INCOMPLETE MARKETS AND THE OBSERVABILITY OF RISK PREFERENCE PROPERTIES

by

H.M. POLEMARCHAKIS and L. SELDEN

Columbia University, New York, N.Y. 10027

ABSTRACT

Polemarchakis, H.M. and Selden, L. -- Incomplete Markets and the Observability of Risk Preference Properties.

In the framework of possibly incomplete asset markets, we derive observable conditions which are necessary and sufficient for an agent's demand function to be compatible with the maximization of some monotone, concave, von Neumann-Morgenstern objective function. On the other hand, we demonstrate that, in general, as long as markets are incomplete, it is not possible to infer from the observed asset demand function whether the generating representation of preferences necessarily satisfies monotonicity, risk aversion, or the expected utility hypothesis. Finally, we suggest extensions of the analysis to multi-attribute allocation problems under uncertainty, and we discuss the implications of the results for prediction and welfare comparisons.

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1. INTRODUCTION

Rational behavior in choice problems under uncertainty is typically associated with the existence of a complete preordering which is defined over some appropriate space of distribution functions, is continuously representable and satisfies the following three properties

(i) monotonicity;

(ii) risk aversion;

(iii) strong independence.

Under these conditions, roughly speaking, the representation can be expressed as an expected utility function, with the NM (von Neumann-Morgenstern) index being increasing and concave. In this paper, we examine conditions under which each of the above three unobservable preference properties corresponds to observable restrictions on consumer demand behavior.

In the standard formulation of the single-period, finite state asset allocation problem, an individual is assumed to divide his initial income, or wealth, among m assets (or complex securities) so as to maximize his expected

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utility $\sum_{s=1}^{n} \pi_s u(c_s)$, where π_s denotes the probability of state s and c_s denotes consumption in state s. The index u is monotone, "state independent" and concave and asset markets need not be complete, $m \leq n$ (see Section 2 below).

Under the supposition that the state probabilities and asset returns are known and the only <u>observable</u> characteristic of the agent is his demand for the assets $x \equiv (x_1,\ldots,x_m)$ as a function of prices and income, one can pose a number of questions:

<u>Question 1</u>: What are necessary and sufficient conditions for x to be derivable from <u>some</u> representation ϕ which satisfies monotonicity and risk aversion and which (ordinally) is an expected utility function, $\phi(c_1,\ldots,c_n) = T[\sum_{s=1}^{n} \pi_s u\ (c_s)]$, $T' > 0$?

<u>Question 2</u>: Suppose that a given asset demand system x is generated by a representation ϕ satisfying monotonicity and risk aversion; when can x also be generated by a representation <u>not</u> satisfying these properties?

<u>Question 3</u>: Suppose that x is generated by an expected utility function; when can x also be generated by a representation which is <u>not</u> an expected utility function?

<u>Question 4</u>: Assuming that there exists a monotone, risk averse expected utility function which generates x, when can it be recovered uniquely from the class of monotone, risk averse expected utility representations?

Question 1 is answered in the next section. Necessary and sufficient conditions are derived for at least one of the utility functions generating a given set of asset demands to be a monotone, concave NM representation. Although restrictive, these conditions provide, if satisfied, the means for (in principle) recovering the entire class of generating NM representations. In Section 3, we show that the answer to both Question 2 and Question 3 is "almost always". Thus under incomplete asset markets, the hypothesis that a given demand system was generated by maximizing some non-NM representation (or one which fails to satisfy monotonicity and risk aversion) can almost never be refuted. The point is simply that very different representations will generally be indistinguishable at the demand level even assuming complete knowledge of the agent's demand function, the state probabilities and the matrix of asset returns. Question 4 is concerned with the issue of recoverability where it is assumed that somehow one knows that the given demand function is indeed generated by a monotone, risk averse expected utility representation. This question was analyzed in Dybvig and Polemarchakis [7] and Green, Lau, and Polemarchakis [9].

Conditions for the existence of a generating utility function in the context of ordinal preferences and complete markets are well known (Hurwicz and Uzawa [12]), as are those for uniqueness (Mas-Colell [14]). The two distinguishing features of the questions considered in the present paper are the requirement that the generating preferences possess an additively separable representation and the possibility that the asset markets may be incomplete. The earlier work of Houthakker [11], deriving observable demand restrictions corresponding to (ordinally) additive utility functions is not immediately applicable since it is cast in the equivalent of complete markets.[1/]

Section 4 first considers briefly the implications of our results for the case of joint income and asset return uncertainty and then raises the analogues of Question 1- Question 4 for the standard two-period consumption-savings and consumption-portfolio decision problems (see Polemarchakis and Selden [16, 17]).

We conclude the paper with a discussion of some of the implications of not being able, in incomplete asset markets, to infer from observable demand behavior whether the generating representation is an expected utility function. On the one hand, we consider the implications for the prediction of agent behavior under different (incomplete) market regimes, and, on the other, for standard welfare questions. This material represents a fairly comprehensive statement of the implications of not only the present study but also of a number of related papers, and thus some readers may wish to begin with Section 5 rather than 2.

2. EXISTENCE

In this Section we give necessary and sufficient conditions for at least one of the representations generating a given asset demand function to be a monotone, concave expected utility function.

Notation and Definitions

Consider an individual who must allocate his initial income $y > 0$ among $m \geq 2$ assets (or complex securities) indexed by the subscript $j = 1,\ldots,m$. There are n states of nature indexed by the subscript $s = 1,\ldots,n$, where, in general, n exceeds m. The vector of state probabilities is denoted by $\pi \equiv (\pi_1,\ldots,\pi_s,\ldots,\pi_n) \in \mathbb{R}^n_{++}$. 2/ The vector $x \equiv (x_1,\ldots,x_j,\ldots,x_m) \in \mathbb{R}^m$

describes the agent's asset holdings. (The fact that his holdings of asset j, x_j, can be negative means that short sales are allowed.) The prices of the m complex securities are given by the price vector $p \equiv (p_1,\ldots,p_j,\ldots,p_m) \in \mathbb{R}^m_{++}$. Let r_{sj} denote the (gross) return of asset j in state of nature s. The payoffs from each asset in each state of nature are summarized in the $n \times m$ return matrix $R = (r_{sj})^{s=1,\ldots,n}_{j=1,\ldots,m}$, for which the following is assumed to hold:

<u>Assumption [R]:</u> The return matrix R satisfies

(i) $r_{sj} \geq 0$ for all $s = 1,\ldots,n$, $j = 1,\ldots,m$,

(ii) the column vector r_j cannot be written as a linear combination of $\{r_k\}$, $k = 1,\ldots,j-1,j+1,\ldots,m$ for all $j = 1,\ldots,m$, and

(iii) for any $s = 1,\ldots,n$, $r_{sj} > 0$ for some $j = 1,\ldots,m$.

The random variable $r \equiv (r_1,\ldots,r_j,\ldots,r_m)$ determines for any vector of asset holdings $x \in \mathbb{R}^m$, random (end-of-period) consumption. We denote by $c \equiv (c_1,\ldots,c_s,\ldots,c_n)$ the contingent commodity consumption vector and by C the strictly positive orthant $\mathbb{R}^n_{++}$.

It is assumed throughout this paper that the consumer possesses a complete preordering over the space of consumption random variables (or contingent commodity vectors)[3/] which is representable by the twice continuously differentiable ordinal index ϕ: $C \to \mathbb{R}$. The representation ϕ fully characterizes an agent.

We will be concerned with the observable demand restrictions corresponding to three further properties imposed on the agent's preferences (which, for simplicity, are stated in terms of the representation ϕ).

Definition: We say that preferences are monotone if ϕ is strictly increasing in each c_s,[4/]

$$D\phi(c) \gg 0 \quad .$$

Definition (Debreu [3, p. 101]): Preferences are said to be (strictly) risk averse if ϕ is (strictly) quasiconcave, ($D^2\phi$ is negative definite on the orthogonal complement of $D\phi$, denoted $[D\phi]^{\perp}$).

Definition: An agent's preferences are said to be NM (von Neumann-Morgenstern) representable if ϕ satisfies

$$\phi(c_1,\ldots,c_n) = T[\sum_{s=1}^{n} \pi_s u(c_s)] \quad , \quad T' > 0 \ ,$$

where the continuous NM index $u\colon \mathbb{R}_{++} \to \mathbb{R}$ is "state independent".[5/]

Remarks: (1) For a discussion of risk aversion measures in the absence of the expected utility hypothesis (but where the state space is finite) and of the use of these measures in analyzing asset demand behavior, see Yaari [23] and Mayshar [15]. (2) It follows from Arrow [1, p. 127] and Stigum [21] that if ϕ is an NM representation, then ϕ will be (strictly) quasiconcave if and only if u is (strictly) concave. (3) Clearly, monotonicity of $\phi \Leftrightarrow$ monotonicity of the NM index u.

When an agent's preferences are NM representable, we shall assume the following:

Assumption [U]: The NM index u satisfies:

(i) $u' > 0$ everywhere on $\mathbb{R}_{++}$ and

(ii) $u'' < 0$ everywhere on $\mathbb{R}_{++}$.

The matrix of asset returns, R, and the vector of state probabilities, π, will be held fixed. The agent's objective function for his asset demand problem is given by

$$\phi(c_1,\ldots,c_s,\ldots,c_n) = \phi(r_1^t x,\ldots,r_s^t x,\ldots,r_n^t x) \tag{1}$$

or, if his preference ordering is NM representable, by

$$\phi(c_1,\ldots,c_s,\ldots,c_n) = T[\sum_{s=1}^{n} \pi_s u(r_s^t x)] \quad , \quad T' > 0 \quad . \tag{2}$$

For the present, we shall only consider the stronger representation hypothesis (2).

Faced with prices $p \in \mathbb{R}^n_{++}$ and income $y \in \mathbb{R}_{++}$, the agent chooses $x(p,y) \in \mathbb{R}^m$ by solving the following maximization problem

$$\max_x \quad \sum_{s=1}^{n} \pi_s u(r_s^t x) \quad \text{s.t.} \quad p^t x = y \quad , \quad r_s^t x \equiv c_s > 0 \quad , \quad s = 1,\ldots,n \tag{3}$$

where the transform T in (2) has been dropped since, of course, it has no effect on the solution.

<u>Constrained Contingent Commodity Formulation</u>

While the agent's maximization problem (3) is stated in terms of observable demands, it will prove not to be the most convenient formulation since the objective function $\psi(x_1,\ldots,x_m) \equiv \sum_{s=1}^{n} \pi_s u(r_s^t x)$ will, in general, <u>fail</u> to inherit the "preference" properties imposed on ϕ. For instance, assuming ϕ to be NM representable implies that ϕ is (ordinally) additively separable across contingent commodities but does not imply that the objective function ψ is (ordinally) additively separable in the choice variables $x_1,\ldots,x_m$. We next consider an alternative formulation of the agent's optimization problem equivalent to (3), but for which the objective function is $\phi(c_1,\ldots,c_n)$ and thus exhibits the properties of monotonicity, quasiconcavity and (ordinal) additive separability. Because of the equivalence between the two formulations, the contingent commodity demand functions derived below will possess a simple relation to the observable demands $x(p,y)$ solving (3) and hence can, without loss of generality, also be viewed as being observable.

In order to ensure that the contingent commodity formulation is, in fact, equivalent to (3) when asset markets are incomplete $(m < n)$, it is necessary for us to constrain the agent to buying only particular linear combinations of the $(c_1,\ldots,c_s,\ldots,c_n)$. Let us begin by noting that since, under Assumption [R], the asset return matrix R has full column rank, we can partition it into a pair of submatrices R_α and R_β, where R_α is an $(m \times m)$ matrix of full rank and R_β is the complementary submatrix (i.e., $R^t = [R_\alpha^t \,\vdots\, R_\beta^t]$). Next define the matrix $A^t \equiv R_\beta R_\alpha^{-1}$ which is $((n - m) \times m)$. Let c_α and c_β denote the corresponding partitions of the state contingent commodity vector $(c_1,\ldots,c_s,\ldots,c_n)$; i.e., $c^t = [c_\alpha^t \,\vdots\, c_\beta^t]$. Clearly $c_\alpha = R_\alpha x$ and $c_\beta = R_\beta x$; without loss of generality, it is assumed that $c_\alpha = (c_1,\ldots,c_m)$ and $c_\beta = (c_{n-m},\ldots,c_n)$. It then follows from the invertibility of R_α that the agent's constraint on the purchase of contingent commodities can be expressed as

$$c_\beta = A^t c_\alpha \quad .$$

The constrained contingent commodity problem can thus be defined as follows:

$$\max_{c} \phi(c) \equiv \sum_{s=1}^{n} \pi_s u(c_s) \quad \text{s.t.} \quad q_\alpha^t c_\alpha = y \;, \quad c_\beta - A^t c_\alpha = 0 \;,$$

$$(c_\alpha, c_\beta) \in \mathbb{R}^n_{++} \;, \tag{4}$$

where $q_\alpha \in \mathbb{R}^m$ is a price vector corresponding to the contingent commodity vector c_α. It is evident that for $q_\alpha = p(R_\alpha^t)^{-1}$, the maximization problems (3) and (4) are equivalent (see Fischer [8]). Observe that the contingent commodity price vector q_α may have negative elements. This does not violate the "no arbitrage condition" common in the literature since, in addition to the budget constraint, the agent faces the constraint $c_\beta - A^t c_\alpha = 0$. Furthermore, the choice--implicit in (4)-- of setting $q_\beta \equiv 0$ is arbitrary but involves no loss of generality. The vector x solves (3) at (p,y) if and only if $(R_\alpha x, A^t R_\alpha x)$ solves (4) at $(q_\alpha, y) = (p(R_\alpha^t)^{-1}, y)$. Thus as suggested above, we may, with no loss of generality, suppose that the <u>observable characteristics</u> of the agent consist of the <u>constrained contingent commodity demand function</u>.

Although (4) need not, of course, possess a solution for an arbitrary $q_\alpha \in \mathbb{R}^m$, we next show that there is an open subset of contingent commodity prices such that a solution exists and is well-behaved. Let $\text{Int}\, Q_\alpha$ denote the interior of the subset Q_α of $\mathbb{R}^m$ with the property that for $q_\alpha \in Q_\alpha$ a solution to (4) exists, is unique and is characterized by the first-order conditions

$$D_\alpha \phi(c) \;-\; q_\alpha \lambda \;-\; A\mu = 0 \tag{5}$$

$$D_\beta \phi(c) \;+\; I\mu = 0$$

$$q_\alpha^t c_\alpha = y$$

$$-A^t c_\alpha + c_\beta = 0 \quad ,$$

where $\lambda \in \mathbb{R}_{++}$, $\mu \in \mathbb{R}_{--}^{n-m}$ are the unique Lagrange multipliers associated with the constraints in (4). The demand function $c(q_\alpha, y)$ has as its domain $Q_\alpha \times \mathbb{R}_{++}$ and thus, by definition, is well defined for any $(q_\alpha, y) \in \text{Int } Q_\alpha \times \mathbb{R}_{++}$ so long as the set $\text{Int } Q_\alpha$ is not empty.

<u>Lemma 1</u>: The set $\text{Int } Q_\alpha \neq \emptyset$.

(The proof is given in the Appendix.)

From now on we work with the demand function $c(q_\alpha, y)$ and the first-order conditions (5) for $(q_\alpha, y) \in \text{Int } Q_\alpha \times \mathbb{R}_{++}$.

<u>"Slutsky Equations"</u>

Since we know from standard demand theory that utility function properties such as (ordinal) additive separability correspond to restrictions on the Slutsky matrix, we next derive the appropriate "Slutsky equations" for our constrained contingent commodity problem. Totally differentiating the system of equations (5) yields:

$$\begin{bmatrix} D^2_{\alpha\alpha}\phi & 0 & -q_\alpha & -A \\ 0 & D^2_{\beta\beta}\phi & 0 & I \\ -q^t_\alpha & 0 & 0 & 0 \\ -A^t & I & 0 & 0 \end{bmatrix} \begin{bmatrix} dc_\alpha \\ dc_\beta \\ d\lambda \\ d\mu \end{bmatrix} = \begin{bmatrix} \lambda dq_\alpha \\ 0 \\ c^t_\alpha dq_\alpha - dy \\ 0 \end{bmatrix} \quad (6)$$

where by the additive separability of ϕ, $\phi(c) \equiv \sum_{s=1}^{n} \pi_s u(c_s)$, both cross terms $D^2_{\alpha\beta}\phi$ and $D^2_{\beta\alpha}\phi$ vanish and

$$D^2_{\alpha\alpha}\phi(c) = \begin{bmatrix} \pi_1 u''(c_1) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \pi_m u''(c_m) \end{bmatrix} \quad \text{and} \quad (7)$$

$$D^2_{\beta\beta}\phi(c) = \begin{bmatrix} \pi_{m+1} u''(c_{m+1}) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \pi_n u''(c_n) \end{bmatrix} .$$

Define the matrix on the LHS of equation (6) to be B. Then it follows from the assumed negative definiteness of the matrix $D^2\phi$ that B is invertible. The demand function is thus continuously differentiable, as are the functions λ and μ. Setting

$$\begin{bmatrix} S_{\alpha\alpha} & S_{\alpha\beta} & -v_\alpha & -w_\alpha \\ S_{\beta\alpha} & S_{\beta\beta} & -v_\beta & -w_\beta \\ -v_\alpha^t & -v_\beta^t & e & d^t \\ -w_\alpha^t & -w_\beta^t & d & G \end{bmatrix} = B^{-1} , \tag{8}$$

equations (6) and (8) imply the following:

$$D_{q_\alpha} c_\alpha = \lambda S_{\alpha\alpha} - v_\alpha c_\alpha^t \; ; \quad D_y c_\alpha = v_\alpha \; ; \tag{9}$$

$$D_{q_\alpha} \lambda = -\lambda v_\alpha^t + e c_\alpha^t \quad ; \quad D_y \lambda = -e \quad ;$$

$$D_{q_\alpha} \mu = -\lambda w_\alpha^t + d c_\alpha^t \quad ; \quad D_y \mu = -d \quad ;$$

where $D_{q_\alpha} c_\alpha$ and $D_y c_\alpha$ denote, respectively, the matrix of partial derivatives of the form $\partial c_s(q_1,\ldots,q_m,y)/\partial q_i \quad s,i \in \{1,\ldots,m\}$ and the vector of marginal propensities to consume of the form $\partial c_s(q_1,\ldots,q_m,y)/\partial y$. The first expression in (9) clearly resembles the standard Slutsky equation. The following notation will be used to denote "compensated" derivatives[6/]

$$D^*_{q_\alpha} c_\alpha = \lambda S_{\alpha\alpha} = D_{q_\alpha} c_\alpha + (D_y c_\alpha) c_\alpha^t \quad ; \tag{10}$$

$$D^*_{q_\alpha} \lambda = -\lambda v_\alpha^t = D_{q_\alpha} \lambda + (D_y \lambda) c_\alpha^t \quad ;$$

$$D^*_{q_\alpha} \mu = -\lambda w_\alpha^t = D_{q_\alpha} \mu + (D_y \mu) c_\alpha^t \quad .$$

The Existence Proposition

In order to prove our basic existence result, we shall require the demand functions for contingent commodities to satisfy two sets of conditions. The first of these, referred to as regularity conditions, can be defined as follows:

Regularity Conditions: A function $c(q_\alpha, y) \equiv (c_\alpha(q_\alpha, y),\ c_\beta(q_\alpha, y))$ is said to be a regular constrained contingent commodity demand function if and only if the following are satisfied everywhere on Int $Q_\alpha \times \mathbb{R}_{++}$

(i) $c_\alpha(q_\alpha, y)$ is positive and continuously differentiable,

(ii) $q_\alpha^t c_\alpha(q_\alpha, y) = y$ and $A^t c_\alpha(q_\alpha, y) = c_\beta(q_\alpha, y)$ and

(iii) $D_y c_s(q_\alpha, y) \neq 0 \quad s = 1, \ldots, n.$

In the presence of complete markets, where m = n, the matrix A vanishes and regularity follows from the monotonicity and concavity of u. The marginal propensity to consume for each contingent commodity would then be strictly positive. In the case of incomplete markets, requirements (i) and (ii) are both straightforward and relatively innocuous while (iii) is more substantive.[7/]

Next we define a consistency requirement to be satisfied by the contingent commodity demand function and by the Lagrange multipliers $\lambda(q_\alpha, y)$ and $\mu(q_\alpha, y)$ which are associated with the constraints in (4). These conditions will be seen in the proof of Proposition 1 to correspond to the stationarity of the agent's utility function over contingent commodities.

Consistency Conditions: The continuously differentiable functions (λ,μ) defined on Int $Q_\alpha \times \mathbb{R}_{++}$ with images in $\mathbb{R}$ and $\mathbb{R}^{n-m}$, respectively, will be said to satisfy the consistency requirement for a given regular demand function $c(q_\alpha,y)$ if and only if the following conditions are satisfied everywhere on Int $Q_\alpha \times \mathbb{R}_{++}$: whenever $c_s(q_\alpha,y) = c_{s'}(q'_\alpha,y')$,

$$\frac{(\lambda q_s + a_s^t\mu)}{\pi_s} = \frac{(\lambda' q_{s'} + a_{s'}^t\mu')}{\pi_{s'}} \quad \text{if} \quad s,s' \in \{1,\ldots,m\}$$

$$\frac{\mu_s}{\pi_s} = \frac{\mu_{s'}}{\pi_{s'}} \quad \text{if} \quad s,s' \in \{m+1,\ldots,n\}$$

$$\frac{(\lambda q_s + a_s^t\mu)}{\pi_s} = \frac{\mu_{s'}}{\pi_{s'}} \quad \text{if} \quad s \in \{1,\ldots,m\}, \quad s' \in \{m+1,\ldots,n\},$$

where a_s^t, $a_{s'}^t$ are rows of the matrix A.

We next define the function

$$\delta_s(q_\alpha,y) \equiv \begin{cases} \dfrac{(D_y\lambda q_s - a_s^t D_y\mu)}{D_y c_s} & s = 1,\ldots,m \\[2ex] \dfrac{-D_y\mu_s}{D_y c_s} & s = m+1,\ldots,n \end{cases} \tag{11}$$

where as long as λ and μ are continuously differentiable and the contingent commodity demand function is regular, the $\{\delta_1,\ldots,\delta_s,\ldots,\delta_n\}$ are well defined and continuous on Int $Q_\alpha \times \mathbb{R}_{++}$. The function δ_s will be seen to correspond in the recovery of the utility function in Proposition 1 to $\pi_s u''(c_s)$. Finally, we shall use the notation[8/]

$$\Delta_\alpha = \text{diag } (\delta_1,\ldots,\delta_m) \quad ; \quad \Delta_\beta = \text{diag } (\delta_{m+1},\ldots,\delta_n) \quad .$$

Proposition 1: Let $c(q_\alpha,y)$ be a constrained contingent commodity demand function which satisfies the regularity condition. Suppose that the matrix of asset returns satisfies Assumption [R]. Then $c(q_\alpha,y)$ is generated by an NM representation satisfying monotonicity and risk aversion if and only if there exist two functions (λ,μ) which satisfy the consistency requirement and the following three conditions hold everywhere on Int $Q_\alpha \times \mathbb{R}_{++}$:

(1) $\lambda(q_\alpha,y)$ is continuously differentiable and strictly positive, $\mu(q_\alpha,y)$ is continuously differentiable and strictly negative (component-wise),

(2) (i) $(1/\lambda)\Delta_\alpha D^*_{q_\alpha} c_\alpha + q_\alpha (D_y c_\alpha)^t - (1/\lambda) A D^*_{q_\alpha} \mu = I$,

(ii) $(1/\lambda)\Delta_\beta A^t (D^*_{q_\alpha} c_\alpha) A + (1/\lambda)(D^*_{q_\alpha} \mu) A = 0$, and

(3) $\delta_s < 0$, $s = 1,\ldots,n$.

(The proof is given in the Appendix.)

Remark: The conditions (1)-(3) are the demand analogues to the preference properties of monotonicity, (ordinal) additive separability and concavity, respectively, and the consistency condition corresponds to stationarity. Condition

(2) is, when A vanishes in the case of complete markets $(m = n)$, precisely the Houthakker [11] demand restrictions for additive separability.

Remark: If the necessary and sufficient conditions in Proposition 1 are satisfied then $\{\delta_s(q_\alpha, y)\}$ will be observable and will permit (in principle) integration of the full set of generating NM representations.[9/]

3. NON-UNIQUENESS

Having derived necessary and sufficient conditions for at least one of the possibly multiple utility functions generating an asset demand function, $x(p,y)$ -- or equivalently, a constrained contingent commodity demand function, $c(q_\alpha y)$ -- to be a monotone, concave (risk averse) NM representation, we next consider two uniqueness questions. First, suppose that a given demand system is generated, under incomplete markets, by a representation ϕ satisfying monotonicity and risk aversion, then when can the demands also be generated by a representation not satisfying these properties? The second question addressed below is when can a given demand function generated by an expected utility function also be generated by a representation which is not an expected utility function.

As we shall show next, when asset markets are incomplete, unique recoverability is too much to hope for. Thus asset demand functions cannot be used to verify whether the properties of monotonicity, risk aversion and strong independence are satisfied by the agent's representation.

Proposition 2: Assume the return matrix satisfies [R] and $m < n$. Let

ϕ be a twice continuously differentiable, strictly monotone and risk averse representation. Then there exists a pair of twice continuously differentiable functions ϕ^* and ϕ^{**}, such that ϕ^* fails to be monotone and ϕ^{**} fails to be risk averse (quasiconcave) and yet both generate the same asset demand function as ϕ.

(The proof is given in the Appendix.)

Proposition 3: Assume the matrix of asset returns satisfies Assumption [R] and $m < n$. Let $\phi(c_1,\ldots,c_n)$ be an NM representation where the NM index satisfies Assumption [U]. Suppose that

$$u''(c_s) \text{ is bounded away from zero on bounded subsets of } \mathbb{R}_{++}. \qquad (12)$$

Then there exists a twice continuously differentiable strictly monotone, strictly quasiconcave objective function $\hat{\phi}$ which is not NM and yet generates the same asset demand function as ϕ.

(The proof is given in the Appendix.)

Remark: The condition (12) invoked in order to establish the non-uniqueness of the representation is really quite mild (and, in fact, could be relaxed somewhat). It should, for instance, be noted that (12) is satisfied by the frequently employed expected utility function

$$\phi(c) = -\sum_{s=1}^{n} \frac{\pi_s}{\alpha} c_s^{-\alpha} \quad , \quad -1 < \alpha < \infty \quad . \qquad (13)$$

One implication of our results is that a given asset demand system exhibiting portfolio separation[10/] and corresponding to (13) might well have been generated by a non-NM representation, perhaps very different from the class of power function-based NM representations.

Remark: Propositions 2 and 3 demonstrate that, as long as markets are incomplete, full knowledge of the agent's asset demand function is not sufficient to ensure that his preferences satisfy monotonicity, risk aversion or the expected utility hypothesis. The question can then be raised whether any additional information--short of "completing the market"--can suffice to guarantee that any of these properties hold. Since, however, the only assumption concerning market structure employed in the argument for Propositions 2 and 3 is the existence of a single non-zero vector in the kernel of R, it follows that "completion of the market" is, in some sense, necessary to guarantee monotonicity, risk aversion or the expected utility hypothesis. (Also see Section 5 below.)

In Proposition 1, we derived necessary and sufficient conditions for the existence of some NM representation satisfying [U] to generate the demand function for assets whose return structure satisfies [R]. Taking for granted that a generating NM objective function does indeed exist, it is then natural to ask whether it need be the only NM generating representation and whether it is recoverable. In earlier papers, uniqueness and recoverability have been attained under alternative additional restrictions on the NM index u or on the return matrix R. Namely, either it is assumed that u is analytic on $\mathbb{R}_+$ (Green, Lau and Polemarchakis [9]), or that some linear combination of the available assets is riskless (Dybvig and Polemarchakis [7]). Furthermore, examples are known where uniqueness fails in the absence of a riskless asset (see Dybvig and Polemarchakis [7]). We want now to show how the existence argument in Proposition 1 can be used to yield uniqueness and recoverability in the presence of a riskless asset.

Proposition 4 (Dybvig and Polemarchakis [7]): Assume there exists an NM representation satisfying [U] which generates a given asset demand function. Let the matrix of asset returns satisfy [R] and suppose some linear combination of the assets is riskless. Then the generating NM utility function is unique and recoverable.

(The proof is given in the Appendix.)

The intuition behind unique recoverability in the presence of a riskless asset is straightforward: Whenever state contingent consumption is constant across states -- $c_s = \bar{c}$ for all s -- the consistency requirement implied by the existence of an NM index u prevents any digression in the specification of the functions $\lambda(q_\alpha, y)$ and $\mu(q_\alpha, y)$ and, thus, $u''(\bar{c})$ is determined unambiguously.

The unique recoverability result based on the analyticity of u can not be immediately derived in the present context. One would, first of all, have to extend the analysis to allow for a zero level of consumption and, furthermore, rely on higher derivatives of the demand function at the origin. Presumably, knowledge of all the derivatives of the demand function at the origin, combined with the analyticity of λ and μ implied by the analyticity of u, eliminates any degrees of freedom involved in the specification of λ and μ.

4. EXTENSIONS

The analysis so far has been carried out in the simple framework of a single time period and a single preference attribute. In this section we discuss, somewhat informally, several possible extensions.

Let us begin by assuming that the allocation problem is the same as before except for the fact that now the agent faces uncertainty not only concerning the returns of the various assets but concerning his income as well. (This income uncertainty will not be resolved until after the asset allocation decision is made.) That is, his income is given by the vector (y,ε) where $y > 0$ is received at the beginning of the period and $\varepsilon_s \geq 0$, $s = 1,\ldots,n$, is the end-of-period income received under state of nature s. The contingent commodity consumption vector corresponding to the asset holdings x is then given by $c = Rx + \varepsilon$ and the budget constraint by $p^t x = y$ -- the agent is required to be solvent in each state s.[11/]

In order to derive the observable demand restrictions corresponding to various preference properties, we shall, as in the previous sections, transform the portfolio problem into an equivalent constrained contingent commodity problem. Partitioning R into a full rank submatrix R_α and the complementary submatrix R_β, and setting $A^t = R_\beta R_\alpha^{-1}$ and $a = -A^t\varepsilon_\alpha + \varepsilon_\beta$, we see that

$$c_\beta = A^t c_\alpha + a \quad ,$$

where $(\varepsilon_\alpha,\varepsilon_\beta)$ and (c_α,c_β) are the corresponding partitions of the vectors ε and c, respectively. Thus, the linear dependence between c_α and c_β has been transformed into an affine relationship. This change leaves the first-order conditions substantially unaltered and, as the reader may verify, Propositions 1, 2 and 3 remain valid with only very minor modifications required. However, the uniqueness result, Proposition 4, no longer holds:

Proposition 5: Assume there exists an NM representation satisfying [U] which generates a given asset demand function. Let the matrix of asset returns satisfy [R]. Then in the presence of income uncertainty, the existence of a riskless asset is not sufficient to guarantee the uniqueness and recoverability of the generating NM utility function.

(The proof is given in the Appendix.)

Alternatively, we could imagine an agent who must allocate his initial wealth, y, between current consumption c_1 and m assets indexed by $j = 1,\ldots,m$. His future (second-period) contingent consumption would then be given by

$$c_{2s} = r_s x + \varepsilon_s \quad , \quad s = 1,\ldots,n \quad ,$$

where r_s is the vector of asset returns in state s and ε_s denotes his uncertain period-two income.

One could raise in this two-period setting a number of questions analogous to those considered for the one-period case:

(1) What are necessary and sufficient conditions for the demand function (c_1,x) to be compatible with the maximization of some monotone, concave, NM representation?

(2) Can monotonicity, risk aversion and the (two-period) expected utility hypothesis be guaranteed in the absence of complete markets?

(3) Finally, the analysis of Sections 2 and 3 could be extended to the more general "Ordinal Certainty Equivalent" preference setting considered in Polemarchakis and Selden [16, 17].

5. PREDICTION AND WELFARE

Our analysis leads to the general conclusion that as long as asset markets are incomplete one is almost never able to infer from observable demand behavior the "true" representation of preferences over contingent commodities. In particular, questions such as whether the agent is characterized by monotonicity and risk aversion or whether he satisfies the von Neumann-Morgenstern expected utility axioms simply can not be answered conclusively. Since asset markets are indeed incomplete, this issue of non-observability raises a number of vexing problems of considerable practical importance.

Non-uniqueness of the generating representation precludes, above all, the general possibility of forecasting the agent's behavior outside a given market regime. In the case of complete markets, we know from Mas-Colell [14] that, under mild regularity conditions, the preferences generating a given demand function are unique and recoverable. But, of course, in this case there is no real issue of prediction: one can readily determine from the agent's "complete markets" demand function what his demand for assets will be under any alternative incomplete market regime. The desirability of prediction arises only in the framework of incomplete markets; but that is precisely when it is not possible. One has no way of telling which of the multiple representations generating the given asset demands is the unique "true" representation in the sense of generating the "complete markets" demand function. And it is only with this "true" utility function that one can predict accurately the agent's behavior under different market regimes. Observations under several different (incomplete market) regimes will generally be of no help in identifying the "true" representation--although some inferences may be possible. More generally, it remains an open question to determine the class of preference properties which on the one hand can be verified in an incomplete market setting, and on the other hand do have predictive content concerning the agent's response to alternative choice sets.[12/]

In addition to raising serious obstacles to predicting individual behavior, the existence of incomplete markets also poses serious problems for standard welfare analysis. Knowledge of an agent's indifference map can serve to determine the compensation necessary for his "welfare" to be unaffected by a change in the opportunities available to him. It is in this context, that Mas-Colell's [14] recoverability result for complete markets is of interest. Suppose alternatively that markets are not complete and that an agent is confronted with an alteration in the random return on one of the assets in which he has a non-zero investment. One interesting example of such a case would be that of a stockholder facing a change in the stochastic production plan of a firm. To determine how--if at all possible[13/]--the agent is to be compensated for the change so that he does not suffer a welfare loss requires knowledge of his representation of preferences over contingent commodities. Again, however, if the only information available, as is likely to be the case, is the agent's asset demand function under incomplete markets, the "true" representation and hence the correct compensation can not be unambiguously determined. It is an open question to determine the individual welfare conclusions that can be inferred from incomplete market data.

In a related vein, one can ask how the Grossman-Hart [10] firm decision criterion for evaluating production plans under incomplete markets is to be implemented. Their criterion is based on a weighted sum of ex ante stockholder marginal rates of substitution between present and future contingent consumption (cf., Diamond [5] and Drèze [6]). In general, it will not be possible to infer an individual stockholder's "true" preferences from complete knowledge of his asset demand function (even assuming that $(\mathbf{R},\pi)$ is given). The alternative tack of simply asking stockholders to supply their marginal rates of substitution, of course, raises many of the standard problems associated with the "revelation of preferences."

Finally, our analysis raises a number of questions concerning the aggregation of preferences and social welfare. In addition to the individual observability problems discussed above, we are now faced with the problems associated with aggregating individual utility functions under uncertainty. In the case of complete markets, even if agents have homothetic NM preferences and constant income shares and thus ordinal aggregation is indeed possible, the aggregator need not be (ordinally) NM representable (cf., Polemarchakis and Selden [16]). Suppose, however, that individual and aggregate demand observations are limited to a set of assets which do not span. Then it may be the case that even though the "true" (i.e., complete markets) aggregator either does not exist or exists but is not NM, the aggregate demand for existing assets happens to be compatible with the maximization of <u>some</u> expected utility function. If now this "false" aggregator is used for social welfare judgements, the results can be erroneous.

We conclude this section with a simple example cast in the context of an optimal tax problem. It illustrates the difficulties raised for welfare judgements by the inability to distinguish between NM and non-NM preferences on the basis of asset demand functions. We note, however, that these difficulties persist even when one assumes away, as we do in the example, the problem of the existence of an aggregator.

<u>Example</u>. Consider the simple setting of one agent (or equivalently, a group of identical agents), three equiprobable states of nature $(\pi_s = \frac{1}{3},$ $s = 1,2,3)$ and two complex securities $(j = 1,2)$ with return matrix

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{14}$$

Assume that the agent's <u>true</u> preferences are representable by the <u>non</u>-NM objective function

$$\phi(c_1,c_2,c_3) = \sum_{s=1}^{3} \frac{c_s^{\frac{1}{2}}}{3} + kc_1^{\frac{1}{2}}c_2^{\frac{1}{2}}c_3^{\alpha} ,$$

where $k > 0$ and $0 < \alpha < 1$. Observe furthermore that the same asset demand function which is generated by the true non-NM objective function is also produced by the NM objective function

$$\hat{\phi}(c_1,c_2,c_3) = \sum_{s=1}^{3} \frac{c_s^{\frac{1}{2}}}{3} .$$

That, for the existing assets and return structure, the functions ϕ and $\hat{\phi}$ generate exactly the same demands at <u>all</u> prices (p_1, p_2) and income levels follows immediately by observing that $c_3 = x_1r_{31} + x_2r_{32} = 0$ given (14), and hence, effectively ϕ reduces to $\hat{\phi}$. Suppose now that the government needs to raise a revenue B and as a consequence taxes the individual. The instruments at the government's disposal are a proportional tax t on capital gains (where the constant rate t, which applies only to gains, is the same for the two assets) and a lump-sum "income maintenance" transfer T which is paid to the taxpayer in state 3. The objective of the government is to choose (t,T) so as to maximize the individual's utility for after-tax (and transfer) consumption subject to raising an expected tax revenue net of the income transfer T equal to the required quantity B. It must be stressed that, of course, this is only one of the possible formulations of the optimal tax problem under uncertainty, but it suffices to illustrate the difficulties which can arise in welfare judgements due to the non-uniqueness of preferences generating asset demands. It is assumed that the agent chooses his portfolio with full knowledge of the tax-income transfer structure and the exogeneously given prices for the securities, p_1 and p_2.[14/] To simplify the argument, we assume that the agent's income equals 1 and that $p_1 = p_2 = p$. It follows that the budget constraint of the agent can be written as

$$x_1 + x_2 = 1/p,$$

and it is clear that the optimal asset holdings are given by $x_1 = x_2 = 1/2p$ and hence

$$c_1 = x_1 - (1 - p)x_1t = [1 - (1 - p)t]/2p$$
$$c_2 = x_2 - (1 - p)x_2t = [1 - (1 - p)t]/2p$$
$$c_3 = T.$$

The expected taxes raised by the government are then given by

$$\tfrac{1}{3}(1 - p)x_1 t + \tfrac{1}{3}(1 - p)x_2 t - \tfrac{1}{3}T = \tfrac{1}{3}\left[\frac{(1 - p)t}{p} - T\right] .$$

Equating this to the revenue requirement B and rearranging yields the condition

$$t = \frac{p}{(1-p)}(3B + T). \tag{15}$$

Using this expression to eliminate t from c_1 and c_2, we can reduce the optimal tax problem to one of choosing T so as to maximize the agent's utility function. Now were his preferences to be representable by the NM function $\hat{\phi}$, then he would seek to maximize

$$\hat{\phi}(T) = \tfrac{2}{3}\left[\tfrac{1}{2}(\tfrac{1}{p} - 3B - T)\right]^{\frac{1}{2}} + \frac{T^{\frac{1}{2}}}{3}$$

which has as its solution

$$T^* = (1/3p) - B, \tag{16}$$

and using (15),

$$t^* = \frac{1}{(1-p)}[2pB + \tfrac{1}{3}]. \tag{17}$$

Now on the other hand based on the agent's true non-NM representation ϕ, proceeding in the same fashion yields in the limit as k gets large

$$T^* = \frac{3\alpha}{(\alpha+1)}\left[\frac{1}{3p} - B\right] \tag{18}$$

$$t^* = \frac{1}{(\alpha+1)(1-p)}\left[3pB + \alpha\right]. \quad \underline{15/} \tag{19}$$

Remark: The example illustrates that optimal taxes can indeed be quite different for the cases of NM and non-NM preferences even when the two different utility functions are indistinguishable at the level of asset demands. As a result, the government is unable to identify the true preference structure and hence will not know which tax policy to follow. Of course under certain tax structures, it can be the case that not only do ϕ and $\hat{\phi}$ yield the same asset demands but they also result in the same optimal tax rates. Then as long as R does not change, it will not matter whether ϕ or $\hat{\phi}$ is the true utility function. But what if the government expects the return matrix R to change to R'? Based on R', ϕ and $\hat{\phi}$ will no longer, in general, yield the same asset demand functions and no longer determine the same tax rates. Once again, the government will be in the position of not knowing which tax policy to follow.

Appendix

Proof of Lemma 1: Since all we want to show is that Int Q_α is not empty, it will suffice for us to do so with regard to the following subsets of contingent commodity vectors:

$$C_\alpha =_{def} \{c_\alpha \in \mathbb{R}^m | c_\alpha = R_\alpha x \text{ where } x \in \mathbb{R}^m_+\} ,$$

$$C^*_\alpha =_{def} \{c_\alpha \in \mathbb{R}^m | c_\alpha = R_\alpha x \text{ for some } x \in \mathbb{R}^m_{++}\} , \text{ and}$$

$$C^* =_{def} \{c_\alpha = (c_\alpha, c_\beta) \in \mathbb{R}^n | c_\alpha \in C^*_\alpha \text{ and } c_\beta = A^t c_\alpha\} .$$

By Assumption [R], C^*_α is a non-empty, open cone in R^m_{++}. Furthermore, if $c_\alpha \in C^*_\alpha$, then by definition of the matrix A^t we have that

$$c_\beta = A^t c_\alpha = R_\beta R_\alpha^{-1} c_\alpha = R_\beta R_\alpha^{-1} R_\alpha x = R_\beta x \Rightarrow c_\beta \in \mathbb{R}^{n-m}_{++} .$$

Consequently, for any $c_\alpha \in C^*_\alpha$, substitution into the first-order conditions (5) yields a unique $(q_\alpha, \lambda, \mu) \in \mathbb{R}^{m+1+(n-m)}$ which supports $c = (c_\alpha, c_\beta) \in C^*$. Existence follows from the concavity of u (and hence the quasiconcavity of ϕ), while uniqueness follows from the differentiability of u. Furthermore, the assumed strict concavity of the NM index u implies that if $c_\alpha \neq c'_\alpha$, $q_\alpha \neq q'_\alpha$ where q_α and q'_α support c_α and c'_α, respectively.

Next, observe that from the first-order conditions (5) and the fact that $u' > 0$ everywhere on $\mathbb{R}_{++}$, $\mu \in \mathbb{R}^{n-m}_{--}$. To see that $\lambda \in \mathbb{R}_{++}$, begin by premultiplying the first equation in (5) by c^t_α, where $c_\alpha \in C^*_\alpha$:

$$0 = c^t_\alpha D_\alpha \phi - c^t_\alpha q_\alpha \lambda - c^t_\alpha A\mu \Rightarrow \lambda = \frac{[c^t_\alpha D_\alpha \phi - c^t_\beta \mu]}{y} > 0 .$$

To see that Int Q_α is not empty, it suffices for us to show that (for $y \equiv 1$)

the price q_α supporting $c \in C^*$ is a continuously differentiable function $q_\alpha(c_\alpha)$ defined on the open cone C^*_α and that $|Dq_\alpha| \neq 0$. Using the first-order conditions again, $q_\alpha = (c^t_\alpha D_\alpha\phi + c^t_\alpha AD_\beta\phi)^{-1}[D_\alpha\phi + AD_\beta\phi]$ from which it follows that

$$Dq_\alpha = (c^t_\alpha D_\alpha\phi + c^t_\alpha AD_\beta\phi)^{-2}\{[c^t_\alpha D_\alpha\phi + c^t_\alpha AD_\beta\phi][D^2_{\alpha\alpha}\phi + AD^2_{\beta\beta}\phi A^t]$$

$$- [D_\alpha\phi + AD_\beta\phi][c^t_\alpha D^2_{\alpha\alpha}\phi + c^t_\alpha AD^2_{\beta\beta}\phi A^t + (D_\alpha\phi + AD_\beta\phi)^t]\} \quad .$$

Now, $|Dq_\alpha| \neq 0$ iff there exist m linearly independent vectors $\{z_1,\ldots,z_m\}$ such that $(Dq_\alpha)z_i \neq 0$, $i = 1,\ldots,m$. To see that this is indeed the case, observe that for $z \in [D_\alpha\phi + AD_\beta\phi]^L$, $z \neq 0$,

$$z^t(Dq_\alpha)z = (c^t_\alpha D_\alpha\phi + c^t_\alpha AD_\beta\phi)^{-2}z^t[D^2_{\alpha\alpha}\phi + AD^2_{\beta\beta}\phi A^t]z < 0$$

from the assumed strict concavity of u. Since $\dim [D_\alpha\phi + AD_\beta\phi] = m - 1$, it remains only to find one $\hat{z} \notin [D_\alpha\phi + AD_\beta\phi]^L$ such that $(Dq_\alpha)\hat{z} \neq 0$. Let $\hat{z} = [D^2_{\alpha\alpha}\phi + AD^2_{\beta\beta}\phi A^t]^{-1}[D_\alpha\phi + AD_\beta\phi]$, where the inverse exists by the strict concavity of u. Furthermore, $[D_\alpha\phi + AD_\beta\phi]^t\hat{z} \neq 0$ from the negative definiteness of $[D^2_{\alpha\alpha}\phi + AD^2_{\beta\beta}\phi A^t]$ and the fact that $[D_\alpha\phi + AD_\beta\phi] = \lambda q_\alpha \neq 0$, since $0 \notin Q_\alpha$, $\lambda > 0$. Finally, since

$$(Dq_\alpha)z = -(c^t_\alpha D_\alpha\phi + c^t_\alpha AD_\beta\phi)^{-2}[D_\alpha\phi + AD_\beta\phi][(D_\alpha\phi + AD_\beta\phi)^t$$

$$\cdot (D^2_{\alpha\alpha}\phi + AD^2_{\beta\beta}\phi A)^{-1}(D_\alpha\phi + AD_\beta\phi)] \quad ,$$

we have that $(Dq_\alpha)\hat{z} \neq 0$. Q.E.D.

<u>Proof of Proposition 1</u>: First, we demonstrate necessity. From the

first-order conditions (5), it follows that $\mu_s = -\pi_s u'(c_s)$, $s \in \{m+1,\ldots,n\}$, and $\lambda = (1/y)[\sum_{s=1}^{m} c_s \pi_s u'(c_s) - \sum_{s=m+1}^{n} c_s \mu_s]$. Consequently, $\mu_s < 0$ and $\lambda > 0$, and both functions are continuously differentiable. That the consistency condition is satisfied follows immediately from the first-order conditions for a maximum and the stationarity of the NM index u. To see why conditions (2.i) and (2.ii) must hold, observe that $BB^{-1} = I$ implies the following:

$$(D^2_{\alpha\alpha}\phi)(S_{\alpha\alpha}) + q_\alpha v^t_\alpha + Aw^t_\alpha = I \quad ,$$

$$(D^2_{\beta\beta}\phi)(S_{\beta\beta}) - w^t_\beta = I \quad ,$$

$$(D^2_{\alpha\alpha}\phi)v_\alpha + q_\alpha e + Ad = 0 \quad ,$$

$$(D^2_{\beta\beta}\phi)v_\beta - d = 0 \quad ,$$

$$v_\beta = A^t v_\alpha \quad ,$$

$$-w_\beta = I - A^t w_\alpha \quad ,$$

$$S_{\beta\alpha} = S^t_{\alpha\beta} = A^t S_{\alpha\alpha} \quad ,$$

$$S_{\beta\beta} = A^t S_{\alpha\alpha} A \quad .$$

Furthermore, as was noted in the text following equation (8), $e = -D_y\lambda$, $d = -D_y\mu$, $v_\alpha = D_y c_\alpha$, $S_{\alpha\alpha} = (1/\lambda)D^*_{q_\alpha} c_\alpha$ and $w^t_\alpha = -(1/\lambda)D^*_{q_\alpha}\mu$. Finally, since v_α and v_β contain no zero elements under the regularity conditions, the above equations together with (11) imply that $\Delta_\alpha = D^2_{\alpha\alpha}\phi$ and $\Delta_\beta = D^2_{\beta\beta}\phi$ and thus $\delta_s = \pi_s u''(c_s) < 0$ (condition (3)). Direct substitution yields the expressions referred to as conditions (2.i) and (2.ii).

To prove sufficiency, we give a constructive argument. Given $c(q_\alpha, y)$,

$\lambda(q_\alpha,y)$ and $\mu(q_\alpha,y)$, we define $e = -D_y\lambda$, $d = -D_y\mu$, $v_\alpha = D_y c_\alpha$, $S_{\alpha\alpha} = (1/\lambda)D^*_{q_\alpha} c_\alpha$ and $w^t_\alpha = -(1/\lambda)D^*_{q_\alpha}\mu$. We can also define $S_{\alpha\beta}$, $S_{\alpha\alpha}$, $S_{\beta\beta}$, v_β and w_β as was done above. By the regularity of the demand function, $D_y c_s = v_s \neq 0$, $s = 1,\ldots,n$ and hence Δ_α and Δ_β can be defined as in (11). Finally, we can set $G = \Delta_\beta w_\beta$. Now all of the entries in the matrix B^{-1} as well as the following matrix have been defined:

$$B^* = \begin{bmatrix} \Delta_\alpha & 0 & -q_\alpha & -A \\ 0 & \Delta_\beta & 0 & I \\ -q^t_\alpha & 0 & 0 & 0 \\ -A^t & I & 0 & 0 \end{bmatrix} .$$

We want to show that $B^* = (B^{-1})^{-1} \equiv B$ by construction. To check that the diagonal elements of B^*B^{-1} are all equal to unity observe that $\Delta_\alpha S_{\alpha\alpha} + q_\alpha v^t_\alpha + Aw^t_\alpha = I$ from condition (2.i) in the proposition while $\Delta_\beta S_{\beta\beta} - w^t_\beta = I$ from condition (2.ii). That $q^t_\alpha v_\alpha = 1$ follows from the fact that $q^t_\alpha c_\alpha(q_\alpha,y) = y$. Finally, the definition of w_β implies that $A^t w_\alpha - w_\beta = I$. Proceeding in the same fashion, one can readily show that all of the off diagonal elements are equal to zero.

Consider next the family of functions $\{\delta_s(q_\alpha,y) | s = 1,\ldots,n$ and $(q_\alpha,y) \in \text{Int } Q_\alpha \times \mathbb{R}_{++}\}$. Composing these functions with the inverse demand relation yields the family $\{\delta_s(c_\alpha) | s = 1,\ldots,n$ and $c_\alpha \in C^*_\alpha\}$ where as in the proof of Lemma 1, we define $C^*_\alpha \equiv \{c_\alpha \in \mathbb{R}^m_{++} | c_\alpha = R_\alpha x,\ x \in \mathbb{R}^m_{++}\}$. But it then follows from the consistency condition and the definition of $\delta_s(q_\alpha,y)$ (in terms of the functions λ and μ) that $\delta_s(c_\alpha)$ is a well defined continuous function of c_s, $s = 1,\dots,n$. Integrating twice, we obtain a family of twice continuously differentiable functions $u_s(c_s)$ such that $u_s''(c_s) = \delta_s(c_s)$ and the objective function defined on $C^* \equiv \{(c_\alpha, c_\beta) | c_\alpha \in C_\alpha^*$ and $c_\beta = A^t c_\alpha\}$ by

$$\phi(c_1,\dots,c_n) = \sum_{s=1}^{n} u_s(c_s)$$

generates the observed demand function. To complete the proof, it remains to be shown that $u_s(c_s)/\pi_s = u_{s'}(c_{s'})/\pi_{s'}$ for all $c_s = c_{s'}$, and hence $\phi(c_1,\dots,c_n) = \sum_{s=1}^{n} \pi_s u(c_s)$, where $0 < \pi \leq 1$ and $\sum_{s=1}^{n} \pi_s = 1$. But this follows from the consistency requirement. Q.E.D.

<u>Proof of Proposition 2</u>: Let $x(p,y)$ be the asset demand function generated by the twice continuously differentiable, strictly monotone ($D\phi >> 0$) and strictly quasiconcave ($D^2\phi$ is negative definite on $[D\phi]^L$) representation ϕ. Define

$$\phi^*(c) =_{def} \phi(c) + \eta\xi^t c \quad ,$$

where η is a scalar and ξ lies in the kernel of the return matrix R and is non-zero (as long as $m < n$, there exists at least one $\xi \in \mathbb{R}^n$ different from zero, such that $\xi^t R = 0$). By an appropriate choice of η, ϕ^* will fail to have an everywhere positive gradient. On the other hand, the induced objective function for assets $\phi^*(Rx) = \phi(Rx) + \eta\xi^t Rx \equiv \phi(Rx)$ since for any choice of asset holdings, $x \in \mathbb{R}^m$, the resulting contingent commodity vector, $c = Rx$, satisfies $\xi^t c = 0$. Thus, the demand function generated by the new representation ϕ^*, denoted $x^*(p,y)$ is identical to $x(p,y)$ which was generated by $\phi(c)$.

Concerning quasiconcavity, or risk aversion, an analogous argument applies. Let

$$\phi^{**}(c) = \phi(c) + \tfrac{1}{2}\eta(c)[c^t(\xi\xi^t)c] \quad ,$$

where ξ is again a non-zero vector in the kernel of R while η is a positive twice continuously differentiable function. Then

$$D\phi^{**} = D\phi + \tfrac{1}{2}D\eta(c)[c^t(\xi\xi^t)c] + \eta(c)[c^t(\xi\xi^t)] \quad ,$$

while

$$D^2\phi^{**} = D^2\phi + \tfrac{1}{2}D^2\eta(c)[c^t(\xi\xi^t)c] + 2D\eta(c)[c^t(\xi\xi^t)] + \eta(c)[\xi\xi^t].$$

Consequently, by appropriate choice of the function $\eta(c)$, $D^2\phi^{**}$ will fail to be negative semi-definite on $[D\phi^{**}]^{\perp}$ everywhere on $\mathbb{R}^n_{++}$. At the same time, $\phi^{**}(Rx) = \phi(Rx) + \eta(Rx)[x^tR(\xi\xi^t)Rx] = \phi(Rx)$, and hence ϕ^{**} generates the same asset demand function as does ϕ. Q.E.D.

<u>Proof of Proposition 3</u>: Let $x(p,y)$ be the asset demand function generated by the NM representation $\phi(c_1,\ldots,c_n) = \sum_{s=1}^{n} \pi_s u(r_s^t x)$, satisfying (12). As before, since $m < n$, we can find $\xi \in \mathbb{R}^n$, $\xi \neq 0$, such that $\xi^t R = 0$, and hence $\xi^t c = 0$ whenever $c = Rx$, any $x \in \mathbb{R}^m$. Let $\eta(\alpha)$ be a twice continuously differentiable function defined on $\mathbb{R}_{++}$, such that $\eta(\alpha) \equiv \eta'(\alpha) \equiv \eta''(\alpha) = 0$ for $\alpha \notin [\underline{\alpha}, \bar{\alpha}]$, where $0 < \underline{\alpha} < \bar{\alpha}$ are fixed constants; that is, the function $\eta(\alpha)$ vanishes outside a non-degenerate compact interval $[\underline{\alpha},\bar{\alpha}]$. Consider now the objective function $\hat{\phi}(c_1,\ldots,c_n) = \Theta\phi(c) + \eta(\|c\|)[\xi^t c]^k$ where $\|c\|$ is the Euclidean norm $\Theta > 0$ and k is an integer ≥ 3. First observe that even though $\hat{\phi}(c)$ is not (ordinally) additively separable--and hence not NM--it does generate the asset demand function $x(p,y)$ for the

existing assets. Second, the following argument shows that for Θ large, $\hat{\phi}(c)$ inherits the strict monotonicity $(D\hat{\phi}(c) >> 0)$ and strict concavity $(D^2\phi(c)$ negative definite) of $\phi(c)$ on $\mathbb{R}^n_{++}$ -- it is clearly twice continuously differentiable. Since for $\|c\| \not\in [\underline{\alpha},\bar{\alpha}]$, $D\hat{\phi}(c) = D\phi(c)$ and $D^2\hat{\phi}(c) = D^2\phi(c)$, we only need to consider $c \in \hat{C} \equiv \{c \in \mathbb{R}^n_{++} | \|c\| \in [\underline{\alpha},\bar{\alpha}]\}$. But $\hat{C}$ is bounded from above. Since the functions $\partial\phi/\partial c_s$, $\partial^2\phi/\partial c_s^2$ $s \in \{1,\ldots,n\}$ are continuous and since $\partial\phi/\partial c_s > 0$ while $\partial^2\phi/\partial c_s^2 < 0$ and both are bounded away from 0 on $\hat{C}$, there exist constants $\gamma > 0$ and $\delta < 0$ such that $\partial\phi/\partial c_s > \gamma$ and $\partial^2\phi/\partial c_s^2 < \delta$ $s \in \{1,\ldots,n\}$ everywhere on $\hat{C}$. Then choosing Θ large guarantees that the first and second derivatives of ϕ will dominate those of $\eta(\|c\|)[\xi^t c]^k$, and hence $\hat{\phi}$ will satisfy $D\hat{\phi} >> 0$ and $D^2\phi << 0$ on $\hat{C}$, and hence on $\mathbb{R}^n_{++}$. Q.E.D.

<u>Proof of Proposition 4</u>: It suffices to demonstrate that $c(q_\alpha,y)$ determines $u''(\cdot)/u'(\cdot)$ everywhere on $\mathbb{R}_{++}$. The existence of a riskless asset is central to the argument. We want therefore, first, to derive the equivalent for the constrained contingent commodity problem of the existence of the riskless asset in the corresponding complex security problem. The contingent commodity vector, c, and the vector of complex security holdings, x, are related by the relation $c = Rx$. Furthermore, given the decomposition of the contingent commodity vector $c = (c_\alpha,c_\beta)$ and the return matrix $R = [R_\alpha, R_\beta]$ defined before, $c_\alpha = R_\alpha x$ and $c_\beta = R_\beta x$. Since a riskless asset exists, and since R has full column rank, there exists a unique x^* such that $Rx^* = e$ -- the unit vector in $\mathbb{R}^n$. But this implies that $e_\alpha = R_\alpha x^*$ and $e_\beta = R_\beta x^*$, where e_α and e_β are the unit vectors in $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$, respectively. But then, $x^* = R_\alpha^{-1} e_\alpha$,

$e_\beta = R_\beta R_\alpha^{-1} e_\alpha$ and, since $A^t = R_\beta R_\alpha^{-1}$, $e_\beta = A^t e_\alpha$, or $e_\beta^t = e_\alpha^t A$. For any $\bar{c} > 0$ choose $(\bar{q}_\alpha, \bar{y})$ such that $c(\bar{q}_\alpha, \bar{y}) = \bar{c}e$ where $e^t = (e_\alpha^t, e_\beta^t) = (1,\ldots,1)$. Since the demand function is observable, so is the inverse demand function $q_\alpha(c)$ on the range of the demand function and, in particular on C^* (as defined in the proof of Lemma 1). From the existence of a riskless asset and the first-order conditions, (5), we can derive the following system of equations:

$$\sum_{s=1}^{n} \pi_s u'(c_s) = \lambda\left(\sum_{s=1}^{m} q_s\right)$$

$$\pi_1 u'(c_1) + \sum_{s'=m+1}^{n} a_{1s'} \pi_{s'} u'(c_{s'}) = \lambda q_1 \quad .$$

In order to derive these two equations, we begin by premultiplying the first two expressions in (5) by the vector $e^t = (e_\alpha^t, e_\beta^t)$ of units in $\mathbb{R}^n$ which yields

$$e_\alpha^t D_\alpha \phi + e_\alpha^t D_\beta \phi - \lambda e_\alpha^t q_\alpha - e_\alpha^t A\mu + e_\beta^t \mu = 0 \quad .$$

Using the fact that $e_\alpha^t A = e_\beta^t$, by the existence of a riskless asset, gives

$$\sum_{s=1}^{n} \pi_s u'(c_s) - \lambda\left(\sum_{s=1}^{m} q_s\right) = 0 \quad .$$

The second equation in the above system is obtained from (5) by substituting $-\pi_{s'} u'(c_{s'})$ for $\mu_{s'}$, $s' = m + 1,\ldots,n$.

By the positivity of u' on $\mathbb{R}_{++}$ and the positivity of λ everywhere on the Int $Q_\alpha \times \mathbb{R}_{++}$, we can derive from the above system of equations the expression for the marginal rate of substitution between the riskless portfolio and the state 1 contingent consumption:

$$\frac{q_1}{\sum_{s=1}^{m} q_s} = \frac{\pi_1 u'(c_1) + \sum_{s'=m+1}^{n} a_{1s'}\pi_{s'}u'(c_{s'})}{\sum_{s=1}^{n} \pi_s u'(c_s)} \quad .$$

The LHS of this equation is observable from the inverse demand function. To compute the value of $u''(\bar{c})$, we differentiate both sides of

$$q_1[\sum_{s=1}^{n} \pi_s u'(c_s)] = (\sum_{s=1}^{m} q_s)[\pi_1 u'(c_1) + \sum_{s'=m+1}^{n} a_{1s'}\pi_{s'}u'(c_{s'})]$$

with respect to c_1 and evaluate at $c = \bar{c}e$. The resulting equation is linear in $u''(\bar{c})$ with coefficient

$$z(\bar{c}) \equiv (\sum_{s=1}^{m} q_s)[(\pi_1 + \sum_{s'=m+1}^{n} \pi_{s'}a_{1s'})^2 - (\pi_1 + \sum_{s'=m+1}^{n} \pi_{s'}a_{1s'}^2)] \quad .$$

Since $m \geq 2$, $z(\bar{c}) \neq 0$ and hence $u''(\bar{c})$ can be recovered without ambiguity.

As was pointed out earlier, the existence of a riskless asset has enabled us to eliminate λ and μ and thus the ambiguity involved in their choice. This is precisely the motivation underlying construction of the ratio $q_1/\sum_{s=1}^{m} q_s$ -- it does not involve λ and μ given that some asset is riskless.

Q.E.D.

<u>Proof of Proposition 5</u>: It suffices to give an example where recoverability fails. Let there be four equiprobable states and two complex securities and assume

$$R^t = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \varepsilon^t = [0,\pi,0,\pi]$$

(where π here does not denote probability). Then the expected utility functions $\sum_{s=1}^{n} \pi_s u(c_s)$ and $\sum_{s=1}^{n} \pi_s \{u(c_s) + k \sin (c_s)\}$, where k is a scalar, generate indistinguishable asset demand functions, and hence constrained contingent commodity demand functions as well. Observe that the structure of the random income component prevents reduction of the state space to just two points, as one might be tempted to do after observing the return matrix, R.

Q.E.D.

Footnotes

1/ The connection between additive utility functions and additive preferences is examined in Debreu [4].

2/ All vectors are column vectors. A superscript "t" denotes the transpose. Let z be some vector and z^i a component of z. Then $z \geq 0$ means $z^i \geq 0$ for every i, $z > 0$ means $z \geq 0$ and $z \neq 0$ and $z >> 0$ means $z^i > 0$ for every i. $\mathbb{R}^n_+ = \{z \in \mathbb{R}^n | z \geq 0\}$ and $\mathbb{R}^n_{++} = \{z \in \mathbb{R}^n | z >> 0\}$.

3/ For a fairly comprehensive treatment of preferences over the space of random variables, see Rossman and Selden [20].

4/ We shall use $D\phi(c)$ for the vector

$$\left(\frac{\partial \phi}{\partial c_1}(c_1,\ldots,c_n),\ldots,\frac{\partial \phi}{\partial c_s}(c_1,\ldots,c_s,\ldots,c_n),\ldots,\frac{\partial \phi}{\partial c_n}(c_1,\ldots,c_n)\right)$$

and analogously $D^2\phi(c)$ for the matrix of second (partial) derivatives.

5/ On the notion of "state independent" preferences, see Malinvaud [13, p. 285] and Rossman and Selden [20].

6/ See Rossman and Selden [19] on the relation between Hicksian and Slutsky compensation under uncertainty.

7/ It is important to stress that the regularity conditions relate to the demand for contingent commodities and not assets. Imposing (iii) on contingent commodity demands does not, of course, imply that it must hold as well for complex securities. In the case of incomplete markets, the requirement (iii) that the marginal propensity to consume for each contingent commodity be non-zero everywhere represents an additional implicit restriction on both the asset return structure and preferences. As can be seen from Eq. (11) in the text, a violation of (iii) causes $\delta_s(q_\alpha, y)$ (which corresponds to $\pi_s u''(c_s)$ in the recovery of the utility function in the proof of Proposition 1) not to be well-defined. We are indebted to the referee for his clarifying comments on this point.

8/ For any vector $z \in \mathbb{R}^k$, diag (z) denotes the corresponding $(k \times k)$ diagonal matrix.

9/ It should be noted that while the functions (λ,μ) are not immediately observable, one can directly verify the existence of Lagrange multipliers (λ,μ) satisfying the required demand restrictions (1)-(3) in Proposition 1 from the observable constrained contingent commodity demand functions.

10/ See, for example, Cass and Stiglitz [2].

11/ Note that this formulation of the agent's problem does not necessarily presume the existence of a riskless asset.

12/ Given the non-uniqueness of the generating representation in incomplete markets, an individual agent's observable asset demand behavior (even assuming complete knowledge of (R,π)) can hardly be viewed as very "informing". Moreover, it is unlikely that matters can be improved by summing over agents. Thus under incomplete markets, aggregate demands or equilibrium prices transmit little if any information concerning investor tastes.

13/ As shown in Rossman and Selden [19], Slutsky income compensation (which seeks to restore the individual to his original optimal allocation) is not generally possible for perfectly standard changes in the random return on assets. We here, of course, are concerned with the alternative Hicksian notion of compensation (which seeks to restore the individual to his original optimal level of utility).

14/ More generally, the asset prices p_1 and p_2 can not be specified exogenously since they will be affected by the optimal tax structure (T^*, t^*). However given the limited purpose of our example, there would seem to be little reason to introduce this additional complexity.

15/ Note that if $p = \frac{1}{2}$, $B = 1/12$ and $\alpha = \frac{1}{3}$, straightforward computation results in (T^*,t^*) equalling $(7/12, 5/6)$ for the NM case of (16) and (17) and $(7/16, 11/16)$ for the non-NM case of (18) and (19) and in both cases $\phi(T)$ and $\hat{\phi}(T)$ are strictly concave over $[0,1]$. The reader will also observe that if $\alpha = \frac{1}{2}$, then (18) and (19) just happen to reduce exactly to (16) and (17).

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