Capital Risk: Precautionary and Excess Saving

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Abstract

Precautionary saving typically refers to the additional investment in a risk free asset when exogenous labor income is risky versus certain. When risky income results endogenously from the investment in a risky asset, the meaning and characterization of precautionary saving change and far less is known about this case. Capital risk is important for macro and finance models as risk often arises from the variability in asset returns, and precautionary saving plays a key role in interpreting the results. We assume KPS (Kreps-Porteus-Selden) preferences with additively separable time preferences and HARA (hyperbolic absolute risk aversion) risk preferences. Necessary and sufficient conditions are derived for saving to increase when investment is in a portfolio of risky and risk free assets versus just a risk free asset. Time preferences play an essential role whereas the frequently referenced risk preference property prudence is irrelevant. In macrofinance analyses, the equilibrium risk free rate is often shown to be less in the presence versus absence of capital risk and this is interpreted as reflecting a precautionary motive. However this interpretation is not in consonance with the corresponding partial equilibrium demand analysis.

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Leland (1968) and Kimball (1990) introduced and refined the notion of precautionary saving in a two period setting with exogenous risky (labor) income where individuals have EU (expected utility) preferences. The basic idea is that a consumer will save more in the form of a risk free asset when period two income is risky than when it is certain. This additional saving is precautionary in the sense that it creates a buffer stock of certain period two income to offset the possible occurrence of a bad outcome for risky income.[1] It follows from Leland (1968) and Kimball (1990) that a consumer will exhibit precautionary saving if and only if the individual's risk preferences satisfy a property referred to as prudence which corresponds to the NM (von Neumann-Morgenstern) index of the assumed EU representation having a positive third derivative. Kimball and Weil (2009) extend this analysis of income risk to the more general KPS (Kreps and Porteus 1978 and Selden 1978) preference structure where time and risk preferences can be fully separated.[2] The characterization of when a consumer exhibits precautionary saving becomes more complicated, but prudence continues to play an important role.

In recent years, a number of authors working on asset pricing and macro issues have characterized their findings in terms of precautionary saving effects. These papers typically assume that individuals face risk emanating from the presence of risky assets rather than exogenous risky income. (Examples include Barsky 1989, Campbell and Cochrane 1999, Yi and Choi 2006, Reis 2009, Gomes and Ribeiro 2015 and Cochrane 2017.) Often formulas for the equilibrium risk free rate are derived which contain risk terms that reduce the risk free return and are interpreted as reflecting "precautionary saving effects" or "precautionary motives".[3] However for the capital risk case these papers are considering, relatively little work has been done on the underlying microfoundations.[4] In this paper, we seek to make progress in filling this void by first examining asset demand and saving behavior in the classic consumption-portfolio setting, second deriving expressions for the equilibrium risk free rate based on the same underlying specification of consumer preferences and third comparing the implications for saving behavior of

[1] See, for example, Carroll, Hall and Zeldes (1992).

[2] See Selden (1978, 1979).

[3] Other models derive expressions for consumption growth which contain a risk term. If the coefficient of this term is positive, then the increase in growth is interpreted as reflecting "precautionary" saving.

[4] Exceptions which do consider precautionary saving in the presence of capital risk include Weil (1990), Langlais (1995), Gollier (2001, chapter 19) and Eeckhoudt and Schlesinger (2008). Each of these assume special cases of the KPS preference model such as EU and are discussed in later sections.

the equilibrium and partial equilibrium demand analyses. (A similar analysis of saving behavior for the simpler single risky asset consumption-saving problem is provided in Supplemental Appendix B.2.)

As Sandmo (1970) warned, characterizing saving behavior is more complicated in the presence of capital risk than income risk. A key difference as emphasized by Gollier (2001, Chapter 19) is that income risk is typically exogenous while capital risk is endogenous.[5] For the latter, the individual's own actions create risk rather than just buffering against risk and thus it does not seem appropriate, at least to us, to refer to saving in this case as being "precautionary". However, it is still of interest to ask whether in a consumption-portfolio setting, saving is larger when there is both a risky asset and risk free asset versus just a risk free asset. This difference in saving will be referred to as (positive) **excess saving**. One can also ask whether the consumer's demand for the risk free asset is larger in the portfolio versus certainty case. Because the extra demand for the risk free asset can be viewed as serving as a buffer against bad outcomes for the simultaneously chosen risky asset, we will refer it as **precautionary saving.**

In addressing the microfoundation analysis of excess and precautionary saving, we seek to separate the effects of time and risk preferences and follow Kimball and Weil (2009) in assuming KPS preferences as well as two time periods. The latter assumption allows us to derive results which hold for arbitrary risky asset return distributions and for additively separable time and HARA (hyperbolic absolute risk aversion) risk preferences.[6] Assuming that the representation of time preferences is additively separable rather than the typical, more restrictive CES (constant elasticity of substitution) form allows us to introduce a generalization of the EIS (elasticity of intertemporal substitution) which we refer to as the $EMRS$ (elasticity of the intertemporal marginal rate of substitution). The latter time preference measure is shown to have a quite intuitive geometric interpretation and plays a central role in this paper.[7] For dynamic demand and equilibrium analyses, in order to obtain tractable solutions, researchers typically make strong assumptions on both the form of preferences and asset return distributions and also introduce small risk approximations. Our use of KPS utility enables us to avoid such strong restrictions but at the cost of considering only two periods.

[5] Recently, however, some work has been done in a consumption-leisure setting where the level of income risk becomes endogenous. See, for instance, Nocetti and Smith (2011).

[6] See Gollier (2001) for a discussion of this class of utilities and their properties.

[7] Our generalization suggests that the assumption of CES or more generally stationary time preferences is overly strong. Additive separability of the time preference utility is all that is required. Indeed the EMRS result highlights that the specific form of the period one utility $u_1(c_1)$ is irrelevant, although $u_1' > 0$ and $u_1'' < 0$ are required.

Assuming a standard exchange economy in our consumption-portfolio setting, we solve for the equilibrium risk free rate in the absence and presence of a risky asset and characterize when the latter rate is smaller or larger than the former rate.

We show that whether saving is larger or smaller in the presence versus absence of a risky asset involves restrictions only on the time preference $EMRS$ for all but one member of the HARA (hyperbolic absolute risk aversion) class of risk preferences. For the CARA (constant absolute risk aversion) member, excess saving is always negative. The restrictions on the $EMRS$ can be interpreted as a competition between the classic substitution and smoothing effects and a second smoothing effect. For the special case of CES time and CRRA (constant relative risk aversion) risk preferences, positive excess saving requires that the $EIS > 1$. If one assumes the EU special case of KPS preferences, then as Gollier (2001, p. 289) observes this condition implies that the Arrow-Pratt relative risk aversion measure must be less than 1 which is inconsistent with much higher values typically assumed. A similar issue arises for the condition for positive excess saving for the EU external habit model of Campbell and Cochrane (1999).[8] In both of these instances, the KPS generalization of EU preferences accommodates positive excess saving when a quite reasonable restriction on the time preference $EMRS$ holds and virtually no restriction is required on the degree of relative risk aversion. In contrast to the case of income risk where prudence plays a central role in determining whether the consumer exhibits precautionary saving, we provide examples showing that for capital risk, prudence is largely irrelevant.[9] This is at odds with assertions in the literature based on EU preferences that prudence is also important for capital risk.[10] We show that the conflicting views on the relevance of prudence result from the well-known confounding of time and risk preference effects by EU preferences.[11]

As observed above, we also investigate the conditions under which the level of demand for the risk free asset (as opposed to saving in the form of both the risky and risk free asset) is larger in the consumption-portfolio or risky setting versus the certainty setting. This form of precautionary demand can only occur if excess saving is positive. Thus the restrictions are even more stringent. An example is given where for CES time and CRRA risk preferences, excess saving is guaranteed if the $EIS > 1$ but precautionary saving requires that the EIS equal

[8]See the discussion following Corollary 2 in Subsection 4.2.

[9]For instance, see the analysis in Example 1.

[10]It is not uncommon to find suggestions in a capital risk setting even when EZ (Epstein and Zin 1989) preferences are assumed that if the third derivative of the risk preference index is positive, a precautionary motive exists. (See, for example, Gomes and Ribeiro 2015, p. 110.)

[11]See the discussions of this issue in Section 4, following Theorem 4 and Corollary 2.

an unrealistic value of about 5.

Comparing the equilibrium risk free rates derived in the risky and certainty settings, we show that the former is always smaller than the latter for the DARA (decreasing absolute risk aversion) and CARA members of the HARA class of time and risk preferences.[12] This result does not depend on the $EMRS$, which plays a pivotal role in the demand conditions for positive excess saving and precautionary saving. Also, our equilibrium result implies that the risk free asset price and demand are larger in the risky setting. However this conclusion is at odds with the partial equilibrium demand analysis which proves that for the same forms of preferences, individual consumers have negative excess and negative precautionary saving. This lack of consonance between the equilibrium and underlying demand conditions for precautionary saving arises due to differences in the definition of precautionary saving and several incompatible assumptions.[13] Finally, to our knowledge all existing results in the literature have the equilibrium risk free rate being lower in the risky versus certainty setting. In contrast, we show that for a specific characterization of HARA time and risk preferences, the equilibrium risk free rate in the risky setting can also be larger than or equal to the rate in the certainty setting.

In the next section, we first review KPS preferences and then introduce notation and definitions associated with the consumption-portfolio optimization. Section 2 provides a motivating example which illustrates key differences between the capital and income risk cases and between assuming EU and KPS preferences. Section 3 considers the certainty consumption-saving problem and derives an important comparative static result characterizing when for an additively separable utility and a linear constraint, saving increases or decreases with the associated rate of return. Section 4 examines excess and precautionary saving in the consumption-portfolio setting. Section 5 analyzes the relationship between the equilibrium risk free rate in the risky versus certainty setting and compares the results with those obtained for excess and precautionary saving in Section 4. The final section offers concluding comments. Proofs are given in Appendix A. Supporting calculations and the analysis of excess saving for the single risky asset consumption-saving problem are provided in the Supplemental Appendix B.

[12] See Theorem 5 for a more complete statement of the result.

[13] See the discussion near the end of Section 5.

1 Preliminaries

In this section, we first review KPS preferences and then formally introduce the consumption-portfolio choice problem.

1.1 KPS preferences

The consumer has preferences over certain first period and random second period consumption pairs, $(c_1, \widetilde{c}_2)$, which can be represented in two equivalent ways. First following Kreps and Porteus (1978), the utility can take the form

$$\mathcal{U}(c_1, EV(\widetilde{c}_2)). \tag{1}$$

The expression $EV(\widetilde{c}_2)$ is the standard single period state independent EU representation over risky period two consumption, where V is the NM index. In general, the representation (1) fails to be linear in probabilities and diverges from the classic two period EU

$$EW(c_1, \widetilde{c}_2). \tag{2}$$

The index $\mathcal{U}$ in (1) can be viewed as a utility over period one consumption and period two EU values.

The second representation, due to Selden (1978), is given by

$$U(c, \widehat{c}_2) = U\left(c_1, V^{-1}EV(\widetilde{c}_2)\right), \tag{3}$$

where U represents time preferences over certain (c_1, c_2)-pairs and $EV(\widetilde{c}_2)$ is a standard one period EU representation of risk preferences defined over distributions of risky period two consumption corresponding to the random variable $\widetilde{c}_2$. The NM index V is continuous and strictly increasing in c_2. In general the index V can depend on c_1, but in this paper we will focus on the standard case where $V(c_2)$ is independent of c_1. The second argument of U in (3) is the period two certainty equivalent associated with random second period consumption

$$\widehat{c}_2 = V^{-1}\left(EV(\widetilde{c}_2)\right). \tag{4}$$

The representation (3) is fully defined by the (U, V)-pair.[14]

Clearly (1) and (3) are equivalent if one defines

$$\mathcal{U}_{c_1}(\cdot) = U_{c_1} \circ V^{-1}(\cdot). \tag{5}$$

[14]See Selden (1978) for the corresponding axiomatic development and representation theorem for (3). He considers the more general case where the NM index can depend on period one consumption.

Since in the ensuing analysis of precautionary saving, considerable attention will be given to the separate roles of risk preferences defined over risky period two consumption and time preferences defined over certain consumption pairs, the representation (3) is more natural than (1) given that $\mathcal{U}$ is defined over consumption-utility pairs. Also, the form (3) is also more intuitive for the two stage optimization process used extensively in Section 4 below. However because the two preference models are fully equivalent, (3) will be referred to as the KPS utility.

The KPS representation can converge to the two period EU (2) as a special case. To see this, suppose

$$U(c_1, c_2) = u_1(c_1) + u_2(c_2) \qquad \text{and} \qquad V(c_2) = u_2(c_2). \tag{6}$$

Then $U(c_1, \widehat{c}_2)$ is ordinally equivalent to the additively separable two period EU function

$$\begin{aligned} U(c_1, \widehat{c}_2) &= u_1(c_1) + u_2 \circ V^{-1}(EV(\widetilde{c}_2)) = u_1(c_1) + Eu_2(\widetilde{c}_2) \\ &= E[u_1(c_1) + u_2(\widetilde{c}_2)]. \end{aligned} \tag{7}$$

Throughout the rest of this paper, the time or certainty preference representation will take the form

$$U(c_1, c_2) = u_1(c_1) + u_2(c_2), \tag{8}$$

where $u_i' > 0$ and $u_i'' < 0$ $(i = 1, 2)$ and $(c_1, c_2) \in \mathbb{R}_+^2$.[15] It is also assumed that the NM index V satisfies $V' > 0$ and $V'' < 0$. Denote the classic Arrow-Pratt measures of absolute and relative risk aversion, respectively, by

$$\tau_A(c_2) = -\frac{V''(c_2)}{V'(c_2)} \qquad \text{and} \qquad \tau_R = -\frac{c_2 V''(c_2)}{V'(c_2)}. \tag{9}$$

We will refer to risk preferences satisfying (i) $\tau_A' <, =, > 0$ as exhibiting DARA, CARA and IARA (increasing absolution risk aversion), respectively and (ii) $\tau_R' <, =, > 0$ as exhibiting DRRA (decreasing relative risk aversion), CRRA and IRRA (increasing relative risk aversion), respectively. It is understood that conditions involving $\tau_A'(c_2)$ and $\tau_R'(c_2)$ hold for all c_2.

One popular version of KPS utility that will be frequently referenced below is defined by the following CES certainty and CRRA NM risk preference forms

$$U(c_1, c_2) = -\frac{c_1^{-\delta_1}}{\delta_1} - \beta\frac{c_2^{-\delta_1}}{\delta_1} \qquad \text{and} \qquad V(c_2) = -\frac{c_2^{-\delta_2}}{\delta_2}, \tag{10}$$

[15] It should be observed that we generalize the commonly assumed discounted utility

$$U(c_1, c_2) = u(c_1) + \beta u(c_2)$$

by allowing the period one and two utilities to differ. If there is a discount function, it is embedded in the function u_2.

where $\delta_i > -1$ $(i = 1, 2)$.

1.2 Consumption-Portfolio Problems

In subsequent sections, we assume that there is one risky asset and one risk free asset when analyzing the consumption-portfolio problem. At the beginning of period one, the consumer chooses a level of certain first period consumption c_1 and a set of asset holdings, where the returns on the latter fund consumption in period two. In the portfolio setting, markets will generally be incomplete with more states than the number of assets. The random variable $\widetilde{\xi} > 0$ denotes the period two payoff on the risky asset and ξ_f is the payoff for the risk free asset. The prices of the risky and risk free assets are denoted respectively by p and p_f. To ensure that there is no arbitrage,

$$\frac{\min\left(\widetilde{\xi}\right)}{p} < \frac{\xi_f}{p_f} < \frac{\max\left(\widetilde{\xi}\right)}{p}. \tag{11}$$

The condition that the risk free (gross) rate of return is less than the expected (gross) rate of return for the risky asset

$$R_f = \frac{\xi_f}{p_f} < \frac{E\widetilde{\xi}}{p} = E\widetilde{R} \tag{12}$$

guarantees a positive demand for the risky asset. The number of units of the risky and risk free assets is denoted by n and n_f, respectively. It follows that random period two consumption is given by

$$\widetilde{c}_2 = \widetilde{\xi}n + \xi_f n_f. \tag{13}$$

The consumption-portfolio optimization problem is given by[16]

$$\max_{c_1, n, n_f} U\left(c_1, \widehat{c}_2\right) = u_1(c_1) + u_2(\widehat{c}_2) \tag{14}$$

subject to

$$\widehat{c}_2 = V^{-1}EV\left(\widetilde{\xi}n + \xi_f n_f\right) \tag{15}$$

and

$$c_1 + pn + p_f n_f \leq I, \tag{16}$$

[16] A general sufficient condition for the existence of a unique solution to the consumption-saving problem is that u_2 be more concave than V (see Kimball and Weil 2009, Appendix A, for a more detailed discussion). A similar restriction can be applied to the consumption-portfolio problem. But if the $\widehat{c}_2(c_1)$ constraint is linear as is typically considered below, it is enough for U to be strictly quasiconcave.

where period one consumption is the numeraire and $p_1 \equiv 1$.

In the next section, we consider a simple example of saving in the presence of risky labor income. Investing in a risk free asset provides certain second period income to offset the risky labor income which naturally fits the classic notion of precautionary saving. However in the consumption-portfolio problem where both a risky asset and a risk free asset are available, investing in the risky asset doesn't match the intuition of precautionary saving. Nevertheless as in the risky income case, it is interesting and important to characterize when in the capital risk setting saving is larger or smaller in the presence of period two risk compared to its absence. Suppose first that one considers optimal saving where there is only a risk free asset paying the (gross) return R_f. Assuming initial income of I, the consumer maximizes her two period utility and solves for optimal period one saving, denoted by $s_1^{certain}$. Period two consumption is determined by the product $s_1^{certain}R_f$. Next suppose the same I, but the consumer has the possibility to invest, or save, in a portfolio containing a risky asset. Let s_1^{risky} denote optimal saving for the consumption-portfolio problem. Following Gollier (2001, Chapter 19), we adopt the following convention.[17]

Definition 1 *The quantity θ denotes (positive) excess saving in the risky versus certainty setting*

$$\theta = s_1^{risky} - s_1^{certain} \, (> 0) \, . \tag{17}$$

For the case of income risk, θ corresponds exactly to the traditional notion of precautionary saving. For capital risk since the decision to invest in a risky asset is endogenous, θ does not fit the intuition of providing period two insurance against a bad consumption outcome. However in Section 4, we will suggest one natural way to modify θ in order to accommodate the intuition of precautionary saving in the presence of risky investment.

2 A Motivating Example

In this section, we demonstrate that the conditions for $\theta > 0$ can be quite different for the income and capital risk cases. We also show that whereas the sign of V''' (or prudence) is central to the EU analysis of income risk, it (i) need not be crucial for income risk assuming more general KPS preferences and (ii) is irrelevant for

[17] Gollier (2001, Chapter 19) is careful never to refer to θ as precautionary saving. Instead in a consumption-portfolio problem, he suggests that in going from only investing in a risk free asset to investing in both a risk free and risky asset, a precautionary motive is key in determining when $\theta > 0$.

capital risk assuming KPS preferences with standard representations of time and risk preferences.

Example 1 *Assume that*

$$U\left(c_1, c_2\right) = q_1 c_1 - c_1^2 + \beta\left(q_1 c_2 - c_2^2\right) \tag{18}$$

and

$$V(c_2) = q_2 c_2 - c_2^2, \tag{19}$$

where $q_1, q_2 > 2c_2 > 0$. *First, consider the two state case of income risk where* I_{21} *and* I_{22} *denote different values of risky period two (labor) income. The KPS utility is given by*

$$u_1\left(c_1\right) + u_2 \circ V^{-1}\left(\pi_{21} V\left(R_f\left(I - c_1\right) + I_{21}\right) + \pi_{22} V\left(R_f\left(I - c_1\right) + I_{22}\right)\right), \tag{20}$$

where

$$\pi_{21} I_{21} + \pi_{22} I_{22} = 0. \tag{21}$$

It can be verified that

$$\theta \gtreqless 0 \Leftrightarrow q_1 \lesseqgtr q_2. \tag{22}$$

This result is very different from the EU analysis of Leland (1968), where $V''' = 0$ *implies that* $\theta = 0$ *always holds. Here both positive and negative precautionary saving are possible depending on the relationship between the time and risk preference parameters* q_1 *and* q_2.[18] *Next consider the capital risk case of a single risky asset where the risky (gross) rate of return* $\widetilde{R}_2$ *realizes values* R_{21} *and* R_{22} *in states one and two, respectively. The KPS utility function is*

$$u_1\left(c_1\right) + u_2 \circ V^{-1}\left(\pi_{21} V\left(R_{21}\left(I - c_1\right)\right) + \pi_{22} V\left(R_{22}\left(I - c_1\right)\right)\right), \tag{23}$$

where

$$\pi_{21} R_{21} + \pi_{22} R_{22} = R_f. \tag{24}$$

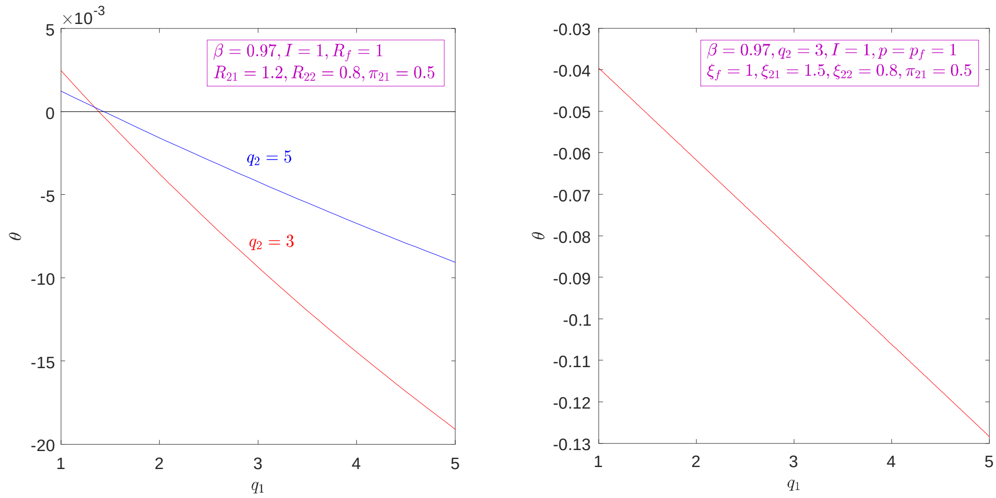
As shown in Figure 1(a), the sign of θ *still depends on the values of* q_1 *and* q_2. *However, unlike the income risk case,* $q_1 = q_2$ *does not imply* $\theta = 0$. *Actually, it*

[18] This is consistent with the result given by Kimball and Weil (2009). We have $V''' = 0$ and IARA. They state that a sufficient condition for positive precautionary saving is that u_2 is more concave than V. Based on (18)-(19), this implies

$$-\frac{u_2''(c_2)}{u_2'(c_2)} = \frac{2}{q_1 - 2c_2} > -\frac{V''(c_2)}{V'(c_2)} = \frac{2}{q_2 - 2c_2},$$

or equivalently,

$$q_1 < q_2.$$



(a) (b)

Figure 1:

can be proved that for the EU case corresponding to $q_1 = q_2$, s_1^{risky} *is always smaller than* $s_1^{certain}$, *implying that* $\theta < 0$.[19] *Finally consider the consumption-portfolio setting. The KPS utility function is*

$$u_1(c_1) + u_2 \circ V^{-1}\left(\pi_{21}V\left(\xi_{21}n + \xi_f n_f\right) + \pi_{22}V\left(\xi_{22}n + \xi_f n_f\right)\right), \tag{25}$$

where

$$\frac{\pi_{21}\xi_{21} + \pi_{22}\xi_{22}}{p} > R_f. \tag{26}$$

As shown in Figure 1(b), for the parameters we choose, θ *is always negative. Therefore for KPS quadratic preferences, prudence seems not to play a role in determining the sign of* θ *for both the income and capital risk cases. (See Appendix B.1 for supporting calculations).*

Given that (i) the capital risk case is so different from that of income risk and (ii) the EU conditions for $\theta > 0$ obtained by Gollier (2001) will be seen not to extend to the more general case of KPS preferences, we derive in Section 4 conditions for positive excess saving for the important class of KPS preferences corresponding to additively separable time preferences and HARA risk preferences in the consumption-portfolio setup.

[19] In contrast to our result, Eeckhoudt and Schlesinger (2008, Example 1) argue that for quadratic EU preferences, saving does not change with a mean preserving spread in the risky interest rate. This conclusion seems inconsistent with their Corollary 2 statement that saving is unchanged for a mean preserving spread if $V'' = 0$ since this is not satisfied by quadratic utility.

3 Certainty Case

In this paper, we focus predominantly on cases where the consumption-portfolio $\widehat{c}_2(c_1)$ constraint (15) is linear. This enables us to exploit key properties of the certainty case. We first review the familiar income and substitution effects and then introduce a comparative static result for this case that will prove to be quite useful.

3.1 Smoothing and Substitution Effects

For the certainty consumption-saving case, assume the consumer solves for the optimal period one consumption which maximizes her additively separable two period utility (8) subject to the budget constraint

$$c_2 = R_f (I - c_1) = R_f s_1. \tag{27}$$

Given additive utility, the effect of an increase in R_f on optimal period one consumption can be expressed as

$$\frac{\partial c_1}{\partial R_f} = \frac{u_2'(c_2) + c_2 u_2''(c_2)}{u_1''(c_1) + R_f^2 u_2''(c_2)}. \tag{28}$$

Following Dreze and Modigliani (1972, Appendix A), the corresponding income and substitution effects are respectively given by[20]

$$\frac{(I - c_1)}{R_f}\left(\frac{\partial c_1}{\partial I}\right)_{R_f = Const.} = \frac{c_2 u_2''(c_2)}{u_1''(c_1) + R_f^2 u_2''(c_2)} > 0 \tag{29}$$

and

$$\left(\frac{\partial c_1}{\partial R_f}\right)_{U = Const.} = \frac{\partial c_1}{\partial R_f} - \frac{(I - c_1)}{R_f}\left(\frac{\partial c_1}{\partial I}\right)_{R_f = Const.} = \frac{u_2'(c_2)}{u_1''(c_1) + R_f^2 u_2''(c_2)} < 0. \tag{30}$$

Increases in R_f make the consumer feel richer and this income effect results in increased period one consumption and decreased saving.[21] This is the smoothing effect.[22] In the next section when considering whether saving is larger or smaller in the presence of risky versus risk free asset returns, we will continue to interpret the income effect generated by a change in asset returns as a smoothing effect. In

[20]As is well-known, the assumed additive separability of U implies that both period one and two consumption are normal goods.

[21]Because $s_1 = I - c_1$, the sign of a change in s_1 with respect to R_f is the reverse of that for a change in c_1.

[22]See Gollier (2001, p. 236).

addition to the income effect, increasing R_f makes period two consumption more attractive and the consumer substitutes period two consumption for period one consumption by saving more. The saving substitution effect, in contrast to the smoothing effect, results in increased saving.

In general, the sign of the overall effect (28) is ambiguous. In the next subsection, we show that in fact for additive utility, one can obtain a very intuitive condition which determines whether the smoothing effect or substitution effect dominates.

3.2 A Comparative Static Result for Additive Utility

The following theorem will play a central role in determining whether $s_1^{risky} \gtreqless s_1^{certain}$ for the consumption-portfolio problem.[23]

Theorem 1 *Assume the optimization problem*

$$\max_{c_1, c_2} U(c_1, c_2) = u_1(c_1) + u_2(c_2) \tag{31}$$

$$S.T.\ c_2 = (I - c_1) R, \tag{32}$$

where $u_i' > 0$, $u_i'' < 0$ $(i = 1, 2)$.[24] *Then*

$$\frac{\partial c_1}{\partial R} \gtreqless 0 \Leftrightarrow \frac{\partial s_1}{\partial R} \lesseqgtr 0 \Leftrightarrow -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \gtreqless 1. \tag{33}$$

For additively separable U, it is easy to see from eqns. (29) and (30) that (33) provides the necessary and sufficient condition for when the c_1 income or smoothing effect dominates (equals, is dominated by) the substitution effect. A simple geometric intuition can be given for Theorem 1. Define the marginal rate of substitution and minus the slope of the constraint (32), respectively, by

$$m_1 =_{def} \frac{u_1'(c_1)}{u_2'(c_2)} \qquad \text{and} \qquad m_2 =_{def} -\frac{c_2}{c_1 - I}. \tag{34}$$

In Figure 2, consider the two constraint lines anchored at a common point. At the tangency between the lower constraint and indifference curve, $m_1 = m_2$. Increasing R in eqn. (32) corresponds to a rotation of the lower constraint line upward

[23] The result in Theorem 1 stated in the context of gross substitutes and complements was given by Wald (1936) and a modern proof was provided by Varian (1985). For completeness, we provide the proof in Appendix A.1. The simple geometric interpretation given below seems to be new. We thank Federico Echenique for pointing out these references to us.

[24] In the constraint (32), R can refer to either R_f or $\widehat{R}_p$, where the latter is defined below for consumption-portfolio problem.

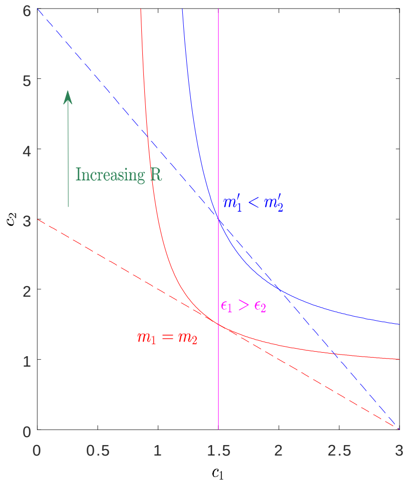


Figure 2:

to the right and can be viewed as changing c_2 for a fixed c_1. The elasticities of the two slope changes with respect to c_2 are given by

$$\epsilon_1 =_{def} \frac{\partial \ln m_1}{\partial \ln c_2} = -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \quad \text{and} \quad \epsilon_2 =_{def} \frac{\partial \ln m_2}{\partial \ln c_2} = \frac{c_2}{c_2} = 1, \tag{35}$$

where ϵ_1 will be referred to as the $EMRS$ with respect to period two consumption. If $\epsilon_1 \gtreqless \epsilon_2$ for all c_2-values then we have $\frac{\partial c_1}{\partial R} \gtreqless 0$ and hence the results in Theorem 1. Returning to the case in Figure 2, because $\epsilon_1 > \epsilon_2$ in response to an increase in R, the higher indifference curve intersects the shifted constraint at the initial optimal c_1, implying that the tangent to the indifference curve is steeper than the shifted constraint. Therefore, the new optimal c_1 is to the right of the initial c_1-value, implying that c_1 increases and s_1 decreases with R.

Three important observations should be made relating to the EMRS condition (33). First, the EMRS condition depends only on u_2 and is independent of u_1. Second, it should be emphasized that the EMRS is in general distinct from the familiar EIS, where the reciprocal of the latter is defined by

$$\frac{1}{EIS} = \frac{d \ln m_1}{d \ln(\frac{c_2}{c_1})}. \tag{36}$$

The quantity $1/EIS$ is often interpreted as an aversion to intertemporal substitution. One special case where the $EMRS$ and EIS are closely related is when U takes the popular CES form in eqn. (10). For this utility, $EMRS = \frac{1}{EIS} = 1 + \delta_1$ and the condition in Theorem 1 can be expressed as

$$\frac{\partial s_1}{\partial R} \lesseqgtr 0 \Leftrightarrow -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \gtreqless 1 \Leftrightarrow \delta_1 \gtreqless 0. \tag{37}$$

Thus if $\delta_1 > 0$, the aversion to intertemporal substitution is greater than the benchmark log utility ($\delta_1 = 0$) resulting in the positive smoothing effect dominating the negative substitution effect and c_1 increases and s_1 decreases with an increase in R.[25] Third, although the formula for the EMRS ϵ_1 in eqn. (35) takes the same mathematical form as the Arrow-Pratt measure of relative risk aversion, these notions are in general quite different. This is true even for the case for the two period EU representation where the indices u_1, u_2 and V are affinely equivalent. The relative risk aversion measure relates to the curvature properties of the NM utility $V(c_2)$, whereas the EMRS measure relates to the change in the MRS for the certainty U corresponding to changes in the value of c_2 with c_1 being held fixed.

4 Consumption-Portfolio Problem

This section focuses on the consumption-portfolio problem where saving takes place via risky and risk free assets. After introducing a useful two stage optimization process, we derive conditions such that excess saving $\theta \gtreqless 0$. As mentioned earlier, it is natural to say that saving is precautionary if the demand for the risk free asset is larger in the consumption-portfolio versus the certainty setting. In addition to being interesting in their own right, we show that the conditions for positive excess saving also are necessary for precautionary saving. We illustrate the applicability of the latter for the important case of CES and CRRA utilities (10).

4.1 Two Stage Optimization

It will be useful to express the consumption-portfolio problem (14)-(16) in the form of an equivalent two stage optimization problem. Without loss of generality, assume that there is one risk free asset and one risky asset. The first stage portfolio problem conditional on c_1 is defined by

$$(n(c_1), n_f(c_1)) = \underset{n, n_f}{\arg\max}\ EV\left(\widetilde{\xi} n + \xi_f n_f\right) \tag{38}$$

subject to

$$pn + p_f n_f \leq I - c_1. \tag{39}$$

[25] It should be emphasized that the popular EIS interpretation for the CES U is potentially misleading since, in fact, saving behavior depends for the more general additive utility U solely on the properties of u_2 and the form of u_1 is irrelevant.

The second stage consumption-saving problem corresponds to

$$c_1 = \arg\max_{c_1} u_1(c_1) + u_2(\widehat{c}_2(c_1)), \tag{40}$$

where

$$\widehat{c}_2(c_1) = V^{-1}\left(EV\left(\widetilde{\xi}n(c_1) + \xi_f n_f(c_1)\right)\right). \tag{41}$$

If the $\widehat{c}_2(c_1)$ constraint for the second stage consumption-saving optimization is linear in c_1, the analysis can be significantly simplified. We will show that this is the case if and only if the period two conditional NM index V is a member of the HARA class. But first, note that the NM indices of the DARA, CARA and IARA members of this class respectively are given by

$$V(c_2) = -\frac{1}{\delta_2}(c_2 - b)^{-\delta_2} \quad (b \gtreqless 0, c_2 > \max(0, b), \delta_2 > -1), \tag{42}$$

$$V(c_2) = -\frac{\exp(-\kappa c_2)}{\kappa} \quad (\kappa > 0) \tag{43}$$

and

$$V(c_2) = \frac{1}{\delta_2}(b - c_2)^{-\delta_2} \quad (b > c_2 > 0, \delta_2 < -1). \tag{44}$$

For the popular DARA case, it is standard to interpret $b > 0$ as a certain subsistence requirement.[26] For the two period EU representation incorporating the DARA case, Campbell and Cochrane (1999) interpret $b > 0$ as an external habit parameter.

As we discuss below, for the DARA, CARA and IARA cases the consumer in general forms two portfolios. The first portfolio's asset demands are dependent on period one investable income $I - c_1$ while the second portfolio is based on a fixed quantity of assets. The first portfolio is comprised of risky and risk free assets depending on which member of the HARA class we are considering and the certainty equivalent return on this portfolio is denoted by $\widehat{R}_p$.[27] The term Δ denotes an implicit period one income translation component (which can be positive or negative) and its precise definition and sign depend on the HARA class member.

[26] For the DARA case we can have $b < 0$, but then the subsistence interpretation does not make sense (see Pollak 1970, p. 748). For the IARA case, b can be interpreted as a bliss point.

[27] It will be important to remember that for HARA preferences, the risky portfolio in general (i) contains both the risky and risk free asset and (ii) has a fixed mixture of assets which is independent of $I - c_1$.

Theorem 2 *$\widehat{c}_2$ is a linear function of c_1 for any distribution $\widetilde{\xi}$ if and only if the NM index $V(c_2)$ is a member of the HARA class of utility functions.*[28]

Based on this theorem, we can always write the $\widehat{c}_2(c_1)$ constraint as

$$\widehat{c}_2 = \widehat{R}_p (I - c_1 - \Delta) . \tag{45}$$

In the next subsection, we give exact definitions for $\widehat{R}_p$ and Δ depending on the specific member of the HARA class and the return distribution for the risky asset. Given Theorem 2 and the two stage optimization facilitated by the KPS separation of time and risk preferences, the analysis of excess saving is greatly simplified. The consumption-portfolio and certainty cases both have the same certainty indifference curves corresponding to U and linear budget constraints. Thus any difference in saving, corresponding to θ, depends on a comparison of the two constraints' slopes, $\widehat{R}_p$ versus R_f, and the translation Δ.

4.2 Excess Saving

Before giving our main results characterizing when excess saving $\theta \gtreqless 0$, we provide some intuition for the important roles played by $\widehat{R}_p$ and Δ in comparing how optimal saving changes when a risky asset is added to the possibility of investing in a risk free asset. Since these roles differ for the DARA, CARA and IARA cases, we will consider them separately.

Given that for HARA risk preferences, the $\widehat{c}_2(c_1)$ constraint always takes the linear form (45), the following establishes its relationship to the certainty constraint $c_2 = R_f(I - c_1)$ where there is just a risk free asset. In particular, it specifies how the constraints differ in terms of a rotation based on $\widehat{R}_p \gtreqless R_f$ and/or a parallel translation or shift corresponding to Δ. This result will play a key role in characterizing necessary and sufficient (or sufficient) conditions for the existence of positive excess saving.

Theorem 3 *The consumer solves the consumption-portfolio problem (38)-(41), where V is a strictly concave member of the HARA class.*

(i) *If V takes the DARA form (42), then*

$$\widehat{R}_p > R_f \quad \text{and} \quad \Delta = \frac{b}{R_f} - \frac{b}{\widehat{R}_p}; \tag{46}$$

[28] It should be noted that Theorem 2 can be applied for the case of multiple risky assets even when markets are incomplete. Since V is a member of the HARA class, following Rubinstein (1974), markets are effectively complete and the financial asset setting can naturally be transformed into the contingent claim setting. Since each conditional contingent claim demand c_{2s} ($s = 1, 2, ..., S$) is linear in c_1, the certainty equivalent function $\widehat{c}_2$ is also linear in c_1.

(ii) *If V takes the CARA form (43), then*[29]

$$\widehat{R}_p = R_f \quad \text{and} \quad \Delta < 0; \tag{47}$$

and

(iii) *If V takes the IARA form (44), then*

$$\widehat{R}_p < R_f \quad \text{and} \quad \Delta = \frac{b}{R_f} - \frac{b}{\widehat{R}_p} < 0. \tag{48}$$

Remark 1 *It may appear counterintuitive that in the above theorem $\widehat{R}_p$ can be less than or equal to R_f. Since both risky and risk free assets are available, it would seem that if $\widehat{R}_p \leq R_f$ the consumer should just hold the risk free asset. For the CARA and IARA members of the HARA class, it is shown in the proof of Theorem 3 that the $\widehat{c}_2(c_1)$ constraint (45) contains a negative Δ term. As a result, $-\Delta$ corresponds to an additional source of income in period two arising from the presence of the risky asset and even though $\widehat{R}_p \leq R_f$ it can be optimal for the consumer to hold the risky asset.*

Before stating our result characterizing when excess saving for the consumption-portfolio case is positive, equal to zero or negative, it will prove useful to provide some additional intuition and simple geometry for comparisons of the DARA and CARA $\widehat{c}_2(c_1)$ linear constraints with the constraint for the certainty saving problem.

First assume the DARA utility (42), where $b > 0$. When solving the conditional portfolio problem, a consumer will form a risk free portfolio to fund the period two subsistence requirement $c_2 = b$ and form a separate risky portfolio comprised of both risky and risk free assets to fund supernumerary consumption (i.e., consumption in excess of the subsistence requirement). Then following Kubler, Selden and Wei (2013, pp. 1043-1044), the minimum level of income such that the period two subsistence requirement can be met is

$$I_{\min} = \frac{b}{R_f}. \tag{49}$$

Let $I - c_1 - \frac{b}{R_f}$ and $\widehat{R}_p$ denote respectively the amount of period one income invested in the risky portfolio and the certainty equivalent return on the risky portfolio. Then the $\widehat{c}_2(c_1)$ constraint is given by

$$\widehat{c}_2 = \widehat{R}_p(I - c_1 - \frac{b}{R_f}) + b, \tag{50}$$

[29] The CARA formula for Δ depends on the form of the risky asset payoff distribution. See eqn. (A.38) in the proof of this theorem in Appendix A.3.

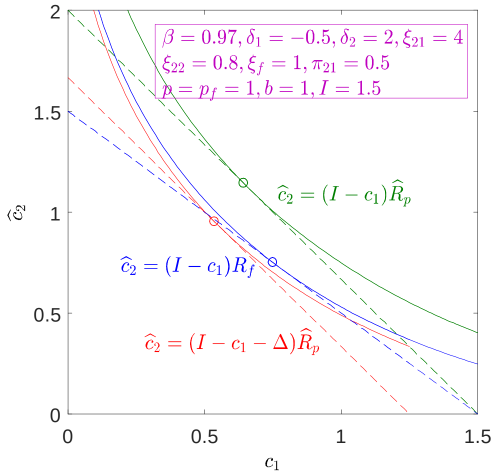


Figure 3:

or equivalently,

$$\widehat{c}_2 = \widehat{R}_p \left(I - c_1 - \Delta\right), \tag{51}$$

where

$$\Delta = \frac{b}{R_f} - \frac{b}{\widehat{R}_p}. \tag{52}$$

Given the assumed DARA utility, we show in the proof of Theorem 3 that it is always optimal for $n > 0$. Risk being larger in the risky portfolio versus holding just a risk free asset, implies that $\widehat{R}_p > R_f$. It then follows from eqns. (51) and (52) that Δ results from the funding of the period two subsistence requirement with the risk free asset rather than the risky portfolio, is positive and can be viewed as an opportunity loss reducing period one investable income $I - c_1$.

The KPS two stage optimization facilitates a very clear geometric distinction between saving when there is just a risk free asset and when there is both a risky asset and a risk free asset. The key is to compare the portfolio $\widehat{c}_2(c_1)$ constraint with the certainty constraint as is illustrated in Figure 3. First there is a rotation of the constraint resulting in its slope steepening corresponding to the change from $-R_f$ to $-\widehat{R}_p$. Then second there is a southwesterly parallel shift in the $\widehat{c}_2(c_1)$ constraint corresponding to $-\Delta$ which follows from the subsistence term b in V. Without the term b, introducing capital risk results in the $\widehat{c}_2(c_1)$ constraint becoming steeper. This change in slope can be decomposed into the income and substitution effects (29) and (30) discussed above in Section 3.1. For the case in Figure 3, since $\delta_1 = -0.5 < 0$, the $EMRS < 1$ and the negative substitution effect dominates and optimal c_1 is less than in the certainty case, implying

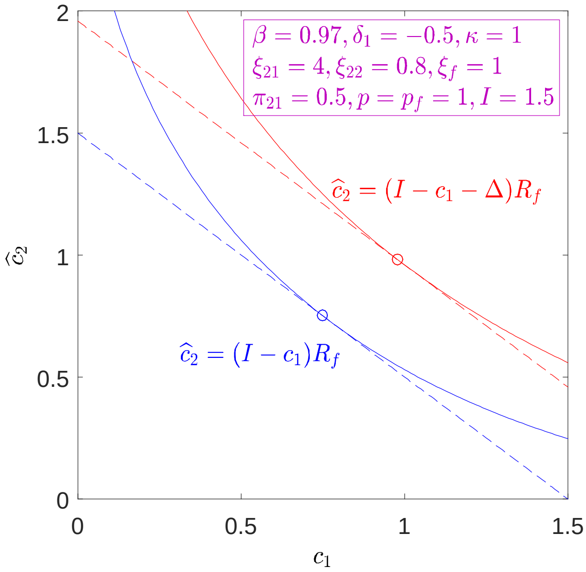


Figure 4:

larger savings. So far this would correspond to the CRRA special case of DARA preferences where $b = 0$. However when including the effect of the opportunity loss, since $-\Delta = -\left(\frac{b}{R_f} - \frac{b}{\widehat{R}_p}\right) < 0$, this effectively reduces the investable period one income and creates a new negative income effect. As a result, optimal c_1 will decrease more as can be seen from comparing the tangencies of the highest and lowest indifference curves in Figure 3. Overall, excess saving θ is given by the corresponding difference in $I - c_1$ for the DARA tangency with the lowest indifference curve and the certainty tangency with the middle indifference curve.

The geometry for CARA utility differs from that of DARA utility since for the former unlike the latter, the risky asset holding is fixed, independent of income. Thus the portfolio that depends on $I - c_1$ is risk free and $\widehat{R}_p = R_f$. However expressing period two consumption as $R_f(I - c_1)$ ignores the fact that part of period one income is actually invested in the higher return risky asset. It will be noted that in Figure 4 unlike Figure 3, $\widehat{R}_p = R_f$ and hence there is no rotation of the constraint leading to the standard income and substitution effects. The only effect is a smoothing effect resulting from the increase in period two income due to Δ. That is, in order to ensure that the implicit increase in period two income doesn't result in just an increase in period two consumption, the consumer smoothes this increase by consuming more in period one and saves less implying that $\theta < 0$.

The IARA case can be analyzed similarly and will not be discussed here.

We next show that by combining Theorems 1 and 3, the sign of θ depends crucially on the time preference $EMRS$ except for the case of CARA risk preferences.

Theorem 4 *Consider the consumption-portfolio optimization problem (38)-(41) where V is a strictly concave member of the HARA class. If V takes the DARA form (42), then*

(i) *assuming V is also DRRA, where $b > 0$,*

$$\theta > 0 \quad \text{if} \quad -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \leq 1; \tag{53}$$

(ii) *assuming V is also CRRA, where $b = 0$,*

$$\theta \gtreqless 0 \Leftrightarrow -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \lesseqgtr 1; \quad \text{and} \tag{54}$$

(iii) *assuming V is also IRRA, where $b < 0$,*

$$\theta < 0 \quad \text{if} \quad -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \geq 1. \tag{55}$$

If V takes the CARA form (43), then

$$\theta < 0. \tag{56}$$

If V takes the IARA form (44), then

$$\theta < 0 \quad \text{if} \quad -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \leq 1. \tag{57}$$

Remark 2 *Although risk preferences do not affect the sign of θ, they can affect the size of excess saving. Consider the CRRA case, Theorem 4(ii) corresponding to Figure 3 where $b = 0$. If the risk preference parameter δ_2 increases, the northeasterly rotation of the $\widehat{c}_2$ constraint from the certainty constraint is less. As a result, optimal c_1 decreases by less compared to the certainty case and excess saving $s_1^{risky} - s_1^{certain}$ declines.*

A key contribution of Theorem 4 is showing that when risk preferences are represented by the HARA class (excluding the CARA case) and time preferences are additively separable, the existence of excess saving depends on the $EMRS$ instead of, as widely believed, prudence. Define absolute prudence by

$$\mathcal{P}_A(c_2) = -\frac{V'''(c_2)}{V''(c_2)}. \tag{58}$$

Then if the KPS utility takes the special case of EU where u_2 and V are equivalent up to a positive affine transformation, Gollier (2001, Proposition 75) proves that there will be positive excess saving if and only if

$$\mathcal{P}_A(c_2) > 2\tau_A(c_2), \tag{59}$$

where τ_A is defined by (9). If one assumes that the EU NM index and time preference u_2 take the following CRRA form

$$V(c_2) = u_2(c_2) = -\frac{c_2^{-\delta}}{\delta} \quad (\delta > -1), \tag{60}$$

then a necessary and sufficient condition for positive excess saving is that relative risk aversion $\tau_R(c_2) = \delta + 1 < 1$. Gollier (2001, p. 289) observes, based on empirical data, that relative risk aversion will not be this small and hence excess saving would be negative. However from the DARA condition in Theorem 4(ii), where risk preferences and time preferences are distinguished, the actual requirement for positive excess saving is that the $EIS > 1$. For the case where u_2 takes the same power utility form as V but with a different exponent

$$u_2(c_2) = -\beta\frac{c_2^{-\delta_1}}{\delta_1} \quad (\delta_1 > -1), \tag{61}$$

the necessary and sufficient condition for positive excess saving is $\delta_1 < 0$ which has nothing to do with the risk aversion parameter δ_2 and corresponding value of $\tau_R(c_2)$. Moreover in terms of empirical estimates, the assumption that $\delta_1 < 0$ or the $EIS > 1$ seems much less objectionable. In fact, Bansal and Yaron (2004) and Epstein, Farhi and Strzalecki (2014) in their discussion of long-run risk argue for an $EIS > 1$ implying that $\delta_1 < 0$. For KPS preferences, there is no requirement that relative risk aversion be unnaturally low. Hence for the important case of CES time preferences and CRRA risk preferences, it seems reasonable to suppose that excess saving is positive (also see Corollary 2(ii)). In summary, the conclusions deriving from a comparison between absolute prudence and absolute risk aversion are the result of the well-known confounding of time and risk preferences inherent in EU preferences.

Remark 3 *It may be natural to think that the question of whether saving increases when there is both a risky and risk free asset versus just a risk free asset is analogous to whether saving increases when there is an increase in the risk of the risky asset. If one considers the CES and CRRA utilities (10), then it follows from Theorem 4(ii) that saving increases if and only if $\delta_1 < 0$. However, assuming the same preferences Selden (1979, p. 80) shows that corresponding to an increase* *in risk saving increases if and only if $\delta_1 > 0$. The key to resolving this seeming paradox is to recognize that Selden considers a mean preserving increase in risk, whereas for the excess saving analysis in Theorem 4 both the expected return and risk are larger for the risky portfolio versus the certainty case. Thus in terms of Figure 3 assuming $b = 0$ (implying that $\Delta = 0$), the excess saving analysis corresponds to northeasterly rotation of the $\widehat{c}_2$ constraint whereas the mean preserving increase in risk corresponds to a southwesterly rotation. Hence it is not surprising, that the necessary and sufficient conditions for increased saving are opposite. One important implication of this observation, is that if a consumer with CES time and CRRA risk preferences were to be observed based on market data or experiments increasing her saving in response to a mean preserving increase in risk in the consumption-saving problem, then it would be impossible for her to also exhibit positive excess saving (and precautionary saving introduced in the next subsection) in the consumption-portfolio problem.*[30]

As noted earlier for DARA utility (42), the risk preference parameter $b > 0$ is often interpreted as an external habit formation parameter. It is then quite natural to ask whether external habit formation increases precautionary saving. Diaz, Pijoan-Mas and Rios-Rull (2003) found that this is indeed the case for an EU, multiplicative habit model. Although our setting is quite different, we reach the conclusion that excess saving increases with the risk preference parameter b.

Corollary 1 *Consider the consumption-portfolio optimization problem (38)-(41) where V is strictly concave and takes the DARA form (42), then*

$$\partial\theta/\partial b > 0. \tag{62}$$

Finally since some of the Theorem 4 conditions are only sufficient, one might argue that other sufficient conditions could depend on prudence. However the following corollary shows that if one assumes a translated origin power utility for the certainty utility u_2, the sufficient conditions in Theorem 4 become necessary and sufficient. This would seem to further argue that at least for the very important HARA class of risk preferences, the test for positive excess saving depends on the time preference measures of $EMRS$ and EIS and not on the risk preference measure of prudence.

[30] Assuming a consumption-saving setting with a single risky asset in Appendix B.2, we show in Theorem 7 that $\delta_1 > 0$ is necessary and sufficient for the positive excess saving. This is consistent with the mean preserving spread result in Selden (1979), since in Theorem 7 we assume $E\widetilde{R} = R_f$. The latter is different from the assumption that $E\widetilde{R} > R_f$ in the consumption-portfolio results in this section.

Corollary 2 *Consider the consumption-portfolio (38)-(41), where V is strictly concave.*

(i) *If V takes the DARA form (42),* $b \gtreqless 0$ *and*

$$u_2(c_2) = -\frac{(c_2 - b)^{-\delta_1}}{\delta_1} \quad (\delta_1 > -1), \tag{63}$$

then

$$\theta \gtreqless 0 \Leftrightarrow \delta_1 \lesseqgtr 0 \qquad \textit{and} \tag{64}$$

(ii) *If V takes the IARA form (44) and*

$$u_2(c_2) = \frac{(b - c_2)^{-\delta_1}}{\delta_1} \quad (\delta_1 < -1), \tag{65}$$

then

$$\theta < 0. \tag{66}$$

Campbell and Cochrane (1999) introduce external habit formation preferences as part of their equilibrium analysis. They assume an EU representation which for two periods can be viewed as a special case of the utility in Corollary 2(i), where $b > 0$ and $\delta_1 = \delta_2$. The preference parameter b is interpreted as an external habit. Corollary 2(i) shows that consumers with the KPS generalization of the external habit model will save more in a risky versus certainty settings only when $\delta_1 < 0$. For the Campbell and Cochrane formulation, the assumption that $\delta_1 = \delta_2$ requires that the consumer's risk preferences also must satisfy $\delta_2 < 0$. This is inconsistent with the δ_2-value of 1 used in Campbell and Cochrane (1999, pp. 218 and 225) and also with typically assumed estimates of $4 - 6$. Using the same KPS utility as in Corollary 2(i), we derive in Proposition 3, in Supplemental Appendix B.4, the closed form expression (B.85) for the equilibrium risk free rate. This formula can be viewed as a two period KPS version of eqn. (3) in Cochrane (2017), where δ_1 and δ_2 can be fully distinguished.

4.3 Precautionary Saving

To define precautionary saving in the consumption-portfolio setting, first denote investment in the risk free asset for the certainty and risky cases, respectively, as $n_f^{certain}$ and n_f^{risky}. Then we have the following definition.

Definition 2 *For the consumption-portfolio problem (38)-(41), let* ϑ *denote the difference in saving in the risk free asset*

$$\vartheta = p_f {n_f}^{risky} - p_f {n_f}^{certain}. \tag{67}$$

Assuming p_f is the same for the certainty and risky cases, precautionary saving corresponds to

$$\vartheta \gtreqless 0 \Leftrightarrow n_f^{risky} \gtreqless n_f^{certain}. \tag{68}$$

When $\vartheta > 0$, this is consistent with the intuition of precautionary savings being the additional investment in a risk free asset to provide certain period two insurance or a buffer against a bad return outcome from the risky asset. The following connects precautionary saving to excess saving.

Proposition 1 *A necessary condition for positive precautionary saving is positive excess saving, i.e.,*

$$\vartheta > 0 \Rightarrow \theta > 0. \tag{69}$$

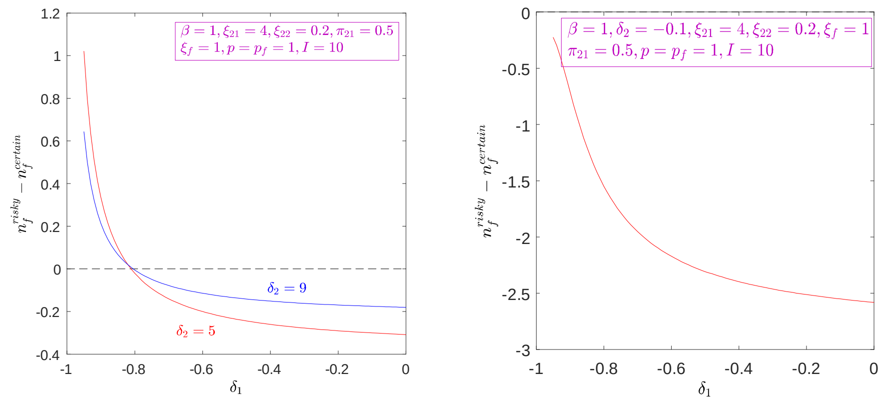
In Theorem 4, the necessary condition for precautionary saving is shown to always be violated for CARA risk preferences assuming an additively separable U and it follows from the contrapositive of Proposition 1 that $\vartheta < 0$ holds.

For the popular CES and CRRA utilities (10), it follows from Theorem 4(ii) that the necessary condition for $\vartheta > 0$ is violated when $\delta_1 \geq 0$ or $EIS < 1$. Next we demonstrate that depending on the asset return payoffs and the value of the risk preference parameter δ_2, precautionary saving can hold or fail when $-1 < \delta_1 < 0$. Some intuition for why this is the case can be gleamed from the two stage formulation of the consumption-portfolio problem. An increase in risk can be thought of as having two potentially conflicting effects on n_f^{risky}. First, there is an investment or saving effect which assumes the portfolio composition is fixed. Second, there is a portfolio reallocation effect. The competition between these two effects is illustrated in the example below.

Example 2 *Assume the CES and CRRA utilities corresponding to (10). It follows from Theorem 4(ii) and Proposition 1 that a necessary condition for $n_f^{risky} > n_f^{certain}$ is $s_1^{risky} > s_1^{certain}$, or $\delta_1 < 0$. Although this is also a sufficient condition for positive excess saving, it is not a sufficient condition for positive precautionary saving. Consider the complete market, two state case as in Kubler, Selden and Wei (2013). Let the random asset payoff $\widetilde{\xi}$ take the values ξ_{21} with probability $1 > \pi_{21} > 0$ and ξ_{22} with probability $\pi_{22} = 1 - \pi_{21}$. Assume $\xi_{21} > \xi_{22} > 0$ and the risk free asset has payoff $\xi_f > 0$. Assume the following parameter values*

$$\beta = 1, \delta_2 = 5 \text{ or } 9, \xi_{21} = 4, \xi_{22} = 0.2, \xi_f = 1, \pi_{21} = 0.5, p = p_f = 1, I = 10. \tag{70}$$

In Figure 5(a), $n_f^{risky} - n_f^{certain}$ is plotted versus δ_1. Precautionary saving occurs only when δ_1 is sufficiently negative. Although positive excess saving is realized



(a) (b)

Figure 5:

when $\delta_1 < 0$ and $EIS > 1$, for the parameters (70), precautionary saving can occur only if the EIS takes the unreasonably high value of about 5. As argued in Appendix B.3, a necessary condition for positive precautionary savings is that optimal asset demands satisfy $n < n_f$. For the return distribution in (70), it can be verified that there exists a special δ_2-value equal to -0.07 such that optimal $n = n_f$. For $\delta_2 > -0.07$, the greater risk aversion ensures that $n < n_f$ and hence $\vartheta > 0$ is possible. Conversely if $\delta_2 = -0.10$, then as illustrated in Figure 5(b) ϑ can never be positive. It should be noted that in Figure 5(a) conditional on δ_1 being low enough that $\vartheta > 0$, an increase in the risk aversion parameter δ_2 results in a decrease in ϑ. The intuition for this result can be understood in terms of the saving and reallocation effects discussed above. As the consumer becomes more risk averse, $\widehat{R}_p$ decreases and the positive excess saving θ decreases (see Figure 3, where $b = 0$). This results in a decrease in the demand for both the risky and risk free assets. But the increase in risk aversion also causes a portfolio reallocation with the consumer reducing her holdings of the risky asset and increasing her demand for the risk free asset. For the case in Figure 5(a), the negative saving effect dominates the positive reallocation effect and n_f^{risky} and ϑ both decline. (For a more complete discussion of the example and for supporting calculations, see Appendix B.3.)

In the prior subsection, we showed that excess saving increases with the external habit risk parameter b. The following proves that the conclusion in Corollary 1 extends to precautionary saving ϑ.

Result 1 *Consider the consumption-portfolio optimization problem (38)-(41) where V is strictly concave and takes the DARA form (42), where $b > 0$. Further assume that optimal risk free asset holdings satisfy $n_f > b/\xi_f$. Then*

$$\partial\vartheta/\partial b > 0. \tag{71}$$

Intuitively, one can think of the requirement that $n_f > b/\xi_f$ as implying that the consumer not only finances her period two consumption requirement b by the risk free asset, but also is long the risk free asset in her risky portfolio. This will be the case if she is sufficiently risk averse.

5 Equilibrium Analysis

As argued in the prior subsection, the assumption of additively separable time preferences and CARA risk preferences implies that the presence of a risky asset in the consumption-portfolio problem always results in the demand for the risk free asset being less versus when there is only a risk free asset, i.e., $\vartheta < 0$. This would seem to imply that in a representative agent equilibrium the price for the risk free asset in the risky setting should be less than in the certainty case and hence the risky equilibrium risk free rate R_f^{risky} should be larger than the certainty $R_f^{certain}$. A similar implication should apply as well for the case in Example 2 where time and risk preferences are represented by CES and CRRA utilities unless the EIS is unreasonably large. However this inference about the relationship between the risky and certainty equilibrium risk free rates is not consistent with standard equilibrium analyses. Assuming the same forms of time and risk preferences are possessed by a representative agent, Theorem 5 show that the opposite conclusion $R_f^{risky} < R_f^{certain}$ holds. Indeed it is standard in the microfinance literature to suggest that the precautionary motive holds when the equilibrium R_f^{risky} is decreasing with risk.[31] After first presenting the equilibrium results, we then discuss the source of this seeming inconsonance between the demand and equilibrium analyses.

Assume the standard representative agent exchange economy setting, where the agent solves the following consumption-portfolio problem[32]

$$\max_{c_1,n,n_f} u_1(c_1) + u_2\left(V^{-1}EV(\widetilde{\xi}n + \xi_f n_f)\right) \tag{72}$$

[31] For example, Campbell and Cochrane (1999, p. 212) interpret a "precautionary saving" term in the expression for R_f^{risky} as showing that "as uncertainty increases, consumers are more willing to save, and this willingness drives down the equilibrium risk free interest rate".

[32] Barsky (1989), Weil (1990), Campbell and Cochrane (1999), Yi and Choi (2006) and Gomes and Ribeiro (2015) all assume a representative agent exchange economy.

subject to

$$c_1 + pn + p_f n_f = \overline{c}_1 + p\overline{n} + p_f \overline{n}_f, \tag{73}$$

where $\overline{c}_1$, $\overline{n}$ and $\overline{n}_f$ denote, respectively, endowments of period one consumption, the risky asset and the risk free asset. Using period one consumption as the numeraire, equilibrium prices (p, p_f) ensure that markets clear,[33]

$$c_1 = \overline{c}_1, n = \overline{n} \text{ and } n_f = \overline{n}_f. \tag{74}$$

Given the single agent setting, it is clear that there will be a unique equilibrium defined by (c_1, n, n_f, p, p_f). This equilibrium corresponds to the fixed parameter set $(\overline{c}_1, \overline{n}, \overline{n}_f, \widetilde{\xi}, \xi_f)$ where equilibrium prices are endogenous. To make the certainty and risky cases comparable, we assume that $E\widetilde{\xi} = \xi_f$ and

$$\overline{c}_1^{certain} = \overline{c}_1^{risky} \text{ and } \overline{n}_t = \overline{n}_f^{certain} = \overline{n}^{risky} + \overline{n}_f^{risky}, \tag{75}$$

where the superscripts "risky" and "certain" are respectively used to distinguish the portfolio setting with a risky asset versus the certainty case with only a risk free asset. Similar notation is used when distinguishing the equilibrium R_f in the two settings. Given that $E\widetilde{\xi} = \xi_f$, our assumption for endowments is necessary and sufficient such that when the random return $\widetilde{\xi}$ in each state goes to ξ_f, the risky case converges to the certainty case as in Barsky (1989), Campbell and Cochrane (1999), Yi and Choi (2006) and Gomes and Ribeiro (2015).[34] It should be noted that $E\widetilde{\xi} = \xi_f$ does not imply $E\widetilde{R} = R_f$. In the equilibrium setting, assuming $\overline{n} > 0$, it can easily be verified that one always has $p < p_f$, implying that

$$E\widetilde{R} = \frac{E\widetilde{\xi}}{p} > \frac{\xi_f}{p_f} = R_f. \tag{76}$$

Then we have the following theorem.

Theorem 5 *Assume (i)*

$$U(c_1, c_2) = -\frac{(c_1 - b)^{-\delta_1}}{\delta_1} - \beta\frac{(c_2 - b)^{-\delta_1}}{\delta_1} \quad (\delta_1 > -1, b \gtreqless 0) \tag{77}$$

[33] It should be noted that the equilibrium results in this section are independent of the endowments of assets. In other words, whether $\overline{n}_f \gtreqless 0$ does not change our conclusions.

[34] To see that the conditions in (75) are necessary and sufficient, assume that $\widetilde{\xi}$ converges in every state to ξ_f. Then we have

$$\widetilde{\xi}\overline{n}^{risky} + \xi_f \overline{n}_f^{risky} = \xi_f \left(\overline{n}^{risky} + \overline{n}_f^{risky}\right),$$

which is equal to $\xi_f \overline{n}_f^{certain}$ if and only if $\overline{n}_f^{certain} = \overline{n}^{risky} + \overline{n}_f^{risky}$.

and

$$V\left(c_2\right) = -\frac{\left(c_2 - b\right)^{-\delta_2}}{\delta_2} \quad \left(\delta_2 > -1\right); \tag{78}$$

or (ii)

$$U\left(c_1, c_2\right) = -\frac{\exp\left(-\kappa_1 c_1\right)}{\kappa_1} - \beta\frac{\exp\left(-\kappa_1 c_2\right)}{\kappa_1} \quad \left(\kappa_1 > 0\right) \tag{79}$$

and

$$V\left(c_2\right) = -\frac{\exp\left(-\kappa_2 c_2\right)}{\kappa_2} \quad \left(\kappa_2 > 0\right). \tag{80}$$

Then,

$$R_f^{risky} < R_f^{certain}. \tag{81}$$

(In Appendix B.4, we derive analytic expressions for R_f^{risky} based on specific distributions for the consumption ratio $\widetilde{c}_2/c_1$ as is done for instance in Campbell and Cochrane 1999, Yi and Choi 2006 and Gomes and Ribeiro 2015. However unlike these papers, given our simpler two period setting the formulas are obtained without having to make approximations.)

Although $R_f^{risky} < R_f^{certain}$ holds for many common forms of time and risk preferences, the following shows that this is not always the case.[35]

Proposition 2 *Assume that*

$$U\left(c_1, c_2\right) = \frac{\left(b - c_1\right)^{-\delta_1}}{\delta_1} + \beta\frac{\left(b - c_2\right)^{-\delta_1}}{\delta_1} \tag{82}$$

and quadratic IARA risk preferences defined by

$$V\left(c_2\right) = -\frac{\left(b - c_2\right)^2}{2}, \tag{83}$$

where $\delta_1 < -1$, $b > 0$ and $c_1, c_2 < b$.[36] *Then we have*[37]

$$R_f^{risky} \lesseqqgtr R_f^{certain} \Leftrightarrow \delta_1 \lesseqqgtr -2. \tag{84}$$

[35]It should be noted that, also in a two period setting, Barsky (1989) obtains a similar conclusion to Theorem 5(i) where $b = 0$. However, he requires the risky asset return distribution to be lognormal and zero risk free asset endowments. Elul (1997) reaches similar conclusions to Theorem 5(ii) and Proposition 2. However, his setting differs in several important ways from what we assume in this section. First, two period EU is assumed rather than our more general KPS utility. Hence he is unable to distinguish the separate roles of risk and time preferences as in the Proposition 2, condition (84). Second, he does not consider the case of DARA risk preferences in Theorem 5(i). Third, he allows for multiple agents with some heterogeneity in contrast to our assumption of a single agent. Fourth, he assumes a different endowment structure which would be incompatible with our representative agent case.

[36]The condition $\delta_1 < -1$ ensures that the utility U is strictly quasiconcave.

[37]The condition (84) would seem to suggest that the prudence measure for u_2 is necessary and

The conclusion in Proposition 2 differs from that of Theorem 5 in two important ways. First, R_f^{risky} can be larger than or equal to $R_f^{certain}$ in contrast to Theorem 5. Second, the comparison between R_f^{risky} and $R_f^{certain}$ depends on the representative agent's time preferences where it does not in Theorem 5. The result (84) has another surprising implication. To see this most clearly, consider the following expression for the equilibrium R_f^{risky} derived in the proof of Proposition 2

$$R_f^{risky} = \frac{\left(b - \overline{c}_1\right)^{-\delta_1 - 1}}{\beta \left(E\left(b - \widetilde{\xi}\overline{n} - \xi_f \overline{n}_f \right)^2 \right)^{-\frac{\delta_1}{2} - 1} E\left(b - \widetilde{\xi}\overline{n} - \xi_f \overline{n}_f \right)}. \tag{85}$$

When $\delta_1 = -2$, the KPS utility defined by (82)-(83) converges to the special EU case and R_f^{risky} converges to the certain risk free return

$$R_f^{certain} = \frac{b - \overline{c}_1}{\beta \left(b - \xi_f \overline{n}_f \right)}. \tag{86}$$

This does not happen for any of the other HARA class members.

Although $R_f^{risky} < R_f^{certain}$ is often interpreted in equilibrium analyses as evidence of a precautionary motive, one cannot use partial equilibrium saving behavior of the representative agent to support this result for two reasons. First, there is an important difference in letting risk go to zero in the equilibrium and demand analyses. For the equilibrium analysis, the assumptions that $E\widetilde{\xi} = \xi_f$ and

$$\overline{c}_1^{certain} = \overline{c}_1^{risky} \quad \text{and} \quad \overline{n}_t = \overline{n}_f^{certain} = \overline{n}^{risky} + \overline{n}_f^{risky}, \tag{87}$$

ensure that if the risk goes to zero, the risky case will converge to the certainty case. However for the demand analysis, partial equilibrium requires that

$$\frac{E\widetilde{\xi}}{p} > \frac{\xi_f}{p_f} \tag{88}$$

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sufficient for the risky equilibrium risk free rate to be less than the certainty rate since

$$u_2''' \gtreqless 0 \Leftrightarrow \delta_1 \lesseqgtr -2.$$

Also, the u_2 corresponding to the two cases in Theorem 5 where $R_f^{risky} < R_f^{certain}$ satisfies the condition for prudence. However, the following example demonstrates that this inequality is not implied by the prudence of u_2. Assume

$$U\left(c_1, c_2\right) = -\left(q_1 - c_1\right)^2 - \beta\left(q_1 - c_2\right)^2 \quad \text{and} \quad V\left(c_2\right) = -\left(q_2 - c_2\right)^2,$$

where $q_1, q_2 > 0$ and $c_1, c_2 < q_1$, $c_2 < q_2$, then

$$R_f^{risky} \gtreqless R_f^{certain} \Leftrightarrow q_1 \gtreqless q_2.$$

be assumed to ensure positive risky asset holdings. If risk goes to zero, the risky asset becomes a risk free asset with a higher return, which introduces arbitrage opportunities. As a result, we cannot take the zero risk limit for demand analysis and the zero risk case must be considered separately. Second, partial equilibrium demand analysis assumes that the exogenously given market risk free rate R_f is the same in the risky and risk free settings, whereas this is not possible in the equilibria for cases (i) and (ii) in Theorem 5. As we show below, these differences play a critical role in explaining the inconsonance in the demand and equilibrium analyses.

Example 3 *Assume that*

$$U(c_1, c_2) = -\frac{\exp(-\kappa_1 c_1)}{\kappa_1} - \beta\frac{\exp(-\kappa_1 c_2)}{\kappa_1} \quad (\kappa_1 > 0) \tag{89}$$

and

$$V(c_2) = -\frac{\exp(-\kappa_2 c_2)}{\kappa_2} \quad (\kappa_2 > 0). \tag{90}$$

We have proved in Theorem 5 that

$$R_f^{risky} < R_f^{certain}. \tag{91}$$

Further assume that $\widetilde{\xi}$ is normally distributed with mean $E\widetilde{\xi} = \xi_f$ and variance σ^2. Since the certainty case can be viewed as the limit case with zero risk, this condition also implies that[38]

$$\frac{\partial R_f^{risky}}{\partial \sigma^2} < 0. \tag{92}$$

Next consider the partial equilibrium demand analysis. Assume $\widetilde{\xi}/p$ is normally distributed with variance σ^2 and mean $E\widetilde{R} > R_f$, where as suggested above the inequality is required to ensure that $n > 0$. It can be verified that

$$n_f = \frac{I - \frac{1}{2}\frac{\left(E\widetilde{R} - R_f\right)\left(E\widetilde{R} + R_f + 2\right)}{\kappa_2 \sigma^2} + \frac{\ln\left(\beta R_f\right)}{\kappa_1}}{\left(1 + R_f\right) p_f} \tag{93}$$

and

$$\frac{\partial n_f^{risky}}{\partial \sigma^2} > 0. \tag{94}$$

Thus, consistent with the equilibrium story, the equilibrium risk free asset return decreases with risk and at the demand level, the consumer's demand for the risk free asset increases. However in the demand analysis when comparing n_f^{risky} and

[38] This conclusion is confirmed by differentiating the equilibrium R_f expression (B.97) in Supplemental Appendix B.4 with respect to σ^2.

$n_f^{certain}$, for the reason indicated above we cannot use the limit argument to show that $n_f^{certain} < n_f^{risky}$ is implied. In fact, as we have proved in Theorem 4 and Proposition 1, when considering the certainty and risky cases separately for the CARA NM index, we always have $n_f^{certain} > n_f^{risky}$, implying that $\vartheta < 0$. (See Appendix B.5 for supporting calculations.)

6 Conclusion

In this paper, the KPS preference model is used to analyze excess and precautionary savings when the consumer faces capital rather than income risk. When risk preferences are represented by the widely assumed HARA class, prudence, contrary to comments in the literature, is neither necessary nor sufficient for positive excess saving behavior. Instead the necessary and sufficient and sufficient conditions depend on the pure time preference measure $EMRS$. The source of this misperception is the use of EU and its inability to separate time and risk preferences. For the popular cases of EU CRRA and external habit formation (translated origin CRRA) risk preferences, the resulting required conditions for positive excess saving impose unreasonable restrictions on risk preferences. However for the corresponding KPS generalizations, positive excess saving can be achieved with separate restrictions on both time and risk preference parameters that are fully consistent with standard empirical and experimental results. We show in a standard pure exchange representative equilibrium model that for common forms of time and risk preferences, the equilibrium risk free rate is less in the risky portfolio setting than in the certainty case. This implies that the representative agent demands more of the risk free asset in the risky setting. However, the demand analysis indicates that for the same forms of utility one obtains the opposite conclusion. This inconsonance results from differences in the definitions of precautionary saving and associated underlying assumptions in the demand and equilibrium settings.

Several extensions of this work seem natural. The first would be to relax one or both of the assumptions that U is additively separable and the second period EU NM index is independent of period one consumption. One potentially interesting way to do this would be to generalize the KPS preference model to incorporate a simple linear habit formation model and a form of risk preference dependence. We have already shown in Corollary 1 and Result 1 that varying the DARA risk preference parameter b increases excess and precautionary saving. It would be interesting to investigate whether θ and ϑ continue increase in versions of the linear habit model and risk preference dependence where respectively both

u_2 and V depend on period one consumption. A second extension would be to merge the consumption-portfolio and consumption-leisure optimizations. In this case, the introduction of capital risk and labor risk would both be endogenous. Although clearly desirable, this would seem to be a challenging problem.

Appendix

A Proofs

A.1 Proof of Theorem 1

Differentiating the first order condition

$$\frac{u_1'(c_1)}{u_2'(c_2)} = R \tag{A.1}$$

with respect to R, yields

$$u_1''(c_1)\frac{\partial c_1}{\partial R} = Ru_2''(c_2)\frac{\partial c_2}{\partial R} + u_2'(c_2). \tag{A.2}$$

Differentiating the constraint

$$c_2 = (I - c_1) R \tag{A.3}$$

with respect to R, it follows that

$$\frac{\partial c_2}{\partial R} = (I - c_1) - R\frac{\partial c_1}{\partial R}. \tag{A.4}$$

Substituting eqn. (A.4) into (A.2) yields

$$\left(u_1''(c_1) + R^2u_2''(c_2)\right)\frac{\partial c_1}{\partial R} = Ru_2''(c_2)(I - c_1) + u_2'(c_2). \tag{A.5}$$

Since we require that the optimal point given by the first order condition be a local maximum, the second order condition ensures that

$$u_1''(c_1) + R^2u_2''(c_2) < 0. \tag{A.6}$$

Therefore,

$$\frac{\partial c_1}{\partial R} \gtreqqless 0 \Leftrightarrow Ru_2''(c_2)(I - c_1) + u_2'(c_2) \lesseqqgtr 0. \tag{A.7}$$

Notice that

$$Ru_2''(c_2)(I - c_1) + u_2'(c_2) \lesseqqgtr 0 \Leftrightarrow -\frac{c_2u_2''(c_2)}{u_2'(c_2)} \gtreqqless 1. \tag{A.8}$$

Then we have

$$\frac{\partial c_1}{\partial R} \gtreqqless 0 \Leftrightarrow -\frac{c_2u_2''(c_2)}{u_2'(c_2)} \gtreqqless 1. \tag{A.9}$$

A.2 Proof of Theorem 2

First prove necessity. Since we require that $\widehat{c}_2$ be a linear function of c_1 for any $\widetilde{\xi}$, consider the two state case. Thus $\widehat{c}_2$ is a linear function of c_1 only if each state c_{2i} is a linear function of c_1. It follows from Pollak (1971) that the NM index must be a HARA member.[39] Next prove sufficiency. If V is a HARA member, it can be easily verified that (Gollier 2001)

$$-\frac{\partial^2 V/\partial c_2^2}{\partial V/\partial c_2} = \frac{1}{a + bc_2}, \tag{A.10}$$

where a and b are arbitrary constants. It follows from Selden (1980, Corollary, p. 440) that $\widehat{c}_2$ is a linear function of c_1.

A.3 Proof of Theorem 3

In this appendix, we prove Theorem 3 without assuming complete markets. We discuss the DARA, CARA and IARA cases separately.

(i) Consider the DARA case

$$V(c_2) = -\frac{(c_2 - b)^{-\delta_2}}{\delta_2}. \tag{A.11}$$

First, assume $b = 0$. The first order condition gives

$$E\left[\left(\widetilde{\xi} - \frac{p}{p_f}\xi_f\right)\left(\widetilde{\xi}n + \xi_f n_f\right)^{-1-\delta_2}\right] = 0. \tag{A.12}$$

Following the covariance inequality, we have

$$\begin{aligned} 0 &= E\left[\left(\widetilde{\xi} - \frac{p}{p_f}\xi_f\right)\left(\widetilde{\xi}n + \xi_f n_f\right)^{-1-\delta_2}\right] \\ &\gtreqless E\left[\widetilde{\xi} - \frac{p}{p_f}\xi_f\right] E\left[\left(\widetilde{\xi}n + \xi_f n_f\right)^{-1-\delta_2}\right] \Leftrightarrow n \lesseqgtr 0. \end{aligned} \tag{A.13}$$

Since

$$E\left[\widetilde{\xi} - \frac{p}{p_f}\xi_f\right] > 0 \quad \text{and} \quad E\left[\left(\widetilde{\xi}n + \xi_f n_f\right)^{-1-\delta_2}\right] > 0, \tag{A.14}$$

optimal demand denoted by n^* is positive. Define

$$\widehat{R}_p = \frac{\widehat{c}_2}{I - c_1} = \frac{\left(E\left[\left(\widetilde{\xi}n + \xi_f n_f\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}}}{I - c_1}. \tag{A.15}$$

[39] It should be noted that the modified Bergson family defined in Pollak (1971) corresponds to the HARA class of NM indices (Gollier 2001 and Rubinstein 1976).

If $n = 0$, then

$$\widehat{R}_p = \frac{\xi_f n_f}{I - c_1} = \frac{\xi_f (I - c_1)}{(I - c_1) p_f} = R_f. \tag{A.16}$$

In the second stage optimization, we consider the problem

$$\max_{n, n_f} \left(E \left[\left(\widetilde{\xi} n + \xi_f n_f \right)^{-\delta_2} \right] \right)^{-\frac{1}{\delta_2}} \quad S.T. \;\; pn + p_f n_f \leq I - c_1. \tag{A.17}$$

Since the preferences are homothetic, the optimal demand $\left(n^*, n_f^*\right)$ must satisfy

$$n^* = \zeta_1 (I - c_1) > 0 \quad \text{ and } \quad n_f^* = \zeta_2 (I - c_1). \tag{A.18}$$

Because $\left(n^*, n_f^*\right)$ is the optimal demand where $n^* > 0$, the resulting $\widehat{c}_2$ must be larger than the case with $n = 0$. Therefore, we have

$$\widehat{R}_p = \frac{\widehat{c}_2}{I - c_1} = const > \frac{\widehat{c}_2|_{n=0}}{I - c_1} = R_f, \tag{A.19}$$

where $\widehat{c}_2|_{n=0}$ denotes $\widehat{c}_2$ evaluated at $n = 0$. Next, assuming $b \neq 0$ and defining

$$n_f^{new} = n_f - \frac{b}{\xi_f}, \tag{A.20}$$

the optimization problem

$$\max_{n, n_f} \left(E \left[\left(\widetilde{\xi} n + \xi_f n_f - b \right)^{-\delta_2} \right] \right)^{-\frac{1}{\delta_2}} \quad S.T. \;\; pn + p_f n_f \leq I - c_1 \tag{A.21}$$

can be converted into

$$\max_{n, n_f} \left(E \left[\left(\widetilde{\xi} n + \xi_f n_f^{new} \right)^{-\delta_2} \right] \right)^{-\frac{1}{\delta_2}} \quad S.T. \;\; pn + p_f n_f^{new} \leq I - \frac{b}{R_f} - c_1. \tag{A.22}$$

Using a similar argument as in the $b = 0$ case, it follows that

$$\widehat{R}_p = const > R_f, \tag{A.23}$$

where

$$\widehat{c}_2^{new} = \left(E \left[\left(\widetilde{\xi} n + \xi_f n_f^{new} \right)^{-\delta_2} \right] \right)^{-\frac{1}{\delta_2}} \tag{A.24}$$

and

$$\widehat{R}_p = \frac{\widehat{c}_2^{new}}{I - \frac{b}{R_f} - c_1} = \frac{\left(E \left[\left(\widetilde{\xi} n + \xi_f n_f^{new} \right)^{-\delta_2} \right] \right)^{-\frac{1}{\delta_2}}}{I - \frac{b}{R_f} - c_1}. \tag{A.25}$$

Since

$$\widehat{c}_2 = \left(E \left[\left(\widetilde{\xi} n + \xi_f n_f - b \right)^{-\delta_2} \right] \right)^{-\frac{1}{\delta_2}} + b, \tag{A.26}$$

following eqn. (A.25), the $\widehat{c}_2(c_1)$ constraint is given by

$$\widehat{R}_p = \frac{\widehat{c}_2 - b}{I - \frac{b}{R_f} - c_1}, \tag{A.27}$$

or equivalently,

$$\widehat{c}_2 = \widehat{R}_p (I - c_1 - \Delta), \tag{A.28}$$

where

$$\Delta = \frac{b}{R_f} - \frac{b}{\widehat{R}_p}. \tag{A.29}$$

(ii) Consider the CARA case

$$V(c_2) = -\frac{\exp(-\kappa c_2)}{\kappa}. \tag{A.30}$$

Based on the first order condition,

$$E\left[\left(\widetilde{\xi} - \frac{p}{p_f}\xi_f\right)\exp\left(-\kappa\left(\widetilde{\xi}n + \xi_f n_f\right)\right)\right] = 0, \tag{A.31}$$

which is also equivalent to

$$E\left[\left(\widetilde{\xi} - \frac{p}{p_f}\xi_f\right)\exp\left(-\kappa\widetilde{\xi}n\right)\right] = 0. \tag{A.32}$$

Since the first order condition is independent of n_f, the optimal demand $n^* = const$. Following the covariance inequality, we also have $n^* > 0$. Then

$$\widehat{c}_2 = -\frac{\ln\left(E\left[\exp\left(-\kappa\left(\widetilde{\xi}n + \xi_f n_f\right)\right)\right]\right)}{\kappa}. \tag{A.33}$$

If $n = 0$, then

$$\widehat{c}_2 = \xi_f n_f, \tag{A.34}$$

implying that

$$\widehat{R}_p = \frac{\xi_f n_f}{I - c_1} = \frac{\xi_f (I - c_1)}{(I - c_1) p_f} = R_f. \tag{A.35}$$

Otherwise, since $n^* = const$, we have

$$\begin{aligned}\widehat{c}_2 &= -\frac{\ln\left(E\left[\exp\left(-\kappa\widetilde{\xi}n^*\right)\right]\right) + \ln\left(-\kappa R_f (I - c_1 - pn^*)\right)}{\kappa} \\ &= R_f (I - c_1) - \left(pR_f n^* + \frac{\ln\left(E\left[\exp\left(-\kappa\widetilde{\xi}n^*\right)\right]\right)}{\kappa}\right)\end{aligned} \tag{A.36}$$

Therefore, the $\widehat{c}_2(c_1)$ constraint is given by

$$\widehat{c}_2 = R_f (I - c_1 - \Delta), \tag{A.37}$$

where

$$\Delta = pn^* + \frac{\ln\left(E\left[\exp\left(-\kappa\widetilde{\xi}n^*\right)\right]\right)}{\kappa R_f}. \tag{A.38}$$

Because $\left(n^*, n_f^*\right)$ is the optimal portfolio and $n^* > 0$, the resulting $\widehat{c}_2$ must be larger than the case with $n = 0$. Thus we have $\Delta < 0$.

(iii) Consider the IARA case

$$V\left(c_2\right) = \frac{\left(b - c_2\right)^{-\delta_2}}{\delta_2}. \tag{A.39}$$

Defining

$$n^{new} = -n \quad \text{and} \quad n_f^{new} = \frac{b}{\xi_f} - n_f, \tag{A.40}$$

the optimization problem

$$\max_{n,n_f}\left[b - \left(E\left[\left(b - \widetilde{\xi}n - \xi_f n_f\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}}\right] \quad S.T.\ \ pn + p_f n_f \le I - c_1 \tag{A.41}$$

can be converted into

$$\min_{n,n_f}\left(E\left[\left(\widetilde{\xi}n^{new} + \xi_f n_f^{new}\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}} \quad S.T.\ \ pn^{new} + p_f n_f^{new} \ge \frac{b}{R_f} - \left(I - c_1\right). \tag{A.42}$$

Since $c_2 < b$, we have

$$\widetilde{\xi}n^{new} + \xi_f n_f^{new} = b - \left(\widetilde{\xi}n - \xi_f n_f\right) > 0. \tag{A.43}$$

Using a similar argument as in the DARA case where $b = 0$ and noticing that we consider the minimum value instead of maximum value, we have

$$\widehat{c}_2^{new} = \left(E\left[\left(\widetilde{\xi}n^{new} + \xi_f n_f^{new}\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}} \tag{A.44}$$

and

$$\widehat{R}_p = \frac{\widehat{c}_2^{new}}{\frac{b}{R_f} - \left(I - c_1\right)} < R_f. \tag{A.45}$$

Since

$$\widehat{c}_2 = b - \left(E\left[\left(b - \widetilde{\xi}n - \xi_f n_f\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}}, \tag{A.46}$$

following eqn. (A.45), the $\widehat{c}_2\left(c_1\right)$ constraint is given by

$$\widehat{R}_p = \frac{b - \widehat{c}_2}{\frac{b}{R_f} - \left(I - c_1\right)}, \tag{A.47}$$

or equivalently,

$$\widehat{c}_2 = \widehat{R}_p\left(I - c_1 - \Delta\right), \tag{A.48}$$

where

$$\Delta = \frac{b}{R_f} - \frac{b}{\widehat{R}_p} < 0. \tag{A.49}$$

A.4 Proof of Theorem 4

For the DARA case, it follows from Theorem 3 that

$$\widehat{c}_2 = \widehat{R}_p \left(I - c_1 - \Delta\right), \tag{A.50}$$

where

$$\widehat{R}_p > R_f \quad \text{and} \quad \Delta = \frac{b}{R_f} - \frac{b}{\widehat{R}_p}. \tag{A.51}$$

If $b = 0$, comparing the constraints

$$\widehat{c}_2 = \widehat{R}_p \left(I - c_1\right) \qquad \text{and} \qquad c_2 = R_f \left(I - c_1\right), \tag{A.52}$$

Theorem 1 implies that

$$\theta \gtreqless 0 \Leftrightarrow -\frac{c_2 u_2'' \left(c_2\right)}{u_2' \left(c_2\right)} \lesseqgtr 1. \tag{A.53}$$

If $b > 0$, compare the constraints

$$\widehat{c}_2 = \widehat{R}_p \left(I - c_1 - \Delta\right) \qquad \text{and} \tag{A.54}$$

and

$$\widehat{c}_2 = \widehat{R}_p \left(I - c_1\right). \tag{A.55}$$

The corresponding first order conditions are

$$u_1' \left(c_1\right) = \widehat{R}_p u_2' \left(\widehat{R}_p \left(I - c_1 - \Delta\right)\right) \qquad \text{and} \qquad u_1' \left(c_1\right) = \widehat{R}_p u_2' \left(\widehat{R}_p \left(I - c_1\right)\right). \tag{A.56}$$

Since $u_1'', u_2'' < 0$, c_1 for the constraint (A.54) is smaller than c_1 for the constraint (A.55). Comparing the constraints

$$\widehat{c}_2 = \widehat{R}_p \left(I - c_1\right) \qquad \text{and} \qquad c_2 = R_f \left(I - c_1\right), \tag{A.57}$$

it follows from Theorem 1 that

$$\theta \gtreqless 0 \Leftrightarrow -\frac{c_2 u_2'' \left(c_2\right)}{u_2' \left(c_2\right)} \lesseqgtr 1. \tag{A.58}$$

Therefore, we have

$$\theta > 0 \quad \text{if} \quad -\frac{c_2 u_2'' \left(c_2\right)}{u_2' \left(c_2\right)} \leq 1. \tag{A.59}$$

If $b < 0$, the c_1-value corresponding to the constraint (A.54) is smaller than the c_1-value corresponding to (A.55). Therefore, we have

$$\theta < 0 \quad \text{if} \quad -\frac{c_2 u_2'' \left(c_2\right)}{u_2' \left(c_2\right)} \geq 1. \tag{A.60}$$

For the CARA case, it follows from Theorem 3 that

$$\widehat{c}_2 = R_f \left(I - c_1 - \Delta \right), \tag{A.61}$$

where

$$\Delta < 0. \tag{A.62}$$

Compare the constraints

$$\widehat{c}_2 = R_f \left(I - c_1 - \Delta \right) \quad \text{and} \quad c_2 = R_f \left(I - c_1 \right). \tag{A.63}$$

The corresponding first order conditions are

$$u_1' \left(c_1 \right) = R_f u_2' \left(R_f \left(I - c_1 - \Delta \right) \right) \quad \text{and} \quad u_1' \left(c_1 \right) = R_f u_2' \left(R_f \left(I - c_1 \right) \right). \tag{A.64}$$

Since $u_1'', u_2'' < 0$, the value of c_1 corresponding to the constraint (A.54) is larger than the value corresponding to (A.55). Therefore, we always have $\theta < 0$. For the IARA case, it follows from Theorem 3 that

$$\widehat{c}_2 = \widehat{R}_p \left(I - c_1 - \Delta \right), \tag{A.65}$$

where

$$\widehat{R}_p < R_f \text{ and } \Delta = \frac{b}{R_f} - \frac{b}{\widehat{R}_p} < 0. \tag{A.66}$$

Clearly, c_1 is larger for the constraint (A.54) than for (A.55). Comparing the constraints

$$\widehat{c}_2 = \widehat{R}_p \left(I - c_1 \right) \quad \text{and} \quad c_2 = R_f \left(I - c_1 \right), \tag{A.67}$$

it follows from Theorem 1 that

$$\theta \lesseqqgtr 0 \Leftrightarrow -\frac{c_2 u_2'' \left(c_2 \right)}{u_2' \left(c_2 \right)} \lesseqqgtr 1. \tag{A.68}$$

Therefore, we have

$$\theta < 0 \quad \text{if} \quad -\frac{c_2 u_2'' \left(c_2 \right)}{u_2' \left(c_2 \right)} \leq 1. \tag{A.69}$$

A.5 Proof of Corollary 1

It follows from Theorem 3 that the $\widehat{c}_2$ constraint is given by

$$\widehat{c}_2 = \widehat{R}_p \left(I - c_1 - \Delta \right). \tag{A.70}$$

The first order condition is

$$u_1' \left(c_1 \right) = \widehat{R}_p u_2' \left(\widehat{R}_p \left(I - c_1 - \Delta \right) \right). \tag{A.71}$$

Since

$$\frac{\partial \widehat{R}_p}{\partial b} = 0 \quad \text{and} \quad \frac{\partial \Delta}{\partial b} > 0 \tag{A.72}$$

and $u_i'' < 0$ $(i = 1, 2)$, we have

$$\frac{\partial c_1^{risky}}{\partial b} < 0. \tag{A.73}$$

Noticing that

$$\frac{\partial c_1^{certain}}{\partial b} = 0, \tag{A.74}$$

we have

$$\frac{\partial \theta}{\partial b} = \frac{\partial \left(s_1^{risky} - s_1^{certain} \right)}{\partial b} = \frac{\partial \left(c_1^{certain} - c_1^{risky} \right)}{\partial b} > 0. \tag{A.75}$$

A.6 Proof of Corollary 2

For (i), as proved in Theorem 3, the constraint is given by

$$\widehat{c}_2 = \widehat{R}_p \left(I - c_1 - \Delta \right), \tag{A.76}$$

where

$$\widehat{R}_p > R_f \ \text{ and } \ \Delta = \frac{b}{R_f} - \frac{b}{\widehat{R}_p}. \tag{A.77}$$

Defining

$$\widehat{c}_2^{new} = \widehat{c}_2 - b, \tag{A.78}$$

we have

$$u_1 \left(c_1 \right) - \frac{\left(\widehat{c}_2^{new} \right)^{-\delta_1}}{\delta_1} \tag{A.79}$$

and the constraint is given by

$$\widehat{c}_2^{new} = \widehat{R}_p \left(I - c_1 - \frac{b}{R_f} \right). \tag{A.80}$$

For the certainty case, the constraint can be rewritten as

$$c_2^{new} = R_f \left(I - c_1 - \frac{b}{R_f} \right). \tag{A.81}$$

Since $\widehat{R}_p > R_f$, it follows from Theorem 1 that

$$\theta \gtreqqless 0 \Leftrightarrow \delta_1 \lesseqqgtr 0. \tag{A.82}$$

For (ii), it follows from Theorem 3 that the constraint is given by

$$\widehat{c}_2 = \widehat{R}_p \left(I - c_1 - \Delta \right), \tag{A.83}$$

where

$$\widehat{R}_p < R_f \quad \text{and} \quad \Delta = \frac{b}{R_f} - \frac{b}{\widehat{R}_p} < 0. \tag{A.84}$$

Defining

$$\widehat{c}_2^{new} = -\left(b - \widehat{c}_2\right) < 0, \tag{A.85}$$

we have

$$U\left(c_1, \widehat{c}_2^{new}\right) = u_1\left(c_1\right) + \frac{\left(-\widehat{c}_2^{new}\right)^{-\delta_1}}{\delta_1} \tag{A.86}$$

and the constraint is given by

$$\widehat{c}_2^{new} = \widehat{R}_p\left(I - c_1 - \frac{b}{R_f}\right). \tag{A.87}$$

It can be easily verified that $U\left(c_1, \widehat{c}_2^{new}\right)$ is strictly increasing and strictly concave in $\widehat{c}_2^{new}$. For the certainty case, the constraint can be rewritten as

$$c_2^{new} = R_f\left(I - c_1 - \frac{b}{R_f}\right). \tag{A.88}$$

Since $\widehat{R}_p < R_f$ and

$$-\frac{c_2^{new} u_2''\left(c_2^{new}\right)}{u_2'\left(c_2^{new}\right)} = 1 + \delta_1 < 0, \tag{A.89}$$

it follows from Theorem 1 that[40]

$$\theta < 0. \tag{A.90}$$

A.7 Proof of Proposition 1

Noting that

$$p_f n_f^{certain} = s_1^{certain}, \tag{A.91}$$

we have

$$\vartheta = p_f {n_f}^{risky} - s_1^{certain}. \tag{A.92}$$

Observe that

$$s_1^{risky} = I - c_1 = pn + p_f n_f^{risky}, \tag{A.93}$$

$V'' < 0$ and $E\widetilde{R} > R_f$ imply $n > 0$ (Kubler, Selden and Wei 2013, p. 1036). It then follows that $p_f n_f^{risky} < s_1^{risky}$. Therefore for precautionary saving to be positive in the consumption-portfolio setting, excess saving must be positive.

[40] Note that the proof of Theorem 1 works even if $c_2 < 0$.

A.8 Proof of Result 1

Since for a given c_1, increasing b will increase the risk aversion, we have

$$\frac{\partial n^{risky}(c_1)}{\partial b} < 0 \text{ and } \frac{\partial n_f^{risky}(c_1)}{\partial b} > 0. \tag{A.94}$$

If the risk free asset holdings satisfy $n_f > b/\xi_f$, then it follows from Proposition 2 and Proposition 5(ii) in Kubler, Selden and Wei (2014) that

$$\frac{\partial n_f^{risky}(c_1)}{\partial I} > 0 \Rightarrow \frac{\partial n_f^{risky}(c_1)}{\partial c_1} < 0. \tag{A.95}$$

Moreover,

$$\frac{\partial n_f^{certain}}{\partial b} = 0. \tag{A.96}$$

Therefore,

$$\frac{\partial \vartheta}{\partial b} = \frac{\partial\left(p_f n_f^{risky} - p_f n_f^{certain}\right)}{\partial b} = p_f \frac{\partial n_f^{risky}(c_1)}{\partial b}\frac{\partial n_f^{risky}(c_1)}{\partial c_1}\frac{\partial c_1^{risky}}{\partial b} > 0. \tag{A.97}$$

A.9 Proof of Theorem 5

For (i), the first order condition is

$$p_f = \frac{\beta \xi_f \left(E\left(\widetilde{\xi n} + \xi_f \overline{n}_f - b\right)^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}-1} E\left(\widetilde{\xi n} + \xi_f \overline{n}_f - b\right)^{-\delta_2 - 1}}{(\overline{c}_1 - b)^{-\delta_1 - 1}}, \tag{A.98}$$

implying that

$$R_f^{risky} = \frac{(\overline{c}_1 - b)^{-\delta_1 - 1}}{\beta\left(E\left(\widetilde{\xi n} + \xi_f \overline{n}_f - b\right)^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}-1} E\left(\widetilde{\xi n} + \xi_f \overline{n}_f - b\right)^{-\delta_2 - 1}}. \tag{A.99}$$

Without the risky asset, we have

$$R_f^{certain} = \frac{(\overline{c}_1 - b)^{-\delta_1 - 1}}{\beta\left(\xi_f \overline{n}_f - b\right)^{-\delta_1 - 1}}. \tag{A.100}$$

Since

$$\left(E\left(\widetilde{\xi n} + \xi_f \overline{n}_f - b\right)^{-\delta_2}\right)^{-\frac{1}{\delta_2}} > \left(E\left(\widetilde{\xi n} + \xi_f \overline{n}_f - b\right)^{-\delta_2 - 1}\right)^{-\frac{1}{1+\delta_2}}, \tag{A.101}$$

we have

$$E\left(\widetilde{\xi n} + \xi_f \overline{n}_f - b\right)^{-\delta_2 - 1} > \left(E\left(\widetilde{\xi n} + \xi_f \overline{n}_f - b\right)^{-\delta_2}\right)^{\frac{1+\delta_2}{\delta_2}}. \tag{A.102}$$

Therefore,

$$\left(E\left(\widetilde{\xi}\overline{n}+\xi_f\overline{n}_f-b\right)^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}-1}E\left(\widetilde{\xi}\overline{n}+\xi_f\overline{n}_f-b\right)^{-\delta_2-1}>\left(E\left(\widetilde{\xi}\overline{n}+\xi_f\overline{n}_f-b\right)^{-\delta_2}\right)^{\frac{1+\delta_1}{\delta_2}}. \tag{A.103}$$

Since

$$\left(E\left(\widetilde{\xi}\overline{n}+\xi_f\overline{n}_f-b\right)^{-\delta_2}\right)^{-\frac{1}{\delta_2}}<\xi_f\overline{n}_f-b, \tag{A.104}$$

we have

$$\left(E\left(\widetilde{\xi}\overline{n}+\xi_f\overline{n}_f-b\right)^{-\delta_2}\right)^{\frac{1+\delta_1}{\delta_2}}>\left(\xi_f\overline{n}_f-b\right)^{-\delta_1-1}. \tag{A.105}$$

Noticing that

$$R_f^{certain}=\frac{\left(\overline{c}_1-b\right)^{-\delta_1-1}}{\beta\left(\xi_f\overline{n}_f-b\right)^{-\delta_1-1}}, \tag{A.106}$$

we always have

$$R_f^{risky}<R_f^{certain}. \tag{A.107}$$

For (ii), it follows from the first order condition that

$$R_f^{risky}=\frac{\exp\left(-\kappa_1\overline{c}_1\right)}{\beta\left(E\exp\left(-\kappa_2\widetilde{c}_2\right)\right)^{\frac{\kappa_1}{\kappa_2}}}. \tag{A.108}$$

Without the risky asset, we have

$$R_f^{certain}=\frac{\exp\left(-\kappa_1\overline{c}_1\right)}{\beta\exp\left(-\kappa_1\xi_f\overline{n}_f\right)}. \tag{A.109}$$

It follows from Jensen's inequality that

$$E\exp\left(-\kappa_2\widetilde{c}_2\right)>\exp\left(-\kappa_2E\widetilde{c}_2\right)=\exp\left(-\kappa_2\xi_f\overline{n}_f\right). \tag{A.110}$$

Therefore, we have

$$\left(E\exp\left(-\kappa_2\widetilde{c}_2\right)\right)^{\frac{\kappa_1}{\kappa_2}}>\left(\exp\left(-\kappa_2\xi_f\overline{n}_f\right)\right)^{\frac{\kappa_1}{\kappa_2}}=\exp\left(-\kappa_1\xi_f\overline{n}_f\right). \tag{A.111}$$

Thus, the following always holds

$$R_f^{risky}<R_f^{certain}. \tag{A.112}$$

A.10 Proof of Proposition 2

It follows from the first order condition that

$$R_f^{risky}=\frac{\left(b-\overline{c}_1\right)^{-\delta_1-1}}{\beta\left(E\left(b-\widetilde{\xi}\overline{n}-\xi_f\overline{n}_f\right)^2\right)^{-\frac{\delta_1}{2}-1}E\left(b-\widetilde{\xi}\overline{n}-\xi_f\overline{n}_f\right)}. \tag{A.113}$$

Without the risky asset, we have

$$R_f^{certain} = \frac{(b - \overline{c}_1)^{-\delta_1 - 1}}{\beta \left(b - \xi_f \overline{n}_f\right)^{-\delta_1 - 1}}. \tag{A.114}$$

Since

$$E\left(b - \widetilde{\xi}\overline{n} - \xi_f \overline{n}_f\right)^2 > \left(E\left(b - \widetilde{\xi}\overline{n} - \xi_f \overline{n}_f\right)\right)^2 = \left(b - \xi_f \overline{n}_f\right)^2, \tag{A.115}$$

we have

$$\left(E\left(b - \widetilde{\xi}\overline{n} - \xi_f \overline{n}_f\right)^2\right)^{-\frac{\delta_1}{2} - 1} \gtreqless \left(b - \xi_f \overline{n}_f\right)^{-2-\delta_1} \Leftrightarrow \delta_1 \lesseqgtr -2. \tag{A.116}$$

Therefore,

$$\left(E\left(b - \widetilde{\xi}\overline{n} - \xi_f \overline{n}_f\right)^2\right)^{-\frac{\delta_1}{2} - 1} E\left(b - \widetilde{\xi}\overline{n} - \xi_f \overline{n}_f\right) \gtreqless \left(b - \xi_f \overline{n}_f\right)^{-1-\delta_1} \tag{A.117}$$

if and only if

$$\delta_1 \lesseqgtr -2. \tag{A.118}$$

Thus,

$$R_f^{risky} \lesseqgtr R_f^{certain} \Leftrightarrow \delta_1 \lesseqgtr -2. \tag{A.119}$$

B Supplemental Appendix

B.1 Supporting Calculations for Example 1

For the income risk case, the first order condition is

$$\begin{aligned} u_1'(c_1) &= \frac{q_1 - 2\widehat{c}_2}{q_2 - 2\widehat{c}_2} \beta R_f \left(\pi_1 \left(q_2 - 2\left(R_f\left(I - c_1\right) + I_{21}\right)\right) + \pi_2 \left(q_2 - 2\left(R_f\left(I - c_1\right) + I_{22}\right)\right)\right) \\ &= \frac{q_1 - 2\widehat{c}_2}{q_2 - 2\widehat{c}_2} \beta R_f \left(q_2 - 2\left(R_f\left(I - c_1\right)\right)\right), \end{aligned} \tag{B.1}$$

where

$$\widehat{c}_2 = V^{-1}\left(\pi_1 V\left(R_f\left(I - c_1\right) + I_{21}\right) + \pi_2 V\left(R_f\left(I - c_1\right) + I_{22}\right)\right). \tag{B.2}$$

For the certainty case, the first order condition is

$$u_1'(c_1) = \beta R_f \left(q_1 - 2\left(R_f\left(I - c_1\right)\right)\right). \tag{B.3}$$

Noticing that

$$\begin{aligned} & \left(q_2 - 2\left(R_f\left(I - c_1\right)\right)\right)\left(q_1 - 2\widehat{c}_2\right) - \left(q_1 - 2\left(R_f\left(I - c_1\right)\right)\right)\left(q_2 - 2\widehat{c}_2\right) \\ = \;& 2\left(q_1 - q_2\right)\left(\widehat{c}_2 - R_f\left(I - c_1\right)\right) \end{aligned} \tag{B.4}$$

and

$$\widehat{c}_2 < R_f (I - c_1), \tag{B.5}$$

we have

$$\frac{q_1 - 2\widehat{c}_2}{q_2 - 2\widehat{c}_2}\beta R_f (q_2 - 2(R_f (I - c_1))) \gtreqless \beta R_f (q_1 - 2(R_f (I - c_1))) \Leftrightarrow q_1 \lesseqgtr q_2. \tag{B.6}$$

Since $u_1'' (c_1) < 0$, we have

$$\theta \gtreqless 0 \Leftrightarrow q_1 \lesseqgtr q_2. \tag{B.7}$$

For the capital risk case, the first order condition is

$$u_1' (c_1) = \frac{u_2' (\widehat{c}_2)}{V' (\widehat{c}_2)} (\pi_{21} R_{21} V' (R_{21} (I - c_1)) + \pi_{22} R_{22} V' (R_{22} (I - c_1))), \tag{B.8}$$

where

$$\widehat{c}_2 = V^{-1} (\pi_{21} V (R_{21} (I - c_1)) + \pi_{22} V (R_{22} (I - c_1))). \tag{B.9}$$

Noticing that

$$u_2' (c_2) = \beta (q_1 - 2c_2) \qquad \text{and} \qquad V' (c_2) = q_2 - 2c_2, \tag{B.10}$$

the first order condition can be rewritten as

$$\begin{aligned} u_1' (c_1) &= \frac{q_1 - 2\widehat{c}_2}{q_2 - 2\widehat{c}_2}\beta (\pi_1 R_{21} (q_2 - 2R_{21} (I - c_1)) + \pi_2 R_{22} (q_2 - 2R_{22} (I - c_1))) \\ &= \frac{q_1 - 2\widehat{c}_2}{q_2 - 2\widehat{c}_2}\beta \left(R_f q_2 - 2\left(\pi_{21} R_{21}^2 + \pi_{22} R_{22}^2\right) (I - c_1)\right). \end{aligned} \tag{B.11}$$

For the certainty case, the first order condition is

$$u_1' (c_1) = \beta \left(R_f q_1 - 2R_f^2 (I - c_1)\right). \tag{B.12}$$

If $q_1 = q_2$, then since

$$\pi_{21} R_{21}^2 + \pi_{22} R_{22}^2 > (\pi_{21} R_{21} + \pi_{22} R_{22})^2 = R_f^2, \tag{B.13}$$

c_1 in the certainty case is less than c_1 in the risky case, implying that the investor will reduce saving when facing risk, i.e., s_1^{risky} is always smaller than $s_1^{certain}$. For the consumption-portfolio setting, the first order condition for the conditional portfolio problem is

$$\frac{q_2 - 2c_{21}}{q_2 - 2c_{22}} = k, \tag{B.14}$$

implying that

$$c_{21} = \frac{k (I - c_1) - 0.5 (k - 1) q_2 p_{22}}{k p_{21} + p_{22}} \qquad \text{and} \qquad c_{22} = \frac{I - c_1 + 0.5 (k - 1) q_2 p_{21}}{k p_{21} + p_{22}}. \tag{B.15}$$

Therefore, the period one optimization problem can be rewritten as

$$\max_{c_1,\widehat{c}_2}\left(q_1c_1 - c_1^2 + \beta\left(q_1\widehat{c}_2 - \widehat{c}_2^2\right)\right) \tag{B.16}$$

subject to

$$\widehat{c}_2 = \frac{\sqrt{\pi_{21}k^2+\pi_{22}}}{p_{21}k+p_{22}}\left(I - c_1 - \frac{q_2}{2}\left(\frac{1}{R_f} - \frac{p_{21}k+p_{22}}{\sqrt{\pi_{21}k^2+\pi_{22}}}\right)\right). \tag{B.17}$$

The first order condition is

$$u_1'(c_1) = \beta\left(\frac{\sqrt{\pi_{21}k^2+\pi_{22}}}{p_{21}k+p_{22}}q_1 - 2\frac{\pi_{21}k^2+\pi_{22}}{(p_{21}k+p_{22})^2}\left(I - c_1 - \frac{q_2}{2}\left(\frac{1}{R_f} - \frac{p_{21}k+p_{22}}{\sqrt{\pi_{21}k^2+\pi_{22}}}\right)\right)\right). \tag{B.18}$$

For the certainty case, the first order condition is still

$$u_1'(c_1) = \beta R_f\left(q_1 - 2\left(R_f(I - c_1)\right)\right). \tag{B.19}$$

There seems to be no simple condition to compare the right hand sides of eqns. (B.18) and (B.19). Thus we consider simulations based on the above first order conditions as shown in the text for Example 1.

B.2 Consumption-Saving Problem

In this appendix, we consider the single risky asset consumption-saving problem for KPS preferences, which is given by

$$\max_{c_1} U(c_1,\widehat{c}_2) = u_1(c_1) + u_2(\widehat{c}_2), \tag{B.20}$$

subject to

$$\widehat{c}_2 = V^{-1}EV\left((I - c_1)\widetilde{R}\right), \tag{B.21}$$

where $\widetilde{R} = \widetilde{\xi}/p$ denotes the return on the single risky asset. We derive conditions for when saving is larger in the presence of a single risky asset versus a risk free asset both where the certainty equivalent constraint (B.21) is linear and several cases where it is not. The sign of θ, as defined in Definition 1, is shown in general to be independent of the risk preference property prudence. For instance in Theorem 7 and Example 4 below, the NM indices always satisfy $V''' > 0$ and yet θ can be positive or negative.[41]

[41] Kimball and Weil (2009) also assume two period KPS preference. In the context of income risk, they show that $V''' > 0$ is not necessary for positive precautionary saving. A consumer will exhibit precautionary saving if her risk preferences satisfy (i) DARA which implies that $V''' > 0$ exhibits prudence or (ii) $V''' > 0$ along with other assumptions. Clearly, DARA can be violated and the consumer can exhibit negative precautionary saving even though $V''' > 0$.

In order to characterize when a consumer saves more in a setting with a single risky asset versus a risk free asset, the most straightforward case is where the period two certainty equivalent constraint is linear in c_1 since then Theorem 1 can be directly used. The following establishes that the certainty equivalent constraint is linear when and only when the NM index takes the CRRA form (B.23).

Theorem 6 *For the optimization (B.20)-(B.21), the period two certainty equivalent constraint satisfies*

$$\widehat{c}_2 = V^{-1}EV\left(\widetilde{\xi}n\right) = V^{-1}EV\left(\widetilde{R}\left(I-c_1\right)\right) = mc_1 + k \tag{B.22}$$

for any $\widetilde{R}$ if and only if

$$V\left(c_2\right) = -\frac{c_2^{-\delta_2}}{\delta_2} \quad \left(\delta_2 > -1\right), \tag{B.23}$$

where V is defined up to a positive affine transformation.[42]

Proof. First prove sufficiency. Assuming that V takes the CRRA form (B.23) we have

$$\widehat{c}_2 = \left(V^{-1}EV\left(\frac{\widetilde{\xi}}{p}\right)\right)\left(I-c_1\right) = \left(V^{-1}EV\left(\widetilde{R}\right)\right)\left(I-c_1\right), \tag{B.24}$$

which is linear in I. Next prove necessity. Define $s_1 = I - c_1$. Since we require that the condition works for any distribution, for necessity, it is enough to consider the two state case. If the $\widehat{c}_2\left(c_1\right)$ constraint is linear in I, we have

$$\widehat{c}_2 = V^{-1}\left(\pi_{21}V\left(R_{21}s_1\right) + \pi_{22}V\left(R_{22}s_1\right)\right) = as_1 + b, \tag{B.25}$$

where a and b are constants which can depend on $\widetilde{R}$. First letting $s_1 \to 0$, one obtains

$$V\left(0\right) = V\left(b\right). \tag{B.26}$$

Since $V' > 0$, we have $b = 0$. Thus

$$\pi_{21}V\left(R_{21}s_1\right) + \pi_{22}V\left(R_{22}s_1\right) = V\left(as_1\right). \tag{B.27}$$

Differentiating both sides of eqn. (B.27) with respect to s_1, yields

$$\pi_{21}R_{21}V'\left(R_{21}s_1\right) + \pi_{22}R_{22}V'\left(R_{22}s_1\right) = aV'\left(as_1\right). \tag{B.28}$$

[42] The assumption that V takes the CRRA form (B.23) together with U being strictly quasi-concave ensures that the problem (B.20)-(B.21) has a unique solution.

Differentiating both sides of eqn. (B.27) with respect to R_{21}, yields

$$\pi_{21} s_1 V'(R_{21} s_1) = s_1 \frac{\partial a}{\partial R_{21}} V'(a s_1). \tag{B.29}$$

Differentiating both sides of eqn. (B.28) with respect to R_{21}, yields

$$\pi_{21} V'(R_{21} s_1) + \pi_{21} R_{21} s_1 V''(R_{21} s_1) = \frac{\partial a}{\partial R_{21}} V'(a s_1) + a s_1 \frac{\partial a}{\partial R_{21}} V''(a s_1). \tag{B.30}$$

Since $V' > 0$, dividing eqn. (B.30) by (B.29), one obtains

$$\frac{1}{s_1} + \frac{R_{21} V''(R_{21} s_1)}{V'(R_{21} s_1)} = \frac{1}{s_1} + \frac{a V''(a s_1)}{V'(a s_1)}, \tag{B.31}$$

or equivalently

$$\frac{R_{21} s_1 V''(R_{21} s_1)}{V'(R_{21} s_1)} = \frac{a s_1 V''(a s_1)}{V'(a s_1)}. \tag{B.32}$$

Similarly, one can obtain

$$\frac{R_{22} s_1 V''(R_{22} s_1)}{V'(R_{22} s_1)} = \frac{a s_1 V''(a s_1)}{V'(a s_1)}. \tag{B.33}$$

Therefore,

$$\frac{R_{21} s_1 V''(R_{21} s_1)}{V'(R_{21} s_1)} = \frac{R_{22} s_1 V''(R_{22} s_1)}{V'(R_{22} s_1)}. \tag{B.34}$$

Since eqn. (B.34) holds for any R_{21}, R_{22} and s_1, we have

$$\frac{c_2 V''(c_2)}{V'(c_2)} = const, \tag{B.35}$$

implying that

$$V(c_2) = -\frac{c_2^{-\delta_2}}{\delta_2} \quad (\delta_2 > -1), \tag{B.36}$$

which is defined up to an affine transformation. ■

When the NM index is defined by (B.23), the certainty equivalent constraint takes the specific form

$$\widehat{c}_2 = (I - c_1) \widehat{R}, \tag{B.37}$$

where

$$\widehat{R} = \left(E\left[\widetilde{R}^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}}. \tag{B.38}$$

We next characterize when excess saving is positive for (i) the popular CRRA risk preference case where the certainty equivalent constraint (B.37) is linear and (ii) the KPS generalization of the important external habit formation EU representation of Campbell and Cochrane (1999).[43] In both instances, the sign of excess saving is shown to depend on the time preference $EMRS$.

[43] The Campbell and Cochrane (1999) representation is a special case of the KPS utility assumed in Theorem 7(ii) where up to suitable affine transformations both the certainty u_2 and risk preference V take the translated CRRA form (B.41). The parameter b is interpreted as a reflection of an external habit.

Theorem 7 *Assume a single risky asset where $E\widetilde{R} = R_f$ and KPS preferences, where U takes the additively separable form (8). Then*

(i) *if*

$$V(c_2) = -\frac{c_2^{-\delta_2}}{\delta_2} \quad (\delta_2 > -1), \tag{B.39}$$

we have

$$\theta \gtreqless 0 \Leftrightarrow -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \gtreqless 1 \tag{B.40}$$

and

(ii) *if*

$$V(c_2) = -\frac{(c_2 - b)^{-\delta_2}}{\delta_2} \quad (b > 0, \delta_2 > -1), \tag{B.41}$$

we have

$$\theta > 0 \ \ if \quad -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \geq 1. \tag{B.42}$$

Proof. For (i), it follows from Theorem 1 that

$$\frac{\partial c_1}{\partial R} \gtreqless 0 \Leftrightarrow -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \gtreqless 1. \tag{B.43}$$

Since $\widehat{R} < R_f$,

$$\theta \gtreqless 0 \Leftrightarrow c_1^{risk} \lesseqgtr c_1^{certain} \Leftrightarrow \frac{\partial c_1}{\partial R} \gtreqless 0 \Leftrightarrow -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \gtreqless 1. \tag{B.44}$$

For (ii), first we need to argue that the optimization problem based on $(c_1, \widehat{c}_2)$ is well-defined. Thus we prove that the period two certainty equivalent constraint

$$\widehat{c}_2 = \left(E\left[\left(\widetilde{R}(I - c_1) - b\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}} + b \tag{B.45}$$

is always concave. Note that

$$\frac{d\widehat{c}_2}{dc_1} = -\left(E\left[\left(\widetilde{R}(I - c_1) - b\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}-1} E\left[\widetilde{R}\left(\widetilde{R}(I - c_1) - b\right)^{-\delta_2-1}\right] < 0 \tag{B.46}$$

and

$$\begin{aligned} \frac{d^2\widehat{c}_2}{dc_1^2} &= (1+\delta_2)\left(E\left[\left(\widetilde{R}(I - c_1) - b\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}-2} \times \\ &\left(\begin{array}{c} \left(E\left[\widetilde{R}\left(\widetilde{R}(I - c_1) - b\right)^{-\delta_2-1}\right]\right)^2 - \\ E\left[\widetilde{R}^2\left(\widetilde{R}(I - c_1) - b\right)^{-\delta_2-2}\right] E\left[\left(\widetilde{R}(I - c_1) - b\right)^{-\delta_2}\right] \end{array}\right) \end{aligned} \tag{B.47}$$

Following the Cauchy–Schwarz inequality, we have

$$\left(E\left[\widetilde{R}\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2-1}\right]\right)^2 < E\left[\widetilde{R}^2\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2-2}\right]E\left[\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2}\right], \tag{B.48}$$

implying that

$$\frac{d^2\widehat{c}_2}{dc_1^2} < 0 \tag{B.49}$$

and hence the constraint (B.45) is always concave. Next we show that the slope of the curve (B.45) decreases with the preference parameter b, i.e.,

$$\frac{\partial \frac{d\widehat{c}_2}{dc_1}}{\partial b} < 0. \tag{B.50}$$

Note that

$$\begin{aligned}\frac{\partial \frac{d\widehat{c}_2}{dc_1}}{\partial b} &= (1+\delta_2)\left(E\left[\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}-2} \\ &\quad \left(\begin{array}{c} E\left[\widetilde{R}\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2-1}\right]E\left[\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2-1}\right]- \\ E\left[\widetilde{R}\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2-2}\right]E\left[\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2}\right]\end{array}\right) \\ &= \frac{b\left(1+\delta_2\right)\left(E\left[\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}-2}\left(\begin{array}{c}\left(E\left[\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2-1}\right]\right)^2- \\ E\left[\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2-2}\right]E\left[\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2}\right]\end{array}\right)}{I-c_1}. \end{aligned} \tag{B.51}$$

Following the Cauchy–Schwarz inequality,

$$\left(E\left[\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2-1}\right]\right)^2 < E\left[\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2-2}\right]E\left[\left(\widetilde{R}\left(I-c_1\right)-b\right)^{-\delta_2}\right], \tag{B.52}$$

implying that

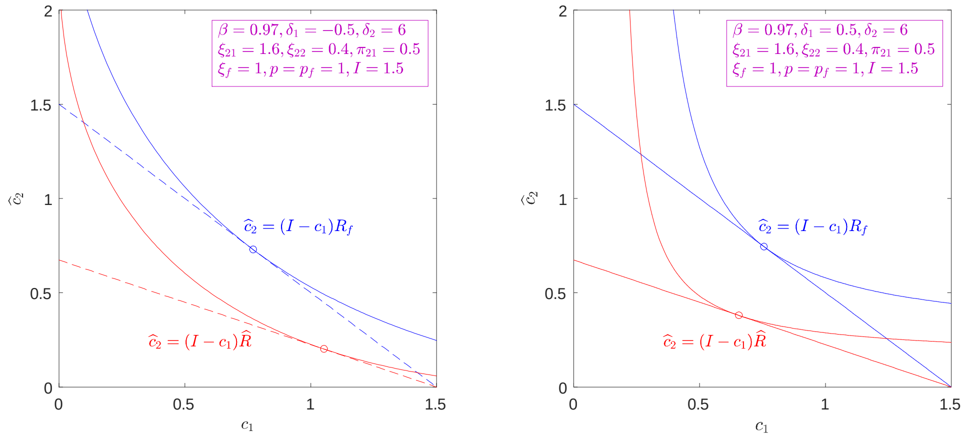
$$\frac{\partial \frac{d\widehat{c}_2}{dc_1}}{\partial b} < 0. \tag{B.53}$$

Therefore, the introduction of b makes the $\widehat{c}_2\left(c_1\right)$ constraint steeper at each value of c_1. Moreover, the relative risk aversion measure τ_R defined by (9) is given by

$$\tau_R = \frac{\left(1+\delta_2\right)c_2}{c_2-b}, \tag{B.54}$$

which is an increasing function of b. It follows that

$$\frac{\partial \widehat{c}_2}{\partial b} < 0. \tag{B.55}$$



(a) (b)

Figure 6:

Therefore, the introduction of b makes the $\widehat{c}_2\,(c_1)$ constraint lower for each value of c_1. For the additively separable utility U, the slope of the indifference curve is flatter when fixing c_1 and decreasing c_2. As a result, the introduction of b will decrease the optimal c_1 and increase s_1. Thus the excess saving θ increases and the result directly follows from part (i) of Theorem 7. ■

The intuition for case (i) in Theorem 7 has a very appealing geometric interpretation. To characterize when θ is positive or negative, it is only necessary to compare the certainty budget constraint

$$c_2 = (I - c_1)\,R_f. \tag{B.56}$$

with the risky constraint (B.37). These constraints have the same c_1-intercept and differ just in their slope. Since we have assumed that the risky asset $E\widetilde{R} = R_f$, it follows that $\widehat{R} < R_f$ and the risk constraint will rotate southwest from the certainty constraint. Moreover in both instances, we have the same additively separable certainty utility U. This is illustrated in Figure 6, where U takes the CES form in (10) and the condition (B.40) specializes to

$$\theta \gtreqless 0 \Leftrightarrow \delta_1 \gtreqless 0 \Leftrightarrow EIS \lesseqgtr 1. \tag{B.57}$$

Figures 6(a) and (b) differ only in whether δ_1 is less or greater than 0. Figure 6(b) illustrates that when $\delta_1 > 0$ and $EIS < 1$, optimal c_1 decreases in the risky versus certainty case and $s_1^{risky} > s_1^{certain}$. Here the consumption smoothing effect is dominated by the negative substitution effect.

In contrast to Theorem 7(i), the $\widehat{c}_2(c_1)$ constraint in Theorem 7(ii) fails to be linear as can be seen from (B.59) in Example 4 below. Nevertheless, the intuition for why precautionary savings is positive is consistent with that of Theorem 7(i). Introduction of the shift parameter $b > 0$ results in the $\widehat{c}_2\left(c_1\right)$ constraint always being concave and having steeper slope than that of the CRRA constraint $-\widehat{R}$**,** as illustrated in Figure 7(a). As a result, optimal c_1 decreases, s_1 increases and θ increases. It follows from Theorem 7(i) that when $b = 0$ and $\delta_1 \geq 0$, we have $\theta \geq 0$. Therefore, if $b > 0$ and $\delta_1 \geq 0$, we must have $\theta > 0$, which is consistent with Theorem 7(ii). The following illustrates an application of Theorem 7(ii).

Example 4 *Assume*

$$U\left(c_1, c_2\right) = -\frac{c_1^{-\delta_1}}{\delta_1} - \beta\frac{c_2^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V\left(c_2\right) = -\frac{\left(c_2 - b\right)^{-\delta_2}}{\delta_2} \quad \left(\delta_1, \delta_2 > -1\right). \tag{B.58}$$

The $\widehat{c}_2$ *constraint is given by*

$$\widehat{c}_2 = \left(E\left[\left(\widetilde{R}\left(I - c_1\right) - b\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}} + b. \tag{B.59}$$

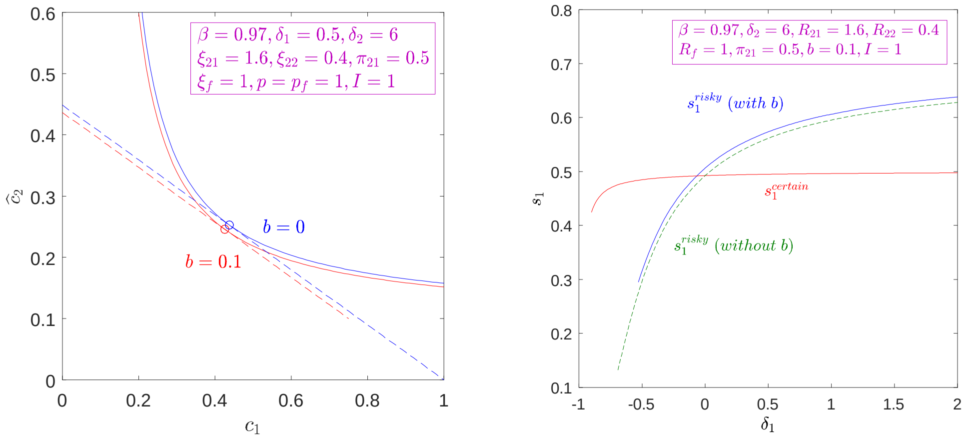
Assume the following specific parameter values

$$\beta = 0.97, \delta_2 = 6, R_{21} = 1.6, R_{22} = 0.4, R_f = 1, \pi_{21} = 0.5, b = 0.1, I = 1. \tag{B.60}$$

Since eqn. (B.59) is not linear in c_1*, we use numerical simulation to compare the optimal* s_1^{risky} *and* $s_1^{certain}$ *in Figure 7(b). Note that* $s_1^{certain}$ *is not affected by* b *and* s_1^{risky} *with* b *is always larger than without* b*. Thus if* $\theta \geq 0$ *for the case with* $b = 0$*, we must have* $\theta > 0$ *for the case with* $b > 0$*. This conclusion is consistent with Theorem 7(ii).*[44]

Remark 4 *Weil (1990) considers a dynamic version of the consumption-saving problem where the consumer is assumed to have generalized isoelastic preferences which in the two period case is equivalent to the KPS utility (10). There is a single risky asset the distribution of which is assumed to be i.i.d. (identically and independently distributed) and takes the lognormal form where* $E\widetilde{R} = R_f$*. Weil (1990) derives a formula, eqn. (10), for consumption growth, where the second term on the right hand side is interpreted as corresponding to the influence of consumption variance on consumption growth. The coefficient of variance is*

[44] For the specific utility assumed in this example, $V''' > 0$ and prudence always holds. However for this case prudence does not imply that θ is always positive. For instance from Figure 7, we see that for δ_1-values in excess of approximately 0.5 the translated CRRA utility results in $\theta < 0$.



(a) (b)

Figure 7:

$\delta_1\delta_2/(\delta_1 + 1)$, restated in our notation. Weil argues that this coefficient must be positive in order for increased saving on the part of the consumer to generate both increased consumption growth and correspondingly increased risk. There is substantial evidence to support that $\delta_2 > 0$ and this implies that $\delta_1 > 0$ must hold. This latter conclusion is consistent with the special case of Theorem 7(i) where U takes the CES form.

Remark 5 *Langlais (1995) investigates what he refers to as "precautionary" saving in a consumption-saving setting with KPS preferences and assumes $E\widetilde{R} = R_f$. However in contrast to our results which hold globally, the analysis of Langlais (1995, Section 3) only holds for a small risk approximation.*

Remark 6 *Using arguments similar to those in Theorem 7, it is possible to generalize the well-known result that when preferences take the CES time and CRRA risk preference KPS form (10), the effect of a mean preserving increase in risk on saving depends on the EIS relative to 1.*[45] *If U takes the far less restrictive additively separable form, then for both CRRA and translated CRRA risk preferences, the effect of a mean preserving increase in risk depends on the $EMRS$ relative to 1. However, it should be stressed that evaluating how changes in risk affect saving behavior is quite different from analyses of excess saving since the former depends on a comparison of particular distributions (such as a mean preserving shift) whereas the latter applies for general distributions.*

[45] See, for instance, Selden (1979).

B.3 Supporting Calculations for Example 2

Let c_{21} and c_{22} denote the contingent claim demands for period two consumption and p_{21} and p_{22} denote the corresponding prices, where

$$p_{21} = \frac{\xi_f p - \xi_{22} p_f}{(\xi_{21} - \xi_{22})\xi_f} \quad \text{and} \quad p_{22} = \frac{\xi_{21} p_f - \xi_f p}{(\xi_{21} - \xi_{22})\xi_f}. \tag{B.61}$$

Define

$$k = \frac{\pi_{22} p_{21}}{\pi_{21} p_{22}} \quad \text{and} \quad \widehat{R}_p = \frac{\left(\pi_{21} + k^{\frac{-\delta_2}{1+\delta_2}} \pi_{22}\right)^{-\frac{1}{\delta_2}}}{p_{21} + k^{\frac{1}{1+\delta_2}} p_{22}}. \tag{B.62}$$

It can be verified that

$$\widehat{c}_2 = \widehat{R}_p \left(I - c_1\right). \tag{B.63}$$

Therefore,

$$c_1^{risky} = \frac{I}{1 + \widehat{R}_p^{-\frac{\delta_1}{1+\delta_1}}}. \tag{B.64}$$

For the certainty case, we have

$$c_1^{certain} = \frac{I}{1 + R_f^{-\frac{\delta_1}{1+\delta_1}}}. \tag{B.65}$$

Then

$$s_1^{risky} \gtreqless s_1^{certain} \Leftrightarrow \delta_1 \lesseqgtr 0. \tag{B.66}$$

Since

$$p_f n_f^{certain} = s_1^{certain} \quad \text{and} \quad p n^{risky} + p_f n_f^{risky} = s_1^{risky} \tag{B.67}$$

and

$$n^{risky} > 0, \tag{B.68}$$

it is clear that a necessary condition for $n_f^{risky} > n_f^{certain}$ is $s_1^{risky} > s_1^{certain}$, or $\delta_1 < 0$ for this example. But since $n^{risky} > 0$, $\delta_1 < 0$ is not a sufficient condition for $n_f^{risky} > n_f^{certain}$. This is confirmed by the following calculation. We have

$$n_f^{certain} = \frac{I}{p_f \left(1 + \beta^{-\frac{1}{1+\delta_1}} R_f^{\frac{\delta_1}{1+\delta_1}}\right)} \tag{B.69}$$

and

$$n_f^{risky} = \frac{\left(\xi_{21} k^{\frac{1}{1+\delta_2}} - \xi_{22}\right) I}{\left(1 + \beta^{-\frac{1}{1+\delta_1}} \widehat{R}_p^{\frac{\delta_1}{1+\delta_1}}\right) \left(p_{21} + p_{22} k^{\frac{1}{1+\delta_2}}\right) \left(\xi_{21} - \xi_{22}\right) \xi_f}. \tag{B.70}$$

Since

$$\frac{\left(p_{21}+p_{22}k^{\frac{1}{1+\delta_2}}\right)(\xi_{21}-\xi_{22})\,\xi_f}{\xi_{21}k^{\frac{1}{1+\delta_2}}-\xi_{22}} = p_f + \frac{\left(1-k^{\frac{1}{1+\delta_2}}\right)\xi_f p}{\xi_{21}k^{\frac{1}{1+\delta_2}}-\xi_{22}}, \tag{B.71}$$

assuming $\xi_{21}k^{\frac{1}{1+\delta_2}}-\xi_{22}>0$, we have

$$n_f^{risky} \gtreqqless n_f^{certain} \Leftrightarrow p_f\beta^{-\frac{1}{1+\delta_1}}R_f^{\frac{\delta_1}{1+\delta_1}} \gtreqqless \left(\begin{array}{c}\left(p_f+\frac{\left(1-k^{\frac{1}{1+\delta_2}}\right)\xi_f p}{\xi_{21}k^{\frac{1}{1+\delta_2}}-\xi_{22}}\right)\beta^{-\frac{1}{1+\delta_1}}\widehat{R}_p^{\frac{\delta_1}{1+\delta_1}}\\ +\frac{\left(1-k^{\frac{1}{1+\delta_2}}\right)\xi_f p}{\xi_{21}k^{\frac{1}{1+\delta_2}}-\xi_{22}}\end{array}\right). \tag{B.72}$$

Since this condition is complicated, another simpler necessary condition for precautionary saving is

$$(\beta R_f)^{-\frac{1}{1+\delta_1}} > \frac{\left(1-k^{\frac{1}{1+\delta_2}}\right)p}{\xi_{21}k^{\frac{1}{1+\delta_2}}-\xi_{22}}. \tag{B.73}$$

Note that the optimal portfolio satisfies

$$\frac{n^*}{n_f^*} = \frac{\left(1-k^{\frac{1}{1+\delta_2}}\right)\xi_f}{\xi_{21}k^{\frac{1}{1+\delta_2}}-\xi_{22}}. \tag{B.74}$$

If we assume that $\beta=\xi_f=p=p_f=1$, the condition (B.73) implies that

$$\frac{n^*}{n_f^*}<1. \tag{B.75}$$

Thus precautionary saving can occur only if the consumer is risk averse enough. Moreover, the condition (B.73) can be simplified to

$$1 > \left(1+\frac{n^*}{n_f^*}\right)\widehat{R}_p^{\frac{\delta_1}{1+\delta_1}} + \frac{n^*}{n_f^*}, \tag{B.76}$$

implying that δ_1 must be less than the threshold value δ_1^*

$$\delta_1 < \delta_1^* = \frac{1}{\frac{\ln \widehat{R}_p}{\ln\left(\left(1-\frac{n^*}{n_f^*}\right)/\left(1+\frac{n^*}{n_f^*}\right)\right)}-1}. \tag{B.77}$$

Therefore, if δ_2 is sufficiently large, i.e., condition (B.75) is satisfied, then there exists a δ_1^* such that positive precautionary saving ϑ will occur if and only if $\delta_1<\delta_1^*$. Another interesting special case is where the consumer shorts the risk free asset, which corresponds to

$$\xi_{21}k^{\frac{1}{1+\delta_2}}-\xi_{22}\le 0 \text{ or } k^{\frac{1}{1+\delta_2}}\le\frac{\xi_{22}}{\xi_{21}}. \tag{B.78}$$

In this case since $n_f^{risky} \leq 0$, by definition precautionary saving can never occur. Thus for some distributions, there is no δ_1 that can make precautionary saving positive.

B.4 Equilibrium Risk Free Rate

In this appendix, we derive analytic forms for R_f which are then compared to formulas for the equilibrium risk free rate in the literature. Following Campbell and Cochrane (1999) and Yi and Choi (2006), specific distributions for the consumption ratio $\widetilde{c}_2/c_1$ will be assumed, where

$$\widetilde{c}_2 = \widetilde{\xi} n + \xi_f n_f. \tag{B.79}$$

For Propositions 3 and 4 below, respectively[46]

$$\ln\left(\frac{\widetilde{c}_2 - b}{c_1 - b}\right) \sim N\left(\ln\frac{c_2^* - b}{c_1^* - b} - \frac{\sigma^2}{2}, \sigma^2\right), \qquad \left(\frac{\widetilde{c}_2}{c_1} \sim N\left(\frac{c_2^*}{c_1^*}, \sigma^2\right)\right), \tag{B.80}$$

where

$$(c_1^*, c_2^*) = \arg\max_{c_1, c_2} U(c_1, c_2) \quad S.T.\ c_2 = (I - c_1)R_f. \tag{B.81}$$

Thus when risk goes to zero, the optimal risky consumption growth converges to the optimal certain consumption growth.

Proposition 3 *Assume that*

$$U(c_1, c_2) = -\frac{(c_1 - b)^{-\delta_1}}{\delta_1} - \beta\frac{(c_2 - b)^{-\delta_1}}{\delta_1} \quad (\delta_1 > -1, b \gtreqless 0), \tag{B.82}$$

and

$$V(c_2) = -\frac{(c_2 - b)^{-\delta_2}}{\delta_2} \quad (\delta_2 > -1). \tag{B.83}$$

Then,

$$\ln R_f^{certain} = \ln\frac{1}{\beta} + (1 + \delta_1)\ln\left(\frac{c_2^* - b}{c_1^* - b}\right) \tag{B.84}$$

and

$$\ln R_f^{risky} = \ln\frac{1}{\beta} + (1 + \delta_1)\ln\left(\frac{c_2^* - b}{c_1^* - b}\right) - \frac{(\delta_1 + 2)(\delta_2 + 1)}{2}\sigma^2, \tag{B.85}$$

---

[46] We take this specific mean value for the lognormal distribution since we want to ensure that the mean of the distribution corresponds to the certain consumption growth c_2^*/c_1^*

$$E\left[\frac{\widetilde{c}_2 - b}{c_1 - b}\right] = \exp\left(-\frac{1}{2}\left(-\sigma^2 - 2\ln\frac{c_2^* - b}{c_1^* - b} + \sigma^2\right)\right) = \frac{c_2^* - b}{c_1^* - b}.$$

The mean value compensation follows from Barsky (1989).

implying that[47]

$$R_f^{risky} < R_f^{certain}. \tag{B.86}$$

Proof. Notice that

$$R_f = \frac{1}{\beta \left(E \left(\frac{\widetilde{c}_2 - b}{c_1 - b} \right)^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2} - 1} E \left(\frac{\widetilde{c}_2 - b}{c_1 - b} \right)^{-\delta_2 - 1}}. \tag{B.87}$$

Since

$$E \left[\left(\frac{\widetilde{c}_2 - b}{c_1 - b} \right)^{-\delta_2} \right] = \exp \left(\frac{\delta_2}{2} \left((\delta_2 + 1) \sigma^2 - 2 \ln \frac{c_2^* - b}{c_1^* - b} \right) \right) \tag{B.88}$$

and

$$E \left[\left(\frac{\widetilde{c}_2 - b}{c_1 - b} \right)^{-\delta_2 - 1} \right] = \exp \left(\frac{\delta_2 + 1}{2} \left((\delta_2 + 2) \sigma^2 - 2 \ln \frac{c_2^* - b}{c_1^* - b} \right) \right), \tag{B.89}$$

we have

$$\begin{aligned} R_f^{risky} &= \frac{1}{\beta \left(E \left(\frac{\widetilde{c}_2 - b}{c_1 - b} \right)^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2} - 1} E \left(\frac{\widetilde{c}_2 - b}{c_1 - b} \right)^{-\delta_2 - 1}} \\ &= \frac{1}{\beta \exp \left(\frac{\delta_1 - \delta_2}{2} \left((\delta_2 + 1) \sigma^2 - 2 \ln \frac{c_2^* - b}{c_1^* - b} \right) + \frac{\delta_2 + 1}{2} \left((\delta_2 + 2) \sigma^2 - 2 \ln \frac{c_2^* - b}{c_1^* - b} \right) \right)} \\ &= \frac{1}{\beta} \exp \left((1 + \delta_1) \ln \frac{c_2^* - b}{c_1^* - b} - \frac{(\delta_1 + 2)(\delta_2 + 1)}{2} \sigma^2 \right) \\ &= \frac{1}{\beta} \left(\frac{c_2^* - b}{c_1^* - b} \right)^{1 + \delta_1} \exp \left(- \frac{(\delta_1 + 2)(\delta_2 + 1)}{2} \sigma^2 \right), \end{aligned} \tag{B.90}$$

implying that

$$\ln R_f^{risky} = \ln \frac{1}{\beta} + (1 + \delta_1) \ln \left(\frac{c_2^* - b}{c_1^* - b} \right) - \frac{(\delta_1 + 2)(\delta_2 + 1)}{2} \sigma^2. \tag{B.91}$$

For the certainty case, it can easily be verified that

$$\ln R_f^{certain} = \ln \frac{1}{\beta} + (1 + \delta_1) \ln \left(\frac{c_2^* - b}{c_1^* - b} \right). \tag{B.92}$$

Thus, we always have

$$R_f^{risky} < R_f^{certain}. \tag{B.93}$$

■

[47] Although the distributional assumptions are different, Proposition 3 is consistent with eqn. (23) in Barsky (1989) where $b = 0$.

Proposition 4 *Assume that*

$$U(c_1, c_2) = -\frac{\exp(-\kappa_1 c_1)}{\kappa_1} - \beta \frac{\exp(-\kappa_1 c_2)}{\kappa_1} \quad (\kappa_1 > 0), \tag{B.94}$$

and

$$V(c_2) = -\frac{\exp(-\kappa_2 c_2)}{\kappa_2} \quad (\kappa_2 > 0). \tag{B.95}$$

Then,

$$\ln R_f^{certain} = \ln \frac{1}{\beta} + \kappa_1 (c_2^* - c_1^*) \tag{B.96}$$

and

$$\ln R_f^{risky} = \ln \frac{1}{\beta} + \kappa_1 (c_2^* - c_1^*) - \frac{1}{2}\kappa_1 \kappa_2 c_1^{*2} \sigma^2, \tag{B.97}$$

implying that

$$R_f^{risky} < R_f^{certain}. \tag{B.98}$$

Proof. Notice that

$$R_f = \frac{\exp(-\kappa_1 c_1)}{\beta (E \exp(-\kappa_2 \widetilde{c}_2))^{\frac{\kappa_1}{\kappa_2}}}. \tag{B.99}$$

Since

$$E \exp(-\kappa_2 \widetilde{c}_2) = \exp\left(\frac{1}{2}\kappa_2 \left(\kappa_2 c_1^{*2} \sigma^2 - 2c_2^*\right)\right), \tag{B.100}$$

we have

$$R_f^{risky} = \frac{\exp(-\kappa_1 c_1)}{\beta \exp\left(\frac{1}{2}\kappa_1 \left(\kappa_2 c_1^{*2}\sigma^2 - 2c_2^*\right)\right)} = \frac{1}{\beta} \exp\left(\kappa_1 (c_2^* - c_1^*) - \frac{1}{2}\kappa_1 \kappa_2 \sigma^2 c_1^{*2}\right), \tag{B.101}$$

implying that

$$\ln R_f^{risky} = \ln \frac{1}{\beta} + \kappa_1 (c_2^* - c_1^*) - \frac{1}{2}\kappa_1 \kappa_2 \sigma^2 c_1^{*2}. \tag{B.102}$$

For the certainty case, it can easily be verified that

$$\ln R_f^{certain} = \ln \frac{1}{\beta} + \kappa_1 (c_2^* - c_1^*). \tag{B.103}$$

Thus, we always have

$$R_f^{risky} < R_f^{certain}. \tag{B.104}$$

■

In both propositions, the first two terms in the expression for $\ln R_f^{risky}$ correspond to $\ln R_f^{certain}$ and the last term depends on the risk σ^2 and the preference parameters. In eqns. (B.85) and (B.97), the last term is negative and reduces $\ln R_f^{risky}$ and for this reason it is standard in the literature to refer to it as reflecting the "precautionary" saving effect or motive. The intuition for this interpretation is that if the the equilibrium risk free rate is less in the risky versus certainty case, then presumably the consumer is saving more in the risky case, driving up the price p_f and lowering the risk free rate. However this interpretation illustrates an inconsonance between the equilibrium and demand results in Proposition 3 and Corollary 2(i), respectively. The latter requires that $\delta_1 < 0$ in order for an individual consumer to save more in the risky versus certainty setting, but the former holds for for any value of δ_1.

For the CARA case considered in Proposition 4 and Theorem 4, the inconsistency between the demand and equilibrium results is even more extreme since both $R_f^{risky} < R_f^{certain}$ and $\vartheta < 0$ hold generally without any restrictions on preference parameters. One reason for this inconsonance is that our endowment assumption (75) that $\overline{n}_f^{certain} > \overline{n}_f^{risky}$ implies that at equilibrium prices the demand for the risk free asset in the risky setting can never be larger than in the certainty setting as required by the precautionary intuition.

The next remark discusses specific applications in the literature where the "precautionary" demand interpretation is given for equilibrium risk free rate formulas.

Remark 7 *Gomes and Ribeiro (2015) (and Yi and Choi 2006) assume a representative agent with EZ (Epstein and Zin 1989) preferences, which in a two period setting converge to the KPS case in Corollary 2(i) where* $b = 0$. *They derive a log-linear form for the Euler equation where* $\log(c_{t+1}/c_t)$ *is expressed as a function of the variance of the error term based on a second-order Taylor expansion. The authors interpret this term as reflecting the consumer's "precautionary" motive since if it is positive, consumption growth increases due to lower* c_t *and increased saving. If one rearranges eqn. (4) in Yi and Choi (2006) or eqn. (8) in Gomes and Ribeiro (2015) to express* $\ln R_i^t$ *as a function of* σ^2 *and considers the risk free asset return, it can be easily seen that the variance term is always negative, implying that* $R_f^{risky} < R_f^{certain}$. *This is consistent with our result in Proposition 3.*[48] *Again, this seems to be at odds with our Corollary 2(i) demand conclusion that positive excess saving for a consumer occurs if and only if* $\delta_1 < 0$.[49]

[48] Yi and Choi (2006) and Gomes and Ribeiro (2015) both assume the EZ recursive preference model. In this framework, second consumption is not simply the investment return from period one asset holdings. Unlike our two period analysis, consumption growth and portfolio allocation need to be handled separately. This difference results in the recursive expression for the risk free rate containing additional terms (such as the market portfolio return) from those in the formula derived in Proposition 3.

[49] Suppose alternatively one defines precautionary saving at the demand level in terms of $\vartheta > 0$. Then it follows from Example 2, that precautionary saving will be positive only if $\delta_1 << 0$.

B.5 Supporting Calculations for Example 3

Since

$$E\left[\exp\left(-\kappa_2\left(\widetilde{\xi}n+\xi_f n_f\right)\right)\right]=\exp\left(-\kappa_2\xi_f n_f+\frac{1}{2}\kappa_2 np\left(\kappa_2 np\sigma^2-2E\widetilde{R}\right)\right), \tag{B.105}$$

the first order condition is

$$\frac{\exp\left(-\kappa_2\xi_f n_f+\frac{1}{2}\kappa_2 np\left(\kappa_2 np\sigma^2-2E\widetilde{R}\right)\right)\left(pE\widetilde{R}-\kappa_2 np^2\sigma^2\right)}{\xi_f\exp\left(-\kappa_2\xi_f n_f+\frac{1}{2}\kappa_2 np\left(\kappa_2 np\sigma^2-2E\widetilde{R}\right)\right)}=\frac{p}{p_f}, \tag{B.106}$$

implying that

$$n=\frac{E\widetilde{R}-R_f}{\kappa_2 p\sigma^2}. \tag{B.107}$$

Thus

$$\begin{aligned}\widehat{c}_2 &= \xi_f n_f-\frac{1}{2}np\left(\kappa_2 np\sigma^2-2E\widetilde{R}\right)\\ &= R_f\left(I-c_1-\frac{E\widetilde{R}-R_f}{\kappa_2\sigma^2}\right)+\frac{1}{2}\frac{E\widetilde{R}-R_f}{\kappa_2\sigma^2}\left(E\widetilde{R}+R_f\right)\\ &= R_f\left(I-c_1\right)+\frac{1}{2}\frac{\left(E\widetilde{R}-R_f\right)^2}{\kappa_2\sigma^2}.\end{aligned} \tag{B.108}$$

Therefore,

$$U\left(c_1,\widehat{c}_2\right)=-\frac{\exp\left(-\kappa_1 c_1\right)}{\kappa_1}-\beta\frac{\exp\left(-\kappa_1\left(R_f\left(I-c_1\right)+\frac{1}{2}\frac{\left(E\widetilde{R}-R_f\right)^2}{\kappa_2\sigma^2}\right)\right)}{\kappa_1}. \tag{B.109}$$

Differentiating the above equation with respect to c_1 and setting it to zero yields

$$\exp\left(-\kappa_1 c_1\right)=\beta R_f\exp\left(-\kappa_1\left(R_f\left(I-c_1\right)+\frac{1}{2}\frac{\left(E\widetilde{R}-R_f\right)^2}{\kappa_2\sigma^2}\right)\right). \tag{B.110}$$

Therefore, we have

$$c_1=\frac{R_f I+\frac{1}{2}\frac{\left(E\widetilde{R}-R_f\right)^2}{\kappa_2\sigma^2}-\frac{\ln\left(\beta R_f\right)}{\kappa_1}}{1+R_f}. \tag{B.111}$$

Combining the above equation with eqn. (B.107) yields

$$n_f=\frac{I-\frac{1}{2}\frac{\left(E\widetilde{R}-R_f\right)\left(E\widetilde{R}+R_f+2\right)}{\kappa_2\sigma^2}+\frac{\ln\left(\beta R_f\right)}{\kappa_1}}{\left(1+R_f\right)p_f}. \tag{B.112}$$

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