just increasing and quasiconcave as in Hurwicz and Uzawa (1971). These stronger properties of the utility are ensured by the requirements on f_{c_1} in condition (ii).[16] *Third, our rationalizing utility is twice continuously differentiable rather than upper semicontinuous in Hurwicz and Uzawa (1971). This difference arises because condition (ii) in Theorem 1 requires $f_{c_1}(c_{21}, k_s)$ to be strictly increasing in k_s which in turn implies that an inverse extended contingent claim demand exists. Since the extended contingent claim demand is twice continuously differentiable, the inverse demand is also twice continuously differentiable, implying that the rationalizing utility is twice continuously differentiable. Fourth, our rationalizing utility is unique while the utility based on the Hurwicz and Uzawa conditions may not be. To guarantee uniqueness, Mas-Colell (1978) gives a sufficient condition which is that the rationalizing utility is strictly increasing and strictly quasiconcave. He introduces a desirability property, which implies that the norm of demand goes to infinity when prices go to the boundary, to achieve the goal. But as discussed above, strict monotonicity and strict quasiconcavity are already ensured by the requirements on f_{c_1} in condition (ii), i.e., $f_{c_1}(c_{21}, k_s)$*

[16] In the classic certainty setting, Mas-Colell (1978) introduces local Lipschitz and desirability conditions on demand to ensure that the rationalizing utility is strictly increasing and strictly quasiconcave and to guarantee uniqueness of the rationalizing utility. If one combines his assumptions with those of Hurwicz and Uzawa (1971), it is possible to show that the resulting conditions are not just sufficient but necessary and sufficient for the existence of a unique rationalizing utility. For further discussion of the merger of the Hurwicz and Uzawa (1971) and Mas-Colell (1978) results, see Kannai, Selden and Wei (2016, Section 2).

is strictly positive and strictly increasing in k_s.

Remark 3 *Kubler, Selden and Wei (2014, Theorem 2) derive a demand test for Expected Utility preferences in a single period setting. They assume that asset markets are complete and asset demand satisfies the necessary and sufficient conditions for the existence of a strictly increasing, three times continuously differentiable and strictly quasiconcave utility. This result is generalized by Theorem 1 in two significant ways. First, asset markets are allowed to be incomplete which significantly complicates the derivation of a demand test for conditional asset demand. Second, restrictions are imposed on asset demand which are necessary and sufficient for the existence of a strictly increasing, twice continuously differentiable and strictly quasiconcave utility rather than simply assuming that it exists. In particular we only need to verify that the contingent claim demand (derived from asset demand) passes the demand restriction $c_{2s} = f_{c_1}(c_{21}, k_s)$ on the restricted set of prices constructed from Theorem 2 below. Then to obtain the existence of a representation, we require that the demand derived from eqns. (46) and (47), which corresponds to the extended contingent claim demand $\overline{\mathbf{c}}_2(\mathbf{p}_2, \boldsymbol{\pi}, I_2)$, satisfy Slutsky symmetry and negative semidefiniteness. Moreover, since the extended contingent claim demand function is derived without considering the interdependence among the contingent claim prices, the utility function rationalizing the extended demand, if it exists, is unique up to an increasing transformation.*

If asset demand satisfies the conditions in Theorems 1 and 2 (below), then eqn. (49) provides a relatively simple differential equation from which to determine each NM index V_{c_1}. This by-product of Theorem 1 represents an attractive alternative to the identification process in Dybvig and Polemarchakis (1981), Green, Lau and Polemarchakis (1979) and Polemarchakis and Selden (1984) which assume that demand is rationalizable by an Expected Utility representation. If this assumption turns out to be erroneous, then as demonstrated in Example 3 above, application of their identification process may give rise to an Expected Utility representation that will not rationalize the given demand. Finally in practice, the process used in this literature for actually deriving the NM index, which is based on the marginal rates of substitution approach following Mas-Colell (1977), often can be more difficult than solving eqn. (49) in Theorem 1.

4.2 A Unique Contingent Claim Price Function

To derive the integrability result in the prior subsection, we postulated the existence of a function that maps asset prices to contingent claim prices. However, the price function $\mathbf{p}_2$, assumed to exist in Theorem 1, is generally impossible to find in practice without additional information. We first give an example illustrating this dilemma.

Example 6 *Assume three states and two assets where the payoffs are given by*

$$\xi_{11} = 1,\ \xi_{12} = 0,\ \xi_{13} = \frac{1}{2},\ \xi_{21} = 0,\ \xi_{22} = 1,\ \xi_{23} = \frac{1}{2}. \tag{51}$$

The period 1 consumption and unconditional asset demands are respectively given by[17]

$$c_1 = \frac{I}{p_1} - \frac{\pi_1 + \pi_2 + \pi_3}{p_1} \left(\begin{array}{c} \frac{1}{p_1} \left(\frac{\pi_1(1-1/B)}{q_1 - q_2} \right)^{\pi_1} \left(\frac{\pi_2(B-1)}{q_1-q_2} \right)^{\pi_2} \\ \times \left(\frac{\pi_3(1-1/B)}{2(q_2 - q_1/B)} \right)^{\pi_3} \end{array} \right)^{-\frac{1}{\pi_1+\pi_2+\pi_3+1}}, \tag{52}$$

$$z_1 = \frac{\pi_1 (1 - 1/B)}{q_1 - q_2} \left(\begin{array}{c} \frac{1}{p_1} \left(\frac{\pi_1(1-1/B)}{q_1 - q_2} \right)^{\pi_1} \left(\frac{\pi_2(B-1)}{q_1-q_2} \right)^{\pi_2} \\ \times \left(\frac{\pi_3(1-1/B)}{2(q_2 - q_1/B)} \right)^{\pi_3} \end{array} \right)^{-\frac{1}{\pi_1+\pi_2+\pi_3+1}} \tag{53}$$

and

$$z_2 = \frac{\pi_2 (B - 1)}{q_1 - q_2} \left(\begin{array}{c} \frac{1}{p_1} \left(\frac{\pi_1(1-1/B)}{q_1 - q_2} \right)^{\pi_1} \left(\frac{\pi_2(B-1)}{q_1-q_2} \right)^{\pi_2} \\ \times \left(\frac{\pi_3(1-1/B)}{2(q_2 - q_1/B)} \right)^{\pi_3} \end{array} \right)^{-\frac{1}{\pi_1+\pi_2+\pi_3+1}}, \tag{54}$$

where

$$A = \sqrt{\left(\left(\pi_2 + \pi_3 \right) q_1 - \left(\pi_1 + \pi_3 \right) q_2 \right)^2 + 4\pi_1 \pi_2 q_1 q_2} \tag{55}$$

and

$$B = \frac{\pi_1 q_1 \left(\left(\pi_2 + \pi_3 \right) q_1 - \left(\pi_1 + 2\pi_2 + \pi_3 \right) q_2 + A \right)}{\pi_2 q_2 \left(\left(2\pi_1 + \pi_2 + \pi_3 \right) q_1 - \left(\pi_1 + \pi_3 \right) q_2 - A \right)}. \tag{56}$$

It can be verified that

$$\frac{\partial c_1}{\partial q_j} + z_j \frac{\partial c_1}{\partial I} = \frac{\partial z_j}{\partial p_1} + c_1 \frac{\partial z_j}{\partial I} \quad (i = 1, 2) \tag{57}$$

and

$$\frac{\partial z_i}{\partial q_j} + z_j \frac{\partial z_i}{\partial I} = \frac{\partial z_j}{\partial q_i} + z_i \frac{\partial z_j}{\partial I} \quad (i \neq j \in \{1, 2\}), \tag{58}$$

implying that the Slutsky matrix is symmetric. The matrix is also negative semi-definite.[18] *However given the quite complicated form of the demand functions, it is very difficult to follow the integrability process proposed by Hurwicz and Uzawa*

[17] It may strike the reader as surprising that asset demand is independent of income I. This will be clarified in Example 10 below. Nevertheless the associated conditional asset demand will be seen to depend on period two income I_2.

[18] Verifying this result is quite tedious although straightforward using Mathematica software. The details are available upon request from the authors.

(1971) to recover the rationalizing utility function. Given that the unconditional demands satisfy the conditions in Lemma 1, it is possible to next derive the corresponding unique conditional asset demand functions. First, solve for the period 2 income

$$I_2 = I - p_1 c_1 = (\pi_1 + \pi_2 + \pi_3) \left(\begin{array}{c} \frac{1}{p_1} \left(\frac{\pi_1(1-1/B)}{q_1-q_2}\right)^{\pi_1} \left(\frac{\pi_2(B-1)}{q_1-q_2}\right)^{\pi_2} \\ \times \left(\frac{\pi_3(1-1/B)}{2(q_2-q_1/B)}\right)^{\pi_3} \end{array} \right)^{-\frac{1}{\pi_1+\pi_2+\pi_3+1}}, \tag{59}$$

from which we obtain

$$p_1 = \left(\frac{I_2}{\pi_1+\pi_2+\pi_3}\right)^{\pi_1+\pi_2+\pi_3+1} \left(\frac{\pi_1(1-1/B)}{q_1-q_2}\right)^{\pi_1} \left(\frac{\pi_2(B-1)}{q_1-q_2}\right)^{\pi_2} \left(\frac{\pi_3(1-1/B)}{2(q_2-q_1/B)}\right)^{\pi_3}. \tag{60}$$

Substituting the above equation into the unconditional asset demand yields

$$z_1 = \frac{\pi_1(1-1/B)\,I_2}{(\pi_1+\pi_2+\pi_3)(q_1-q_2)} \quad \text{and} \quad z_2 = \frac{\pi_2(B-1)\,I_2}{(\pi_1+\pi_2+\pi_3)(q_1-q_2)}. \tag{61}$$

Using (51), one can derive the following relations between assets and contingent claims

$$c_{21} = \xi_{11} z_1 + \xi_{21} z_2 = z_1, \quad c_{22} = \xi_{12} z_1 + \xi_{22} z_2 = z_2 \tag{62}$$

and

$$c_{23} = \xi_{13} z_1 + \xi_{23} z_2 = \frac{1}{2}(z_1 + z_2). \tag{63}$$

Substituting (51) into eqn. (2), we obtain the following relation between asset and contingent claim prices

$$q_1 = p_{21} + \frac{p_{23}}{2} \quad \text{and} \quad q_2 = p_{22} + \frac{p_{23}}{2}. \tag{64}$$

Given the two equations in (64) and the three contingent claim prices, there is no way to find the unknown function $\mathbf{p}_2$ *needed for the integrability result in Theorem 1. In the following discussion, we show how to use (61) to construct this function.*

As the above example illustrates, the difficulty in applying results derived in the contingent claim space to the asset space is that when markets are incomplete ($J < S$), $p_{21}, ..., p_{2S}$ cannot be uniquely recovered from asset prices based on eqn. (2) since there are S variables but only J independent equations. Without knowing (expressions for) contingent claim prices, it is not possible to utilize the integrability result in Theorem 1. However, we next show that it is possible to derive a candidate contingent claim price vector based on an Expected Utility representation by considering the effects of varying probabilities on the inverse demand. Before stating our result, it will prove useful to consider the inverse demand function which maps asset demand, probabilities and income into a supporting price vector.

Denote the Jacobian matrix of derivatives of the vector function $(c_1, \mathbf{z})$ with respect to $(p_1, \mathbf{q})$ as

$$J_u = \frac{\partial (c_1, z_1, ..., z_J)}{\partial (p_1, q_1, ..., q_J)}. \tag{65}$$

Then the following ensures the global existence of unique inverse demand.

Lemma 2 *Assume $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and $z(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ are twice continuously differentiable over prices, probabilities and income. If (i) $\forall(\boldsymbol{\pi}, I) \in \Pi \times \mathcal{I}$, $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and $z(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ are proper maps with respect to $(p_1, \mathbf{q})$ and (ii) $\forall(p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$, $\det J_u \neq 0$, then $\forall(p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$, there exists unique twice continuously differentiable inverse demands $p_1(c_1, \mathbf{z}, \boldsymbol{\pi}, I)$ and $q_i(c_1, \mathbf{z}, \boldsymbol{\pi}, I)$ $(i = 1, ..., J)$.*

Denote the conditional inverse demand function by $q_j(\mathbf{z}, \boldsymbol{\pi}, I_2)$ $(j = 1, 2, ..., J)$.[19] It should be noted that we use $q_j(\cdot)$ to refer to both the unconditional and conditional inverse demand. However, these can easily be distinguished, respectively, by the inclusion of I or I_2 as an argument. For the analysis below, we require taking the partial derivatives of $\mathbf{q}(\mathbf{z}, \boldsymbol{\pi}, I_2)$ with respect to probabilities. Although Lemma 2 is stated in terms of the unconditional demands, one can prove that conditional asset demand, if it exists, inherits the properties (i) and (ii) as well as being twice continuous differentiability.[20] Therefore, if the conditions in Lemma 2 are satisfied, the conditional demand is also globally invertible. Then we have the following theorem.

Theorem 2 *Given the conditional asset demand functions $z_j(\mathbf{q}, \boldsymbol{\pi}, I_2 | \; c_1)$ $(j = 1, ..., J)$ and the payoff matrix $(\xi_{js})_{J\times S}$, suppose that for a given $j \in \{1, ..., J\}$, $\xi_{js} - \frac{c_{2s}q_j}{I_2} \neq 0$ $(\forall s = 1, ..., S)$ and the conditions in Lemma 2 are satisfied. Then the following set of equations*

$$p_{2s} = \frac{\pi_s \left(\frac{\partial q_j}{\partial \pi_s} - \sum_{l=1}^{S-1} \pi_l \frac{\partial q_j}{\partial \pi_l} \right)}{\xi_{js} - \frac{c_{2s}q_j}{I_2}} \quad (s = 1, ..., S-1) \tag{66}$$

[19] Since the asset price $\mathbf{q}$ is exogenously given and independent of income, it may seem strange to state $\mathbf{q}(\mathbf{z}, \boldsymbol{\pi}, I_2)$. However it should be emphasized that this function corresponds to inverse demand. Indeed when we consider the left hand side of eqns. (66) - (67) and in Theorem 2, $\mathbf{q}$ denotes inverse demand and is a function of asset demand, probabilities and income. For the right hand side of eqns. (66) - (67), if c_{2s} is not proportional to I_2, $(c_{2s}q_j)/I_2$ will also be a function of I_2. Therefore, the contingent claim inverse demand (price) $\mathbf{p}_2$ will be a function of I_2 as well.

[20] The inheritance of twice continuous differentiability is obvious. For a formal proof for the inheritance of properties (i) and (ii), refer to Kannai, Selden and Wei (2016, Claims 2 and 3 in the proof of Theorem 3).

and

$$p_{2S} = -\frac{\pi_S \sum_{l=1}^{S-1} \pi_l \frac{\partial q_j}{\partial \pi_l}}{\xi_{jS} - \frac{c_{2S} q_j}{I_2}} \tag{67}$$

(derived under the assumption that the utility in the first stage optimization (19) takes the Expected Utility form (43)) define a unique solution $\mathbf{p}_2(\mathbf{q}, \boldsymbol{\pi}, I_2)$ *corresponding to the candidate set of contingent claim prices.*

In Theorem 2, we conjecture that the conditional preferences are represented by an Expected Utility and then derive a unique set of contingent claim prices. If the conjecture is correct, then the conditions in Theorem 1 will be satisfied using these contingent claim prices. If the conjecture is wrong, one can still derive a unique set of contingent claim prices based on eqns. (66) - (67). However, the conditions in Theorem 1 will fail with these prices. It should be emphasized that the S contingent claim prices constructed in Theorem 2 are in general not independent of each other since they are functions of the J asset prices, where $J < S$.

Remark 4 *In Example 7 below, we illustrate that the derivative conditions in Theorem 2 allow one to extend the integrability result for Expected Utility preferences from complete to incomplete markets. Applying similar logic, it is relatively straightforward to show that for revealed preference tests of Expected Utility based on contingent claim demand, one can use an independent set of discrete demand, price and probability observations to play the role of eqns. (66) - (67) in Theorem 2 to pin down a candidate vector of contingent claim prices. While feasible, the informational demand is non-trivial and realistically could only be applied in a laboratory setting such as in the non-parametric asset demand tests of Choi, et al. (2007). We return to this issue in Section 6.*

The following continuation of Example 6 illustrates first how to use Theorem 2 to derive a unique set of contingent claim prices. Then second, these prices are used in an application of Theorems 1 and 2 to verify that the given conditional asset demand is in fact generated as the result of an Expected Utility maximization.

Example 7 *To find a unique set of contingent claim prices consistent with (64) in Example 6, we use the results from Theorem 2, where the inverse demand function is based on the conjecture that the conditional asset demand was generated as the result of Expected Utility maximization. Deriving the inverse conditional asset* *demand functions from eqn. (61) yields*

$$q_1 = \frac{(\pi_1 + \pi_3) z_1 + \pi_1 z_2}{(\pi_1 + \pi_2 + \pi_3)(z_1 + z_2) z_1} I_2 \quad \text{and} \quad q_2 = \frac{(\pi_2 + \pi_3) z_2 + \pi_2 z_1}{(\pi_1 + \pi_2 + \pi_3)(z_1 + z_2) z_2} I_2. \tag{68}$$

Computing the derivatives of q_1 with respect to π_1 and π_2 as in Theorem 2, and performing some algebraic calculations, it can be shown that

$$p_{21} = \frac{q_1 - q_2}{1 - 1/B}, \quad p_{22} = \frac{q_1 - q_2}{B - 1} \quad \text{and} \quad p_{23} = \frac{2(q_2 - q_1/B)}{1 - 1/B}. \tag{69}$$

Clearly this set of contingent claim prices satisfies (64) and is uniquely determined based on the given asset prices and state probabilities.

Next we show that the given conditional asset demand is consistent with the maximization of an Expected Utility function. Using the contingent claim price relations (69), it can be easily seen that

$$\frac{p_{21}}{p_{22}} = B \quad \text{and} \quad \frac{p_{21}}{p_{23}} = \frac{q_1 - q_2}{2(q_2 - q_1/B)}. \tag{70}$$

Then utilizing eqn. (62) to consider the relationship between c_{21} and c_{22}, we have[21]

$$\begin{aligned} \frac{c_{22}}{c_{21}} &= \frac{z_2}{z_1} = \frac{q_1((\pi_2 + \pi_3) q_1 - (\pi_1 + 2\pi_2 + \pi_3) q_2 + A)}{q_2((2\pi_1 + \pi_2 + \pi_3) q_1 - (\pi_1 + \pi_3) q_2 - A)} \\ &= \frac{\pi_2}{\pi_1} B = \frac{\pi_2 p_{21}}{\pi_1 p_{22}}. \end{aligned} \tag{71}$$

Similarly, utilizing eqn. (63), the relationship between c_{21} and c_{23} is given by

$$\frac{c_{23}}{c_{21}} = \frac{z_1 + z_2}{2z_1} = \frac{1}{2} + \frac{\pi_2 B}{2\pi_1}. \tag{72}$$

Noticing that

$$\frac{\left(1 + \frac{\pi_2 B}{\pi_1}\right)(q_2 - q_1/B)}{q_1 - q_2} = \frac{\pi_3}{\pi_1}, \tag{73}$$

it follows that

$$\frac{c_{23}}{c_{21}} = \frac{z_1 + z_2}{2z_1} = \frac{\pi_3 p_{21}}{\pi_1 p_{23}}. \tag{74}$$

Thus, $c_{2s} = k_s c_{21}$ and $c_{2s} = c_{21}$ when setting $k_s = 1$ $(s = 2, 3)$. Combining eqns. (71) and (74) with the budget constraint

$$p_{21} c_{21} + p_{22} c_{22} + p_{23} c_{23} = I_2, \tag{75}$$

[21] To obtain the relation between c_{21} and c_{22} in eqn. (71), first solve for I_2 as a function of c_{21} (this is possible since contingent claims must be normal goods in an Expected Utility setting). Next substitute for I_2 in the c_{22} demand function, yielding c_{22} as a function of c_{21}, q_1, q_2 and probabilities. Using eqn. (69) to solve for q_1 and q_2 as functions of p_{21} and p_{22}, c_{22} can be expressed as a function of c_{21}, p_{21}, p_{22} and probabilities.

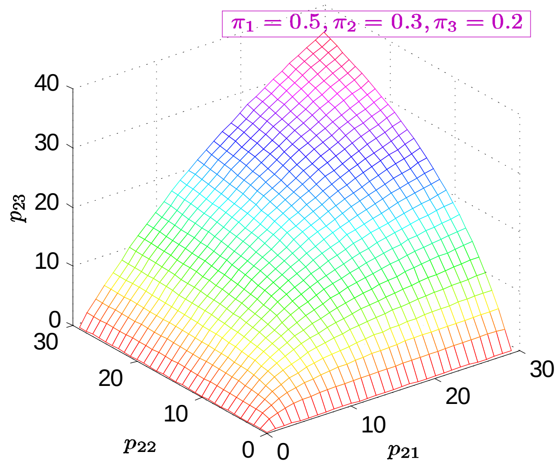


Figure 1:

yields the unique set of extended contingent claim demands

$$\bar{c}_{2s} = \frac{\pi_s I_2}{p_{2s}\left(\pi_1 + \pi_2 + \pi_3\right)} \quad (s = 1, 2, 3)\,. \tag{76}$$

These demand functions satisfy Slutsky symmetry and negative semidefiniteness. Then it follows from Theorem 1 that the conditional asset demand (61) is consistent with Expected Utility maximization where the contingent claim prices are given by (69). Moreover since

$$f_{c_1}\left(c_{21}, k_s\right) = k_s c_{21} = V_{c_1}^{\prime -1}\left(\frac{V_{c_1}^{\prime}\left(c_{21}\right)}{k_s}\right) \tag{77}$$

implies

$$V_{c_1}^{\prime}\left(k_s c_{21}\right) = \frac{V_{c_1}^{\prime}\left(c_{21}\right)}{k_s}, \tag{78}$$

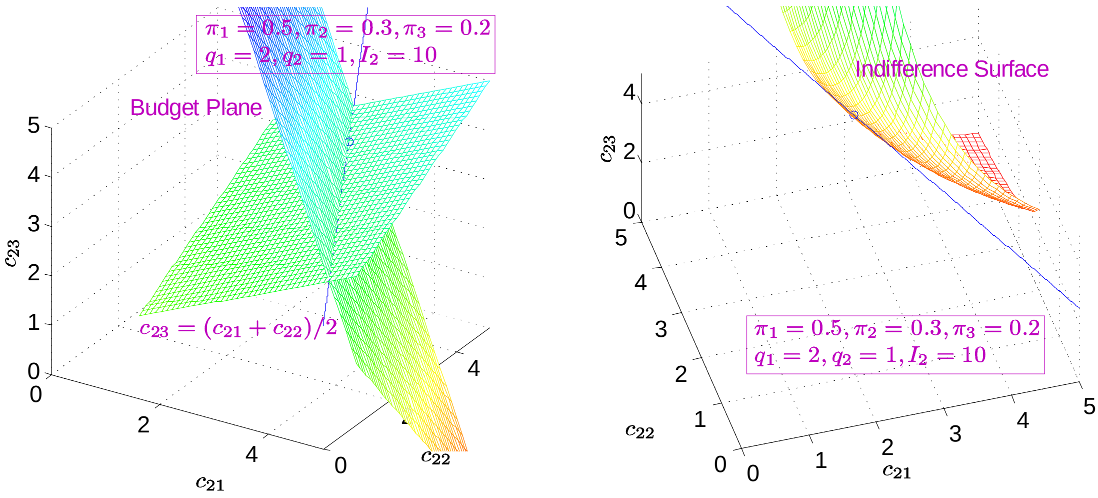
$V_{c_1}^{\prime}\left(c_2\right)$ *is homogeneous with degree* -1 *and hence* $V_{c_1}\left(c_2\right) = \ln c_2$. *(Supporting calculations can be found in Appendix B.2.)*

Note that in Example 7, because the contingent claim prices are not independent, (p_{21}, p_{22}, p_{23}) cannot take any value in $\mathbb{R}^3_{++}$. It follows from eqns. (70), (72) and (74) that

$$p_{23} = \frac{2\pi_1 p_{21} p_{22}}{\pi_1 p_{22} + \pi_2 p_{21}}, \tag{79}$$

which corresponds to a surface in $\mathbb{R}^3_{++}$. This is shown in Figure 1, where it is assumed that

$$\left(\pi_1, \pi_2, \pi_3\right) = \left(0.5, 0.3, 0.2\right). \tag{80}$$



(a) (b)

Figure 2:

Similarly, c_{21}, c_{22} and c_{23} are not independent of each other and it follows from eqns. (62) - (63) that the optimal demand lies on the plane defined by

$$c_{23} = \frac{1}{2} (c_{21} + c_{22}) . \tag{81}$$

This plane is illustrated along with the budget plane in Figure 2(a). The optimal demand vector lies on the line formed by the intersection of these two planes. In Figure 2(b), this line is seen to be tangent to an Expected Utility indifference surface at the optimal demand.

As demonstrated in Example 7, the quite complex conditional asset demand (61) can be rationalized by the following familiar Expected Utility

$$\mathcal{V}_{c_1} (z_1, z_2; \pi_1, \pi_2, \pi_3) = \sum_{s=1}^{3} \pi_s \ln (\xi_{1s} z_1 + \xi_{2s} z_2) . \tag{82}$$

In contingent claim space, the corresponding Expected Utility is given by

$$\mathcal{V}_{c_1} (c_{21}, c_{22}, c_{23}; \pi_1, \pi_2, \pi_3) = \sum_{s=1}^{3} \pi_s \ln c_{2s}. \tag{83}$$

However as observed earlier, we cannot conclude that (83) holds in the full contingent claim space, since the contingent claim demands c_{21}, c_{22} and c_{23} are not independent. For Example 7, $c_{23} = \frac{1}{2} (c_{21} + c_{22})$. Therefore, in the contingent claim setting, we can only conclude that the preferences corresponding to the asset demand (61) are Expected Utility representable in the subspace $\left(c_{21}, c_{22}, \frac{1}{2}\left(c_{21}+c_{22}\right)\right)$. Outside this subspace, we do not know what the preferences are due to the lack of information in the incomplete market case (see Example 8 below). With respect to the axiom system derived in Kubler, Selden and Wei (2016), we can also only conclude that the axioms for Expected Utility hold for this contingent claim subspace.

Remark 5 *The only other work we are aware of which addresses similar questions to those considered in this paper is Polemarchakis and Selden (1981). They also assume an incomplete market setting (although with just a single time period) and ask when asset demand can be rationalized by an Expected Utility representation defined over contingent claims. However Polemarchakis and Selden (1981) differs in assuming that asset demands can be rationalized by a utility function and investigate whether one of the possible multiple representations takes the Expected Utility form. Their approach to dealing with incomplete markets is quite different in that they consider two constraints. The first is the budget constraint based on the first J contingent claims and the second is the interdependence of the first J contingent claims and the rest of the $S-J$ contingent claims. As a result of these two constraints, multiple Lagrange multipliers are introduced which differs from the traditional case of a single multiplier corresponding to the budget constraint involving the S contingent claims. In addition to not addressing the fundamental integrability question of whether a representation exists, Polemarchakis and Selden (1981) also do not suggest how in an application such as Examples 6 and 7 one might "guess" what the set of multipliers are from the given asset demand as would be required in order to establish that the rationalizing utility takes the Expected Utility form.*

The following example inspired by discussion in Polemarchakis and Selden (1981) shows that there may exist multiple utilities which generate the same conditional asset demand. Moreover, the role of the extended versus spanned contingent claim space is clarified in this example.

Example 8 *Assuming the same conditional asset demand functions as in Example 7, consider the following utility function*

$$\mathcal{V}_{c_1}\left(c_{21}, c_{22}, c_{23}; \pi_1, \pi_2, \pi_3\right)=\sum_{s=1}^{3} \pi_s \ln c_{2s}+\sqrt{c_{21}+c_{22}-2 c_{23}}, \tag{84}$$

which is not an Expected Utility function. However, using eqns. (62) - (63),

$$c_{21}+c_{22}-2 c_{23}=z_1+z_2-\left(z_1+z_2\right)=0, \tag{85}$$

implying that

$$\mathcal{V}_{c_1}(z_1, z_2; \pi_1, \pi_2, \pi_3) = \sum_{s=1}^{3} \pi_s \ln(\xi_{1s} z_1 + \xi_{2s} z_2). \tag{86}$$

Therefore, the non-Expected Utility function (84) generates the same conditional asset demand (61) as the Expected Utility function (83). It should be emphasized that the extended contingent claim demand specified in Theorem 1 is defined over the full contingent claim space. But it is derived and naturally extended to the full contingent claim space from the subspace, where the conditional asset demand span. This is based on the assumption of preferences being Expected Utility representable since the conditional asset demand already passes condition (ii) in Theorem 1. Therefore for Example 7, the extended contingent claim demand is the optimal demand corresponding to the utility function (83). It cannot be rationalized by the utility function (84) unless one considers the subspace $\left(c_{21}, c_{22}, \frac{1}{2}(c_{21} + c_{22})\right)$.

5 Integrability and Identification of Unconditional Demand

Assume unconditional period 1 consumption and asset demand functions satisfy the conditions in Theorems 1 and 2 guaranteeing that there exists an Expected Utility representation of conditional risk preferences. Then to verify the existence of an OCE utility (6) which rationalizes the unconditional demands, it would seem that one could directly apply the standard certainty integrability test including Slutsky symmetry and negative semidefiniteness to the unconditional demand functions $c_1(p_1, \mathbf{p}_2, \boldsymbol{\pi}, I)$, $c_{21}(p_1, \mathbf{p}_2, \boldsymbol{\pi}, I)$, ..., $c_{2S}(p_1, \mathbf{p}_2, \boldsymbol{\pi}, I)$. However this is not possible as the Slutsky matrix is not well-defined. Even though Theorem 2 may guarantee the existence of a unique contingent claim price vector, one still confronts the problem that the contingent claim prices $p_{21}, ..., p_{2S}$ are not independent of one another. More specifically, although one can obtain a unique expression for the extended conditional contingent claim demand based on condition (iii) in Theorem 1, $c_1(p_1, \mathbf{p}_2, \boldsymbol{\pi}, I)$ can take an infinite number of forms. For instance in Examples 6 and 7, the interdependence of contingent claim prices (p_{21}, p_{22}, p_{23}) is given by eqn. (79). It then follows that the period 1 consumption demand function (52), can equivalently be expressed in terms of contingent claim prices as

$$c_1 = \frac{I}{p_1} - \frac{\pi_1 + \pi_2 + \pi_3}{p_1} \left(\frac{1}{p_1} \left(\frac{\pi_1}{p_{21}}\right)^{\pi_1} \left(\frac{\pi_2}{p_{22}}\right)^{\pi_2} \left(\frac{\pi_3}{p_{23}}\right)^{\pi_3} \right)^{-\frac{1}{\pi_1+\pi_2+\pi_3+1}} \tag{87}$$

or

$$c_1 = \frac{I}{p_1} - \frac{\pi_1 + \pi_2 + \pi_3}{p_1} \left(\begin{array}{c} \frac{1}{p_1} \left(\frac{\pi_1}{p_{21}}\right)^{\pi_1} \left(\frac{\pi_2}{p_{22}}\right)^{\pi_2} \\ \times \left(\frac{\pi_3(\pi_1 p_{22} + \pi_2 p_{21})}{2\pi_1 p_{21} p_{22}}\right)^{\pi_3} \end{array} \right)^{-\frac{1}{\pi_1+\pi_2+\pi_3+1}} . \quad (88)$$

Combining the above two different period 1 consumption expressions with the conditional asset demand (61) respectively, one can derive two different unconditional demand systems. The system associated with eqn. (87) satisfies Slutsky symmetry but the system associated with eqn. (88) does not. Hence, there is no way to determine which of the infinite number of expressions should be used to apply the Slutsky symmetry and negative semidefiniteness tests.

There are two special cases where the problem of multiple expressions for $c_1(p_1, \mathbf{p}_2, \boldsymbol{\pi}, I)$ does not arise. First, if preferences are representable by an additively separable Expected Utility

$$\phi(c_1, \mathbf{c}_2; \boldsymbol{\pi}) = u_1(c_1) + \sum_{s=1}^{S} \pi_s u\left(\sum_{j=1}^{J} \xi_{js} z_j\right) = u_1(c_1) + \sum_{s=1}^{S} \pi_s u(c_{2s}), \quad (89)$$

the following additional restrictions on c_1

$$c_{2s} = h\left(c_1, \frac{\pi_s p_{21}}{p_{2s}}\right) \quad (s = 1, ..., S), \quad (90)$$

result in a unique expression for period one consumption. Second, if preferences are myopic separable (see Kannai, Selden and Wei 2014), where

$$\frac{\partial c_1}{\partial q_j} = 0 \quad (j = 1, ..., J), \quad (91)$$

then c_1 is just a function of p_1 and the multiplicity of demand functions is avoided.

In order to avoid this indeterminacy problem, we build on Theorems 1 and 2. Suppose the conditions in these results are satisfied and thus there exists conditional risk preferences representable by an Expected Utility function. Then the only question of whether unconditional demands can be rationalized by OCE preferences is whether there also exists the representation of time preferences U in the OCE utility

$$U\left(c_1, V_{c_1}^{-1} \sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})\right). \quad (92)$$

The possible range of the period 1 consumption C_1 can be defined as follows

$$C_1 = \{c_1 \in \mathbb{R}_{++} | \ c_1 = c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I), (p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}\}. \quad (93)$$

Given V_{c_1}, the domain of c_2 is given by

$$C_2 = \left\{ \widehat{c}_2 \in \mathbb{R}_{++} | \ \widehat{c}_2 = V_{c_1}^{-1} \sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s}), \mathbf{c}_2 \in \mathcal{C} \right\}. \quad (94)$$

We require $U : C_1 \times C_2 \to \mathbb{R}$ to be twice continuously differentiable, strictly increasing and strictly quasiconcave. To prove the existence of U, we follow a process introduced in Polemarchakis and Selden (1984) which in effect converts the consumer's choice problem from one choosing over $(c_1, c_{21}, ..., c_{2S})$ into one choosing over the certain consumption vector (c_1, c_2). In so doing, the indeterminacy among the contingent claim prices $(p_{21}, ..., p_{2S})$ is avoided since each of the contingent claim prices converges to the price for certain period two consumption p_2.

In order to convert the choice problem over $(c_1, c_{21}, ..., c_{2S})$ into one over (c_1, c_2), it is first necessary to derive, based on the unconditional demands, q_j/p_1 $(j = 1, ..., J)$ as a function of $(c_1, \mathbf{z}, \boldsymbol{\pi})$.[22] Then without loss of generality, the risk free asset z_1 is assumed to pay 1 in each contingent claim state. Then letting $z_1 = 1$ and $z_2 = ... = z_J = 0$, it follows that period 2 consumption $c_2 = z_1$ is certain and the corresponding period 2 consumption price $p_2 = q_1$.[23]

Lemma 3 *Assume that the conditional asset demand $\mathbf{z}(\mathbf{q}, \boldsymbol{\pi}, I_2 |\ c_1)$ satisfies the conditions in Theorem 1. Asset 1 is risk free and pays off 1 in each contingent claim state and assets $2, ..., J$ are risky.*[24] *Further assume that the conditions in Lemma 2 hold. Then there exists (i) a twice continuously differentiable inverse demand vector (p_1, p_2), which satisfies*

$$\frac{p_2}{p_1} = \left. \frac{q_1(c_1, z_1, ..., z_J, \boldsymbol{\pi})}{p_1(c_1, z_1, ..., z_J, \boldsymbol{\pi})} \right|_{z_1 = c_2, z_2 = ... = z_J = 0} \tag{95}$$

and (ii) an optimal demand vector $(c_1, c_2) \in C_1 \times C_2$ which solves the pair of equations, (95) and the budget constraint

$$p_1 c_1 + p_2 c_2 = I. \tag{96}$$

Although in the above result the risky assets satisfy $z_2 = ... = z_J = 0$, the following illustrates that this does not ensure that the optimal demands c_1 and

[22] In order for the representation of time preferences to be consistent with Kreps-Porteus-Selden preferences, we want to ensure that it is probability independent. The simplex normalization of probabilities assumed in this paper is necessary although not sufficient for this independence.

[23] It should be emphasized that because Polemarchakis and Selden (1984) assume the existence of an OCE representation of unconditional demands, they are able to use the first order conditions for an OCE optimization to identify the utility function directly from the inverse demand ratio p_2/p_1. However since we do not make this assumption and are instead providing necessary and sufficient conditions for integrability based on the demand functions, we alternatively need to solve for the optimal demand (c_1, c_2) from combining the inverse demand ratio and the budget constraint.

[24] Lemma 3 also holds if instead of assuming a risk free asset, there is an effectively risk free asset.

c_2 are independent of probabilities. As a consequence if a representation of time preferences exists which rationalizes these demands, it can inherit the probability dependence from the demands for c_1 and c_2.

Example 9 *Assume the same asset payoffs (51) as in Example 6 and the following slightly modified unconditional demands*

$$c_1 = \frac{I}{p_1} - \frac{\pi_1+\pi_2+\pi_3}{p_1}\left(\begin{array}{c} \frac{1}{\pi_1 p_1}\left(\frac{\pi_1(1-1/B)}{q_1-q_2}\right)^{\pi_1}\left(\frac{\pi_2(B-1)}{q_1-q_2}\right)^{\pi_2} \\ \times\left(\frac{\pi_3(1-1/B)}{2(q_2-q_1/B)}\right)^{\pi_3}\end{array}\right)^{-\frac{1}{\pi_1+\pi_2+\pi_3+1}}, \quad (97)$$

$$z_1 = \frac{\pi_1(1-1/B)}{q_1-q_2}\left(\begin{array}{c} \frac{1}{\pi_1 p_1}\left(\frac{\pi_1(1-1/B)}{q_1-q_2}\right)^{\pi_1}\left(\frac{\pi_2(B-1)}{q_1-q_2}\right)^{\pi_2} \\ \times\left(\frac{\pi_3(1-1/B)}{2(q_2-q_1/B)}\right)^{\pi_3}\end{array}\right)^{-\frac{1}{\pi_1+\pi_2+\pi_3+1}} \quad (98)$$

and

$$z_2 = \frac{\pi_2(B-1)}{q_1-q_2}\left(\begin{array}{c} \frac{1}{\pi_1 p_1}\left(\frac{\pi_1(1-1/B)}{q_1-q_2}\right)^{\pi_1}\left(\frac{\pi_2(B-1)}{q_1-q_2}\right)^{\pi_2} \\ \times\left(\frac{\pi_3(1-1/B)}{2(q_2-q_1/B)}\right)^{\pi_3}\end{array}\right)^{-\frac{1}{\pi_1+\pi_2+\pi_3+1}}. \quad (99)$$

Following the same process as in Example 6, it is easy to see that the conditional asset demand is the same, and so is the rationalizing conditional Expected Utility function NM index

$$V_{c_1}(c_2) = \ln c_2. \quad (100)$$

Using the above demand functions to derive the corresponding unconditional inverse demand functions and the fact that an effectively risk free asset exists, the following price ratio is implied

$$\frac{p_2}{p_1} = \frac{q_1+q_2}{p_1} = \pi_1(\pi_1+\pi_2+\pi_3)\,c_2^{-\pi_1-\pi_2-\pi_3-1} = \frac{\pi_1}{c_2^2}, \quad (101)$$

where q_1+q_2 is the price of the effectively risk free asset. Combining eqn. (101) with the budget constraint

$$p_1c_1 + p_2c_2 = I \quad (102)$$

yields

$$c_1 = \frac{I-\sqrt{\pi_1 p_1 p_2}}{p_1} \quad \text{and} \quad c_2 = \sqrt{\frac{\pi_1 p_1}{p_2}}. \quad (103)$$

Clearly the demands for periods 1 and 2 consumption are probability dependent. Since the demand functions in eqn. (103) satisfy the conditions in Theorem 3 below, there exists a utility which rationalizes the probability dependent demands. Applying the Hurwicz and Uzawa (1971) recovery process to (103), one obtains the following probability dependent utility

$$U(c_1,c_2) = c_1 - \frac{\pi_1}{c_2}. \quad (104)$$

(Supporting calculations for this example can be found in Appendix B.3.)

To ensure that the representation of time preferences U is probability independent, as assumed by Kreps-Porteus-Selden preferences, it is necessary to introduce an additional restriction on the unconditional demands $(c_1, \mathbf{z})$. The corresponding inverse demand ratio q_1/p_1 must satisfy the following Certainty Regularity Property.

Property 1 *(Certainty Regularity)* $\forall (c_1, c_2) \in C_1 \times C_2$ *and* $\forall \boldsymbol{\pi} \in \Pi$,

$$\frac{\partial \left(\left. \frac{q_1(c_1, z_1, ..., z_J, \boldsymbol{\pi})}{p_1(c_1, z_1, ..., z_J, \boldsymbol{\pi})} \right|_{z_1 = c_2, z_2 = ... = z_J = 0} \right)}{\partial \pi_s} = 0 \quad (s = 1, ..., S-1)\,. \tag{105}$$

Setting $z_2 = ... = z_J = 0$ and $c_2 = z_1$, the resulting inverse demand ratio q_1/p_1 is required to be independent of probabilities. It is clear from eqn. (101) in the above example, that this property is not satisfied and as a result c_1 and c_2 are both probability dependent.

Then we have the following theorem.

Theorem 3 *Assume that $S > 2$ and the conditions in Theorems 1 and 2 are satisfied for the conditional asset demand $\mathbf{z}(\mathbf{q}, \boldsymbol{\pi}, I_2 |\ c_1)$ and the conditions in Lemma 2 are satisfied. Further assume that there exists a risk free or an effectively risk free asset. Then there exists a unique twice continuously differentiable, strictly increasing, strictly quasiconcave and probability independent representation of time preferences function $U(c_1, c_2) : C_1 \times C_2 \to \mathbb{R}$ rationalizing the demands if and only if Certainty Regularity is satisfied and the Slutsky matrix associated with the demand function (c_1, c_2) derived from Lemma 3 is negative semidefinite.*

If the conditions in Theorem 3 are satisfied by a given $(c_1, \mathbf{z})$, then it follows immediately that they are rationalizable by an OCE utility function (16).

Remark 6 *It will be noted that in Theorem 3, we obtain uniqueness of the representation without having to assume the local Lipschitz condition required in Mas-Colell (1977). As noted in the proof of the theorem, this is because local Lipschitz continuity is implied by continuous differentiability.*

Remark 7 *Once $(U, \{V_{c_1}\})$ have been uniquely identified, eqn. (9) can be used to test whether the OCE utility specializes to the two period Expected Utility case.*

The application of Theorem 3 is next illustrated building on Examples 6 and 7.

Example 10 *Deriving the inverse demand functions from the given demands (52) - (54), in Example 6, and using the fact that an effectively risk free asset exists, the following price ratio is implied*

$$\frac{p_2}{p_1} = \frac{q_1 + q_2}{p_1} = (\pi_1 + \pi_2 + \pi_3)\, c_2^{-\pi_1 - \pi_2 - \pi_3 - 1} = \frac{1}{c_2^2}, \tag{106}$$

where $q_1 + q_2$ is the price of the effectively risk free asset. This verifies that Certainty Regularity is satisfied. Combining the above equation with the budget constraint

$$p_1 c_1 + p_2 c_2 = I \tag{107}$$

yields

$$c_1 = \frac{I - \sqrt{p_1 p_2}}{p_1} \quad \text{and} \quad c_2 = \sqrt{\frac{p_1}{p_2}}. \tag{108}$$

Since we construct the effectively risk free asset and erase all uncertainty by considering the portfolio with the same payoffs in each contingent claim state, the demand (108) corresponds to solving the certainty optimization problem

$$\max_{c_1, c_2} U(c_1, c_2) \quad S.T. \;\; p_1 c_1 + p_2 c_2 \le I. \tag{109}$$

The demand functions (108) satisfy the requirements for the existence of a rationalizing utility and the Hurwicz and Uzawa (1971) recovery process can be used to obtain the following probability independent representation of time preferences

$$U(c_1, c_2) = c_1 - \frac{1}{c_2}, \tag{110}$$

which is defined up to an increasing transformation. Since the utility function (110) is quasilinear in c_1, it is not surprising that the demand functions (53) and (54) are independent of income I. (Supporting calculations can be found in Appendix B.4.)

Together Examples 6, 7 and 10, provide a comprehensive demonstration of how to use Theorems 1, 2 and 3 to verify the existence an OCE rationalization of period 1 consumption and asset demand and to recover the corresponding representations of conditional risk and time preferences.

6 Conclusion

In this paper, we consider a two period incomplete market setting, where period 1 consumption and period 2 asset demand are given. Then we (i) provide necessary and sufficient conditions such that these demands are rationalizable by

Kreps-Porteus-Selden preferences and (ii) if this is the case, provide a means for identifying the representations of the underlying risk and time preferences. Two key innovations are introduced. First, the optimization problem is decomposed into two stages and the integrability problem is considered for each stage separately. Second, a unique contingent claim price vector is derived by varying probabilities which facilitates conducting the analysis in an extended contingent claim setting.

In terms of practical applications of our results, it would seem quite difficult to obtain analytic expressions for period 1 consumption and asset demand. However, it is possible to extend the analysis in this paper to a setting where one is given finite demand, price and probability observations rather than demand functions. The two stage optimization could still be used as the basis for conducting separate revealed preference tests for the representations of risk and time preferences. To duplicate the process in Theorem 2 in a discrete setting and derive a unique set of contingent claim prices, it would be necessary to generate a set of asset price-state probability observations $\{(\mathbf{q}', \boldsymbol{\pi}')\}$. These observations could be obtained from a given data set $\{(c_1, \mathbf{z}, p_1, \mathbf{q}, \boldsymbol{\pi})\}$, where for different observations of prices and probabilities (and a fixed asset payoff matrix and income), the demands $(c_1, \mathbf{z})$ are unchanged. Then the revealed preference test for an Expected Utility representation of risk preferences based on the contingent claim setting in Kubler, Selden and Wei (2014) could be applied to the (conditional) asset demand. Using this method, it may be possible to answer in a two period consumption-portfolio setting the question raised by Andreoni and Sprenger (2012), and cited in Section 1, as to whether risk and time preferences are independent.

Appendix

A Proofs

A.1 Proof of Lemma 1

Consider the following equations

$$c_1 = c_1 (p_1, \mathbf{q}, \boldsymbol{\pi}, I) \quad \text{and} \quad I_2 = I - p_1 c_1 (p_1, \mathbf{q}, \boldsymbol{\pi}, I). \tag{A.1}$$

If $\forall (p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$,

$$\det \frac{\partial (c_1, I_2)}{\partial (p_1, I)} \neq 0, \tag{A.2}$$

and the vector map $(c_1, I_2)(p_1, I)$ is proper, then following Gordon (1972) and Wagstaff (1975), (p_1, I) can be solved for as a unique twice continuously differentiable function of $(c_1, \mathbf{q}, \boldsymbol{\pi}, I_2)$ from the set of equations (A.1). Substituting

$$p_1(c_1, \mathbf{q}, \boldsymbol{\pi}, I_2) \quad \text{and} \quad I(c_1, \mathbf{q}, \boldsymbol{\pi}, I_2) \tag{A.3}$$

into the unconditional asset demand $z_i(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ $(i = 1, ..., J)$, $\forall (p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$, one obtains the unique continuously differentiable conditional demand $z_i(\mathbf{q}, \boldsymbol{\pi}, I_2 | \, c_1)$ $(i = 1, ..., J)$.

A.2 Proof of Theorem 1

Since conditions (i) and (ii) in Lemma 1 are satisfied, there exists unique continuously differentiable conditional asset demand $z_i(\mathbf{q}, \boldsymbol{\pi}, I_2 | \, c_1)$ $(i = 1, ..., J)$, which can be transformed into the conditional contingent claim demand using

$$c_{2s}(\mathbf{q}, \boldsymbol{\pi}, I_2 | \, c_1) = \sum_{j=1}^{J} \xi_{js} z_j(\mathbf{q}, \boldsymbol{\pi}, I_2 | \, c_1). \tag{A.4}$$

Since there exists a twice continuously differentiable price function $\mathbf{p}_2$ satisfying

$$q_j = \sum_{s=1}^{S} p_{2s}(\mathbf{q}, \boldsymbol{\pi}, I_2) \, \xi_{js}, \tag{A.5}$$

we have

$$c_{2s}(\mathbf{q}, \boldsymbol{\pi}, I_2 | \, c_1) = c_{2s}(\mathbf{p}_2, \boldsymbol{\pi}, I_2 | \, c_1). \tag{A.6}$$

Then necessity follows directly from the first order conditions. Next we prove sufficiency. We first assume that there exists a utility function over contingent claims representing the risk preferences conditional on a given c_1, which is denoted by $\mathcal{V}_{c_1}(\mathbf{c}_2; \boldsymbol{\pi})$. The existence of $\mathcal{V}_{c_1}$ will be verified below. Assuming the existence of a $\mathcal{V}_{c_1}$, we then show that for $S > 2$, the condition $c_{2s} = f_{c_1}(c_{21}, k_s)$ implies that $\mathcal{V}_{c_1}$ is additively separable in contingent claims. The first order conditions for the conditional optimization problem are

$$\frac{\partial \mathcal{V}_{c_1} / \partial c_{21}}{\partial \mathcal{V}_{c_1} / \partial c_{2s}} = \frac{p_{21}}{p_{2s}} \qquad (s = 2, ..., S). \tag{A.7}$$

For each $s \in \{2, ..., S\}$, since $c_{2s} = f_{c_1}(c_{21}, k_s)$ and $f_{c_1}(c_{21}, k_s)$ is a strictly increasing function of k_s, we have

$$\frac{\pi_s p_{21}}{\pi_1 p_{2s}} = k_s = f^{-1}_{c_1, c_{21}}(c_{2s}), \tag{A.8}$$

or equivalently

$$\frac{\partial \mathcal{V}_{c_1}/\partial c_{2s}}{\partial \mathcal{V}_{c_1}/\partial c_{21}} = \frac{\pi_s}{\pi_1 f^{-1}_{c_1,c_{21}}(c_{2s})}, \tag{A.9}$$

implying that

$$\frac{\partial \left(\frac{\partial \mathcal{V}_{c_1}/\partial c_{2s}}{\partial \mathcal{V}_{c_1}/\partial c_{21}}\right)}{\partial c_{2i}} = 0 \;\; (\forall i, s \in \{2, ..., S\}, i \neq s). \tag{A.10}$$

Therefore,

$$\mathcal{V}_{c_1}(\mathbf{c}_2; \boldsymbol{\pi}) \;\; = \sum_{s=1}^{S} V_s(c_{2s}; \boldsymbol{\pi} | \;\; c_1). \tag{A.11}$$

Following the same argument as in Kubler, Selden and Wei (2014), we can prove that

$$V_s(c_{2s}; \boldsymbol{\pi} | \;\; c_1) = K(\boldsymbol{\pi}) \pi_s V_{c_1}(c_{2s}) \quad (s = 1, ..., S). \tag{A.12}$$

Thus it immediately follows that the given conditional demands can be rationalized by a utility function that is ordinally equivalent to a state independent Expected Utility function

$$\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s}). \tag{A.13}$$

If the above test passes, then one can derive the extended contingent claim demands using the relation $c_{2s} = f_{c_1}(c_{21}, k_s)$ and the budget constraint. If the extended demands satisfy the Slutsky symmetry and negative semidefiniteness, then the presumption of the existence of $\mathcal{V}_{c_1}$ is satisfied. Next we show how to identify V_{c_1} uniquely. First prove the uniqueness of f_{c_1}. Assume that there exist two functions $f^{(1)}_{c_1}$ and $f^{(2)}_{c_1}$ such that $\forall s \in \{2, ..., S\}$,

$$c_{2s} = f^{(1)}_{c_1}(c_{21}, k_s) \quad \text{and} \quad c_{2s} = f^{(2)}_{c_1}(c_{21}, k_s), \tag{A.14}$$

implying that

$$f^{(1)}_{c_1}(c_{21}, k_s) = f^{(2)}_{c_1}(c_{21}, k_s). \tag{A.15}$$

Since the above equation holds for any $c_{21} \in \mathbb{R}_{++}$ and $k_s \in \mathbb{R}_{++}$, $f^{(1)}_{c_1}$ and $f^{(2)}_{c_1}$ are the same function, i.e., $f^{(1)}_{c_1} = f^{(2)}_{c_1} = f_{c_1}$. Hence there exists an NM index V_{c_1}, where $V'_{c_1} > 0$ and $V''_{c_1} < 0$, such that

$$f_{c_1}(c_{21}, k_s) = V'^{-1}_{c_1}\left(\frac{V'_{c_1}(c_{21})}{k_s}\right). \tag{A.16}$$

Assume that there exists another NM index H_{c_1}, where $H'_{c_1} > 0$ and $H''_{c_1} < 0$, such that

$$f_{c_1}(c_{21}, k_s) = H'^{-1}_{c_1}\left(\frac{H'_{c_1}(c_{21})}{k_s}\right). \tag{A.17}$$

Since $H'_{c_1} > 0$, $H_{c_1}^{-1}$ exists and we can define

$$T = V_{c_1} \circ H_{c_1}^{-1}, \tag{A.18}$$

implying that

$$V_{c_1} = T \circ H_{c_1}. \tag{A.19}$$

Notice that

$$c_{2s} = f_{c_1}(c_{21}, k_s) = H_{c_1}^{\prime -1}\left(\frac{H'_{c_1}(c_{21})}{k_s}\right) = V_{c_1}^{\prime -1}\left(\frac{V'_{c_1}(c_{21})}{k_s}\right), \tag{A.20}$$

implies that

$$V'_{c_1}(c_{2s}) = \frac{V'_{c_1}(c_{21})}{k_s}, \tag{A.21}$$

or equivalently

$$T'(H_{c_1}(c_{2s}))\, H'_{c_1}(c_{2s}) = \frac{T'(H_{c_1}(c_{21}))\, H'_{c_1}(c_{21})}{k_s}. \tag{A.22}$$

Thus

$$c_{2s} = H_{c_1}^{\prime -1}\left(\frac{T'(H_{c_1}(c_{21}))\, H'_{c_1}(c_{21})}{T'(H_{c_1}(c_{2s}))\, k_s}\right) = H_{c_1}^{\prime -1}\left(\frac{H'_{c_1}(c_{21})}{k_s}\right). \tag{A.23}$$

Since $H_{c_1}^{\prime -1}$ is a strictly decreasing function, we have

$$\frac{T'(H_{c_1}(c_{21}))\, H'_{c_1}(c_{21})}{T'(H_{c_1}(c_{2s}))\, k_s} = \frac{H'_{c_1}(c_{21})}{k_s}, \tag{A.24}$$

or equivalently,

$$T'(H_{c_1}(c_{21})) = T'(H_{c_1}(c_{2s})). \tag{A.25}$$

Since the above equation holds for any $c_{21}, c_{2s} \in \mathbb{R}_{++}$, we can conclude that

$$T'(x) = const, \tag{A.26}$$

or equivalently,

$$T(x) = ax + b, \tag{A.27}$$

where a and b are constants. Therefore, H_{c_1} and V_{c_1} are affinely equivalent. Thus V_{c_1} can be uniquely identified up to an affine transformation as follows.

$$f_{c_1}(c_{21}, k_s) = V_{c_1}^{\prime -1}\left(\frac{V'_{c_1}(c_{21})}{k_s}\right). \tag{A.28}$$

Noticing that $f_{c_1}(c_{21}, k_s)$ is twice continuously differentiable, strictly positive and strictly increasing in k_s, V_{c_1} is twice continuously differentiable, strictly increasing and strictly concave.

A.3 Proof of Lemma 2

Consider the following set of equations

$$c_1 = c_1\left(p_1, \mathbf{q}, \boldsymbol{\pi}, I\right) \quad \text{and} \quad \mathbf{z} = \mathbf{z}\left(p_1, \mathbf{q}, \boldsymbol{\pi}, I\right). \tag{A.29}$$

If $\forall(p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$,

$$\det \frac{\partial\left(c_1, z_1, ..., z_J\right)}{\partial\left(p_1, q_1, ..., q_J\right)} \neq 0, \tag{A.30}$$

and $\forall(\boldsymbol{\pi}, I) \in \Pi \times \mathcal{I}$, $c_1\left(p_1, \mathbf{q}, \boldsymbol{\pi}, I\right)$ and $z\left(p_1, \mathbf{q}, \boldsymbol{\pi}, I\right)$ are proper maps with respect to $(p_1, \mathbf{q})$, then following Gordon (1972) and Wagstaff (1975), $(p_1, \mathbf{q})$ can be solved for as a unique twice continuously differentiable function of $(c_1, \mathbf{z}, \boldsymbol{\pi}, I)$ from the set of equations (A.29).

A.4 Proof of Theorem 2

If preferences are represented by a state independent Expected Utility function, then we have

$$\mathcal{V}_{c_1}\left(c_{21}, .., c_{2s}; \pi_1, .., \pi_S\right) = \sum_{s=1}^{S} \pi_s V_{c_1}\left(c_{2s}\right), \tag{A.31}$$

where

$$c_{2s} = \sum_{j=1}^{J} \xi_{js} z_j. \tag{A.32}$$

$\forall j \in \{1, ..., J\}$, the first order condition for the optimization problem

$$\max_{\mathbf{c}_2, \mathbf{z}} \mathcal{V}_{c_1}\left(c_{21}, .., c_{2s}; \pi_1, .., \pi_S\right) \quad S.T. \ c_{2s} = \sum_{j=1}^{J} \xi_{js} z_j \text{ and } \sum_{j=1}^{J} q_j z_j = I_2, \tag{A.33}$$

is

$$\sum_{s=1}^{S} \pi_s \xi_{js} V'_{c_1}\left(c_{2s}\right) = \mu q_j, \tag{A.34}$$

where μ is the Lagrange multiplier. Since the conditions in Lemma 2 hold, there exists a unique twice continuously differentiable function $\mathbf{q}(\mathbf{z}, \boldsymbol{\pi}, I_2 | \ c_1)$. Using $\pi_S = 1 - \sum_{s=1}^{S-1} \pi_s$ and given the inverse demand function $\mathbf{q}(\mathbf{z}, \boldsymbol{\pi}, I_2 | \ c_1)$, differentiating eqn. (A.34) with respect to π_s ($s \in \{1, ..., S-1\}$), one obtains

$$\xi_{js} V'_{c_1}\left(c_{2s}\right) - \xi_{jS} V'_{c_1}\left(c_{2S}\right) = \frac{\partial \mu}{\partial \pi_s} q_j + \frac{\partial q_j}{\partial \pi_s} \mu. \tag{A.35}$$

Differentiating the budget constraint with respect to π_s ($s \in \{1, ..., S-1\}$), it follows that

$$\sum_{j=1}^{J} \frac{\partial q_j}{\partial \pi_s} z_j = 0. \tag{A.36}$$

Combining eqn. (A.35) with (A.36) yields

$$V'_{c_1}(c_{2s})\,c_{2s} - V'_{c_1}(c_{2S})\,c_{2S} = \frac{\partial \mu}{\partial \pi_s}\sum_{j=1}^{J} q_j z_j = \frac{\partial \mu}{\partial \pi_s} I_2. \tag{A.37}$$

Substituting the above equation into (A.35) one obtains

$$\xi_{js}V'_{c_1}(c_{2s}) - \xi_{jS}V'_{c_1}(c_{2S}) = \left(V'_{c_1}(c_{2s})\,c_{2s} - V'_{c_1}(c_{2S})\,c_{2S}\right)\frac{1}{q_j}I_2 + \frac{\partial q_j}{\partial \pi_s}\mu, \tag{A.38}$$

Defining

$$p_{2s} = \frac{\pi_s V'_{c_1}(c_{2s})}{\mu}, \tag{A.39}$$

we obtain

$$\frac{\partial q_j}{\partial \pi_s} = \left(\xi_{js} - \frac{c_{2s}q_j}{I_2}\right)\frac{p_{2s}}{\pi_s} - \left(\xi_{jS} - \frac{c_{2S}q_j}{I_2}\right)\frac{p_{2S}}{\pi_S} \quad (s = 1, ..., S-1). \tag{A.40}$$

Using $\sum_s p_{2s}c_{2s} = I_2$ and $\sum_s \xi_{js}p_{2s} = q_j$ and summing over all $s = 1, \ldots, S-1$, it follows that

$$\sum_{s=1}^{S-1}\pi_s\frac{\partial q_j}{\partial \pi_s} = q_j - \xi_{jS}p_{2S} - \frac{q_j}{I_2}(I_2 - p_{2S}c_{2S}) - (1-\pi_S)\left(\xi_{jS} - \frac{c_{2S}q_j}{I_2}\right)\frac{p_{2S}}{\pi_S} \tag{A.41}$$

or

$$p_{2S} = \frac{\sum_{s=1}^{S-1}\pi_s\frac{\partial q_j}{\partial \pi_s}}{-\frac{1}{\pi_S}(\xi_{jS} - \frac{c_{2S}q_j}{I_2})} = -\frac{\pi_S\sum_{l=1}^{S-1}\pi_l\frac{\partial q_j}{\partial \pi_l}}{\xi_{jS} - \frac{c_{2S}q_j}{I_2}}. \tag{A.42}$$

Substituting the above equation into (A.40) yields

$$\frac{\partial q_j}{\partial \pi_s} = \left(\xi_{js} - \frac{c_{2s}q_j}{I_2}\right)\frac{p_{2s}}{\pi_s} + \sum_{l=1}^{S-1}\pi_l\frac{\partial q_j}{\partial \pi_l}, \tag{A.43}$$

implying that

$$p_{2s} = \frac{\pi_s\left(\frac{\partial q_j}{\partial \pi_s} - \sum_{l=1}^{S-1}\pi_l\frac{\partial q_j}{\partial \pi_l}\right)}{\xi_{js} - \frac{c_{2s}q_j}{I_2}} \quad (s = 1, ..., S-1). \tag{A.44}$$

A.5 Proof of Lemma 3

Since conditions in Lemma 2 hold, $(c_1, z_1, ..., z_J)$ are globally invertible. Therefore, we have

$$\frac{p_1}{q_1} = \frac{p_1(c_1, z_1, ..., z_J, \boldsymbol{\pi})}{q_1(c_1, z_1, ..., z_J, \boldsymbol{\pi})}. \tag{A.45}$$

Assuming that $z_1 = c_2$ and $z_2 = ... = z_J = 0$, it can be easily verified that

$$c_{21} = ... = c_{2S} = c_2, \tag{A.46}$$

implying that

$$\widehat{c}_2 = V_{c_1}^{-1} \left(\sum_{s=1}^{S} \pi_{2s} V_{c_1} \left(c_{2s} \right) \right) = c_2. \tag{A.47}$$

Therefore,

$$p_1 c_1 + q_1 z_1 = p_1 c_1 + q_1 c_2 = p_1 c_1 + p_2 c_2 \tag{A.48}$$

and thus $q_1 = p_2$. Hence eqn. (A.45) can be rewritten as

$$\frac{p_1}{p_2} = \left. \frac{p_1 \left(c_1, z_1, ..., z_J, \boldsymbol{\pi} \right)}{q_1 \left(c_1, z_1, ..., z_J, \boldsymbol{\pi} \right)} \right|_{z_1 = c_2, z_2 = ... = z_J = 0}. \tag{A.49}$$

Together with the budget constraint

$$p_1 c_1 + p_2 c_2 = I, \tag{A.50}$$

one can derive the optimal demand (c_1, c_2).

A.6 Proof of Theorem 3

Necessity is obvious. Next prove sufficiency. Since (c_1, c_2) is twice continuously differentiable and the corresponding Slutsky matrix is symmetric (symmetry always holds for the two good case if $p_1 c_1 + p_2 c_2 = I$) and negative semidefinite, it follows from Jehle and Reny (2011, Theorem 2.6) that there exists an increasing and quasiconcave utility function U defined on the strictly positive orthant of a Euclidean space that generates the given demand. Since the conditions in Lemma 2 hold, it follows from Lemma 3 that the inverse demand associated with (c_1, c_2) exists and is also twice continuously differentiable. Then it follows from Debreu (1972, 1976) and Katzner (1970) that U is twice continuously differentiable. Since $p_2 = q_1$, $(p_1, p_2)(c_1, c_2)$ is also a proper map. It follows from Mas-Colell (1978, Lemma 8) that properness, continuity and quasiconcavity together imply that U is strictly increasing. Then following Mas-Colell (1978, Lemma 9) continuity, quasiconcavity and strict monotonicity together imply strict quasiconcavity. Therefore, U is twice continuously differentiable, strictly increasing and strictly quasiconcave. Since continuous differentiability implies local Lipschitz as in Mas-Colell (1978, Definition 9), it follows from Mas-Colell (1978, Theorem 4) that U is unique (up to an increasing transformation). Finally, since Certainty Regularity is satisfied, U is probability independent (ignoring a monotone transformation which could be based on probabilities).

B Supporting Calculations

B.1 Supporting Calculations for Example 3

Assume conditional risk preferences are represented by

$$\sum_{s=1}^{3} \pi_s \ln\left(z_1 + \left(\xi_{2s} + 1\right) z_2\right). \tag{B.1}$$

Noting that

$$c_{2s} = z_1 + \xi_{2s} z_2 \quad \left(s = 1, 2, 3\right), \tag{B.2}$$

we have

$$z_2 = \frac{c_{22} - c_{21}}{\xi_{22} - \xi_{21}}. \tag{B.3}$$

Therefore, one way to write the utility function (B.1) as a function defined over contingent claims is

$$\sum_{s=1}^{3} \pi_s \ln\left(c_{2s} + \frac{c_{22} - c_{21}}{\xi_{22} - \xi_{21}}\right), \tag{B.4}$$

which is not strictly increasing in (c_{21}, c_{22}, c_{23}) in the full choice space. The MRS (marginal rate of substitution) between the risky asset and the risk free asset is given by

$$m_{21}\left(z_1, z_2\right) = \frac{q_2}{q_1} = \frac{\sum_{s=1}^{3} \frac{\pi_s(\xi_{2s}+1)}{z_1+(\xi_{2s}+1)z_2}}{\sum_{s=1}^{3} \frac{\pi_s}{z_1+(\xi_{2s}+1)z_2}} = \frac{\sum_{s=1}^{3} \frac{\pi_s \xi_{2s}}{z_1+(\xi_{2s}+1)z_2}}{\sum_{s=1}^{3} \frac{\pi_s}{z_1+(\xi_{2s}+1)z_2}} + 1. \tag{B.5}$$

It follows from Dybvig and Polemarchakis (1981) that

$$-\frac{u''(c_2)}{u'(c_2)} = \frac{\frac{\partial m_{21}(c_2,0)}{\partial z_2}}{\left(\sum_{s=1}^{3} \pi_s \xi_{2s}\right)^2 - \sum_{s=1}^{3} \pi_s \xi_{2s}^2}, \tag{B.6}$$

which can be simplified to

$$-\frac{u''(c_2)}{u'(c_2)} = \frac{1}{c_2}, \tag{B.7}$$

implying that

$$u\left(c_2\right) = \ln c_2, \tag{B.8}$$

which is defined up to a positive affine transformation. Then following the identification process in Dybvig and Polemarchakis (1981), one is led to conclude that the agent's conditional risk preferences over assets are represented by

$$\sum_{s=1}^{3} \pi_s \ln\left(z_1 + \xi_{2s} z_2\right). \tag{B.9}$$

The MRS corresponding to the representation (B.9) is

$$m_{21}(z_1, z_2) = \frac{q_2}{q_1} = \frac{\sum_{s=1}^{3} \frac{\pi_s \xi_{2s}}{z_1 + \xi_{2s} z_2}}{\sum_{s=1}^{3} \frac{\pi_s}{z_1 + \xi_{2s} z_2}}, \tag{B.10}$$

which is clearly different from that given by eqn. (B.5). Therefore, the asset demand generated by the utility functions (B.9) and (B.1) will be different.

B.2 Supporting Calculations for Example 7

It follows from Theorem 2 that

$$\frac{\partial q_1}{\partial \pi_1} = \frac{z_2 I_2}{(z_1 + z_2) z_1} = \left(\xi_{11} - \frac{c_{21} q_1}{I_2}\right) \frac{p_{21}}{\pi_1} - \left(\xi_{13} - \frac{c_{23} q_1}{I_2}\right) \frac{p_{23}}{1 - \pi_1 - \pi_2}, \tag{B.11}$$

and

$$\frac{\partial q_1}{\partial \pi_2} = -\frac{I_2}{z_1 + z_2} = \left(\xi_{12} - \frac{c_{22} q_1}{I_2}\right) \frac{p_{22}}{\pi_2} - \left(\xi_{13} - \frac{c_{23} q_1}{I_2}\right) \frac{p_{23}}{1 - \pi_1 - \pi_2}. \tag{B.12}$$

After some algebra, the conditional asset demand (61) can be simplified to

$$z_1 = \frac{(1 + \pi_1) q_1 - (1 - \pi_2) q_2 - A}{2 q_1 (q_1 - q_2)} I_2 \text{ and } z_2 = \frac{(1 - \pi_1) q_1 - (1 + \pi_2) q_2 + A}{2 q_2 (q_1 - q_2)} I_2, \tag{B.13}$$

where A is defined by eqn. (55). Combining eqns. (B.11) and (B.12) with the budget constraint

$$\sum_{s=1}^{3} p_{2s} c_{2s} = I_2 \tag{B.14}$$

and noticing that

$$c_{21} = z_1, \quad c_{22} = z_2 \text{ and } c_{23} = \frac{1}{2}(z_1 + z_2), \tag{B.15}$$

where z_1 and z_2 are given by eqn. (B.13), one can obtain

$$p_{21} = \frac{q_1 - q_2}{1 - 1/B}, \quad p_{22} = \frac{q_1 - q_2}{B - 1} \text{ and } p_{23} = \frac{2(q_2 - q_1/B)}{1 - 1/B}, \tag{B.16}$$

where B is defined by eqn. (56).

B.3 Supporting Calculations for Example 9

Using the demand functions (97) - (99), the unconditional inverse demand functions can be computed as follows[25]

$$\frac{q_1}{p_1} = \pi_1 \left(\pi_1 z_1^{-\pi_1 - 1} z_2^{-\pi_2} \left(\frac{1}{2} z_1 + \frac{1}{2} z_2 \right)^{-\pi_3} + \frac{1}{2} \pi_3 z_1^{-\pi_1} z_2^{-\pi_2} \left(\frac{1}{2} z_1 + \frac{1}{2} z_2 \right)^{-\pi_3 - 1} \right) \tag{B.17}$$

and

$$\frac{q_2}{p_1} = \pi_1 \left(\pi_2 z_1^{-\pi_1} z_2^{-\pi_2 - 1} \left(\frac{1}{2} z_1 + \frac{1}{2} z_2 \right)^{-\pi_3} + \frac{1}{2} \pi_3 z_1^{-\pi_1} z_2^{-\pi_2} \left(\frac{1}{2} z_1 + \frac{1}{2} z_2 \right)^{-\pi_3 - 1} \right). \tag{B.18}$$

Since

$$rank \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ 1 & 1 & 1 \end{pmatrix} = 2, \tag{B.19}$$

based on eqn. (18), there exists an effectively risk free asset. It follows that one can construct a portfolio which has the same payoff in each of the contingent claim state. Defining

$$z_1 = z_2 = c_2, \tag{B.20}$$

one can obtain

$$\frac{q_1}{p_1} = \pi_1 \left(\pi_1 + \frac{1}{2} \pi_3 \right) c_2^{-\pi_1 - \pi_2 - \pi_3 - 1} \text{ and } \frac{q_2}{p_1} = \pi_1 \left(\pi_2 + \frac{1}{2} \pi_3 \right) c_2^{-\pi_1 - \pi_2 - \pi_3 - 1}. \tag{B.21}$$

Since the price for this portfolio is $p_2 = q_1 + q_2$, we have

$$\frac{p_2}{p_1} = \frac{q_1 + q_2}{p_1} = \pi_1 \left(\pi_1 + \pi_2 + \pi_3 \right) c_2^{-\pi_1 - \pi_2 - \pi_3 - 1} = \frac{\pi_1}{c_2^2}. \tag{B.22}$$

B.4 Supporting Calculations for Example 10

Becaue there does not exist a risk free asset, we need to construct the effectively risk free asset first. Since

$$rank \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ 1 & 1 & 1 \end{pmatrix} = 2, \tag{B.23}$$

[25] This can be solved for using Mathematica software. Although computations are quite messy, eqns. (B.17) - (B.18) can be more readily verified from the first order conditions using the utility derived in Example 9

$$\mathcal{U}(c_1, \mathbf{z}) = c_1 - \frac{\pi_1}{z_1^{\pi_1} z_2^{\pi_2} \left(\frac{1}{2} z_1 + \frac{1}{2} z_2 \right)^{\pi_3}}.$$

based on eqn. (18), there exists an effectively risk free asset. It follows that one can construct a portfolio which has the same payoff in each of the contingent claim state. To obtain the price of the effectively risk free asset, we first need to derive the inverse demand for each asset. For the demands (52) - (54), it can be verified that

$$\frac{q_1}{p_1} = \pi_1 z_1^{-\pi_1 - 1} z_2^{-\pi_2} \left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)^{-\pi_3} + \frac{1}{2}\pi_3 z_1^{-\pi_1} z_2^{-\pi_2} \left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)^{-\pi_3 - 1} \tag{B.24}$$

and

$$\frac{q_2}{p_1} = \pi_2 z_1^{-\pi_1} z_2^{-\pi_2 - 1} \left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)^{-\pi_3} + \frac{1}{2}\pi_3 z_1^{-\pi_1} z_2^{-\pi_2} \left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)^{-\pi_3 - 1}. \tag{B.25}$$

Following a similar procedure as in Example 9, the price ratio for periods 1 and 2 consumption is given by

$$\frac{p_2}{p_1} = \frac{q_1 + q_2}{p_1} = (\pi_1 + \pi_2 + \pi_3)\, c_2^{-\pi_1 - \pi_2 - \pi_3 - 1} = \frac{1}{c_2^2}. \tag{B.26}$$

Given the presence of an effectively risk free asset, the price ratio in the Certainty Regularity Property corresponds to $(q_1 + q_2)/p_1$, which is clearly probability independent. Combining the above equation with the budget constraint

$$p_1 c_1 + p_2 c_2 = I \tag{B.27}$$

yields

$$c_1 = \frac{I - \sqrt{p_1 p_2}}{p_1} \qquad \text{and} \qquad c_2 = \sqrt{\frac{p_1}{p_2}}. \tag{B.28}$$

Defining

$$m = \frac{I}{p_2} \qquad \text{and} \qquad p = \frac{p_1}{p_2}, \tag{B.29}$$

period 1 demand function can be rewritten as

$$c_1 = \frac{m}{p} - \sqrt{\frac{1}{p}}. \tag{B.30}$$

Following Hurwicz and Uzawa (1971), the income compensation function μ satisfies

$$\frac{d\mu}{dp} = \frac{\mu}{p} - \sqrt{\frac{1}{p}}. \tag{B.31}$$

Solving the above partial differential equation yields

$$\mu = \left(K + \frac{2}{\sqrt{p}}\right) p = Kp + 2\sqrt{p}. \tag{B.32}$$

Therefore, the indirect utility function is

$$V(p,m) = \frac{m}{p} - \frac{2}{\sqrt{p}}. \tag{B.33}$$

Since

$$c_1 = \frac{m}{p} - \sqrt{\frac{1}{p}} \quad \text{and} \quad c_2 = \sqrt{p}, \tag{B.34}$$

we have

$$m = \left(c_1 + \frac{1}{c_2}\right) c_2^2 \quad \text{and} \quad p = c_2^2. \tag{B.35}$$

Substituting the above relations into the indirect utility function (B.33) yields the following direct utility function

$$U(c_1,c_2) = \frac{m}{p} - \frac{2}{\sqrt{p}} = \frac{\left(c_1 + \frac{1}{c_2}\right) c_2^2}{c_2^2} - \frac{2}{c_2} = c_1 - \frac{1}{c_2}, \tag{B.36}$$

which is defined up to an increasing transformation.

Remark 8 *The above process does not work if there is no risk free asset (or an effectively risk free asset) in the demand system. If the derived contingent claim prices are independent of income,*[26] *it is still possible to obtain $U(c_1,c_2)$ by assuming $\pi_i = 1$ and $\pi_s = 0$ $(s \neq i, s = 1,...,S)$ in the contingent claim setting.*[27] *However, since this process assumes specific probability structure, even if the demand (c_1,c_2) is independent of probabilities, $U(c_1,c_2)$ may still have probability dependence. For example, when assuming $\pi_i = 1$ and $\pi_s = 0$ $(s \neq i, s = 1,...,S)$, the following two utilities will generate the same demands*

$$u(c_1) + u(c_2) \tag{B.37}$$

and

$$u(c_1) + (u(c_2))^{1+\prod_{s=1}^{S}(1-\pi_s)}. \tag{B.38}$$

One way to resolve this problem is as follows. Once a probability independent U is recovered, then together with $\{V_{c_1}\}$, one can solve the optimal demands. If the optimal demands coincide with the given demands, one can conclude that U is probability independent.

[26] This requirement is used to ensure that the optimization problem for $U(c_1,c_2)$ is well defined. If the price for c_2 depends on c_1, we cannot use the first order condition to solve the optimization problem.

[27] We consider the contingent claim demands here since assuming $\pi_i = 1$ and $\pi_s = 0$ $(s \neq i, s = 1,...,S)$ may result in asset demands not being well defined.

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