# Welfare Analysis of Dark Pools[*]

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**Abstract**

We investigate the welfare implications of operating alternative market structures known as electronic crossing networks or "dark pools" alongside traditional "lit" markets. We study equilibria of a market where intrinsic traders and speculators, endowed with heterogeneous fine-grained information, endogenously choose between dark and lit venues. We establish that while the dark pool attracts relatively uninformed investors, the orders therein experience adverse selection. Moreover, the informational segmentation created by a dark pool leads to greater transaction costs in the lit market. Taken together, we conclude that there exist reasonable parameter regimes where introducing a dark pool *decreases* the overall welfare.

# Disclosure Statement for Krishnamurthy Iyer

I have nothing to disclose.

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# Disclosure Statement for Ramesh Johari

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# 1.  Introduction

Crossing networks, more commonly known as dark pools, are a market mechanism that seeks to directly facilitate trade between buyers and sellers of an asset outside of traditional exchanges or dealer markets. Dark pools operate by having buyers and sellers submit orders that are hidden from the overall market. Trades occur from direct matches between orders in the pool with complementary trading needs. One touted advantage of dark pools is that they can often directly match natural buyers with natural sellers of an asset without intermediating market makers as in a dealer market, or with market orders on an exchange. Thus, in the absence of transaction costs charged by liquidity providers, trade can be cheaper in a dark pool. On the other hand, trade is uncertain in a dark pool: *a priori*, an agent faces a risk that their order will not be matched (or, "filled"), and they may face more disadvantageous prices at a later time. In this way, dark pools offer investors a choice that exposes a fundamental trade-off between *price* and *uncertainty of trade.*

Dark pools have grown in popularity as a financial market mechanism in recent years. Currently, in the U.S. equity market, there are 40+ dark pools[1] handling approximately one in seven trades.[2] Recently, however, these alternative markets have received significant attention from regulators and prosecutors. One focus of this attention is overt fraud, such as possible misrepresentations regarding the operations of these opaque and largely unregulated entities. However, beyond this, there has also been a worry that dark pools may negatively impact the broader market in more subtle ways, e.g.,[3]

> *A rise in off-exchange trading could hurt investors … The reason: With more investors trading in the dark, fewer buy and sell orders are being placed on exchanges. That can translate into worse prices for stocks, because prices for stocks are set on exchanges.*

In the present paper, we explore this phenomenon and study the interactions between contingent trade in a dark pool and certain trade in a lit market, with a particular view toward characterizing welfare effects.

To understand the trade-off between price and execution risk, a critical element is the role of *information*, specifically, private, asymmetric information possessed by market participants as to the short-term value of an asset. The role of information in the analysis of market microstructure has been a longstanding topic of study, dating back to the seminal work of Glosten and Milgrom (1985) and Kyle (1985). In our setting, information is important for

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[1]Ganchev et al. (2010).

[2]S. Patterson. Regulator probes dark pools. *The Wall Street Journal*, February 15, 2013.

[3]Ibid.

two fundamental reasons: First, if an agent has information or beliefs about short-term price changes, this will clearly impact their individual decision-making. For example, an agent might justify paying a premium for certain trade given commensurate certainty in the value of the asset. Second, however, even in the absence of any such information, an agent needs to reason about the informational characteristics of those with whom they trade. In considering contingent trade, agents with less information about the value of an asset face *adverse selection* or a "winner's curse": by systematically trading with more informed investors, trade occurs in situations where it is in fact *ex post* undesirable to the agent. Not surprisingly, informational considerations affect all aspects of market operation, from trader behavior to the profitability of market making. Given the complex strategic interactions involved, qualitative insights and operational guidelines can be elusive.

In this paper, we address this challenge. Specifically, we consider a stylized, one period model, where a continuum of infinitesimal, risk-neutral agents can choose to trade an asset. Agents seeking to buy or sell an asset can do so on the *open market*. Here, they may trade the asset immediately and with certainty by trading with market-making intermediaries (i.e., a traditional, competitive dealer market), in exchange for paying a premium in the form of a transaction cost (i.e., the bid-offer spread). Alternatively, agents may choose to trade in a *dark pool* market where they are directly matched with other agents. Trade in the dark pool occurs by reference to the mid-market price in the open market, and hence does not incur a transaction cost.[4] On the other hand, a dark pool presents a risky opportunity for trade: if there is a mismatch between the overall populations of buyers and sellers in a dark pool, then a subset of the agents' orders will not be filled.

Agents make this decision based on private information as to the short-term future price of the asset (i.e., the common value or fundamental value), as well as their own intrinsic demand for the asset (i.e., their individual hedging demand or idiosyncratic value). A central contribution of the paper is to allow a rich and high resolution model of information. All

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[4]Note that one claimed benefit of dark pools is that they mitigate the pre-trade information leakage that might arise from the placement of a large order in the open market. In a typical dark pool, only trades that occur are reported to the broader market — the underlying orders are confidential — and these are only reported after the fact. In this paper, we do not focus on this aspect of dark pools. While the information leakage of a large execution is an important issue in electronic markets, this is typically managed by spreading out a large order into smaller orders across time (see, e.g. Moallemi et al., 2012), rather than executing such large orders on a dark pool. Indeed, in U.S. equity markets, order and trade sizes in a typical dark pool are largely comparable to those in lit markets (Tuttle, 2013). This suggests that information leakage from large orders is a second order effect in the choice of trade venue.

Similarly, we consider a stylized, "crossing network" view of dark pools as venues where trade occurs only through mid-point matches with reference to a lit market. Strictly speaking, many dark pools are internally organized as (hidden) electronic limit order books, and more complex mechanisms for trade are possible. Our focus here is on the trade-off between price and execution risk that dark pools present, and our modeling choices are made to highlight this trade-off while maintaining tractability.

agents in our setting possess private information as to the asset value. However, their private signals are heterogeneous and vary in strength across agents.

Agents are also endowed with an idiosyncratic value for the asset. In this regard, there are three classes of agents: *intrinsic buyers*, with positive idiosyncratic value, *intrinsic sellers*, with negative idiosyncratic value, and *speculators*, with no idiosyncratic value — the latter class seek only to maximize their expected wealth. Note that speculators trade only on the basis of the private information they possess. Thus by varying the mass of speculators present in the market, our model admits exogenous variation of the level of information present in the market.

Our equilibrium concept involves a Bayes-Nash equilibrium among the agents, with transaction costs in the open market set through a zero profit condition (i.e., competitive market makers). In our analysis, we compare two market structures: (1) one where agents choose between the dark pool and the open market; and (2) one where agents can trade only in the open market.

We derive several analytic and numerical insights:

1. **Information segmentation.** We find that, in general, traders are segmented by their level of information. More precisely, in equilibrium, when all else is equal, the dark pool is utilized by relatively uninformed or mildly informed traders, whereas highly informed traders will trade in the open market so as to exploit their short-term information through guaranteed profitable executions. This implies that, via an information segmentation mechanism, trade in the dark pool will alter the informational characteristics of trade in the open market.

2. **Adverse selection.** We establish that, in equilibrium, traders in the dark pool experience adverse selection. Specifically, conditional on their order being filled in the dark pool, a buyer's (resp., seller's) expectation of the asset's fundamental value is lower (resp., higher) than their prior, unconditional expectation. This arises from the fact that the execution of an order in the dark pool is correlated with the fundamental value of the asset in way that is to the detriment of most dark pool participants: buyers in the dark pool are more likely to be filled when the dark pool price is above the fundamental value, while sellers are more likely to be filled when the dark pool price is below. The presence of speculators in the market further exacerbates this effect.

   However, unlike adverse selection in other settings, this detrimental correlation cannot be explained by the traditional mechanism of information asymmetry: trade with a highly informed counterparty. Indeed, as mentioned earlier, the more informed traders trade in the open market. Instead, in our model, adverse selection is created through

4

the aggregate behavior of the *group* of investors participating in the dark pool. These investors are all relatively uninformed, but in the case where the fundamental value is higher (resp., lower) than the dark pool transaction price, there are more investors with a slight positive (resp., negative) signal than the opposite. In other words, adverse selection endogenously occurs in the dark pool through the aggregation of diffuse information from a cross section of marginally informed agents.

3. **Welfare.** The tractability of our model allows us to provide significant insights regarding the welfare effect of a dark pool. Intuitively, in our setting, monetary transfers are zero-sum when viewed in aggregate from a systemic perspective — the gain of one agent is the loss of another. Transfers of the asset, however are not: the sale of the asset from an agent with low idiosyncratic value to an agent with high idiosyncratic value can be a Pareto improvement. Welfare quantifies such gains from trade.

   A naive view of the introduction of the dark pool suggests that welfare can only increase. After all, traders have an additional venue for trade, and more choice should imply higher welfare. Of course, this naive view ignores the role of strategic interactions, and the effect of the introduction of a dark pool is significantly more subtle when these interactions are considered.

   Indeed, one of our most striking conclusions is that in reasonable parameter regimes, the introduction of a dark pool can actually *decrease* overall welfare. Intuitively, to see why this occurs, note that with the introduction of a dark pool, agents are facing a choice of two transaction costs: the explicit spread in the open market, and the implicit adverse selection cost in the dark pool. We show that the introduction of a dark pool can cause the transaction cost in the open market to *rise* — a direct outcome of the strategic choices made by traders and the market maker in the presence of the dark pool. (Since the most informed traders head to the open market, the market maker widens the spread to combat their informational advantage.) At the same time, the adverse selection cost to traders who transact in the dark pool is shown to be high. The combination of these effects leads to a welfare loss, as more intrinsic traders avoid trade in the presence of a dark pool.

   We do note that welfare losses may not obtain for all values of the model parameters. As one example, we demonstrate that when the mass of speculators increases, the introduction of a dark pool may not materially alter the transaction cost in the open market. Though the adverse selection cost in the dark pool remains high, if intrinsic buyers and sellers have a sufficiently high idiosyncratic value for the asset, then the introduction of the dark pool increases welfare. In this particular regime, the naive

intuition described above is approximately correct: the dark pool functions as a complementary venue of trade for the (highly motivated) intrinsic traders. However, we note numerically that this region is sensitive to the presence of a sufficiently large mass of speculators, as well as a sufficiently large idiosyncratic value for intrinsic traders: in the absence of either condition, we find that the dark pool decreases welfare.

In terms of testable implications, many of our model predictions are consistent with observations from the empirical literature on dark pool trading. The information segmentation of relatively uninformed trades to dark pools has been observed (Comerton-Forde and Putniņš, 2015), nevertheless dark pool trades do have informational content (Nimalendran and Ray, 2014) and there can be substantial adverse selection costs when trading in a dark pool (Næs and Ødegaard, 2006). Since we identify two parameter regimes, one where dark pool trading is welfare decreasing and the other is welfare increasing, a natural and important empirical question is to ask which regime real world financial markets fall under. As is the case with many of the theoretical models in the literature, our model is stylized, and many of the parameters (e.g., idiosyncratic values) cannot be directly calibrated. That said, the parameter regime in which welfare decreases in our model largely corresponds to cases where the introduction of dark pools results in an increase in open market transaction costs, and this latter phenomenon has been observed in a number of studies (e.g., Comerton-Forde and Putniņš, 2015; Degryse et al., 2014; Foley et al., 2012). As a result, our model suggests that today's markets are best captured by the parameter regime where the introduction of the dark pool reduces welfare overall.

While others have developed theoretical models for dark pools (e.g., Hendershott and Mendelson, 2000; Ye, 2011; Zhu, 2014), to our knowledge, ours is the first model offering concrete theoretical welfare implications for dark pool trading in the presence of asymmetric information. Our results are dependent on a number of technical assumptions and our model is stylized. However, it does offer evidence that regulators are rightly concerned about the real world welfare impact of dark pools.

## 1.1. Literature review

Our underlying open market setting here is reminiscent of the model of Glosten and Milgrom (1985) for studying dealer markets with asymmetric information. In this way, our model shares some similarities with Zhu (2014), who also builds a information-based framework for studying the relationship between an open market and a dark pool. However, there are key differences between our work and his; Zhu (2014) considers a coarse model of information where traders are either fully informed or completely uninformed, and seeks to

understand the impact of a dark pool on incentives for the costly acquisition of information. Consistent with our results, he establishes that the dark pool attracts uninformed traders, and that this may reduce liquidity in the open market. Ultimately, however, Zhu (2014) focuses on the role of dark pools in price discovery, but without insight into the welfare implications. To contrast, our work does not speak to price discovery and instead focuses on welfare. Ye (2011) develops a model for deciding between an open market and a dark pool in the setting of Kyle (1985), but allows for only a single, fully informed trader with an endogenous choice of mechanism, and the model is limited by the fact that uninformed traders exogenously choose a venue. Hendershott and Mendelson (2000) consider a model where the fully informed traders have an exogenously specified strategy. Baldauf and Mollner (2015) consider a related but different problem where asymmetrically informed traders choose between multiple exchanges.

A number of other authors develop theoretical models for dark pools in the absence of asymmetric information as to the asset value (e.g., Dönges and Heinemann, 2006; Degryse et al., 2009; Afèche et al., 2014). Notably, Buti et al. (2014) model a dark pool that operates in parallel with a limit order book, and make welfare predictions in a symmetric information setting. Their model considers a trade-off between execution risk and cost that is similar to that in the present paper. However, as their paper is in a symmetric information setting, the authors pose it as a challenge to understand the impact of dark pools in the presence of asymmetric information, an important driver of real world financial markets. Our paper resolves this challenge, as our primary concern is to precisely understand how asymmetric information and adverse selection impact welfare in the presence of a dark pool.

Dark pools have also been studied in the optimal execution literature (e.g., Ganchev et al., 2010; Klöck et al., 2011; Kratz and Schöneborn, 2013, 2014), where the goal is to formulate and solve an individual agent's decision problem of how to trade in order to efficiently liquidate a large portfolio. In such settings, however, the behavior of other agents in the market is described through non-strategic, reduced form specifications. Hence, such models are complementary to our work: their models are not meant to be used to reason about market structure counterfactuals.

## 2.  Model

We study a market organized for trading shares of a single security. We assume two types of marketplaces exist to conduct trade: (1) an intermediated *open market* and (2) a *dark pool* market. A continuum of traders decide, based on their private information, whether to trade in the open market or to enter the dark pool. In this section, we describe the asset, the

marketplaces, the intermediating market maker, the traders' types, and their private signals, utilities, and strategies.

## 2.1. Asset

A single security is traded in the market at time $t = 0$. The common value of the security at $t = 1$ is unknown, and we model the uncertainty as a random variable $\sigma$. More specifically, we assume that $\sigma$ takes values in the set $\{-1, +1\}$, with either value equally likely. This captures, in a stylized manner, the notion that the security value may undergo either a positive or a negative jump of equal magnitude, and that the prior belief about this change in value is uninformed. We assume that all agents in the market are risk-neutral, and hence (ignoring idiosyncratic value and private information effects to be discussed shortly) the mid-market value of the security at time $t = 0$ is zero. We assume that $\sigma$ is fully revealed to all agents in the market at time $t = 1$.

## 2.2. Traders

The market has a continuum of infinitesimal traders, each seeking to buy or sell at most one share of the security at time $t = 0$. Each trader $i$ is characterized by an idiosyncratic value $v_i$, that, along with the common value $\sigma$, determines the value the trader attaches to a single unit of the security. More precisely, we assume that, at time $t = 1$, a trader with idiosyncratic value $v_i$ values the security at $\sigma + v_i$. Here, the idiosyncratic value $v_i$ can capture, for example, the hedging demand particular to trader $i$.

We fix a parameter $V \in (0, 1]$ and we assume the idiosyncratic value $v_i$ can take one of three values: $+V$; $-V$; or zero. Our model admits and results hold for more general distributions of the idiosyncratic value; we adopt this formulation to make the analysis and the exposition simpler. In particular, depending on their idiosyncratic value, we characterize the traders into three groups: (1) *intrinsic buyers*, i.e., those traders with a positive idiosyncratic value $(v_i = +V)$; (2) *intrinsic sellers*, i.e., those with a negative idiosyncratic value, $(v_i = -V)$; and (3) *speculators*, i.e., those with a zero idiosyncratic value $(v_i = 0)$.

Intuitively, in the absence of any private information, intrinsic buyers (resp., sellers) arrive at the market with the inclination to buy (resp., sell) one unit of the security. We emphasize that this initial inclination may get altered due to their private information (which we describe below). On the other hand, for a speculator, their motivation to trade arises only out of their private information. As a result, by varying the mass of speculators in the market (relative to the mass of intrinsic traders), we are able to parametrically vary the amount of information entering the market.

8

Finally, we assume that the mass or measure of intrinsic buyers and sellers is equal and normalized to 1, while the mass of speculators is $\mu \geq 0$. Thus, the total mass of traders in the market is $2 + \mu$.

## 2.3. Marketplaces

The market is composed of two distinct types marketplaces:

1. **Open market**. We envision the open market as an dealer market where, at time $t = 0$, any trader may enter to buy or sell a single unit of the security. The open market is intermediated by a market maker. We assume the market maker is risk-neutral, has no idiosyncratic value for the asset, and has an uninformed prior belief (via-à-vis the private information of Section 2.4) on the common asset value $\sigma$. Therefore, the mid-market value of the asset to the market maker at time $t = 0$ is zero. The market maker charges an additional cost $\delta \in [0, 1]$ over this mid-market value in order to transact. In other words, at time $t = 0$, the security is bid at a price of $-\delta$ and offered at a price of $\delta$ in the open market. (One may also think of $2\delta$ as analogous to the bid-ask spread incurred by the market orders in specialist markets or limit order book markets.) Thus, the open market offers guaranteed immediate execution at the cost of an additional transaction cost $\delta$.

2. **Dark pool**. At time $t = 0$, traders may choose to enter the dark pool to conduct trade by seeking to buy or sell a single unit of the security. The dark pool is a parallel marketplace without intermediating market makers. It clears trades an instant after time $t = 0$, at a price determined by reference to the prevailing mid-market price in the open market at time $t = 0$, i.e., a price of zero. Thus, a trade carried out at the dark pool market does not incur the transaction cost $\delta$ that is imposed in the open market.

   The orders in the dark pool are cleared using a uniform random matching process between the two sides of the market, i.e., those traders seeking to buy and those traders seeking to sell. Each buy order has an equal chance of getting matched with one of the sell orders and vice versa. Thus, if the mass of buy orders in the dark pool is $m_b$ and the mass of sell orders is $m_s$, then when $m_b > m_s$, all sell orders get filled, while only $m_s$ of the buy orders get filled. Moreover, each sell order gets filled independently with probability $m_s/m_b$ — this quantity is known as the *fill rate* for buy orders. The situation is analogous when $m_s > m_b$.

Unless the mass of buy orders exactly equals the mass of sell orders, orders in the dark pool suffer from a risk of non-execution. Thus, a trader, while deciding between the open market and the dark pool, has to balance the trade-off between zero transaction cost and possibility of non-execution.

Each trader in the market is strategic in selecting the marketplace to trade at. Each trader makes the entry and trade decisions based on her idiosyncratic value as well as her private information about $\sigma$. We next describe the private information structure among the traders.

## 2.4. Private information

Recall that the value of the security $\sigma$ at time $t = 1$ is an unknown random quantity at time $t = 0$. We assume that each trader $i$ at time $t = 0$ (before making any strategic decision) receives a private signal $s_i \in \mathcal{S}$ informing her about the common value $\sigma$.

Formally, we assume that the price movement $\sigma$ is distributed according to a common prior $\mathbf{P}$, where

$$\mathbf{P}(\sigma = +1) = \mathbf{P}(\sigma = -1) = 1/2. \tag{1}$$

Further, we assume a *conditionally independent* signal structure: conditional on the price movement $\sigma$, the signals $\{s_i\}$ are distributed independently and identically.[5]

Since the only uncertainly in our model arises from the realization of the common value $\sigma$, which has a Bernoulli distribution, without loss of generality, we assume that each signal $s$ directly represents the posterior probability that $\sigma = 1$, and that the set of possible signals is the unit interval $\mathcal{S} \triangleq [0, 1]$. In other words, for all $s \in [0, 1]$,

$$\mathbf{P}(\sigma = 1|s) = s. \tag{2}$$

Hence, a signal of $s = 1/2$ corresponds to a trader who is uninformed beyond the prior, while more informed traders would see signals closer to 0 or 1. As we will see in what follows, this representation of private information allows us to express traders' utility for various actions as a linear function of their signals, thereby simplifying our analysis.

For tractability of analysis, we further assume that, given $\sigma$, the signals are distributed on $\mathcal{S} = [0, 1]$ according to a distribution $F_\sigma$, with the following cumulative distribution function

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[5]To be precise, since we consider a model with a continuum of traders, a conditionally independent signal structure requires appealing to an exact law of large numbers argument (Qiao et al., 2014). We suppress these technical details for clarity.

(cdf):[6]

$$F_1(x) = x^2, \quad \text{and} \quad F_{-1} = 1 - (1-x)^2, \quad \text{for all } x \in [0,1]. \tag{3}$$

A simple application of Bayes' rule verifies that, given the prior distribution (1), an observation of a signal from the distribution (3) yields the posterior belief (2).

Our model can extended to a one-parameter family of signal distributions indexed by $\kappa > 0$, based on a sub-class of *beta distributions*. In particular, the signal distributions $F_\sigma^\kappa$ are given by

$$F_1^\kappa \sim \mathsf{Beta}(\kappa + 1, \kappa), \quad \text{and} \quad F_{-1}^\kappa \sim \mathsf{Beta}(\kappa, \kappa + 1).$$

The quantity $\kappa$ parameterizes the *informativeness* of traders' signal distribution in the following sense. As $\kappa \to \infty$, the signal distribution becomes more uninformative: $F_\sigma^\kappa$ concentrates around 0.5 irrespective of the value of $\sigma$. On the other hand, as $\kappa \to 0$, the signal distribution becomes more informative: for small enough $\kappa > 0$, $F_1^\kappa$ is concentrated around 1, while $F_{-1}^\kappa$ is concentrated around 0. Note that our choice of signal distribution in (3) corresponds to the case $\kappa = 1$. All our results in Section 4 continue to hold for general values of $\kappa$, while those in Section 5 require $\kappa = 1$.

## 2.5. Trader actions and utility

As the traders are risk-neutral, we assume that they seek to maximize the expected value of their position at $t = 1$ while deciding where and how to trade. We explain in detail below the decision problem faced by a trader at time $t = 0$.

At time $t = 0$, a trader with idiosyncratic value $v$ and private signal $s$, has to decide among the following actions:

1. $(OM, B)$: Buy a share at the open market, yielding an expected utility of

$$u_{OM,B}(v,s) = \mathsf{E}[\sigma + v | s] - \delta = 2s - 1 + v - \delta. \tag{4}$$

2. $(OM, S)$: Sell a share at the open market, yielding an expected utility of

$$u_{OM,S}(v,s) = -\mathsf{E}[\sigma + v | s] - \delta = 1 - 2s - v - \delta. \tag{5}$$

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[6]Here, we abuse the notation $F_\sigma$ to denote both the distribution as well as its cdf.

3. $(DP, B)$: Enter a buy order into the dark pool market, yielding an expected utility of

$$u_{DP,B}(v, s) = \mathsf{E}[(\sigma + v)\mathbf{I}\{\text{buy fill}\}|s], \tag{6}$$

where $\mathbf{I}\{\text{buy fill}\}$ denotes the indicator variable corresponding to the event that the buy order gets filled in the dark pool market.

4. $(DP, S)$: Enter a sell order into the dark pool market, yielding an expected utility of

$$u_{DP,S}(v, s) = -\mathsf{E}[(\sigma + v)\mathbf{I}\{\text{sell fill}\}|s], \tag{7}$$

where $\mathbf{I}\{\text{sell fill}\}$ denotes the indicator variable corresponding to the event that the sell order gets filled in the dark pool market.

5. $N$: Do not trade, yielding an expected utility of $u_N(v, s) = 0$.

We let $\mathcal{A} \triangleq \{(OM, B), (OM, S), (DP, B), (DP, S), N\}$ denote the action set available to a trader. Thus, a trader's decision problem is to choose an action $a \in \mathcal{A}$ that will maximize her expected utility, given her idiosyncratic value and her private signal:

$$\underset{a \in \mathcal{A}}{\text{maximize}} \ \ u_a(v, s).$$

Observe that the expected utility of a trader on entering an order (buy or sell) into the dark pool depends on the likelihood of the order getting filled, which depends on how many other orders are present in the dark pool. This, in turn, depends on the actions of other traders in market. Thus, we see that the traders' decision problems are intricately coupled.

## 2.6. Trader strategies

Having defined the traders' decision problem, we now look at their strategies. A strategy for a trader specifies, for every signal $s \in \mathcal{S}$, the action to take in the market. Formally, a strategy for a trader is a map $\lambda \colon \mathcal{S} \to \mathcal{A}$. (Here and throughout the paper, we only consider *pure* strategies for the market participants.)

We will restrict our attention to the case where all traders with the same idiosyncratic value use the same strategy. In other words, we require the collection of strategies used by the traders to satisfy an *anonymity* condition, where the strategy used by a trader does not depend on her identity. With this condition in place, we denote the strategy of a trader with idiosyncratic value $v$ as $\lambda_v \colon \mathcal{S} \to \mathcal{A}$, and we define a strategy profile $\lambda \triangleq (\lambda_{-V}, \lambda_0, \lambda_V)$ as the tuple of strategies employed by all the traders in the market.

## 2.7.   Competitive market makers

We will be interested in studying market-making in the open market in a state of *perfect competition* with *free entry*. In such a situation, if the market maker made a positive expected profit, competitors would enter the open market and undercut the market maker by reducing the transaction costs. This suggests that in any equilibrium under perfect competition and free entry, the market maker's expected utility should be zero.

Formally, we capture this competitive limit by assuming that the open market is organized by a risk-neutral market maker, with the uninformed prior **P**, who receives the entire transaction cost $\delta$ for each trade in the open market. Along with this revenue, she also faces the risk of adverse selection due to the presence of informed traders. The cost due to this risk depends on the traders' strategies, which in turn depend on $\delta$. We will then assume that the the transaction cost $\delta$ in the open market is set so that the market maker's total expected utility is zero.

# 3.   Equilibrium

From the description of the market model in the preceding section, we observe that the traders' decision problems are coupled due to the order matching process in the dark pool. In particular, a trader's optimal action depends on how many other traders enter orders in the dark pool. For any fixed transaction cost $\delta$, this defines a game among the traders, for which we define a partial equilibrium concept. As described in the preceding section, the transaction cost $\delta$ is then set so that the market maker earns zero profit; we call a resulting solution a competitive equilibrium. In this section we formalize the definitions of both partial and competitive equilibria.

## 3.1.   Partial equilibrium

In this subsection we formulate a partial equilibrium concept for the game among the traders, given a fixed transaction cost $\delta$ in the open market. We first specify in more detail how a trader's utility depends on the actions of all other traders in the market, and in particular, on the *fill rate* (i.e., the fraction of buy or sell orders that are executed) in the dark pool for each realization of the common value $\sigma$. We then introduce the appropriate equilibrium concept for the game among traders: *Bayes-Nash equilibrium* (BNE). In a BNE, each trader's strategy specifies for any signal $s$ an optimal action, holding fixed the other traders' strategies.

### 3.1.1. Fill rates

In order to calculate the fill rates, as described in Section 2.3, we need to know the mass of buy and sell orders in the dark pool. These are determined by the strategies of the traders.

Suppose the traders' strategy profile is given by $\lambda = (\lambda_{-V}, \lambda_0, \lambda_V)$. For each idiosyncratic value $v \in \{0, \pm V\}$ and each action $a \in \mathcal{A}$, we define the set

$$\Theta_{v,a}^\lambda \triangleq \{s \in [0,1] \; : \; \lambda_v(s) = a\}. \tag{8}$$

Thus, $\Theta_{v,a}^\lambda \subset [0,1]$ denotes the set of signal values for which a trader with idiosyncratic value $v$ chooses action $a$ under the strategy profile $\lambda$. In the rest of the paper, we restrict our attention to those strategies for which the preceding sets are $F_\sigma$-measurable.

Conditional on the security value $\sigma$, the mass of buy orders in the dark pool is given by

$$m_B^\lambda(\sigma) = \sum_{v \in \{\pm V\}} F_\sigma(\Theta_{v,(DP,B)}^\lambda) + \mu F_\sigma(\Theta_{0,(DP,B)}^\lambda). \tag{9}$$

Similarly, the mass of sell orders is given by

$$m_S^\lambda(\sigma) = \sum_{v \in \{\pm V\}} F_\sigma(\Theta_{v,(DP,S)}^\lambda) + \mu F_\sigma(\Theta_{0,(DP,S)}^\lambda). \tag{10}$$

Observe that $m_B^\lambda(1)$ is positive if and only if $m_B^\lambda(-1)$ is positive. Similarly, $m_S^\lambda(1)$ is positive if and only if $m_S^\lambda(-1)$ is. This follows from the fact that the measures $F_1$ and $F_{-1}$ are equivalent.

Now, suppose both $m_B^\lambda(\sigma)$ and $m_S^\lambda(\sigma)$ are positive. As a buy order in the dark pool is matched uniformly with a sell order, if $m_B^\lambda(\sigma) \leq m_S^\lambda(\sigma)$, a buy order in the dark pool gets filled with certainty, while a sell order gets filled with probability $m_B^\lambda(\sigma)/m_S^\lambda(\sigma)$. On the other hand, if $m_B^\lambda(\sigma) \geq m_S^\lambda(\sigma)$, a sell order gets filled with certainty, while a buy order gets filled with probability $m_S^\lambda(\sigma)/m_B^\lambda(\sigma)$. Thus for every value of $\sigma$, we can define the buy fill rate $\phi_B^\lambda(\sigma)$ and sell fill rate $\phi_S^\lambda(\sigma)$ as

$$
\begin{aligned}
\phi_B^\lambda(\sigma) &\triangleq \mathsf{E}[\mathbf{I}\{\text{buy fill}\}|\sigma] = \min\left(1, \frac{m_S^\lambda(\sigma)}{m_B^\lambda(\sigma)}\right), \\
\phi_S^\lambda(\sigma) &\triangleq \mathsf{E}[\mathbf{I}\{\text{sell fill}\}|\sigma] = \min\left(1, \frac{m_B^\lambda(\sigma)}{m_S^\lambda(\sigma)}\right),
\end{aligned}
\tag{11}
$$

where $m_B^\lambda(\sigma)$ and $m_S^\lambda(\sigma)$ are as defined in (9) and (10). For completeness, we set $\phi_B^\lambda(\sigma) \triangleq 0$ if $m_B^\lambda(\sigma) = 0$ and we set $\phi_S^\lambda(\sigma) \triangleq 0$ if $m_S^\lambda(\sigma) = 0$.

### 3.1.2. Trader expected utility

From the fill rates, we obtain refined expressions for a trader's utility, in terms of other traders' actions. Suppose the strategy profile in the market is given by $\lambda = (\lambda_{-V}, \lambda_0, \lambda_V)$. In order to explicitly specify the dependence of the trader's utility on the strategy profile, we qualify them with a superscript $\lambda$. Then, from (6), we have

$$
\begin{aligned}
u^\lambda_{DP,B}(v, s) &= \mathsf{E}[(\sigma + v)\mathbf{I}\{\text{buy fill}\}|s] \\
&= \mathsf{E}\left[\mathsf{E}[(\sigma + v)\mathbf{I}\{\text{buy fill}\}|\sigma]|s\right] \\
&= \mathsf{E}\left[(\sigma + v)\phi^\lambda_B(\sigma)|s\right] \\
&= s(1 + v)\phi^\lambda_B(1) + (1 - s)(-1 + v)\phi^\lambda_B(-1).
\end{aligned}
\tag{12}
$$

Here, we have used the tower property of conditional expectation in the second equality. Similarly, from (7), we have

$$
\begin{aligned}
u^\lambda_{DP,S}(v, s) &= -\mathsf{E}[(\sigma + v)\mathbf{I}\{\text{sell fill}\}|s] \\
&= -\mathsf{E}\left[(\sigma + v)\phi^\lambda_S(\sigma)|s\right] \\
&= -s(1 + v)\phi^\lambda_S(1) - (1 - s)(-1 + v)\phi^\lambda_S(-1).
\end{aligned}
\tag{13}
$$

Finally, observe that, in the open market, a trader transacts directly with the market maker. Thus, the utility of actions involving trade in the open market (or the explicit choice not to trade) do not depend on the actions of other traders or on the strategy profile $\lambda$. Hence, we have, as in (4)–(5),

$$
u^\lambda_{OM,B}(v, s) = 2s - 1 + v - \delta, \quad u^\lambda_{OM,S}(v, s) = 1 - 2s - v - \delta, \quad u^\lambda_N(v, s) = 0.
$$

for all $v \in \{0, \pm V\}$ and $s \in [0, 1]$.

### 3.1.3. Bayes-Nash equilibrium

We use Bayes-Nash equilibrium as the partial equilibrium solution concept for the game among traders, for a fixed transaction cost $\delta$.

**Definition 1** (Partial equilibrium). *A strategy profile $\lambda = (\lambda_{-V}, \lambda_0, \lambda_V)$ and a fixed open market transaction cost $\delta$ together constitute a partial equilibrium if, assuming the open market transaction cost is fixed at $\delta$, the strategy profile $\lambda$ satisfies the Bayes-Nash equilibrium (BNE)*

*condition given by*

$$\lambda_v(s) \in \operatorname*{argmax}_{a \in \mathcal{A}}\ u_a^\lambda(v, s), \tag{14}$$

*for all $v \in \{0, \pm V\}$ and $s \in [0, 1]$.*

Thus, in a partial equilibrium, each trader's strategy employs an optimal action for every signal, fixing all other traders' strategies and assuming a given open market transaction cost.

In what follows, it will be useful to adopt the following convention breaking ties among various actions: $N > (DP, B) > (DP, S) > (OM, B) > (OM, S)$. This is arbitrary, and is done primarily so that there is a unique best response at each signal for any trader. (As will be seen later, tie-breaking is only needed on a set of measure zero, and hence does not alter fundamental characteristics of a strategy profile such as the associated fill rates in the dark pool or the adverse selection experienced by the market maker.) With the tie-breaking rule in place, the inclusion in the definition of a partial equilibrium can be replaced with an equality. Further, given any strategy profile $\lambda$, we can now define a best response strategy profile $\Lambda[\lambda] \triangleq (\Lambda_{-V}[\lambda], \Lambda_0[\lambda], \Lambda_V[\lambda])$, where for any $v \in \{0, \pm V\}$, $\Lambda_v[\lambda]$ is the unique strategy defined by

$$\Lambda_v[\lambda](s) \triangleq \operatorname*{argmax}_{a \in \mathcal{A}}\ u_a^\lambda(v, s),$$

for all $s \in [0, 1]$. The definition of a partial equilibrium $(\lambda, \delta)$ can now be simplified to require that the strategy profile $\lambda$ satisfy $\lambda = \Lambda[\lambda]$, i.e., a fixed point of the best response map $\Lambda$ assuming a transaction cost $\delta$. Note that the map $\Lambda$ implicitly depends on the mass of speculators $\mu$, the idiosyncratic value $V$, and the transaction charge $\delta$; we will make these dependencies explicit when the context demands.

## 3.2. Competitive equilibrium

In this subsection we define a competitive equilibrium by adding the zero profit condition for the market maker to the preceding definition of a partial equilibrium. We start by fixing the transaction cost $\delta$, and compute the expected utility of the market maker given a partial equilibrium among the traders. We then use this calculation to formalize the zero profit condition.

### 3.2.1. Market maker profit

Suppose the transaction cost in the open market is $\delta$, and consider a partial equilibrium $(\lambda, \delta)$. Let $m^\lambda_{(OM,B)}(\sigma)$ (resp., $m^\lambda_{(OM,S)}(\sigma)$) denote the volume of buy orders (resp., sell orders) in the open market, conditional on the security value $\sigma$. As the market maker receives the transaction cost on each trade in the open market, the total revenue to the market maker is the product of the transaction cost $\delta$ and the total volume of trade in the open market. Thus, the total expected revenue for the market maker due to the transaction cost is given by

$$u_{tr}(\delta, \lambda) = \delta \mathsf{E} \left[ m^\lambda_{(OM,B)}(\sigma) + m^\lambda_{(OM,S)}(\sigma) \right].$$

Next, note that conditional on $\sigma$, the market maker's net position in the security is given by the difference $m^\lambda_{(OM,S)}(\sigma) - m^\lambda_{(OM,B)}(\sigma)$. Ignoring the transaction cost, the expected mark-to-market profit from this position between time $t = 0$ and $t = 1$ is given by

$$u_{mm}(\delta, \lambda) = \mathsf{E} \left[ \sigma \left( m^\lambda_{(OM,S)}(\sigma) - m^\lambda_{(OM,B)}(\sigma) \right) \right].$$

Observe that if the net quantity of the security bought by the market maker had been independent of the security value $\sigma$, then in expectation this mark-to-market profit would have value zero. In general, however, if according to the strategy profile $\lambda$, only the more informed traders choose to trade with the market maker, it will likely be the case that $u_{mm}(\delta, \lambda) \leq 0$. Indeed, $-u_{mm}(\delta, \lambda)$ represents the cost incurred by the market maker due to adverse selection.

Putting these together, the total expected utility of the market maker is

$$
\begin{aligned}
u(\delta, \lambda) &= u_{tr}(\delta, \lambda) + u_{mm}(\delta, \lambda) \\
&= \mathsf{E} \left[ (\delta + \sigma) m^\lambda_{(OM,B)}(\sigma) + (\delta - \sigma) m^\lambda_{(OM,S)}(\sigma) \right].
\end{aligned}
\tag{15}
$$

### 3.2.2. Zero profit condition

We are now ready to formally define a competitive equilibrium, which combines our BNE conditions with a zero profit condition for the market maker:

**Definition 2.** *A strategy profile $\lambda$ and a transaction cost $\delta$ together constitute a* competitive equilibrium *if and only if*

  1. *For a transaction cost of $\delta$, the strategy profile $\lambda$ constitutes a BNE; and*

2. *Given the strategy profile $\lambda$ and transaction cost $\delta$, the market maker's utility $u(\delta, \lambda)$,*
   *as defined in* (15)*, is zero.*

Observe that, from (15),

$$u(\delta, \lambda) \geq \mathsf{E}\left[\left(\delta - |\sigma|\right)\left(m^\lambda_{(OM,B)}(\sigma) + m^\lambda_{(OM,S)}(\sigma)\right)\right].$$

Then, if $\delta > |\sigma| = 1$ and there is any trade in the open market, the market maker's utility will be strictly positive. Therefore, when the transaction cost is large $(\delta > 1)$, there can be no competitive equilibrium involving trade in the open market. To avoid this uninteresting situation, we require that $\delta \in [0, 1]$.

# 4.  Structural results on partial equilibrium

In this section, we introduce a natural symmetry condition on the equilibrium strategies that lets us obtain results on the structure of a partial equilibria. First, we show that partial equilibria involving such strategies are completely specified by the buy fill rate in the dark pool. Second, from the structure of the equilibrium strategy, we show how the traders' choice of the trading venue depends on their private information. Finally, we demonstrate that, in equilibrium, trade in the dark pool experiences adverse selection.

## 4.1.  Symmetric strategy profiles

As the model we consider is symmetric with respect to change in the asset value $\sigma$, the idiosyncratic valuations of intrinsic buyers and sellers, and the mass of intrinsic buyers or sellers in the market, we focus our attention on a class of strategy profiles that satisfy a natural symmetry requirement. We begin with the following definition:

**Definition 3.** *A strategy profile* $\lambda = (\lambda_{-V}, \lambda_0, \lambda_V)$ *is* symmetric *if the following holds, for all* $v \in \{0, \pm V\}$ *and* $s \in [0, 1]$*:*

$$\begin{aligned}
\lambda_v(s) = (OM, B) \quad &\text{if and only if} \quad \lambda_{-v}(1 - s) = (OM, S), \\
\lambda_v(s) = (DP, B) \quad &\text{if and only if} \quad \lambda_{-v}(1 - s) = (DP, S), \\
\lambda_v(s) = N \quad\quad\;\; &\text{if and only if} \quad \lambda_{-v}(1 - s) = N.
\end{aligned} \quad (16)$$

In words, in a symmetric strategy profile, we require that if an intrinsic buyer with signal $s$ enters an order into the dark pool market, then an intrinsic seller with signal $1 - s$ enters an opposite order into the dark pool market. Similarly, if an intrinsic buyer with signal $s$

18

trades in the open market, then an intrinsic seller with signal $1 - s$ trades in the open market in the opposite direction. Finally, we require analogous conditions to hold for speculators with signals $s$ and $1 - s$.

As our first result shows, the class of symmetric strategy profiles is closed under the best response map. The proof is deferred to the appendix.

**Lemma 1.** *Suppose* $\lambda = (\lambda_{-V}, \lambda_0, \lambda_V)$ *is a symmetric strategy profile. Then the best response strategy profile* $\Lambda[\lambda]$ *is symmetric.*

The preceding result allows us to focus on partial equilibria with a symmetric strategy profile. This is useful because, as we show next, in any partial equilibrium with symmetric strategy profile, the fill rates are symmetric and can be characterized by a single parameter.

## 4.2.  Fill rates

Observe that, following Definition 3, in a symmetric strategy profile $\lambda$, for any $v \in \{0, \pm V\}$ and any $s \in [0, 1]$, we have $s \in \Theta^\lambda_{v,(DP,B)}$ if and only if $1 - s \in \Theta^\lambda_{-v,(DP,S)}$, where $\Theta^\lambda_{v,(DP,B)}$ and $\Theta^\lambda_{v,(DP,S)}$ are the sets of signals resulting in buy and sell orders in the dark pool from a trader with idiosyncratic value $v$, as defined in (8). Thus, from (9), we obtain for $\sigma \in \{-1, 1\}$,

$$
\begin{aligned}
m^\lambda_B(\sigma) &= \sum_{v \in \{\pm V\}} F_\sigma(\Theta^\lambda_{v,(DP,B)}) + \mu F_\sigma(\Theta^\lambda_{0,(DP,B)}) \\
&= \sum_{v \in \{\pm V\}} F_\sigma(1 - \Theta^\lambda_{-v,(DP,S)}) + \mu F_\sigma(1 - \Theta^\lambda_{0,(DP,S)}) \\
&= \sum_{v \in \{\pm V\}} F_{-\sigma}(\Theta^\lambda_{-v,(DP,S)}) + \mu F_{-\sigma}(\Theta^\lambda_{0,(DP,S)}) \\
&= m^\lambda_S(-\sigma).
\end{aligned}
$$

Here, in the second line, we have defined $1 - A \triangleq \{1 - x : x \in A\}$ for any set $A \subset \mathbb{R}$. The equality in the third line follows from the symmetry of the signal structure. In particular, we have $f_1(s) = f_{-1}(1 - s)$ for all $s \in [0, 1]$, where $f_\sigma$ is the density of $F_\sigma$ for $\sigma \in \{-1, +1\}$. The last equality follows from (10).

Thus, when the strategy profile is symmetric and $m^\lambda_B(\sigma)$ is positive for all $\sigma$, the fill rates, as defined in (11), satisfy

$$
\phi^\lambda_B(\sigma) = \phi^\lambda_S(-\sigma) = \min\left(1, \frac{m^\lambda_S(\sigma)}{m^\lambda_B(\sigma)}\right) > 0, \quad \text{for } \sigma \in \{-1, 1\}.
$$

This implies that, when $m^\lambda_B(\sigma)$ is positive for all $\sigma$ and $\lambda$ is symmetric, the fill rates must

satisfy one of the following two possibilities:

$$\phi_B^\lambda(1) = \phi_S^\lambda(-1) = 1, \quad 0 < \phi_B^\lambda(-1) = \phi_S^\lambda(1) < 1; \qquad \text{OR}$$
$$0 < \phi_B^\lambda(1) = \phi_S^\lambda(-1) \leq 1, \qquad \phi_B^\lambda(-1) = \phi_S^\lambda(1) = 1.$$

In the former case, a buy order in the dark pool is more likely to be filled when the asset value increases, and less likely when it decreases. On the other hand, in the latter case, a buy order in the dark pool is less likely to be filled with the asset value increases, and more likely when it decreases. The following result shows that, in any partial equilibrium with trade in the dark pool, the fill rates must satisfy the latter condition. The proof is deferred until the appendix.

**Theorem 1.** *For any $\delta \in [0, 1]$, suppose there is a partial equilibrium $(\lambda, \delta)$ with symmetric strategy profile $\lambda$ involving trade in the dark pool.[7] Then,*

$$0 < \phi_B^\lambda(1) = \phi_S^\lambda(-1) \leq 1, \quad \phi_B^\lambda(-1) = \phi_S^\lambda(1) = 1.$$

The preceding result establishes that in any partial equilibrium in symmetric strategies with trade in the dark pool, a buy order in the dark pool will get filled with certainty if the asset value decreases. As we show later in Section 4.5, this aspect of a partial equilibrium plays an important role in generating an adverse selection cost that is incurred by traders in the dark pool.

## 4.3. Threshold strategies

Next, we show that the strategies in a partial equilibrium take a particularly simple form: there are thresholds that completely determine behavior of intrinsic traders and speculators. Formally, for each transaction cost $\delta \in [0, 1]$, fill rate $f \in [0, 1]$ and $v \in \{-V, 0, V\}$, define

---

[7]Note, if the equilibrium symmetric strategy profile $\lambda$ does not involve trade in the dark pool, then the fill rates satisfy $\phi_B^\lambda(\sigma) = \phi_S^\lambda(\sigma) = 0$ by definition.

the following sub-intervals of the unit interval $[0, 1]$:

$$\Theta^{f,\delta}_{v,(OM,B)} \triangleq \left( \max \left\{ \frac{1-v+\delta}{2}, \frac{\delta}{(1+v)(1-f)} \right\}, 1 \right]$$

$$\Theta^{f,\delta}_{v,(DP,B)} \triangleq \left( \frac{1-v}{1-v+f(1+v)}, \frac{\delta}{(1+v)(1-f)} \right],$$

$$\Theta^{f,\delta}_{v,N} \triangleq \left[ \max \left\{ \frac{1-v-\delta}{2}, \frac{f(1-v)}{f(1-v)+1+v} \right\}, \min \left\{ \frac{1-v}{1-v+f(1+v)}, \frac{1-v+\delta}{2} \right\} \right]$$

$$\Theta^{f,\delta}_{v,(DP,S)} \triangleq \left[ 1 - \frac{\delta}{(1-v)(1-f)}, \frac{f(1-v)}{f(1-v)+1+v} \right)$$

$$\Theta^{f,\delta}_{v,(OM,S)} \triangleq \left[ 0, \min \left\{ \frac{1-v-\delta}{2}, 1 - \frac{\delta}{(1-v)(1-f)} \right\} \right) \tag{17}$$

Note that we allow the possibility for some of these intervals to be empty.[8] For each $\delta \in [0, 1]$, define $\mathcal{H}(\delta)$ as a one parameter family of symmetric strategy profiles

$$\mathcal{H}(\delta) \triangleq \left\{ \lambda^{f,\delta} \ : \ f \in [0, 1] \right\},$$

where for any $f \in [0, 1]$, and $v \in \{0, \pm V\}$, we define

$$\lambda^{f,\delta}_v(s) \triangleq a, \quad \text{if } a \in \mathcal{A}, \ s \in \Theta^{f,\delta}_{v,a}. \tag{18}$$

In words, the strategy $\lambda^{f,\delta}_v$ picks the action for a given signal according to membership in the intervals defined in (17). In the case where one of the intervals in (17) is empty, the corresponding action is not used.

We refer to $\mathcal{H}(\delta)$ as the set of *threshold strategies*, since, for a fixed idiosyncratic value $v$, these strategies determine actions through a set of thresholds (the endpoints of the intervals defined above) that partition the set of possible signals. The following lemma establishes the importance of these strategies in equilibrium. The proof is deferred to the appendix.

**Lemma 2.** *Suppose $\delta \in [0, 1]$. Then,*

(i) *For any $f \in [0, 1]$ and $v \in \{0, \pm V\}$ the intervals $\{\Theta^{f,\delta}_{v,a}\}_{a \in \mathcal{A}}$ form a partition of the signal set $\mathcal{S} = [0, 1]$, i.e., they are mutually exclusive and collectively exhaustive.*

(ii) *Suppose $\lambda$ is a symmetric strategy profile with fill rates $\phi^\lambda_B(1) \triangleq f \in (0, 1]$ and $\phi^\lambda_S(1) = 1$, then $\Lambda[\lambda] = \lambda^{f,\delta} \in \mathcal{H}(\delta)$. For a symmetric strategy profile $\lambda$ with fill rates $\phi^\lambda_B(1) = \phi^\lambda_S(1) = 0$, we have $\Lambda[\lambda] = \lambda^{0,\delta} \in \mathcal{H}(\delta)$.*

---

[8]We adopt the convention that an interval is defined to be the empty set if its endpoints are not ordered, i.e., $[a, b] \triangleq \emptyset$ if $a > b$, and $(a, b) \triangleq \emptyset$, $(a, b] \triangleq \emptyset$ and $[a, b) \triangleq \emptyset$ if $a \geq b$.

Part (i) simply establishes that the threshold strategies described above are well-defined. Part (ii) has two important implications. First, observe that Theorem 1 guarantees that any partial equilibrium with a symmetric strategy profile satisfies the hypotheses of Part (ii). Therefore, Part (ii) guarantees that in any such partial equilibria, the traders employ threshold strategies. Second, observe that, for a fixed $\delta$, a threshold strategies is uniquely characterized by the parameter $f \in [0, 1]$. Then, combined with first implication, in any partial equilibrium, the Bayes-Nash fixed point condition takes the form $\Lambda[\lambda^{f,\delta}] = \lambda^{f,\delta}$. This is a one parameter fixed point equation involving a single unknown, the buy fill rate $f$. The second implication turns out to be crucial for the downstream analysis of the market in Section 5, as well as in numerical analysis through the use of line search methods for equilibrium computations; we discuss this in detail in Section 6.1.

Before continuing, we note that, for any transaction cost $\delta \in [0, 1]$ there always exists a trivial partial equilibrium where no trader enters the dark pool. This corresponds to $\lambda^{0,\delta} \in \mathcal{H}(\delta)$. This arises due to the fact that trade in the dark pool occurs only through matching. In particular, from a single trader's point of view, if no other trader enters the dark pool, then the fill rate is zero, and hence there is no benefit to unilaterally deviating to enter the dark pool. Observe that this is the same outcome that arises in a market without the presence of a dark pool. Subsequently, when evaluating how the presence of a dark pool affects the market, we will compare market characteristics in this equilibrium with those in an equilibrium where there is trade in the dark pool.

Further, a partial equilibrium with trade in the dark pool may not exist for certain values of the model parameters. Typically, such a partial equilibrium may not exist if the traders' idiosyncratic values are small, there are too many speculators, or if the transaction cost in the open market is too low. Intuitively, in the former two scenarios, if trade were to occur in the dark pool, it would primarily be based on the differences in traders' private information. However, such a situation is precluded by the no-trade theorem of Milgrom and Stokey (1982), which prohibits any trade among rational (risk-neutral) traders who differ only in their beliefs. In the latter scenario of low transaction cost in the open market, the traders may prefer to trade with certainty with the market maker in the open market over entering orders in the dark pool.

## 4.4. Information segmentation

As established in Section 4.3, in any partial equilibrium with a symmetric strategy profile, the traders must employ threshold strategies. In such strategies, traders with equal idiosyncratic value are segmented according to their private information in order to determine actions.

This segmentation creates a relationship between a trader's choice of venue to trade and their informedness and takes a particular form, as evidenced in the following corollary to Lemma 2. The proof is straightforward, and we omit it for brevity.

**Corollary 1** (Information segmentation)**.** *Define the following total order on the action set $\mathcal{A}$,*

$$(OM, S) \prec (DP, S) \prec N \prec (DP, B) \prec (OM, B).$$

*Given a transaction cost $\delta \in [0, 1]$, suppose there exists a partial equilibrium with symmetric strategy profile $\lambda$. Then, for all $v \in \{0, \pm V\}$ and $0 \leq s \leq s' \leq 1$, we have that $\lambda_v(s) \preceq \lambda_v(s')$.*

Corollary 1 is depicted pictorially in Figure 1. It establishes that a particular ordering must hold in the actions chosen in all partial equilibria. Two observations can be made about this ordering:

1. Regarding the direction of trade, observe that for a fixed idiosyncratic value $v$, if a trader with a signal $s$ chooses to sell, all traders with signals less than $s$ will also sell. Similarly, if a trader with a signal $s$ chooses to buy, all traders with signals greater than $s$ will also buy. Loosely speaking, traders with high signals will buy, while traders with low signals will sell, all else being equal.

2. Regarding the choice of venue, observe that for a fixed idiosyncratic value $v$, if a trader with a signal $s$ chooses to sell (resp., buy) in the open market, all traders with signals less (resp., greater) than $s$ will also sell (resp., buy) in the open market. Loosely speaking, traders who are more informed (i.e., with signals closer to 0 or 1 depending on the direction of trade) will prefer the open market to the dark pool. In other words, the dark pool will be populated with traders who are relatively less informed than those who choose the open market.

The latter observation suggests that, via an information segmentation mechanism, trade in the dark pool will alter the informational characteristics of trade in the open market. This fact has important downstream implications for transactions costs in the open market in competitive equilibrium, which we will see in Section 5.

## 4.5.   Adverse selection

Consider the following definition:

**Definition 4** (Adverse selection)**.** *A trader submitting a buy (resp., sell) order in the dark pool suffers from adverse selection if her expectation of the value of the asset, conditional*
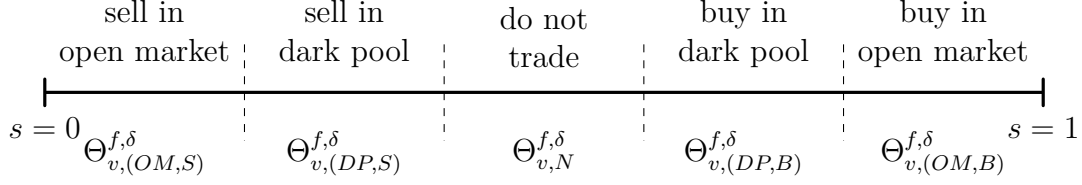
**Figure 1:** Threshold strategies in partial equilibrium. For any given $\delta$, the thresholds depend on the idiosyncratic value $v$, and some intervals may be empty.

*on the order being filled, is lower (resp., higher) than her (unconditional) expectation of the value of the asset. Formally, we have for any $s \in [0,1]$,*

$$\mathsf{E}\left[\sigma | s, \text{buy fill}\right] \leq \mathsf{E}\left[\sigma | s\right] \leq \mathsf{E}\left[\sigma | s, \text{sell fill}\right].$$

The following result, whose proof is deferred until the appendix, states that adverse selection in the dark pool is pervasive in any partial equilibrium:

**Theorem 2.** *In any partial equilibrium $\lambda$ involving symmetric strategies with trade in the dark pool, all traders in the dark pool suffer from adverse selection. Defining the* adverse selection cost $Adv(\lambda)$ *as*

$$Adv(\lambda) \triangleq \mathsf{E}\left[\sigma \,\Big|\, s = \tfrac{1}{2}\right] - \mathsf{E}\left[\sigma \,\Big|\, s = \tfrac{1}{2}, \text{buy fill}\right],$$

*we have $Adv(\lambda) = \mathsf{E}\left[\sigma \,\Big|\, s = \tfrac{1}{2}, \text{sell fill}\right] - \mathsf{E}\left[\sigma \,\Big|\, s = \tfrac{1}{2}\right] \geq 0$. Furthermore, the adverse selection cost under such an equilibrium is strictly decreasing with the buy (sell) fill rate.*

Adverse selection arises from the fact that the execution of an order in the dark pool is correlated with the movement in the value of the asset in a detrimental way: a buy (resp., sell) order is more likely to be executed when the value of the asset moves down (resp., up). However, this detrimental correlation *cannot* be directly attributed to information asymmetry in the dark pool; in fact, as mentioned earlier, the more informed traders trade in the open market. Rather, adverse selection is created through the aggregate behavior of the group of overall traders participating in the dark pool. In particular, consider the behavior of a speculator ($v = 0$). In a symmetric equilibrium, by Lemma 2, the subset of signals $\Theta_{0,(DP,B)}^{f,\delta} \subset [0,1]$ for which a speculator chooses to buy in the dark pool and the subset of signals $\Theta_{0,(DP,S)}^{f,\delta} \subset [0,1]$ for which a speculator chooses to sell in the dark pool are sub-intervals of the real line of equal length, and are symmetric about the point $s = 1/2$. In the case where $\sigma = +1$ (resp., $\sigma = -1$), then, clearly the *probability mass* of speculators who choose to buy (resp., sell) in the dark pool is larger than those who choose to sell (resp., buy)

24

in the dark pool. This mismatch of masses creates adverse selection amongst speculators in the dark pool, and this intuition extends as well to intrinsic buyers and seller. In this way, adverse selection endogenously occurs in the dark pool through the aggregation of diffuse information from a cross section of marginally informed agents.

Thus, adverse selection is an intrinsic characteristic of any partial equilibrium in the market. Note that this adverse selection imposes an *implicit* transaction cost on the trader in the dark pool. This explains why there may not be a partial equilibrium involving trade in the dark pool for sufficiently low values of the (explicit) transaction cost in the open market.

## 5.    Welfare analysis of competitive equilibria

In this section we exploit our structural understanding of partial equilibria to analyze competitive equilibria. Our emphasis is on understanding the welfare consequences of the introduction of a dark pool. A naive view of the introduction of the dark pool would suggest welfare can only increase with the presence of the dark pool, since participants are afforded greater opportunities for trade than before. When traders and market makers are strategic and adapt to the presence of the dark pool, however, it is not *a priori* evident whether the dark pool does indeed increase welfare in equilibrium.

Before we begin, we clarify the definition of welfare in our model. As usual, we define welfare to be the sum of the expected utility of all the agents in the market. (Here the expectation is with respect to the uninformed common prior **P**.) Since all agents in the market are risk neutral, it follows that the monetary transfers among the agents in the market do not affect the welfare. Hence, the welfare in the market depends solely on the final allocation of the security in the market. In particular, the market welfare is higher if more intrinsic buyers end up holding the security at time $t = 1$, and more intrinsic sellers end up being short the security at time $t = 1$. Similarly, the welfare is lower when fewer intrinsic buyers hold and fewer intrinsic sellers have sold the security at time $t = 1$. From this discussion, one can also consider the market welfare as the degree to which the intrinsic tendencies of the traders (prior to receiving any private information) are actualized.

In our welfare analysis, a central theme is the role of two potential transaction costs faced by the traders. One transaction cost is *explicit* in our model: any trader in the open market faces a transaction cost $\delta$ set by the market maker. The second transaction cost is *implicit*: any trader in the dark pool faces an adverse selection cost (cf. Section 4.5); from Theorem 2, we know this cost is higher when the fill rate in the dark pool is lower. To a large extent, our welfare results are driven by the role of these transaction costs in shaping the trading decisions of intrinsic buyers and sellers. We compare two types of equilibria:

those where a dark pool is present, and those where a dark pool is absent. By comparing the two transaction costs in the former with the open market transaction cost in the latter, we can get a sense of how welfare is affected by introduction of a dark pool.

Our main result in this section uses this approach to show in a benchmark case that introduction of the dark pool can actually *decrease* welfare. In particular, in Section 5.2 we consider a setting where there are no speculators ($\mu = 0$). Regardless of the intrinsic value parameter $V$, we first show that the introduction of a dark pool *raises* the transaction cost set by the market maker in the open market, relative to the competitive equilibrium without a dark pool. Now this suggests that the only way welfare can increase is if the dark pool itself is sufficiently attractive to motivate more intrinsic traders to transact than in the absence of a dark pool. We show that in fact, the adverse selection cost in the dark pool is sufficiently high that the opposite occurs: welfare is lower in the presence of a dark pool.

In general, we find the same result holds numerically for a wide range of the problem parameter values $(\mu, V)$ — our numerical investigation is presented in Section 6. However, we also note that there are specific combinations of parameter values where the introduction of a dark pool can increase welfare. In Section 5.3, we consider the behavior of the market as the mass of speculators increases, and in particular as $\mu \to \infty$. In this regime, the market is dominated by speculators trading to exploit private information. Such traders often eschew the dark pool in favor of the certainty of the open market, and we establish that the explicit transaction costs in the open market in the equilibria with and without the dark pool are qualitatively similar. Since the introduction of the dark pool does not materially impact the transaction cost in the open market, we find ourselves closer to a regime where the "naive" intuition described above is correct; namely, the dark pool functions as a new venue for trade for the intrinsic traders, and can only lead to more intrinsic traders participating in the market.

There is a caveat, however: as $\mu$ increases, the (buy) fill rate in the dark pool drops, i.e., the adverse selection cost in the dark pool increases. Thus, in order to actually see a welfare gain, the intrinsic value parameter $V$ must be sufficiently high that intrinsic traders of moderate signal are sufficiently motivated to take advantage of the trading possibilities created by the introduction of the dark pool. We validate this intuition in by showing that if both $\mu$ and $V$ are large (i.e., many speculators and a high idiosyncratic motivation for intrinsic buyers and sellers to trade), welfare can in fact increase with the introduction of a dark pool.

## 5.1. Preliminaries: No dark pool

In this section, we define a reference competitive equilibrium NODP with no trade in the dark pool, against which we compare all equilibria with trade in the dark pool. The following result shows that such an equilibrium exists. The proof is deferred to the appendix.

**Lemma 3.** *For any $\mu \geq 0$ and $0 < V \leq 1$, there exists a unique competitive equilibrium, denoted by* NODP*, with transaction cost $\delta_{\mathsf{NODP}}(\mu, V) = 1 - \frac{V}{\sqrt{1+\mu}} < 1$, and where the symmetric strategy profile involves trade only in the open market (i.e., the fill rate in the dark pool is zero).*

Note that, without the presence of the dark pool, the transaction cost in the open market would be set at $\delta_{\mathsf{NODP}}(\mu, V)$ in a perfectly competitive market. In order to study how the presence of the dark pool affects the transaction costs in the open market, we will compare the transaction cost in any competitive equilibrium with trade in the dark pool to $\delta_{\mathsf{NODP}}$[9]. Similarly, the welfare implications of trade in the dark pool will be assessed by comparing the welfare of any competitive equilibrium with trade in the dark pool to the welfare of the NODP equilibrium.

## 5.2. A benchmark case: No speculators

In this section we consider a benchmark model, where there are no speculators ($\mu = 0$). We have two results. First, we show that the transaction cost in the presence of a dark pool is *higher* than $\delta_{\mathsf{NODP}}$. We use this insight to show our main result: the introduction of a dark pool decreases welfare.

We start with the following theorem, which states that in any competitive equilibrium where there is trade in the dark pool, the transaction cost in the open market would be greater than or equal to $\delta_{\mathsf{NODP}}$. The proof is deferred to the appendix.

**Theorem 3.** *Suppose $\mu = 0$ and $0 < V \leq 1$. For any transaction cost $\delta < \delta_{\mathsf{NODP}}$, the market maker's expected utility in any partial equilibrium $\lambda$ is negative.*

Thus, we obtain the following corollary, stating that the presence of the dark pool increases the transaction cost set by a competitive market makers:

**Corollary 2.** *Suppose $\mu = 0$ and $0 < V \leq 1$, in any competitive equilibrium where the fill rate in the dark pool is positive, the transaction cost is greater than or equal to that in the* NODP *competitive equilibrium with no trade in the dark pool.*

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[9]We suppress the dependence on the parameters $\mu$ and $V$ when the context is clear.

The intuition behind the preceding result is that the presence of a dark pool increases the adverse selection cost faced by a competitive market maker. In particular, as relatively uninformed traders move to trade in the dark pool, the population of traders in the open market becomes relatively better informed and collectively increases the adverse selection cost for the market maker. In order to compensate for this, a competitive market maker sets a higher transaction cost.

We next investigate how the presence of the dark pool affects welfare. Our main result in this section implies that the presence of the dark pool reduces the welfare of the market. The proof is available in Appendix B.

**Theorem 4.** *Suppose $\mu = 0$ and $0 < V \leq 1$. Suppose the transaction cost $\delta$ satisfies $\delta \geq \delta_{\mathsf{NODP}}$, and $\lambda$ is a corresponding partial equilibrium. Then, the welfare under $\lambda$ is less than or equal to that in the* $\mathsf{NODP}$ *competitive equilibrium with no trade in the dark pool.*

Taken together with Theorem 3, we obtain the following result:

**Corollary 3.** *Suppose $\mu = 0$ and $0 < V \leq 1$. The welfare under any competitive equilibrium with a positive fill rate in the dark pool is less than or equal to the welfare of the* $\mathsf{NODP}$ *equilibrium with no trade in the dark pool.*

We briefly describe the intuition behind the preceding result. Note that as emphasized above, welfare is improved if more intrinsic buyers hold the security (and symmetrically, more intrinsic sellers have sold the security). Now observe that the transaction cost in the open market rises with the introduction of a dark pool; this effect leads to fewer intrinsic buyers buying in the open market, a negative welfare effect. The only potential countervailing force is that some of these traders are enticed to buy in the dark pool instead: if that volume is sufficient, welfare may actually rise. However, in equilibrium, traders in the dark pool suffer an adverse selection cost; and we show that this is cost is sufficient to actually cause a net reduction in the fraction of intrinsic buyers ultimately holding the security.

## 5.3. High levels of speculation and high idiosyncratic value

The preceding section illustrates that the introduction of a dark pool can actually decrease welfare. In this section we illustrate that this result may not obtain over all parameter values. In particular, we show that   when the mass of speculators $\mu$ is high enough and the idiosyncratic value $V$ is high, there exist competitive equilibria involving trade in the dark pool that attain a higher welfare than that in the competitive equilibrium $\mathsf{NODP}$ with no trade the dark pool.

Throughout this section, we assume that the idiosyncratic value $V$ is at its maximum $V = 1$. We begin by showing that, for all large enough $\mu$, there exists a competitive equilibrium where both the intrinsic traders and the speculators trade in both the dark pool and the open market:

**Theorem 5.** *Suppose $V = 1$. For all large enough $\mu \geq 0$, there exists a competitive equilibrium, denoted by* BOTH*, where both the intrinsic traders and the speculators trade in both the dark pool and the open market; the equilibrium fill rate $f_{\text{BOTH}}(\mu)$ and the equilibrium transaction cost $\delta_{\text{BOTH}}(\mu)$ in this equilibrium satisfy:[10]*

$$f_{\text{BOTH}}(\mu) = \frac{C_1}{\sqrt{\mu}} + o\left(\frac{1}{\sqrt{\mu}}\right), \quad \delta_{\text{BOTH}}(\mu) = 1 - \frac{C_2}{\sqrt{\mu}} + o\left(\frac{1}{\sqrt{\mu}}\right),$$

*where $C_1, C_2 > 0$ are constants.*

For comparison, we have the following result for the NODP equilibrium:

**Lemma 4.** *Suppose $V = 1$. Then, for any $\mu \geq 0$, the transaction cost $\delta_{\text{NODP}}(\mu)$ and the welfare $w_{\text{NODP}}(\mu)$ in the* NODP *competitive equilibrium with no trade in the dark pool are given by*

$$\delta_{\text{NODP}}(\mu) = 1 - \frac{1}{\sqrt{1 + \mu}}, \quad w_{\text{NODP}}(\mu) = 1 + \frac{1}{\sqrt{1 + \mu}}.$$

Finally, we compare the resulting welfare $w_{\text{BOTH}}(\mu)$ in the BOTH equilibrium with $w_{\text{NODP}}(\mu)$, the welfare in the NODP equilibrium without trade in the dark pool:

**Theorem 6.** *Suppose $V = 1$. Then, we have*

$$\lim_{\mu \to \infty} \frac{w_{\text{BOTH}}(\mu)}{w_{\text{NODP}}(\mu)} = \frac{7}{4}.$$

The preceding theorem states that, when the mass of speculators is large and the idiosyncratic value $V$ is high, the presence of the dark pool improves the welfare of the market. To see the intuition behind this result, consider the traders strategies in the limit when $\mu = \infty$. In the NODP equilibrium, all intrinsic buyers with signals above $s = 0.5$ and all intrinsic sellers with signals below $s = 0.5$ trade with the market maker in the open market. This leads to a welfare of 1. On the other hand, in the BOTH equilibrium, in addition to these

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[10]In what follows, given functions $f, g, q \colon \mathbb{R}_+ \to \mathbb{R}_+$, we say that $f = g + o(q)$ if $\limsup_{\mu \to \infty} |f(\mu) - g(\mu)|/q(\mu) = 0$, i.e., if the difference between $f$ and $g$ converges to 0 at a faster rate than $q$. Similarly, we say that $f = g + \Theta(q)$ if $0 < \liminf_{\mu \to \infty} |f(\mu) - g(\mu)|/q(\mu) \leq \limsup_{\mu \to \infty} |f(\mu) - g(\mu)|/q(\mu) < \infty$, i.e., if the difference between $f$ and $g$ converges to 0 at the same rate than $q$.

preceding trades, all intrinsic buyers below the signal $s = 0.5$ enter buy orders in the dark pool and all intrinsic sellers above the signal $s = 0.5$ enter sell orders in the dark pool. When $\sigma = 1$, all the intrinsic sellers orders' get filled with the buy orders from the speculators, and essentially no intrinsic buyers' orders get filled. (The situation is symmetric when $\sigma = -1$.) These trades in the dark pool contribute an additional $3/4$ to the welfare in the BOTH equilibrium.

Note that as $\mu \to \infty$, Theorem 5 and Lemma 4 reveal that the transaction cost in the open market scales as $1 - \Theta(1/\sqrt{\mu})$, for the BOTH *and* the NODP equilibria. In other words, the introduction of the dark pool does not materially alter the transaction cost in the open market. As a result, informally, we should expect that at best the introduction of the dark pool can only induce additional intrinsic traders to participate (which would increase welfare). However, traders may not enter, because the fill rate in the dark pool is very low (i.e., the adverse selection cost is very high). As long as traders have high enough intrinsic value (i.e., $V = 1$), however, traders of weak-to-moderate signals will find it advantageous to enter the dark pool and trade. This leads to the welfare increase observed in Theorem 6. Informally, this suggests that the welfare increase we observe is dependent on both the presence of a sufficiently large mass of speculators, *and* a sufficiently high intrinsic value for trade. Indeed, we investigate this phenomenon numerically in Section 6, and our results confirm this intuition.

# 6.  Computational experiments

In this section, we augment our analytical results with supporting numerical evidence from extensive equilibrium computations  over a broad parameter regime. We begin by describing the numerical approach, and provide illustrations of the threshold strategies of the traders. We then provide three sets of numerical results for the comparative statics of competitive equilibria. The first set studies the benchmark case with no speculators, and, in particular, provides comparative statics with respect to the intrinsic value $V$; as our theoretical results in Section 5.2 demonstrate, in this regime introduction of the dark pool causes welfare to fall. The second set studies the effect of increasing the mass of speculators $\mu$, when traders have a high intrinsic value for trade; as the results in Section 5.3 suggest, in this regime, welfare increases for sufficiently large values of $\mu$. Finally, the last set of results investigates the regime where the mass of speculators is nonzero, but traders have moderate intrinsic value for trade. In these results, the introduction of the dark pool leads to a fall in welfare—suggesting that both a sufficiently large mass of speculators *and* a high intrinsic value for trade must be present for a dark pool to increase welfare.

## 6.1. Numerical approach

Recall that for any fixed values of the the mass of speculators $\mu$, the intrinsic value $V$, and the transaction cost $\delta$, a partial equilibrium with symmetric strategy profile is a strategy profile in the class $\mathcal{H}(\delta)$ that is the fixed point of the best response map $\Lambda[\mu, V, \delta; \cdot]$ — here, we make explicit the dependence of $\Lambda[\cdot]$ on the model parameters. Since the class of strategy profiles $\mathcal{H}(\delta)$ is parameterized by a single (buy) fill rate parameter $f \in [0, 1]$, we can search numerically for all partial equilibria with symmetric strategy profile by searching over the possible values for the fill rate in the interval $[0, 1]$. In particular, we iterate over a discrete set $\mathcal{F} \subseteq [0, 1]$ of values for the fill rate $f$, and compute the resulting fill rate parameter of the best response strategy profile $\Lambda[\lambda^{f, \delta}]$. We store all values of $f$ for which the absolute value of the difference between $f$ and the fill rate parameter of $\Lambda[\lambda^{f, \delta}]$ is below a small threshold $\epsilon_1$. This set of values of $f$ then identifies approximately all partial equilibria with symmetric strategy profile for any transaction cost $\delta \in [0, 1]$ and for given values of the model parameters $\mu$ and $V$.

We perform another numerical search to identify those values of $\delta \in [0, 1]$ that lead to zero profit for the market maker in at least one of the corresponding partial equilibria. More precisely, we iterate over a discrete set $\mathcal{D} \subseteq [0, 1]$ of values for the transaction cost $\delta \in [0, 1]$, and for each of the corresponding partial equilibria computed earlier, we compute the market maker's profit. We store those values of $(\delta, f)$ of the partial equilibria for which the absolute value of the market maker's expected profit is below a small threshold $\epsilon_2$. The value of the transaction cost $\delta$ along with corresponding partial equilibrium fill rate $f$ together then identify a competitive equilibrium for the given values of the model parameters, up to a small numerical tolerance.[11]

Note that in the numerical results that follow, for a given set of model parameters, there are multiple equilibria. We distinguish between the equilibria as follows: the NODP equilibrium, which involves no trade in the dark pool, is labeled "NODP". The equilibria with trade in the dark pool vary along continuous curves as model parameters are change. We label these equilibria as belonging to one of two branches, in order to clarify the relationship between the sets of equilibria in different subfigures.

## 6.2. The benchmark case: No speculators

In Figure 2, we consider, for different idiosyncratic values $V \in (0, 1]$, the benchmark case of no speculation, i.e., $\mu = 0$. In Figure 2(b), we plot the adverse selection cost faced by

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[11]We make the following specific choices for the thresholds and the discrete sets for our numerical results: $\mathcal{F} = \{k/10000 : k = 0, 1, \cdots, 10000\}$, $\mathcal{D} = \{k/1000 : k = 0, 1, \cdots, 1000\}$, $\epsilon_1 = 10^{-3}$, and $\epsilon_2 = 4 \cdot 10^{-4}$.

an uninformed trader ($s = 0.5$) in the dark pool in the different competitive equilibria with trade in the dark pool, for different values of the idiosyncratic value $V$. In Figures 2(a) and 2(c), we plot respectively the transaction cost in the open market and the welfare, in different competitive equilibria, for different values of the idiosyncratic value $V$. Note that for values of $V$ below approximately 0.42, there are no competitive equilibria with trade in the dark pool. Furthermore, given any equilibrium with no trade in the open market, as the transaction cost increases, equilibrium is maintained. Hence, we have the shaded region in the upper right corner of Figure 2(a).

First and foremost, we see from these figures that, for a given $V$, the transaction cost in any competitive equilibria with trade in the dark pool is higher than that in the NODP equilibrium. Similarly, for a given $V$, the welfare in an equilibrium with trade in the dark pool is lower than that in the NODP equilibrium. This is consistent with our analytical results in Section 5.2.

Second, from Figure 2(b), we see that the adverse selection cost in the dark pool decreases as the idiosyncratic value $V$ increases. Furthermore, we see that for values of $V$ greater than approximately 0.6, there are two sets of competitive equilibria with trade in the dark pool. In the lower set of equilibria (the first branch), the adverse selection cost decreases to zero as $V$ increases to one. In the upper set of equilibria (the second branch), although the adverse selection cost decreases to a positive value approximately equal to 0.33, Figure 2(d) reveals that the volume of orders in the dark pool converges to zero.

These figures reveal the intricate connection between transaction costs, adverse selection costs, and welfare. In particular, because the dark pool leads the transaction cost in the open market to increase, and the adverse selection cost is significant, welfare falls. Notice that the relative decrease in welfare is lower as $V \rightarrow 1$; this results because both the transaction cost and the welfare in the competitive equilibria with trade in the dark pool approach that in the NODP equilibrium. As intrinsic traders become more highly motivated to trade, the welfare losses incurred by introduction of the dark pool are naturally mitigated.

## 6.3. The market with speculators: The case of high idiosyncratic value

Next, we turn our attention to the market with speculators. We first consider, in Figure 3, the case of high idiosyncratic value, with $V = 0.9$. In Figure 3(b), we plot the adverse selection cost faced by an uninformed trader ($s = 0.5$) in the dark pool in different competitive equilibria with trade in the dark pool, for different values of the mass of speculators $\mu$. In Figures 3(a) and 3(c), we plot respectively the transaction cost in the open market and the welfare, in different competitive equilibria, for different values of the mass of speculators $\mu$.
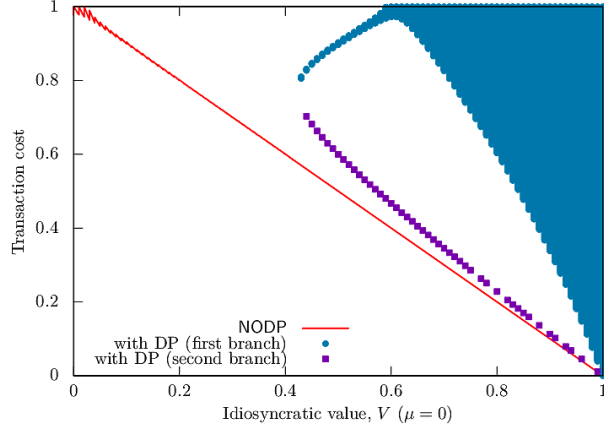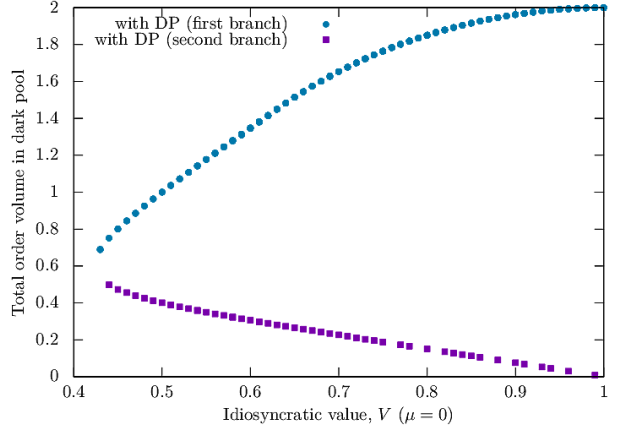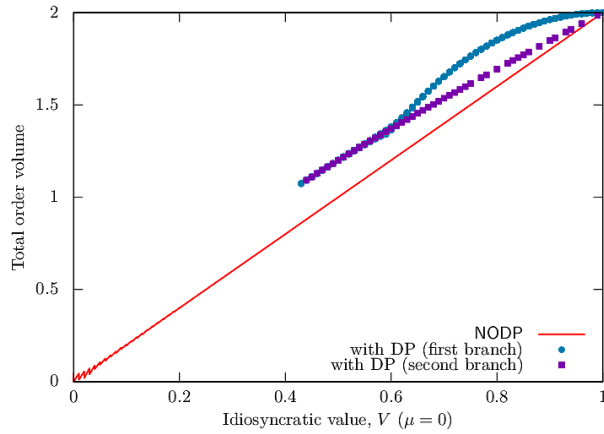
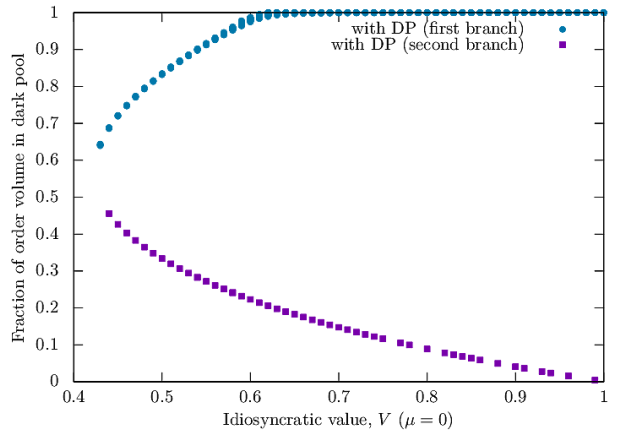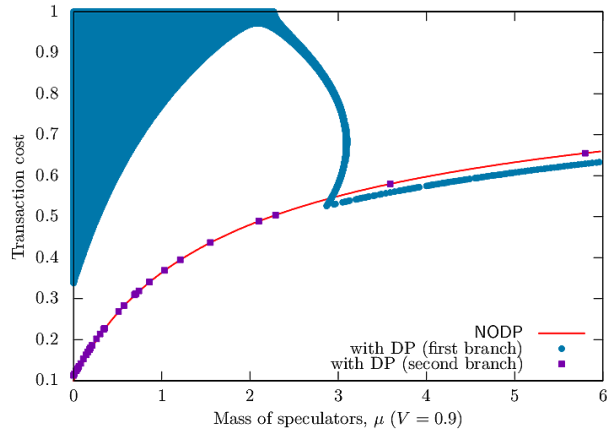**(a)** Transaction cost.
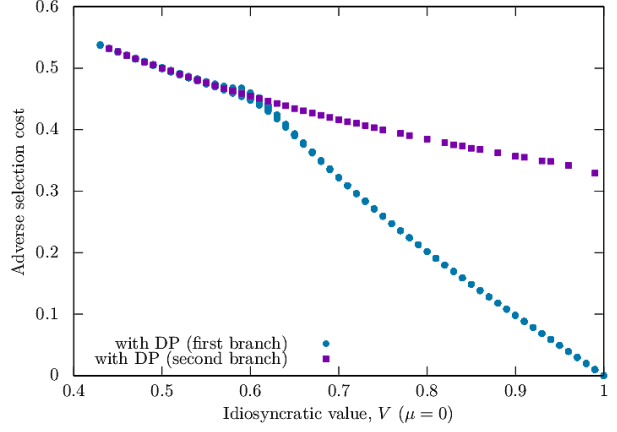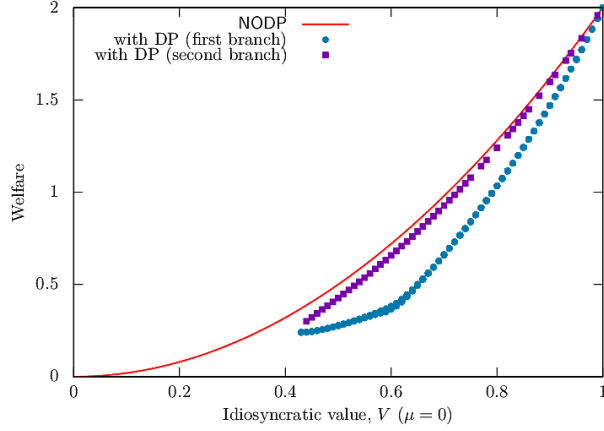
**(b)** Adverse selection in the dark pool.

**(c)** Welfare.

**(d)** Volume of orders in dark pool.

**(e)** Total volume of orders.

**(f)** Fraction of order volume in dark pool.

**Figure 2:** Competitive equilibria in the benchmark case with no speculators ($\mu = 0$).

33

As before, given any equilibrium with no trade in the open market, as the transaction cost increases, equilibrium is maintained. Hence, we have the shaded region in the upper left corner of Figure 3(a).

From Figure 3(c), we see that for $\mu$ greater than approximately 3, there exists a competitive equilibrium with trade in the dark pool (in the first branch) such that the welfare is higher as compared to the NODP equilibrium. This observation expands on the asymptotic analytical results in Section 5.3; in this case, $\mu$ is large but finite, and $V$ is high, but strictly less than one, and the welfare increases on introduction of the dark pool.

If we try to understand the roots of this effect, we are led to study the transaction cost in the open market (cf. Figure 3(a)) and the adverse selection cost on introduction of the dark pool (cf. Figure 3(b)). We have two main observations. First, on the first branch, the transaction cost in the open market is higher with the presence of the dark pool when $\mu$ is low, but eventually behaves similarly to (and is slightly lower than) the transaction cost in the open market without the dark pool. Second, the adverse selection cost is significant throughout the range of $\mu$ we consider. However, because $V$ is large, intrinsic buyers and sellers are still motivated to trade. Since the dark pool does not materially impact the transaction costs in the open market, the presence of the dark pool merely acts as another venue for intrinsic buyers and sellers to trade, resulting in welfare gains. This matches the analytical findings in Section 5.3.

## 6.4. The market with speculators: The case of moderate idiosyncratic value

We conclude by considering a regime where the mass of speculators is nonzero, but the idiosyncratic value $V$ is moderate; for concreteness we use the value $V = 0.6$, but our results remain qualitatively similar for other values of $V$. In Figure 4(b), we plot the adverse selection cost faced by an uninformed trader ($s = 0.5$) in the dark pool in different competitive equilibria with trade in the dark pool, for different values of the mass of speculators $\mu$. In Figures 4(a) and 4(c), we plot respectively the transaction cost in the open market and the welfare, in different competitive equilibria, for different values of the mass of speculators $\mu$.

From these figures, we see that for large enough mass of speculators $\mu$, there is no competitive equilibrium with trade in the dark pool. This observation supports our assertion that, in markets with high levels of speculation, for the welfare gains from the presence of dark pool to be realized, the intrinsic traders must have significant outside incentives to trade. In the absence of this effect, at best the dark pool will not be a sufficiently attractive venue for trade to raise welfare; and at worst, it will lead to a loss of welfare, as in Section 5.2.
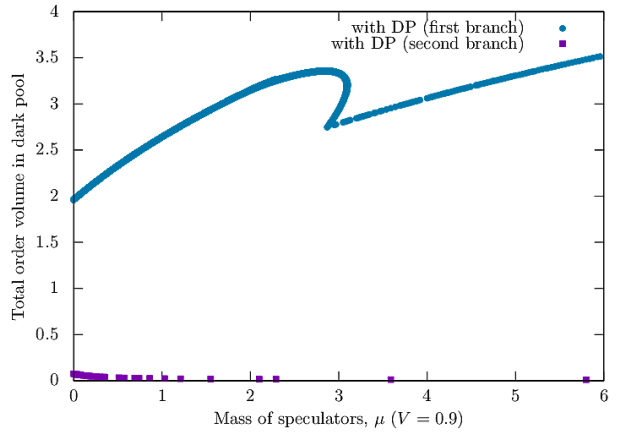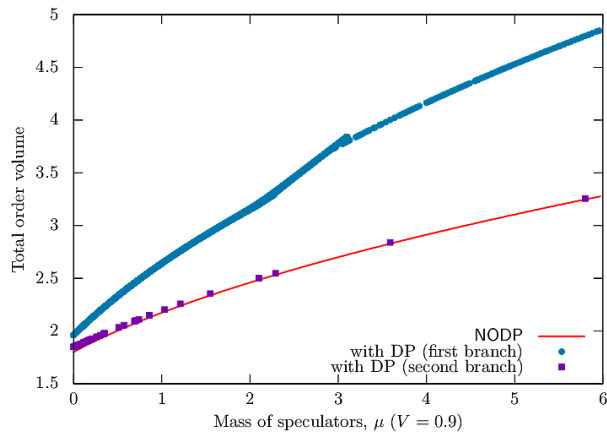
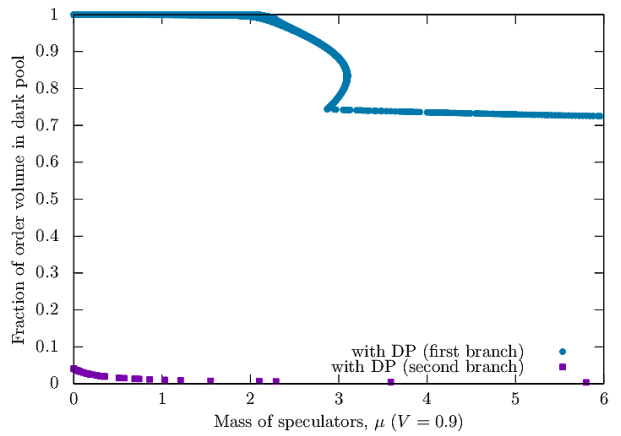**(a)** Transaction cost.
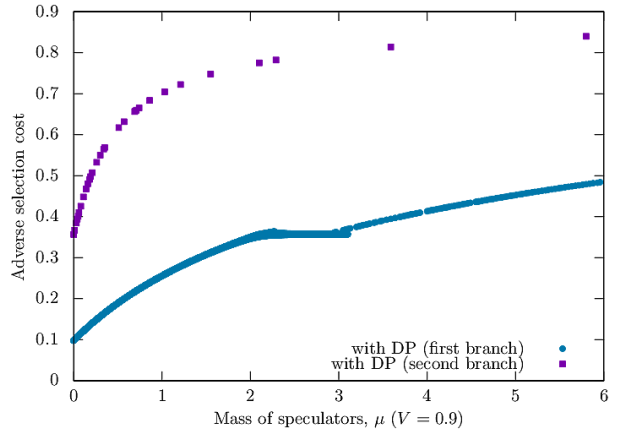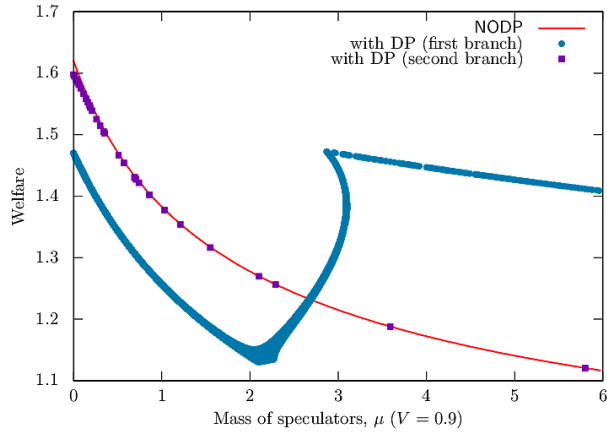
**(b)** Adverse selection in the dark pool.

**(c)** Welfare.
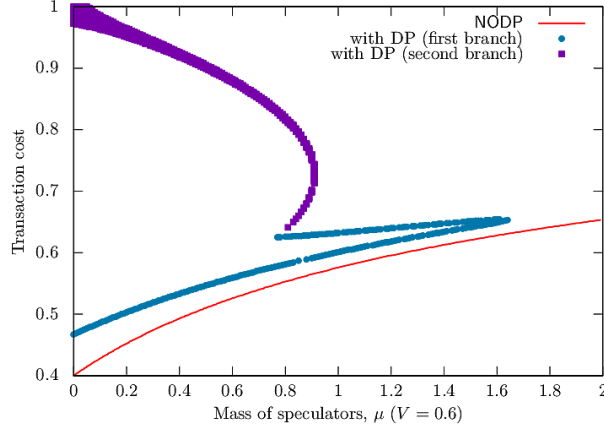
**(d)** Volume of orders in dark pool.
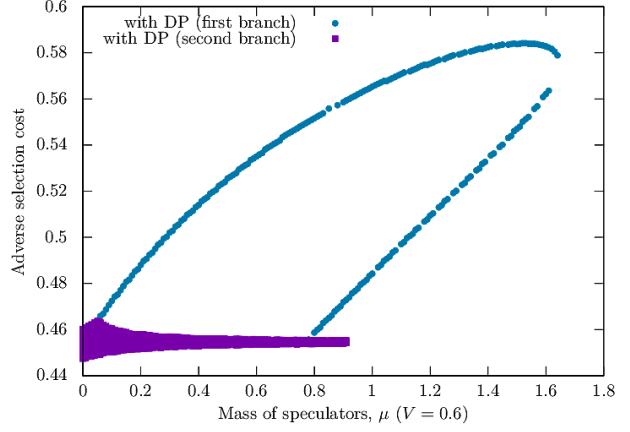
**(e)** Total volume of orders.

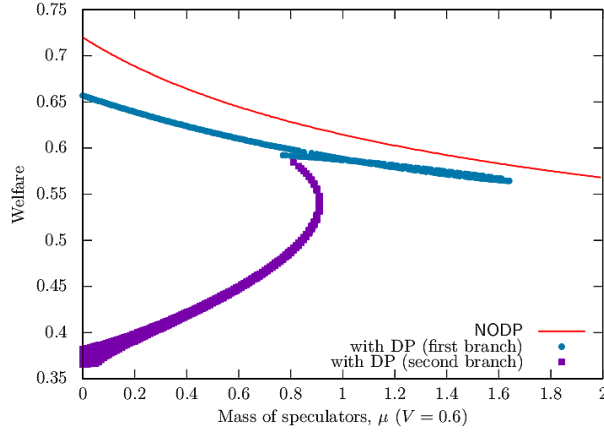**(f)** Fraction of order volume in dark pool.

**Figure 3:** Competitive equilibria with speculators, for high idiosyncratic value ($V = 0.9$).

**(a)** Transaction cost.

**(b)** Adverse selection in the dark pool.

**(c)** Welfare.

**(d)** Volume of orders in dark pool.

**(e)** Total volume of orders.

**(f)** Fraction of order volume in dark pool.

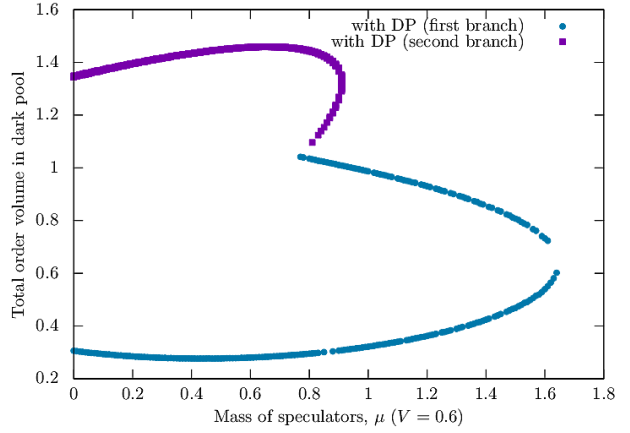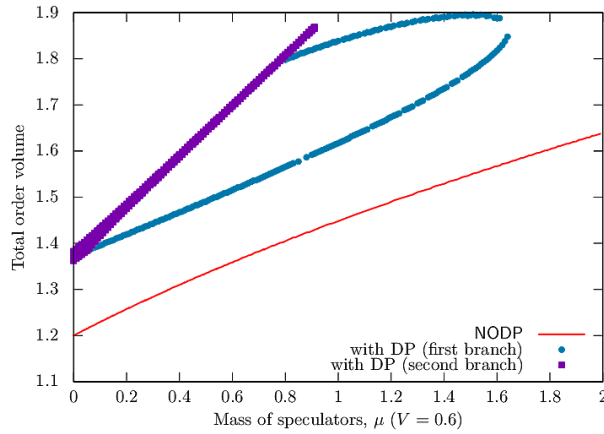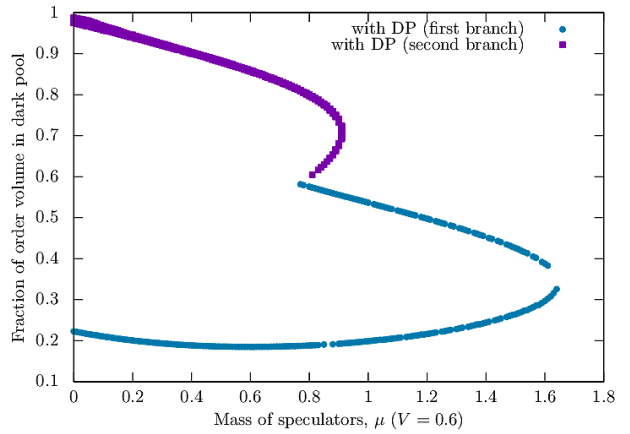**Figure 4:** Competitive equilibria with speculators, for moderate idiosyncratic value ($V = 0.6$).

36

# 7. Conclusion

Our main goal in this paper is to analyze the welfare implications of operating a dark pool alongside traditional lit markets. We consider a stylized model of a competitive market where traders have heterogeneous fine-grained private information about the short-term future price of the asset. The main conclusion is that in reasonable parameter regimes, the introduction of a dark pool can *decrease* market welfare. This welfare decrease can be attributed primarily to the fact that in equilibrium, the orders in the dark pool face an implicit transaction cost in the form of adverse selection. We show that this adverse selection occurs despite the fact that highly informed traders trade in the open market, whereas the dark pool is populated with orders from relatively moderately informed traders. This implicit transaction cost, combined with a higher transaction cost in the open market, leads to a decrease in the overall welfare of the market.

Our analysis also shows the existence of parameter regimes where the presence of dark pool can be welfare improving, in particular, when the level of speculation is high, and when intrinsic buyers and sellers have sufficiently high idiosyncratic value for the asset. However, as noted in our welfare analysis, the welfare gains obtain in this regime precisely because the dark pool does not significantly raise the transaction cost in the open market; thus it functions as a complementary venue of trade for highly motivated intrinsic traders. This finding is important because it is in contrast to empirical findings that suggest that the introduction of a dark pool typically *raises* transaction costs in the lit market (e.g., Comerton-Forde and Putniņš, 2015; Degryse et al., 2014; Foley et al., 2012). Our model suggests that in parameter regimes where transaction costs in the open market increase on introduction of a dark pool, welfare decreases. As a result, our paper finds that the regulators have a legitimate concern about the potentially negative welfare implications of the introduction of dark pools alongside traditional lit markets.

# References

P. Afèche, A. Diamant, and J. Milner. Double-sided batch queues with abandonment: Modeling crossing networks. *Operations Research*, 62(5):1179–1201, 2014.

M. Baldauf and J. Mollner. Competition among exchanges. Working paper, available at `http://stanford.edu/~baldauf/files/JMP_baldauf.pdf`, 2015.

S. Buti, B. Rindi, and I. M. Werner. Dark pool trading strategies, market quality and welfare. Working paper, available at `http://ssrn.com/abstract=1630499`, 2014.

C. Comerton-Forde and T. J. Putniņš. Dark trading and price discovery. Working paper, available at `http://ssrn.com/abstract=2183392`, 2015.

H. Degryse, M. Van Achter, and G. Wuyts. Dynamic order submission strategies with competition between a dealer market and a crossing network. *Journal of Financial Economics*, 91(3):319–338, 2009.

H. Degryse, F. De Jong, and V. V. Kervel. The impact of dark trading and visible fragmentation on market quality. *Review of Finance*, 2014.

J. Dönges and F. Heinemann. Crossing network versus dealer market: Unique equilibrium in the allocation of order flow. Working paper, 2006.

S. Foley, K. Malinova, and A. Park. Dark trading on public exchanges. Working paper, available at `http://ssrn.com/abstract=2182839`, 2012.

K. Ganchev, Y. Nevmyvaka, M. Kearns, and J. W. Vaughan. Censored exploration and the dark pool problem. *Communications of the ACM*, 53:99–107, May 2010. URL `http://doi.acm.org/10.1145/1735223.1735247`.

L. R. Glosten and P. R. Milgrom. Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics*, 14(1):71–100, 1985.

T. Hendershott and H. Mendelson. Crossing networks and dealer markets: Competition and performance. *Journal of Finance*, pages 2071–2115, 2000.

F. Klöck, A. Schied, and Y. S. Sun. Price manipulation in a market impact model with dark pool. Working paper, available at `http://ssrn.com/abstract=1785409`, 2011.

P. Kratz and T. Schöneborn. Portfolio liquidation in dark pools in continuous time. *Mathematical Finance*, 25(3):496–544, 2013.

P. Kratz and T. Schöneborn. Optimal liquidation in dark pools. *Quantitative Finance*, 14 (9):1519–1539, 2014.

A. S. Kyle. Continuous auctions and insider trading. *Econometrica: Journal of the Econometric Society*, pages 1315–1335, 1985.

P. Milgrom and N. Stokey. Information, trade and common knowledge. *Journal of Economic Theory*, 26(1):17–27, 1982.

C. C. Moallemi, B. Park, and B. Van Roy. Strategic execution in the presence of an uninformed arbitrageur. *Journal of Financial Markets*, 15(4):361–391, 1 2012.

R. Næs and B. A. Ødegaard. Equity trading by institutional investors: To cross or not to cross? *Journal of Financial Markets*, 9(2):79 – 99, 2006.

M. Nimalendran and S. Ray. Informational linkages between dark and lit trading venues. *Journal of Financial Markets*, 17(0):230–261, 2014. URL `http://www.sciencedirect.com/science/article/pii/S1386418113000128`.

L. Qiao, Y. Sun, and Z. Zhang. Conditional exact law of large numbers and asymmetric information economies with aggregate uncertainty. Working paper, available at `http://arxiv.org/abs/1410.1147`, 2014.

L. Tuttle. Alternative trading systems: Description of ATS trading in national market system stocks. Technical report, U.S. Security and Exchange Commission, Division of Economic and Risk Analysis, October 2013.

M. Ye. A glimpse into the dark: Price formation, transaction cost and market share of the crossing network. Working paper, available at `http://ssrn.com/paper=1521494`, 2011.

H. Zhu. Do dark pools harm price discovery? *Review of Financial Studies*, 27(3):747–789, 2014.

# A.  Proofs

**Proof of Lemma 1.** Suppose the strategy profile $\lambda$ is symmetric. Then, it follows directly from the definition that, for any $v \in \{0, \pm V\}$ and any $s \in [0, 1]$, we have $s \in \Theta^\lambda_{v,(DP,B)}$ if and only if $1 - s \in \Theta^\lambda_{-v,(DP,S)}$. Thus, we obtain for all $v \in \{0, \pm V\}$ and $\sigma \in \{\pm 1\}$,

$$F_\sigma(\Theta^\lambda_{v,(DP,B)}) = F_\sigma(1 - \Theta^\lambda_{-v,(DP,S)}) = F_{-\sigma}(\Theta^\lambda_{-v,(DP,S)}).$$

Here, for any set $A$, we have defined $1 - A \triangleq \{1 - x : x \in A\}$. The second line follows from the symmetry of the signal structure. In particular, we have $f_1(s) = f_{-1}(1 - s)$ for all $s \in [0, 1]$, where $f_\sigma$ is the density of $F_\sigma$. This implies that for any set $A$, $F_\sigma(A) = F_{-\sigma}(1 - A)$.

Thus, from (9), we obtain

$$m_B^\lambda(\sigma) = \sum_{v \in \{\pm V\}} F_\sigma(\Theta_{v,(DP,B)}^\lambda) + \mu F_\sigma(\Theta_{0,(DP,B)}^\lambda)$$

$$= \sum_{v \in \{\pm V\}} F_\sigma(1 - \Theta_{-v,(DP,S)}^\lambda) + \mu F_\sigma(1 - \Theta_{0,(DP,S)}^\lambda)$$

$$= \sum_{v \in \{\pm V\}} F_{-\sigma}(\Theta_{-v,(DP,S)}^\lambda) + \mu F_{-\sigma}(\Theta_{0,(DP,S)}^\lambda)$$

$$= m_S^\lambda(-\sigma).$$

The last equality follows from (10). Thus, when the strategy profile is symmetric and $m_B^\lambda(\sigma)$ is positive for all $\sigma$, the fill rates, as defined in (11), satisfy

$$\phi_B^\lambda(\sigma) = \phi_S^\lambda(-\sigma) = \min\left(1, \frac{m_S^\lambda(\sigma)}{m_B^\lambda(\sigma)}\right), \quad \text{for } \sigma \in \{-1, 1\}.$$

From this, (12) and (13), it is straightforward to verify that

$$u_{(OM,B)}^\lambda(v, s) = 2s - 1 + v - \delta$$

$$= 1 - 2(1 - s) - (-v) - \delta$$

$$= u_{(OM,S)}^\lambda(-v, 1 - s).$$

$$u_{(DP,B)}^\lambda(v, s) = s(1 + v)\phi_B^\lambda(1) + (1 - s)(-1 + v)\phi_B^\lambda(-1)$$

$$= s(1 + v)\phi_S^\lambda(-1) + (1 - s)(-1 + v)\phi_S^\lambda(1)$$

$$= -(1 - s)(1 - (-v))\phi_S^\lambda(1) - s(-1 + (-v))\phi_S^\lambda(-1)$$

$$= u_{(DP,S)}^\lambda(-v, 1 - s).$$

Furthermore, we have $u_N(v, s) = 0 = u_N(-v, 1 - s)$. Thus, while computing the best response strategies, the decision problem faced by a trader with idiosyncratic value $v$ at signal $s$ is equivalent to that of a trader with idiosyncratic value $-v$ at signal $1 - s$, with the qualification that whenever the former buys, the latter sells (and vice versa). This suffices to conclude that the best response strategy profile $\Lambda^\lambda$ is symmetric. ∎

**Proof of Theorem 1.** Consider a symmetric strategy profile $\lambda$ such that $\phi_B^\lambda(-1) = \phi_S^\lambda(1) =$

$f < 1$ and $\phi_B^\lambda(1) = \phi_S^\lambda(-1) = 1$. The traders' expected utility can be simplified as

$$u_{OM,B}^\lambda(v,s) = 2s - 1 + v - \delta,$$
$$u_{OM,S}^\lambda(v,s) = 1 - 2s - v - \delta,$$
$$u_{DP,B}^\lambda(v,s) = s(1+v) + (1-s)f(-1+v)$$
$$= s(1 + v + f(1-v)) - f(1-v),$$
$$u_{DP,S}^\lambda(v,s) = sf(-1-v) + (1-s)(1-v)$$
$$= 1 - v - s\left(f(1+v) + 1 - v\right),$$
$$u_N^\lambda(v,s) = 0.$$

From this, it is straightforward to conclude that

$$\Lambda_v^\lambda(s) = \begin{cases} (DP, B) & \text{for } s \geq \frac{1-v}{2}; \\ (DP, S) & \text{for } s < \frac{1-v}{2}. \end{cases}$$

This implies that, under the strategy profile $\gamma \triangleq \Lambda^\lambda$, the mass of buy orders in the dark pool when $\sigma = 1$ satisfies

$$m_B^\gamma(1) = F_1\left[\frac{1-V}{2}, 1\right] + F_1\left[\frac{1+V}{2}, 1\right] + \mu F_1\left[\frac{1}{2}, 1\right]$$
$$= F_{-1}\left[0, \frac{1+V}{2}\right] + F_{-1}\left[0, \frac{1-V}{2}\right] + \mu F_{-1}\left[0, \frac{1}{2}\right]$$
$$\geq F_1\left[0, \frac{1+V}{2}\right] + F_1\left[0, \frac{1-V}{2}\right] + \mu F_1\left[0, \frac{1}{2}\right]$$
$$= m_S^\gamma(1).$$

Here, we have used in the second line the fact that $F_\sigma(A) = F_{-\sigma}(1 - A)$ for any set $A$. The inequality follows from the observation that for any $x$, we have $F_1[0, x] = x^2 \leq 1 - (1-x)^2 = F_{-1}[0, x]$.

Thus, we obtain $m_B^\gamma(1) \geq m_s^\gamma(1)$. Furthermore, as $V \in (0, 1]$, we have $m_B^\gamma(1) > 0$. Hence, we obtain $\phi_S^\gamma(1) = \min\{1, \frac{m_B^\lambda(1)}{m_S^\lambda(1)}\} = 1$. Since $\phi_S^\lambda(1) < 1$, this shows that $\lambda \neq \gamma = \Lambda^\lambda$, and hence $\lambda$ is not a BNE.

This implies that any partial equilibrium $(\lambda, \delta)$ with symmetric strategy profile $\lambda$ involving trade in the dark pool must satisfy $0 < \phi_B^\lambda(1) = \phi_S^\lambda(-1) \leq 1$ and $\phi_B^\lambda(-1) = \phi_S^\lambda(1) = 1$. ∎

**Proof of Lemma 2.** Let $f \in [0, 1]$ and $v \in \{-V, 0, V\}$. It is straightforward to verify that $\Theta_{v,(OM,B)}^{f,\delta} \cap \Theta_{v,(DP,B)}^{f,\delta} = \emptyset$, $\Theta_{v,(OM,B)}^{f,\delta} \cap \Theta_{v,N}^{f,\delta} = \emptyset$, and $\Theta_{v,(DP,B)}^{f,\delta} \cap \Theta_{v,N}^{f,\delta} = \emptyset$. Similarly,

$\Theta^{f,\delta}_{v,(OM,S)} \cap \Theta^{f,\delta}_{v,(DP,S)} = \emptyset$, $\Theta^{f,\delta}_{v,(OM,S)} \cap \Theta^{f,\delta}_{v,N} = \emptyset$, and $\Theta^{f,\delta}_{v,(DP,S)} \cap \Theta^{f,\delta}_{v,N} = \emptyset$. Now, for any $s \in \Theta^{f,\delta}_{v,(OM,B)} \cap \Theta^{f,\delta}_{v,(OM,S)}$ we have $(1-v-\delta)/2 > s > (1-v+\delta)/2$, which contradicts $\delta \geq 0$. Also, for any $s \in \Theta^{f,\delta}_{v,(DP,B)} \cap \Theta^{f,\delta}_{v,(DP,S)}$ we have $f(1-v)/(f(1-v)+1+v) > s > (1-v)/(1-v+f(1+v))$, which contradicts $f \leq 1$. Similarly, for $s \in \Theta^{f,\delta}_{v,(OM,B)} \cap \Theta^{f,\delta}_{v,(DP,S)}$, we have $f(1-v)/(f(1-v)+1+v) > s > (1-v+\delta)/2$, which contradicts $\delta(1-f) \geq 0$. Similar argument shows that $s \in \Theta^{f,\delta}_{v,(DP,B)} \cap \Theta^{f,\delta}_{v,(OM,S)} = \emptyset$. This completes the proof of part (i).

Next, suppose $\lambda$ is a symmetric strategy profile with $\phi^\lambda_B(1) = f \in (0,1]$, and $\phi^\lambda_S(1) = 1$. Given this, it is straightforward to verify that a trader's utility functions for different actions are linear in $s$. This implies that the best response strategy profile $\Lambda[\lambda]$ has a simple threshold structure, where the thresholds correspond to those signal values where two (or possibly more) actions yield the same expected utility. Finally, given the fill rate $f = \phi^\lambda_B(1)$, one can verify through straightforward calculations that these thresholds correspond exactly to those of $\lambda^{f,\delta} \in \mathcal{H}(\delta)$.

Finally, suppose the symmetric strategy profile $\lambda$ has fill rates $\phi^\lambda_B(1) = \phi^\lambda_B(1) = 0$. Since this can arise only if there is a zero mass of buy (or sell) orders in the dark pool, the best response strategy $\Lambda[\lambda]$ would never involve submitting an order to the dark pool, and would only involve orders in the open market. Given this and using the fact that a trader's utility for different actions are linear in $s$, again we obtain through straightforward computation that the best response strategy profile $\Lambda[\lambda]$ has a threshold structure, with thresholds corresponding exactly to that of $\lambda^{0,\delta} \in \mathcal{H}(\delta)$. This completes the proof of part (ii). ∎

**Proof of Theorem 2.** Consider a partial equilibrium $\lambda$ involving symmetric strategies, with trade in the dark pool. A trader with signal $s \in [0,1]$ submitting a buy order in the dark pool, has the following belief about the value of the asset upon her order being filled:

$$
\begin{aligned}
\mathbf{P}\left(\sigma=1|s, \text{buy fill}\right) &= \frac{\mathbf{P}\left(\text{buy fill}|s, \sigma=1\right)\mathbf{P}\left(\sigma=1|s\right)}{\mathbf{P}\left(\text{buy fill}|s, \sigma=1\right)\mathbf{P}\left(\sigma=1|s\right) + \mathbf{P}\left(\text{buy fill}|s, \sigma=-1\right)\mathbf{P}(\sigma=-1|s)} \\
&= \frac{s\mathbf{P}\left(\text{buy fill}|\sigma=1\right)}{s\mathbf{P}\left(\text{buy fill}|\sigma=1\right)+(1-s)\mathbf{P}(\text{buy fill}|\sigma=-1)}.
\end{aligned}
$$

Here, the first equation follows from Bayes' rule, the second equation follows from the fact that the signal structure satisfies $\mathbf{P}(\sigma=1|s) = s$, and that conditional on $\sigma$, the event $\mathbf{I}\{\text{buy fill}\}$ is independent of the signal $s$, as the trade in the dark pool is through uniform matching.

Now, by definition of the fill rates, we have $\mathbf{P}(\text{buy fill}|\sigma) = \phi_B^\lambda(\sigma)$. This implies that,

$$\mathbf{P}\left(\sigma = 1|s, \text{buy fill}\right) = \frac{s\phi_B^\lambda(1)}{s\phi_B^\lambda(1) + (1-s)\phi_B^\lambda(-1)}.$$

Since $\mathsf{E}[\sigma|s] = 2s - 1$, and $\mathsf{E}\left[\sigma|s, \text{buy fill}\right] = 2\mathbf{P}\left(\sigma = 1|s, \text{buy fill}\right) - 1$, we obtain

$$\mathsf{E}\left[\sigma|s\right] - \mathsf{E}\left[\sigma|s, \text{buy fill}\right] = 2s - \frac{s\phi_B^\lambda(1)}{s\phi_B^\lambda(1) + (1-s)\phi_B^\lambda(-1)}$$
$$= \frac{2s(1-s)}{s\phi_B^\lambda(1) + (1-s)\phi_B^\lambda(-1)}\left(\phi_B^\lambda(-1) - \phi_B^\lambda(1)\right).$$

From Theorem 1, we know that in the partial equilibrium $\lambda$, the fill rates satisfy $\phi_B^\lambda(-1) = 1$ and $\phi_B^\lambda(1) \le 1$. Thus, we obtain that

$$\mathsf{E}\left[\sigma|s\right] - \mathsf{E}\left[\sigma|s, \text{buy fill}\right] \ge 0, \quad \text{for all } s \in [0,1].$$

Thus all traders submitting a buy order in the dark pool suffer from adverse selection. The proof for a trader submitting a sell order follows symmetrically.

Finally, we have

$$Adv(\lambda) \triangleq \mathsf{E}\left[\sigma \,\middle|\, s = \tfrac{1}{2}, \text{buy fill}\right] - \mathsf{E}\left[\sigma \,\middle|\, s = \tfrac{1}{2}\right] = \frac{1 - \phi_B^\lambda(1)}{1 + \phi_B^\lambda(1)}.$$

From this expression, using Theorem 1, it is straightforward to show that $Adv(\lambda) = \mathsf{E}\left[\sigma \,\middle|\, s = \tfrac{1}{2}, \text{sell fill}\right] - \mathsf{E}\left[\sigma \,\middle|\, s = \tfrac{1}{2}\right] \ge 0$. Furthermore, again from the expression, we obtain that $Adv(\lambda)$ is a strictly decreasing function of the buy (sell) fill rate $\phi_b^\lambda(1) = \phi_s^\lambda(-1)$. ∎

**Proof of Lemma 3.** For any $\delta \in [0,1]$, consider $\lambda^{0,\delta}$, the partial equilibrium involving trade only in the open market. The market maker's payoff in this partial equilibrium is given by,

$$u(\delta, \lambda^{0,\delta}) = \delta\left(1 - F_1\left(\frac{1-V+\delta}{2}\right) + F_1\left(\frac{1-V-\delta}{2}\right) + 1 - F_1\left(\frac{1+V+\delta}{2}\right) + F_1\left(\frac{1+V-\delta}{2}\right)\right)$$
$$+ \mu\delta\left(1 - F_1\left(\frac{1+\delta}{2}\right) + F_1\left(\frac{1-\delta}{2}\right)\right)$$
$$- \left(1 - F_1\left(\frac{1-V+\delta}{2}\right) - F_1\left(\frac{1-V-\delta}{2}\right)\right) - \left(1 - F_1\left(\frac{1+V+\delta}{2}\right) - F_1\left(\frac{1+V-\delta}{2}\right)\right)$$
$$- \mu\left(1 - F_1\left(\frac{1+\delta}{2}\right) - F_1\left(\frac{1-\delta}{2}\right)\right).$$

Here, the first two lines represent the market maker's revenue from trade, and the last two

lines represent the cost due to adverse selection. These expressions for the revenue and the adverse selection cost follow from the threshold structure of $\lambda^{0,\delta}$, and from the fact that $F_1(x) = 1 - F_{-1}(1-x)$.

Using the fact that $F_1(x) = x^2$ for $x \in [0, 1]$, $F_1(x) = 0$ for $x < 0$ and $F_1(x) = 1$ for $x > 1$, we obtain that,

$$u(\delta, \lambda^{0,\delta}) = \begin{cases} \frac{1}{2}\left(V^2 - (1+\mu)(1-\delta)^2\right) & \text{for } \delta \geq 1 - V; \\ \frac{1}{2}\left(2V^2 - (2+\mu)(1-\delta)^2\right) & \text{for } \delta < 1 - V. \end{cases}$$

From this, we observe that $u(\delta, \lambda^{0,\delta})$ is strictly increasing in $\delta$, and negative for $\delta < 1 - V$. Since $u(1, \lambda^{0,1}) = V^2/2 > 0$, we obtain that there is a unique $\delta = \delta_{\mathsf{NODP}}(\mu, V) \in [0, 1)$ such that $u(\delta, \lambda^{0,\delta}) = 0$. Thus, there exists a unique competitive equilibrium $(\delta, \lambda^{0,\delta})$ with trade only in the open market, where $\delta = \delta_{\mathsf{NODP}}(\mu, V)$. From a straightforward calculation, we obtain

$$\delta_{\mathsf{NODP}}(\mu, V) = 1 - \frac{V}{\sqrt{1+\mu}}. \qquad \blacksquare$$

Our results in Section 5.2 make use of the following lemma, which states that for values of the transaction cost lower than $\delta_{\mathsf{NODP}}$, there is always trade in the open market in any partial equilibrium with a symmetric strategy profile.

**Lemma 5.** *Let $\mu = 0$ and $V \in (0, 1]$. For some fixed transaction cost $\delta$, suppose there exists a partial equilibrium in symmetric strategy profile with no trade in the open market. Then, we must have $\delta \geq \delta_{\mathsf{NODP}}(0, V)$.*

**Proof of Lemma 5.** For a fixed $\delta \in [0, 1]$, consider a partial equilibrium with symmetric strategy profile $\lambda^{\mathsf{DP}}$ where there is no trade in the open market. The strategy $\lambda^{\mathsf{DP}}_{+V}(\cdot)$ is given by

$$\lambda^{\mathsf{DP}}_{+V}(s) = \begin{cases} (DP, B) & \text{if } s \geq b_-; \\ (DP, S) & \text{if } s \leq a; \\ N & \text{otherwise,} \end{cases}$$

where

$$b_- \triangleq \frac{1-V}{1-V+f(1+V)}, \qquad a \triangleq \frac{f(1-V)}{f(1-V)+1+V},$$

44

and $f$ is the equilibrium fill rate, satisfying the equation

$$f = \frac{a^2 + (1 - b_-)^2}{1 - b_-^2 + 1 - (1 - a)^2}. \tag{19}$$

Now, observe that

$$\frac{a^2}{1 - (1 - a)^2} = \frac{a}{2 - a} \leq \frac{a}{1 - a} = \frac{f(1 - V)}{1 + V} \leq f.$$

Thus, since $f$ satisfies (19), we must have

$$\frac{(1 - b_-)^2}{1 - b_-^2} \geq f.$$

This implies that

$$f \leq \frac{1 - b_-}{1 + b_-} = \frac{f(1 + V)}{2(1 - V) + f(1 + V)}.$$

Since $f > 0$, we obtain,

$$(1 + V)(1 - f) \geq 2(1 - V) \geq 1 - V = \delta_{\mathsf{NODP}}(0, V). \tag{20}$$

Finally, observe that since in equilibrium there is no trade in the open market, an intrinsic buyer with signal $s = 1$ must prefer submitting a buy order in the dark pool over buying in the open market. Thus, we must have $u_{OM,B}(V, s) \leq u_{DP,B}(V, s)$ for $s = 1$. This implies that, in equilibrium, we have, $\delta \geq (1 + V)(1 - f)$. Thus, using (20), we obtain $\delta \geq \delta_{\mathsf{NODP}}(0, V)$. ∎

**Proof of Theorem 3.** From Lemma 5, we know that for $\delta < \delta_{\mathsf{NODP}}$, there does not exists a partial equilibrium in symmetric strategy profiles with no trade in the open market. We prove the theorem by considering the two possible types of partial equilibrium for $\delta < \delta_{\mathsf{NODP}}$, and showing the statement holds for each case.

1. *Case 1. A partial equilibrium with no trade in the dark pool.* Note that when the transaction cost is zero, in the partial equilibrium where there is no trade in the dark pool, the market maker's expected utility is negative, as she faces only the adverse selection risk, and receives no revenue from the transaction charge. For any $\delta < \delta_{\mathsf{NODP}}$, consider the partial equilibrium $\lambda_\delta^{\mathsf{OM}}$ where there is no trade in the dark pool. (Such a partial equilibrium always exists.) It is straightforward to show that the market maker's utility in such a partial equilibrium is continuous in $\delta$. If the market maker's expected utility at $(\delta, \lambda_\delta^{\mathsf{OM}})$ were positive, by the continuity of the market maker's

utility and the intermediate value theorem, there exists a value of the transaction cost between 0 and $\delta$, where the market maker's expected utility, in the corresponding partial equilibrium with no trade in the dark pool market, is zero. However, this contradicts the uniqueness of the NODP competitive equilibrium with no trade in the dark pool. Hence, the market maker's expected utility in the partial equilibrium with the symmetric strategy profile $\lambda_\delta^{\text{OM}}$ is negative.

2. *Case 2. A partial equilibrium with trade in both the dark pool and the open market.* We split the market maker's expected utility into four components: (1) that from intrinsic buyers who buy in the open market; (2) from intrinsic buyers who sell in the open market, (3) that from intrinsic sellers who sell in the open market; and (4) from intrinsic sellers who buy in the open market. We show that for $\delta < \delta_{\text{NODP}}$, in a partial equilibrium with trade in both the dark pool and the open market, each of the four components of the market maker's expected utility is negative. We focus on the first two parts, as the analysis for the last two parts is symmetric.

First, consider a trader with signal $s$ who buys the asset from the market maker. From such a trade, the market maker's utility is $\delta - 1$ if $\sigma = 1$ and $\delta + 1$ if $\sigma = -1$. Thus, the contribution to the market maker's expected utility from all intrinsic buyers buying in the open market and having signal $s$ is given by $u(B, s) \triangleq (\delta - 1)f_1(s)\mathbf{P}(\sigma = 1) + (\delta + 1)f_{-1}(s)\mathbf{P}(\sigma = -1) = \delta - 2s + 1$. (Here, $f_\sigma$ is the density of $F_\sigma$; we have $f_1(s) = 2s$ and $f_{-1}(s) = 2(1 - s)$.) Similarly, the contribution to the market maker's expected utility from all intrinsic buyers selling in the open market having signal $s$ is given by $u(S, s) \triangleq (\delta + 1)f_1(s)\mathbf{P}(\sigma = 1) + (\delta - 1)f_{-1}(s)\mathbf{P}(\sigma = -1) = \delta - 1 + 2s$. (Same expressions hold for intrinsic sellers.) Moreover, from these expressions, we see that $u(B, s)$ is strictly decreasing in $s$, and $u(S, s)$ is strictly increasing in $s$.

Second, for $\delta < \delta_{\text{NODP}}$, consider the partial equilibrium with no trade in the dark pool, $\lambda_\delta^{\text{OM}}$. From the threshold structure of $\lambda_\delta^{\text{OM}}$, the contribution to the market maker's expected utility from intrinsic buyers who buy in the open market under $\lambda_\delta^{\text{OM}}$ is given by

$$\mathsf{E}[u(B, s)\mathbf{I}\{\lambda_{\delta,V}^{\text{OM}}(s) = (OM, B)\}] = \mathsf{E}[(\delta - 2s + 1)\mathbf{I}\{s > \frac{1 + V - \delta}{2}\}]$$
$$= \frac{1}{4}(1 + V - \delta)(\delta + V - 1).$$

Similarly, the contribution to the market maker's expected utility from intrinsic buyers

who sell in the open market under $\lambda_\delta^{\mathsf{OM}}$ is given by

$$\mathsf{E}[u(S,s)\mathbf{I}\{\lambda_{\delta,V}^{\mathsf{OM}}(s)=(OM,S)\}] = \mathsf{E}[(\delta+2s-1)\mathbf{I}\{s < \frac{1-V-\delta}{2}\}]$$
$$= \frac{1}{4}(1-V-\delta)(\delta-V-1).$$

(Symmetric expressions hold for intrinsic sellers.) Since $\delta < \delta_{\mathsf{NODP}} = 1 - V$, we see that the two quantities are negative.

Next, for $\delta < \delta_{\mathsf{NODP}}$, let $\lambda$ be a partial equilibrium involving trade in both the dark pool and the open market. Under the strategy profile $\lambda$, the dark pool draws some traders away from the open market, as compared to the partial equilibrium $\lambda_\delta^{\mathsf{OM}}$. From the threshold structure of the strategies in a partial equilibrium, it is straightforward to verify that all intrinsic buyers who buy in the open market under $\lambda_\delta^{\mathsf{OM}}$ but enter a buy order in the dark pool under $\lambda$ have lower value of the signal than that of any intrinsic buyer who buys in the open market in both the partial equilibria. Similarly, all intrinsic buyers who sell in the open market under $\lambda_\delta^{\mathsf{OM}}$ but enter a sell order in the dark pool under $\lambda$ have higher value of the signal than that of any intrinsic seller who sells in the open market in both the partial equilibria. (Same statement holds respectively for intrinsic sellers.) From the discussion in the preceding paragraph and the strict monotonicity of $u(B,s)$ and $u(S,s)$, it follows that the contribution to the market maker's expected utility from intrinsic buyers who buy in the open market is negative, and from those intrinsic buyers who sell in the open market is again negative. (Same statements hold for intrinsic sellers.) As there is no other contribution to the market maker's expected utility, we obtain that the market maker's expected utility is negative under $\lambda$. ∎

**Proof of Theorem 5.** Suppose in such an equilibrium, the (buy) fill rate in the dark pool is given by $f > 0$, and let $\delta$ be the transaction cost in the open market. For such an equilibrium to exist, it must be the case that a speculator with a high enough signal (in particular, $s = 1$) must be willing to trade in the open market as opposed to the dark pool. This implies that we must have

$$\delta < 1 - f. \tag{21}$$

Similarly, since there are some speculators that enter the dark pool, we must have the

following condition between the thresholds of the speculators' strategies:

$$\frac{1}{1+f} < \frac{\delta}{1-f}. \tag{22}$$

Supposing these conditions hold between the transaction cost $\delta$ and the fill rate $f$, the best response strategies of the traders are as follows: an intrinsic buyer with signal $s > \frac{\delta}{2(1-f)}$ enters the open market, whereas all other intrinsic buyers enter buy orders in the dark pool. Speculators with signals greater that $\delta/(1-f)$ buy in the open market, those with signals between $1/(1+f)$ and $\delta/(1-f)$ enter buy orders in the dark pool, those with signals between $1 - \delta/(1-f)$ and $f/(1+f)$ enter sell orders in the dark pool, those with signals less than $1 - \delta/(1-f)$ sell in the open market, and finally, the rest do not trade. (The strategy for intrinsic sellers is symmetric.)

Under this strategy profile for the traders, the market maker's expected utility is given by

$$u(\delta, f) = 2\left(1 - \frac{\delta}{2(1-f)}\right)\left(\delta - \frac{\delta}{2(1-f)}\right) + 2\mu\left(1 - \frac{\delta}{1-f}\right)\left(\delta - \frac{\delta}{1-f}\right).$$

As the market maker sets the transaction cost such that her expected profit is zero, we obtain the following expression for the transaction cost in terms of the buy fill rate:

$$\delta = 2(1-f)\left(\frac{2f(1+\mu) - 1}{2f(1+2\mu) - 1}\right). \tag{23}$$

Next, note that, in equilibrium, the fill rate $f$ is given by the ratio of the mass of sell orders to that of buy orders in the dark pool, when $\sigma = 1$. Thus, we obtain

$$f = \frac{\mu\left(\left(\frac{f}{1+f}\right)^2 - \left(1 - \frac{\delta}{1-f}\right)^2\right) + \left(1 - \left(1 - \frac{\delta}{2(1-f)}\right)^2\right)}{\mu\left(\left(\frac{\delta}{1-f}\right)^2 - \left(\frac{1}{1+f}\right)^2\right) + \left(\frac{\delta}{2(1-f)}\right)^2}$$

Substituting the value of $\delta$ from (23), canceling non-zero factors, and simplifying, we obtain that, in equilibrium, the fill rate must satisfy $g(f, \mu) = 0$, where $g(f, \mu)$ is defined as follows:

$$g(f, \mu) \triangleq f^4\left(16\mu^3 + 36\mu^2 + 24\mu + 4\right) - f^3\left(8\mu^2 + 12\mu + 4\right)$$
$$- f^2\left(20\mu^2 + 20\mu + 3\right) + f\left(8\mu + 4\right) + \mu - 1.$$

We observe that $g(1/\sqrt{\mu}, \mu) < 0$ and $g(2/\sqrt{\mu}, \mu) > 0$ for large enough $\mu$. Thus, for large enough $\mu$, there exists a root $f \in [1/\sqrt{\mu}, 2/\sqrt{\mu}]$ of the polynomial $g(\cdot, \mu)$.

To conclude there exists an equilibrium, we must verify that the root $f$ of the polynomial $g(\cdot, \mu)$ and the corresponding transaction cost in (23) satisfy the necessary conditions in (21) and (22). This is readily verified to be true for all large enough $\mu$. Thus, for all large enough $\mu$, there exists a competitive equilibrium, denoted by BOTH, where both the intrinsic traders and the speculators trade in both the dark pool and the open market.

Let $f_{\mathsf{BOTH}}(\mu)$ denote the buy fill rate, and $\delta_{\mathsf{BOTH}}(\mu)$ denote the transaction cost in the open market in the BOTH equilibrium. Since $f_{\mathsf{BOTH}}(\mu) \in [1/\sqrt{\mu}, 2/\sqrt{\mu}]$ for all large enough $\mu$, we let $t(\mu) = \mu f_{\mathsf{BOTH}}^2(\mu)$ for $t(\mu) \in [1, 4]$. Choose any subsequence $\mu_n \to \infty$ such that $\lim_{n \to \infty} t(\mu_n)$ exists, and let the limit be equal to $t$. As we have $g(f_{\mathsf{BOTH}}(\mu_n), \mu_n) = 0$ for all large enough $\mu_n$, on taking limits as $n \to \infty$ we obtain that $t$ satisfies

$$16t^2 - 20t + 1 = 0.$$

As $t(\mu_n) \in [1, 4]$ for each $\mu_n$, this implies $t = \frac{5+\sqrt{21}}{8}$. Since any converging subsequence has the same limit, we obtain that $t(\infty) = \lim_{\mu \to \infty} \mu f_{\mathsf{BOTH}}^2(\mu)$ exists, and satisfies $t(\infty) = \frac{5+\sqrt{21}}{8}$.

This implies that, for all large enough $\mu$, we have,

$$f_{\mathsf{BOTH}}(\mu) = \sqrt{\frac{t(\infty)}{\mu}} + o\left(\frac{1}{\mu}\right).$$

Finally, substituting for $f_{\mathsf{BOTH}}(\mu)$ in the expression for the transaction cost (23), we obtain

$$\delta_{\mathsf{BOTH}}(\mu) = 1 - \left(1 + \frac{1}{4t(\infty)}\right)\sqrt{\frac{t(\infty)}{\mu}} + o\left(\frac{1}{\mu}\right). \qquad \blacksquare$$

**Proof of Lemma 4.** Suppose $V = 1$ and $\mu \geq 0$. From the proof of Lemma 3, we obtain that the transaction cost in the NODP equilibrium satisfies

$$\delta_{\mathsf{NODP}}(\mu) = 1 - \frac{1}{\sqrt{1+\mu}}.$$

Given this, the traders' strategies in equilibrium are as follows: all intrinsic buyers with signals $s \geq \delta_{\mathsf{NODP}}(\mu)/2$ buy at the open market; all intrinsic sellers with signals $s \leq 1 - \delta_{\mathsf{NODP}}(\mu)/2$ sell at the open market; speculators with signals $s \geq (1 + \delta_{\mathsf{NODP}}(\mu))/2$ buy at the open market; speculators with signals $s \leq (1 - \delta_{\mathsf{NODP}}(\mu))/2$ see at the open market; and all other traders do not trade. Using the structure of the equilibrium strategies and the

expression for $\delta_{\text{NODP}}(\mu)$, we obtain that the welfare in this equilibrium is given by

$$
\begin{aligned}
w_{\text{NODP}}(\mu) &= 1 - \left( \frac{\delta_{\text{NODP}}(\mu)}{2} \right)^2 + \left( 1 - \frac{\delta_{\text{NODP}}(\mu)}{2} \right)^2 \\
&= 1 + \frac{1}{\sqrt{1+\mu}}.
\end{aligned}
$$

Here the first line follows from the fact that, when $\sigma = 1$, the mass of intrinsic buyers who buy at the open market is given by the first two terms, and the mass of intrinsic sellers who sell at the open market is given by the third term. (The expressions are symmetrically interchanged when $\sigma = -1$.) This completes the proof. ∎

**Proof of Theorem 6.** The welfare in the BOTH equilibrium is given by

$$
\begin{aligned}
w_{\text{BOTH}}(\mu) &= 1 + \left( 1 - \left( \frac{\delta_{\text{BOTH}}(\mu)}{2(1 - f_{\text{BOTH}}(\mu))} \right)^2 \right) + f_{\text{BOTH}}(\mu) \left( \frac{\delta_{\text{BOTH}}(\mu)}{2(1 - f_{\text{BOTH}}(\mu))} \right)^2 \\
&= \frac{7}{4} + \frac{1}{4} \left( 1 + \frac{1}{2t(\infty)} \right) \sqrt{\frac{t(\infty)}{\mu}} + o\left( \frac{1}{\mu} \right).
\end{aligned}
$$

Here the first two terms in the first equality follow from the fact that, when $\sigma = 1$, all intrinsic sellers sell the asset, and all intrinsic buyers in the open market buy the asset. The third term in the first equality follows from the fact that only a fraction $f(\mu)$ of the intrinsic buyers in the dark pool end up holding the asset, when $\sigma = 1$. All such traders contribute $V = 1$ to the welfare. (The case when $\sigma = -1$ is symmetric.) We obtain the second line after substituting for $f_{\text{BOTH}}(\mu)$ and $\delta_{\text{BOTH}}(\mu)$ using expressions from the proof of Theorem 5.

Since, for large enough $\mu$, we have $w_{\text{NODP}}(\mu) = 1 + \frac{1}{\sqrt{1+\mu}}$, the result in the theorem statement follows. ∎

## B.  Welfare comparisons for the benchmark case $\left( \mu = 0 \right)$

In the following, we assume that the mass of speculators $\mu = 0$ and $V \in (0, 1]$. For $\delta = \delta_{\text{NODP}}$, no intrinsic buyer sells in the open market. Thus, the strategy of an intrinsic buyer can be represented as follows:

do not
trade

buy in
open market

Intrinsic
buyer

$s = 0$

$b$

$s = 1$
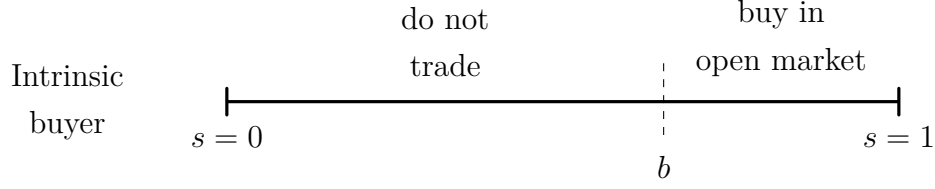
Without DP

It is straightforward to verify that $b = (1 - V + \delta_{\mathsf{NODP}})/2 = 1 - V$. For $\delta \geq \delta_{\mathsf{NODP}}$, in any partial equilibrium involving the dark pool, the thresholds for an intrinsic buyer's strategy can be represented as follows:

sell in
dark pool

do not
trade

buy in
dark pool

buy in
open market

Intrinsic
buyer

$s = 0$

$a$

$b_-$

$b_+$

$s = 1$
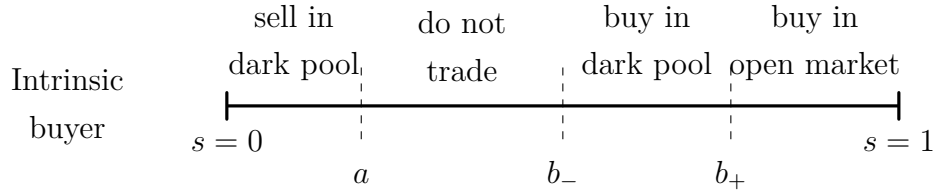
With DP

The thresholds for an intrinsic buyer's strategy are given by

$$a = \frac{f(1 - V)}{f(1 - V) + 1 + V}, \qquad b_+ = \min\left\{\max\left\{\frac{\delta}{(1 - f)(1 + V)}, \frac{1 - V + \delta}{2}\right\}, 1\right\},$$

$$b_- = \min\left\{\frac{1 - V}{f(1 + V) + 1 - V}, b_+\right\}. \tag{24}$$

These thresholds satisfy one of two conditions:

Case (i)   $b \leq b_- \leq b_+$;   OR   Case (ii)   $b_- < b \leq b_+$.

In Case (i), it is straightforward to see that the welfare is lower, as fewer intrinsic buyers end up holding the security. Hence, hereafter we will focus on Case (ii). From $b_- < b$, we obtain that the fill rate has to satisfy $f > V/(1 + V)$.

In this case, the change in the welfare from the introduction of the dark pool can be written as

$$\text{Change in welfare} = F_1(b, b_+)(-V) + F_1(b_-, b_+)f(+V) + F_1(0, a)(-V)$$
$$+ F_{-1}(b_-, b)(+V) + F_{-1}(0, a)f(-V). \tag{25}$$

Here the first line corresponds to the net change in welfare when $\sigma = +1$, and the second line corresponds to the case when $\sigma = -1$. The first term on the first line represents those

traders who initially were trading in the open market, but now have decided to enter the dark pool. By forgoing trading in the open market, these traders each contribute a welfare loss of $-V$. The mass of such traders is $F_1(b, b_+)$. This is offset by the trade in the dark pool: each buyer in the dark pool contributes a welfare gain of $+V$ with probability $f$ equal to the fill rate. The mass of such buyers is $F_1(b_-, b_+)$. Finally, we have those intrinsic buyers who were initially not trading, but now have decided to enter an sell order in the dark pool. As $\sigma = 1$, these orders are filled with probability one, and the mass of such orders is $F_1(0, a)$. Each such order contributes a welfare loss of $-V$. The terms on the second line are obtained in a similar manner.

Rewriting (25), we obtain that the change in welfare is equal to

$$V\left(-F_1(b, b_+) + F_{-1}(b_-, b) - F_1(0, a)\right) + Vf\left(F_1(b_-, b_+) - F_{-1}(0, a)\right).$$

Using the fact that $F_1(x, y) = y^2 - x^2$ and $F_{-1}(x, y) = (1 - x)^2 - (1 - y)^2$, we obtain

Change in welfare

$$= V\left(F_1(b) - F_1(b_+) + F_{-1}(b) - F_{-1}(b_-) - F_1(a)\right) + Vf\left(F_1(b_+) - F_1(b_-) - F_{-1}(a)\right)$$

$$= V\left(b^2 - b_+^2 + 1 - (1 - b)^2 - 1 + (1 - b_-)^2 - a^2\right) + Vf\left(b_+^2 - b_-^2 - 1 + (1 - a)^2\right)$$

$$= V\left(\left(-b_+^2 + 2b - 2b_- + b_-^2 - a^2\right) + f\left(b_+^2 - b_-^2 - 2a + a^2\right)\right).$$

Next, observe that in a partial equilibrium, the (buy) fill rate $f$ is given by

$$f \triangleq \frac{\text{mass of sell orders}}{\text{mass of buy orders}} = \frac{F_1(0, a) + F_{-1}(b_-, b_+)}{F_{-1}(0, a) + F_1(b_-, b_+)} = \frac{a^2 + 2b_+ - 2b_- - b_+^2 + b_-^2}{2a - a^2 + b_+^2 - b_-^2}.$$

Thus, letting $\Delta W \triangleq (\text{change in welfare})/V$, we obtain the change in welfare is given by

$$\Delta W = -b_+^2 + 2b - 2b_- + b_-^2 - a^2 + \left(\frac{a^2 + 2b_+ - 2b_- - b_+^2 + b_-^2}{2a - a^2 + b_+^2 - b_-^2}\right)(b_+^2 - b_-^2 - 2a + a^2).$$

Observing from (24) that $b, a, b_-, b_+$ are functions of $V, f, \delta$, we define the following functions,

$$
\begin{aligned}
Q(V, f, \delta) &\triangleq -b_+^2 + 2b - 2b_- + b_-^2 - a^2, \\
R(V, f, \delta) &\triangleq b_+^2 - b_-^2 - 2a + a^2, \\
N(V, f, \delta) &\triangleq a^2 + 2b_+ - 2b_- - b_+^2 + b_-^2 \\
D(V, f, \delta) &\triangleq 2a - a^2 + b_+^2 - b_-^2.
\end{aligned}
\tag{26}
$$

In a partial equilibrium, the fill rate satisfies $f = N/D$. Moreover, taken as a function of $V, f, \delta$, we have $\Delta W = Q + (N/D)R$ for all $V \in [0, 1]$, $f \in (V/(1+V), 1]$ and $\delta \geq \delta_{\text{NODP}}$.

**Theorem 7.** *Let $\mu = 0$. For any $\delta \geq \delta_{\text{NODP}}$, the welfare in any partial equilibrium involving the dark pool is lower than that in the competitive no-dark-pool equilibrium.*

**Proof.** Since if $b_- \geq b$, fewer intrinsic buyers end up holding the security, hereafter we assume that $b_- < b$. Thus, from Lemma 6, we have $Q + R \leq 0$. Now, if $Q \leq 0$, we obtain that in any partial equilibrium,

$$\Delta W = Q + (N/D)R \leq \max\{Q + R, Q\} \leq 0,$$

where the first inequality follows from the fact that, in any partial equilibrium, the fill rate $N/D \in [0, 1]$, and the last inequality follows from Lemma 6. Thus, for the rest of the proof, we further assume that $Q > 0$.

Next, as we assume $b_- < b$, this implies $f > V/(1+V)$. Further, since $Q > 0$, from Lemma 7, we obtain that $b_+ < 1$. This in turn implies that

$$1 > b_+ = \frac{\delta}{(1-f)(1+V)} \geq \frac{\delta_{\text{NODP}}}{(1-f)(1+V)} = \frac{1-V}{(1-f)(1+V)}, \tag{27}$$

from which we obtain that $f < 2V/(1+V)$. These two bounds allow us to write $f = V(1+u)/(1+V)$ for some $u \in (0, 1)$. From the definition of the thresholds, (24), we have

$$a = \frac{f(1-V)}{f(1-V)+1+V} = \frac{V(1-V)(1+u)}{1+3V+uV-uV^2},$$
$$b_- = \frac{1-V}{f(1+V)+1-V} = \frac{1-V}{1+uV}, \tag{28}$$

and from (27), we obtain the following lower-bound on $b_+$:

$$b_+ \geq \frac{1-V}{(1-f)(1+V)} = \frac{1-V}{1-uV}. \tag{29}$$

From Lemma 8, we obtain an upper-bound on $b_+$, namely, $b_+ < 2b - b_- - 2a^2 + a$. Taken together, we write $b_+$ as

$$b_+ = (1-t)\left(\frac{1-V}{1-uV}\right) + t\left(2b - b_- - 2a^2 + a\right)$$

for some $t \in [0, 1)$. Using the preceding relations, we can now express $\Delta W$ as a function of $V \in (0, 1]$, $u \in (0, 1)$ and $t \in [0, 1)$. However, we can further restrict the domain of $V, u$. To

see, note that we have

$$\frac{1-V}{1-uV} \le b_+ < 2b - b_- - 2a^2 + a.$$

Using the expressions for $b_-$ and $a$ from (28), and the fact that $b = 1 - V$, we obtain

$$\frac{1-V}{1-uV} < 2(1-V) - \frac{1-V}{1+uV} - 2\frac{V^2(1-V)^2(1+u)^2}{(1+3V+uV-uV^2)^2} + \frac{V(1-V)(1+u)}{1+3V+uV-uV^2}.$$

Rearranging and canceling non-negative factors, we obtain

$$0 \ge u^4(2V^5 - 3V^4 + V^3) + u^3(-9V^4 + 8V^3 + 5V^2) + u^2(2V^4 + 19V^3 + 12V^2 + 3V)$$
$$+ u(-3V^2 - 1) - 2V^2 - V - 1.$$

In Lemma 9, we show that this implies that $V \le \min(\frac{37-30u}{25}, 1)$. Thus, it suffices to show that $\Delta W \le 0$ for $u \in (0,1)$, $0 < V \le \min(\frac{37-30u}{25}, 1)$ and $t \in [0,1)$. Furthermore, since $D > 0$ in any partial equilibrium with trade in the dark pool, this is equivalent to showing $QD + NR \le 0$ for the same values of $u, V$ and $t$.

We show this by splitting the analysis into two cases, namely when (1) $u \in (0, 2/5]$ and $V \in [0,1]$; and (2) $u \in (2/5, 1)$ and $0 < V \le \frac{37-30u}{25}$.

*Case 1.* $u \in (0, 2/5]$ *and* $V \in (0,1]$: We make the following substitution:

$$u = \frac{2}{5(1+x^2)}, \qquad\qquad V = \frac{1}{1+y^2}, \qquad\qquad t = \frac{1}{1+z^2},$$

where $x, y, z \in \mathbb{R}^3$. On making the substitution, and canceling non-negative factors from the denominator, we are left with a polynomial in $x^2, y^2$, and $z^2$ with all monomial terms non-positive. From this, we obtain that $QD + NR$ is non-positive.

*Case 2.* $u \in (2/5, 1)$ *and* $0 < V \le \frac{37-30u}{25}$: In this case, we make the following substitution:

$$u = 1 - \frac{3}{5(1+x^2)}, \qquad\qquad V = \frac{1}{1+y^2}\left(\frac{37-30u}{25}\right), \qquad\qquad t = \frac{1}{1+z^2},$$

where $x, y, z \in \mathbb{R}^3$. Again, on making the substitution, and canceling non-negative factors from the denominator, we are left with a polynomial in $x^2, y^2$, and $z^2$ with all monomial terms non-positive. From this, we obtain that $QD + NR$ is non-positive. ∎

The following lemma is useful for proving the main result.

**Lemma 6.** *For $\delta \ge \delta_{\mathsf{NODP}}$, in any partial equilibrium involving the dark pool with $b_- < b$, we have $Q + R \le 0$.*

**Proof**. From (26), we have $Q + R = 2b - 2b_- - 2a$. Since, $b_- < b$, we have from (24),

$$b_- = \frac{1 - V}{f(1 + V) + 1 - V}, \qquad a = \frac{f(1 - V)}{f(1 - V) + 1 + V}, \qquad b = 1 - V.$$

Thus, we get

$$\begin{aligned} Q + R &= 2(1 - V)\left(1 - \frac{1}{f(1 + V) + 1 - V} - \frac{f}{f(1 - V) + 1 + V}\right) \\ &= -\frac{2V(1 - V)(1 - f)^2(1 + V)}{(f(1 + V) + 1 - V)(f(1 - V) + 1 + V)} \\ &\leq 0. \qquad\blacksquare \end{aligned}$$

**Lemma 7.** *For $\delta \geq \delta_{\mathsf{NODP}}$, in any partial equilibrium involving the dark pool with $b_- < b$ and $b_+ = 1$, we have $Q \leq 0$.*

**Proof**. Using the definition of $Q$ from (26) for $b_+ = 1$, we have

$$\begin{aligned} Q &\triangleq -b_+^2 + 2b - 2b_- + b_-^2 - a^2, \\ &= -1 + 2b - 2b_- + b_-^2 - a^2, \\ &= 2b - 2 + (1 - b_-)^2 - a^2, \\ &= -2V + \frac{f^2(1 + V)^2}{(f(1 + V) + 1 - V)^2} - \frac{f^2(1 - V)^2}{(f(1 - V) + 1 + V)^2} \\ &= -\frac{2V\left(f^4(1 - V^2)^2 + 4f^3V^2(1 - V^2) + 2f^2(1 + 3V^4) + 4f(1 - V^4) + (1 - V^2)^2\right)}{(f(1 + V) + 1 - V)^2(f(1 - V) + 1 + V)^2} \\ &\leq 0, \end{aligned}$$

where the fourth equality follows from the definition of the thresholds (24), and the final inequality follows from the fact that the term inside the parenthesis in the numerator is always non-negative. $\qquad\blacksquare$

**Lemma 8.** *For $\delta \geq \delta_{\mathsf{NODP}}$, in any partial equilibrium involving the dark pool, if $Q > 0$, then we have $b_+ < 2b - b_- - 2a^2 + a$.*

**Proof**. Observe that $Q > 0$ implies

$$b_+^2 < 2(b - b_-) + b_-^2 - a^2. \tag{30}$$

Furthermore, in a partial equilibrium, we have $N/D \leq 1$. This implies that

$$a^2 + 2b_+ - 2b_- - b_+^2 + b_-^2 \leq 2a - a^2 + b_+^2 - b_-^2$$

which leads to

$$
\begin{aligned}
b_+ &\leq b_+^2 - b_-^2 + b_- - a^2 + a \\
&< (2(b - b_-) + b_-^2 - a^2) - b_-^2 + b_- - a^2 + a \\
&= 2b - b_- - 2a^2 + a,
\end{aligned}
$$

where the second inequality follows from (30). ∎

**Lemma 9.** *For $u, V \in [0,1]$ with $V > min((37 - 30u)/25, 1)$, we have*

$$
\begin{aligned}
&u^4(2V^5 - 3V^4 + V^3) + u^3(-9V^4 + 8V^3 + 5V^2)+ \\
&\quad u^2(2V^4 + 19V^3 + 12V^2 + 3V) + u(-3V^2 - 1) - 2V^2 - V - 1 > 0.
\end{aligned}
$$

**Proof**. The statement is trivially true for $u \leq 2/5$, as in that case $(37 - 30u)/25 \geq 1$. Thus, we only need to consider $u \in (2/5, 1]$ and $V > (37 - 30u)/25$. We make the following change of variables:

$$u = \frac{2 + 3x}{5}, \qquad V = (1 - y) + y\left(\frac{37 - 30u}{25}\right) = 1 - \frac{18}{25}xy,$$

where $x \in (0, 1]$ and $y \in [0, 1)$. Substituting, we obtain the polynomial in $x$ and $y$ as

$$
\begin{aligned}
P(x, y) = {}& y^5 \left( \frac{-306110016x^9}{6103515625} - \frac{816293376x^8}{6103515625} - \frac{816293376x^7}{6103515625} - \frac{362797056x^6}{6103515625} - \frac{60466176x^5}{6103515625} \right) \\
&+ y^4 \left( \frac{59521392x^8}{244140625} + \frac{31177872x^7}{244140625} - \frac{49128768x^6}{244140625} - \frac{36531648x^5}{244140625} - \frac{5038848x^4}{244140625} \right) \\
&+ y^3 \left( \frac{-4251528x^7}{9765625} + \frac{10707552x^6}{9765625} - \frac{2676888x^5}{9765625} - \frac{22884768x^4}{9765625} - \frac{10054368x^3}{9765625} \right) \\
&+ y^2 \left( \frac{26244x^6}{78125} - \frac{148716x^5}{78125} + \frac{813564x^4}{78125} + \frac{1241244x^3}{78125} + \frac{335664x^2}{78125} \right) \\
&+ y \left( \frac{-1458x^5}{15625} + \frac{972x^4}{15625} - \frac{366768x^3}{15625} - \frac{451548x^2}{15625} - \frac{81198x}{15625} \right) \\
&+ \frac{108x^3}{125} + \frac{1836x^2}{125} + \frac{2004x}{125} + \frac{52}{125}.
\end{aligned}
$$

When $y = 0$, we see trivially that $P(x, y) > 0$ for $x \in (0, 1]$. Next, for $x \in (0, 1]$ and

$y \in (0, 1)$, we again make the following change of variables:

$$x = \frac{1}{1+t^2}, \quad y = \frac{1}{1+s^2},$$

where $s, t \in \mathbb{R}$. Making the substitution, and writing the polynomial $P$ as functions of $s$ and $t$, we obtain,

$$
\begin{aligned}
(1+s^2)^5 &(1+t^2)^9 P(t,s) \\
= \; & \frac{4s^{10}\,(t^2+1)^6}{125}\left(13t^6 + 540t^4 + 1500t^2 + 1000\right) \\
& + \frac{2s^8\,(t^2+1)^4}{15625}\left(16250t^{10} + 666901t^8 + 2853080t^6 + 4570700t^4 + 3169000t^2 + 800000\right) \\
& + \frac{4s^6\,(t^2+1)^3}{78125}\left(81250t^{12} + 3212760t^{10} + 15539976t^8 + 30302525t^6 \right. \\
& \qquad\qquad \left. + 28784820t^4 + 13201200t^2 + 2317000\right) \\
& + \frac{4s^4\,(t^2+1)^2}{9765625}\left(10156250t^{14} + 386376875t^{12} + 2071716000t^{10} + 4709497408t^8 \right.\\
& \qquad\qquad \left. + 5564360065t^6 + 3554905650t^4 + 1148445500t^2 + 142835000\right) \\
& + \frac{4s^2\,(t^2+1)}{244140625}\left(126953125t^{16} + 4639484375t^{14} + 27135400000t^{12} \right.\\
& \qquad\qquad + 69809011025t^{10} + 97143948313t^8 + 77507176490t^6 \\
& \qquad\qquad \left. + 34774505800t^4 + 7864505000t^2 + 654250000\right) \\
& + \frac{1}{6103515625}\left(2539062500t^{18} + 88985156250t^{16} + 559961250000t^{14} \right.\\
& \qquad\qquad + 1594118520000t^{12} + 2518282168800t^{10} + 2354402950374t^8 \\
& \qquad\qquad \left. + 1295555779740t^6 + 387607598400t^4 + 49432614000t^2 + 350540000\right).
\end{aligned}
$$

From this, it follows that $P(t,s) > 0$ for all $s, t \in \mathbb{R}$. Hence $P(x,y) > 0$ for all $x \in [0,1]$ and $y \in [0,1)$. This completes the proof. $\blacksquare$