CONSUMPTION TREES, OCE UTILITY AND THE CONSUMPTION/SAVINGS DECISION*

by

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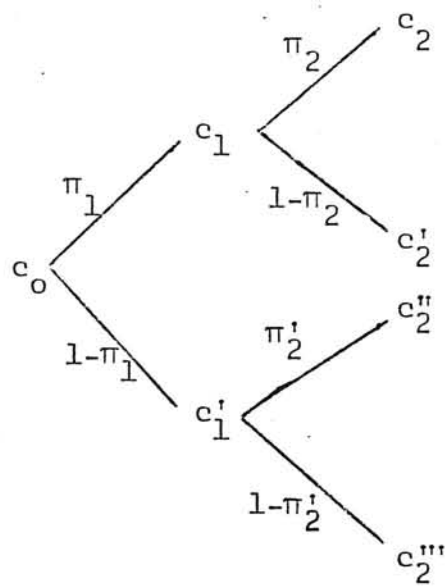
1. INTRODUCTION AND SUMMARY

Following the important contribution of Phelps [20], the last ten years have seen extensive research effort devoted to problems of multiperiod resource allocation in an <u>uncertain</u> setting. Two primary areas of study have been the theory of optimal growth (e.g., Brock and Mirman [3], Mirrlees [17] and Mirman and Zilcha [16]) and the multiperiod consumption/savings (portfolio) problem (e.g., Levhari and Srinivasan [14], Hahn [8], Hakansson [9], Samuelson [28] and Merton [15]). In both cases, especially the latter, it has been standard to assume first that the consumer possesses a complete preordering over joint distributions for consumption flows which is representable by a continuous multiattribute expected utility function and second that the NM (von Neumann-Morgenstern) index is additively separable and stationary.[1] To obtain concrete answers to specific questions such as the effect of increased capital risk on savings, still another assumption is generally added: the consumer's conditional risk preferences (for each time-period) exhibit constant relative risk aversion.

The purpose of this paper is essentially two-fold. First of all, we present a preference-utility model which is more general than the additively separable multiperiod expected utility paradigm in that it allows one significantly greater freedom in prescribing risk preferences and time preferences. Secondly, we utilize this freedom in an application of the resulting theory to a simple multiperiod consumption/savings problem.

The present paper extends the two-period OCE (Ordinal Certainty Equivalent) representation developed in Selden [30] to T periods. To see what is involved, let us define $c_t \in C_t$ and $F_t, G_t \in \overline{X}_t$ to be, respectively, real generalized consumption in time period t and c.d.f.'s (cumulative distribution functions) on C_t. For more than two periods, we shall further require the notion of a "consumption tree":



(1.1)

where π_1, π_2 and π_2' are simple probabilities. More generally, $\mathcal{T}$ will denote the set of (T+1)-period trees. Then three different preference settings can be distinguished

(i) Single-period preferences over $\overline{\underline{X}}_1$

(ii) Two-period preferences over $C_o \times \overline{\underline{X}}_1$

(iii) (T+1)-period preferences over $\mathcal{T}$ (in general, not expressible as $C_o \times \overline{\underline{X}}_1 \times \ldots \times \overline{\underline{X}}_T$)

In each case, one would like to obtain a numeric representation of preferences which is both relatively general and easy to use in standard resource allocation problems. For case (i), the well-known NM (von Neumann-Morgenstern) axioms ensure existence of a continuous expected utility representation (e.g., Grandmont [7]).

The essential idea underlying the two-period OCE approach is one of characterizing preferences over $C_o \times \overline{\underline{X}}_1$ in terms of (1) <u>conditional risk preferences</u> defined over each "cross section" $\{c_o\} \times \overline{\underline{X}}_1$ and (2) basic <u>time preferences</u> defined over $C_o \times C_1$ and representable by a continuous ordinal index

U. Assuming orderings over each $\{c_o\} \times \underline{\overline{X}}_1$ to be NM representable (where the corresponding conditional NM index V_{c_o} is allowed to depend continuously on time-period zero consumption), the choice between any pair (c_o', F_1) and (c_o'', G_1) in $C_o \times \underline{\overline{X}}_1$ can be decomposed into two steps. First, using just the conditional risk preferences ($V_{c_o'}$ and $V_{c_o''}$) convert into equivalent certain current, certainty equivalent period-one consumption pairs, $(c_o', \hat{c}_1(c_o', F_1))$ and $(c_o'', \hat{c}_1(c_o'', G_1))$. Then evaluate the resulting pairs using time preferences: (c_o', F_1) is not preferred to (c_o'', G_1) if and only if $U(c_o', \hat{c}_1(c_o', F_1)) \leq U(c_o'', \hat{c}_1(c_o'', G_1))$.

Since this representation can be shown to include the standard two-period expected utility model as a special case, one is led to ask what additional axiom must be added to obtain the latter? That is, some additional assumptive input is required in order to go from NM preferences on each "cross section" $\{c_o\} \times \underline{\overline{X}}_1$ as assumed under the OCE approach to an expected utility representation over all of $C_o \times \underline{\overline{X}}_1$. This additional axiom, referred to as "coherence" in Rossman and Selden [24], establishes a strong interdependence between the agent's conditional risk preference and time preference indices: $V(c_o, c_1)$ and $U(c_o, c_1)$ must be ordinally equivalent. In contrast, the OCE representation allows one complete freedom in modelling this relationship. It is further shown in [24] that given (U,V)-pairs, although quite standard, may not be compatible behaviorally even if the coherence axiom is assumed.

In attempting to extend the OCE representation to more than two time-periods, a number of new issues arise. First of all, as suggested above, the choice space can not, in general, be viewed as simply a product space involving C_o and the sets of marginal c.d.f.'s $\underline{\overline{X}}_1, \underline{\overline{X}}_2, \ldots, \underline{\overline{X}}_T$. Rather, we need to introduce the notion of consumption trees (e.g., (1.1)). Secondly, to avoid significantly complicating our analysis, it is necessary to introduce a "risk preference independence" axiom which, in essence, states that the choice among one-period lotteries involving c_t can be made independently of both the history $(c_0,\ldots,c_{t-1})$ and the consumption possibilities for the future periods $t+1,\ldots,T$. (Most multiperiod growth and consumption/savings (portfolio) models, such as those cited above, employ expected utility representations satisfying this axiom.) Thirdly, as noted by Kreps and Porteus in [13], one must consider explicitly how multiperiod preferences are affected by the time of resolution of uncertainty. Implicit in the multiperiod expected utility representation is the assumption that the consumer will be indifferent between any pair of consumption trees which differ only in the time when uncertainty is resolved. In this paper we make the same assumption explicitly. However, one might quite naturally ask whether this "temporal resolution indifference" assumption is consistent with the intuitive idea that early resolution ought to be desirable because it implies the receipt of additional information which should be of value to the consumer in his planning process. It is important to distinguish between a preference for early (or late) resolution in consumption trees, which turns out to be a purely psychological matter, and a preference for early resolution in allocation problems, which relates to the possiblity of using the additional information to form a new optimal allocation that is less constrained to hedge against diverse possibilities. In the context of a simple consumption/savings problem (section 4), we show that the temporal resolution indifference for consumption trees, while eliminating the psychological factor, does not preclude our obtaining a general preference for early resolution in allocation.

Despite these three important differences from the two-period case, the basic idea of an OCE representation extends to a multiperiod setting. After formally defining the consumption tree structure and introducing the required set of axioms in the next section, we present our principal preference and representation results in section 3. Consider a pair of 3-period trees, similar in form to (1.1). Let conditional risk preferences and time preferences be given. First using just the former, one can convert each tree into a unique indifferent string of the form $(c_o, \hat{c}_1, \hat{c}_2)$ where each certainty equivalent $\hat{c}_t$ is defined with respect to the appropriate joint probability measure (over state outcomes for periods one through t). Then time preferences are employed to order the resulting pair of strings.

We further establish that under the axioms employed, there exists a unique decomposition of the ordering over consumption trees into time and risk preferences. It is also shown that under risk preference independence the multiperiod expected utility model is but a special case of the OCE representation.

Some related issues are addressed by Kreps and Porteus in their interesting paper [13] on dynamic choice theory and the temporal resolution of uncertainty.

In the final section, we apply the multiperiod OCE preference theory to a simple three-period consumption/savings problem. Our primary focus is on the respective roles of risk and time preferences in determining the optimal level of time-period zero savings.

2. CONSUMPTION TREES

Notation and Formulation

Generally, we shall be interested in consumption trees which are defined for T time-periods and which allow either a finite or infinite number of branches at each node. $C = R_+^{T+1}$ will be the set of all certain consumption streams $(c_o, c_1, \ldots, c_T)$. We shall sometimes find it convenient to express the plan $(c_o, c_1, \ldots, c_T)$ as ${}_oc_T$ and the truncated sequence $(c_t, \ldots, c_{t+n})$ as ${}_tc_{t+n}$. Denote by Θ the set of all possible states of nature which can effect consumption in any period t. θ is an element in Θ. Let B be a collection of subsets of Θ on which probabilities are defined. B is assumed to be a σ-algebra of subsets of Θ. Let P be a probability measure defined on B which assigns a probability to each element of B. Together, the triple (Θ, B, P) constitutes a probability space. Let M be the space of all probability measures on B.

Generally, preceding each time-period $t = 1, \ldots, T$ in a consumption tree there will be chance nodes. The state uncertainty for a particular node will be summarized by a probability measure $P \in M$ having its support in the universal state space Θ. Uncertainty resolves at a given node when the actual state or branch of the tree becomes known. We shall indicate when this uncertainty resolves by subscripting the appropriate probability measure. Thus P_t will denote a measure in M resolving at the beginning of time-period t. In general, a particular P_t will depend on the sequence of previous state outcomes. This will be made explicit by referring to $P_t(\theta; \theta^{(1)}, \ldots\ldots, \theta^{(t-1)})$ as the probability measure resolving in time-period t, which is conditional upon the sequence of <u>specific</u> state outcomes $\theta^{(1)}, \ldots, \theta^{(t-1)}$. The vector $\theta^{(1,t-1)} \equiv (\theta^{(1)}, \ldots, \theta^{(t-1)})$ describes the complete history leading up to the time-period t node corresponding to P_t.

The mapping $\tilde{c}_t \colon \Theta \to R_+$ is a random variable on the probability space (Θ, B, P) which associates to each state θ a nonnegative <u>consumption value for time-period t</u>. ($\tilde{c}_t$ is a random variable only when it is measurable with respect to the σ-algebra of subsets B.) More generally, the consumption value in period t will also depend on state outcomes for earlier periods and will be denoted by $c_t(\theta; \theta^{(1,t-1)})$. We shall additionally be concerned with random vectors; thus let ${}_t\tilde{c}_T$ denote the vector $(\tilde{c}_t, \ldots, \tilde{c}_T)$ defined jointly on the sinble probability space (Θ, B, P). (The symbol ~ will be used to indicate a random variable (or vector) only when the state θ is suppressed.)

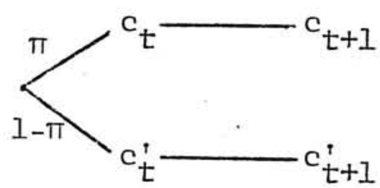
Together, a pair $(P_t, \tilde{c}_t)$ will be referred to as a "<u>one-period lottery</u>"

where the payoffs $\{c_t(\theta)\}$ are consumption amounts for time-period t. Each such pair determines a c.d.f. (cumulative distribution function) on nonnegative consumption values defined by $F_t(c_t) = P_t\{\theta | c_t(\theta) \leq c_t\}$. Let $\underline{X}_t$ be the space of time t (one-period) c.d.f.'s.

More generally, a probability measure P_t and a random vector ${}_t\tilde{c}_{t+n}$ define a <u>vector lottery</u> denoted

$$(P_t, {}_t\tilde{c}_{t+n}) \equiv (P_t, (\tilde{c}_t, \tilde{c}_{t+1}, \ldots, \tilde{c}_{t+n})).$$

Note that for this type of lottery all of the uncertainty concerning the vector of consumption payoffs $\{c_t(\theta),\ c_{t+1}(\theta), \ldots, c_{t+n}(\theta)\}$ is resolved at the beginning of time-period t when the specific state outcome $\theta^{(t)}$ is known. A simple example of such a lottery would be the following:



The special case of a vector lottery characterized by each of its last T-t random variables being constant valued will be denoted

$$(P_t, (\tilde{c}_t, {}_{t+1}c_T)) = (P_t, {}_t\tilde{c}_T)$$

where $c_s(\theta) = c_s$ for $s = t+1, \ldots, T$.

In this framework a (T+1)-period consumption tree, denoted τ, will consist of a certain time-period zero consumption amount followed by a single "one-period lottery" $(P_1, \tilde{c}_1)$ each branch of which is in turn followed by a $(P_2, \tilde{c}_2)$ and so on for T periods. <u>Schematically</u> this can be summarized in the following way

$$(c_o, \{P_1, \tilde{c}_1\}, \{P_2, \tilde{c}_2\}, \ldots, \{P_T, \tilde{c}_T\}),$$

where

$$\{P_t, \tilde{c}_t\} =_{\text{def}} \{(P_t(\theta;\ \theta^{(1,t-1)}),\ c_t(\theta;\ \theta^{(1,t-1)}))\}$$

represents the collection of one-period lotteries which both pay off and resolve at the beginning of time period t. It is important to note that $\theta^{(1,t-1)}$ represents a (t-1)-tuple of parameters determining the t-period chance node (and P_t) while θ is the "active" state variable to be resolved at the beginning of period t. Let $\mathcal{T}$ be the set of all (T+1)-period consumption trees.

If, for a given tree, each of the one-period lotteries is characterized by a one-point or degenerate measure, then the tree will be referred to as a "string" and denoted τ^*. It follows from this definition that a string is essentially identical to a certain consumption stream $(c_o, c_1, \ldots, c_T) \in C$. Letting $\mathcal{T}^* \subset \mathcal{T}$ denote the set of all "strings", we shall use C and $\mathcal{T}^*$ interchangeably.

Assumptions

Let us next set forth the axiomatic structure which will serve as the foundation for the preference and utility results developed in section 3.

Assumption 1:

The space of all (countably additive) probability measures, M, is endowed with the topology of weak convergence and the σ-algebra on (Θ, B) is rich enough to contain all one-point (degenerate) subsets, so that the set of measures with one-point support is contained in M.

Assumption 2:

There exists a complete preordering on the set of trees $\mathcal{T}$ denoted $\preccurlyeq$.

As suggested in the introduction, we shall assume that the ordering over $\mathcal{T}$ exhibits r.p.i. ("risk preference independence"). Let $({}_oc_{t-1};\ (P_t, {}_t\tilde{c}_T))$ denote a (T+1)-period tree in which for the first t time periods one has the consumption vector $(c_o, c_1, \ldots, c_{t-1})$ followed by the vector lottery $(P_t, {}_t\tilde{c}_T)$.

Corresponding to each possible state outcome $\theta^{(t)}$ to be resolved in t is a consumption vector $(c_t(\theta^{(t)}), c_{t+1}(\theta^{(t)}),\ldots,c_T(\theta^{(t)}))$.

Definition:

The preference ordering $\preccurlyeq$ will be said to exhibit r.p.i. iff.

(i) $({}_oc_{t-1};(P_t,(\tilde{c}_t,{}_{t+1}\tilde{c}_T))) \sim ({}_oc_{t-1};(P_t,(\tilde{c}'_t,{}_{t+1}\tilde{c}_T)))$

$\Rightarrow ({}_oc'_{t-1};(P_t,(\tilde{c}_t,{}_{t+1}\tilde{c}'_T))) \sim ({}_oc'_{t-1};(P_t,(\tilde{c}'_t,{}_{t+1}\tilde{c}'_T)))$

for any ${}_oc_{t-1},{}_oc'_{t-1},P_t,\tilde{c}_t,\tilde{c}'_t,{}_{t+1}\tilde{c}_T,{}_{t+1}\tilde{c}'_T$ and $1 \leq t \leq T$;

(ii) $({}_oc_{t-1};(P_t,(\tilde{c}_t,{}_{t+1}c_T))) \preccurlyeq ({}_oc_{t-1};(Q_t,(\tilde{c}'_t,{}_{t+1}c_T)))$

$\Rightarrow ({}_oc'_{t-1};(P_t,(\tilde{c}_t,{}_{t+1}c'_T))) \preccurlyeq ({}_oc'_{t-1};(Q_t,(\tilde{c}'_t,{}_{t+1}c'_T)))$

for any ${}_oc_{t-1},{}_oc'_{t-1},P_t,Q_t,\tilde{c}_t,\tilde{c}'_t,{}_{t+1}c_T,{}_{t+1}c'_T$ and $1 \leq t \leq T$;

and

(iii) given any pair of trees $\tau,\tau' \in \mathcal{T}$ which are identical except at the single chance node defined by the history $\theta^{(1,t-1)}$ where, respectively, the one-period lotteries are $(Q_t,\tilde{c}_t)$ and $(Q_t,\tilde{c}'_t)$, then $(Q,\tilde{c}_t)$ indifferent to $(Q,\tilde{c}'_t)$ (under the restriction of $\preccurlyeq$ to one-period lotteries implied by (ii) above --Cf., Remark below) implies $\tau \sim \tau'$.

Assumption 3:

The preordering $\preccurlyeq$ exhibits r.p.i.

Remarks. (1) Since under condition (ii) of the risk preference independence definition the restriction of the preordering $\preccurlyeq$ to the space of trees

$$\underline{Y}_t({}_oc_{t-1},{}_{t+1}c_T) =_{def} \{({}_oc_{t-1},(P_t,(\tilde{c}_t,{}_{t+1}c_T)))\}$$

is independent of $({}_0c_{t-1}, {}_{t+1}c_T)$, $\leqslant|\underline{\overline{Y}}_t({}_0c_{t-1}, {}_{t+1}c_T)$ <u>induces a preordering on the space of time t (one-period) c.d.f.'s</u> $\underline{\overline{X}}_t$ (corresponding to $(P_t, \tilde{c}_t)$-pairs). Denote this new ordering $\overset{t}{\leqslant}$. Conversely, if a preordering $\overset{t}{\leqslant}$ is given and condition (ii) is assumed, then this preference ordering on $\underline{\overline{X}}_t$ induces the same ordering on the subset of trees $\underline{\overline{Y}}_t({}_0c_{t-1}, {}_{t+1}c_T)$ for any fixed $({}_0c_{t-1}, {}_{t+1}c_T)$.

(2) Condition (i) is closely related to the "independence" axioms of Fishburn ([6], chapter 11) and Pollak [21] for multiattribute NM preferences. However, as will become clear shortly, the assumption that $\leqslant$ is NM representable automatically carries with it a good deal of structure. In particular, it implies that consumption trees can be evaluated simply in terms of joint c.d.f.'s on consumption vectors. In the absence of this assumption, we require additional properties in our definition of risk preference independence. (In a similar vein, note that condition (iii) will hold automatically for any multiperiod expected utility representation of $\leqslant$ satisfying (i) and (ii).)

Together, the collection of orderings corresponding to the T sets of c.d.f.'s (one-period "lotteries"), $\{\underline{\overline{X}}_t\}$, will be referred to as the consumer's "risk preferences". Next, we assume that each of these "primitive" orderings possesses a standard representation. $\overset{t}{\leqslant}$ will be said to be "NM representable" if there exists an order-preserving continuous index $\Lambda_t: \underline{\overline{X}}_t \to R$ and a continuous von Neumann-Morgenstern utility V_t defined by

$$\Lambda_t(F_t) = \int V_t(c_t)dF_t(c_t) = \int V_t(c_t(\theta))dP_t(\theta).$$

<u>Assumption 4</u>:

Each $\overset{t}{\leqslant}$ $(t = 1,\ldots.,T)$ induced by $\leqslant|\underline{\overline{Y}}_t({}_0c_{t-1}, {}_{t+1}c_T)$ is "NM representable" with the continuous "NM index" V_t being strictly monotonically increasing.[2]

Given V_t, the time-period t certainty equivalent consumption flow associated with the lottery $F_t \in \underline{\overline{X}}_t$, $\hat{c}_t$, is defined by the equality

$$V_t(\hat{c}_t) = \int V_t(c_t)\,dF_t(c_t) = \int V_t(c_t(\theta))\,dP_t(\theta). \tag{2.1}$$

Next, after defining the notion of "temporal resolution indifference", we assume that the consumer's preference ordering $\preccurlyeq$ possesses this property.

Definition:

The preference ordering $\preccurlyeq$ will be said to exhibit "temporal resolution indifference" (t.r.i.) iff.

$$({}_0c_{t-k-1};(P_{t-k},({}_{t-k}c_{t-1},{}_t\tilde{c}_T))) \sim ({}_0c_{t-1};(P_t,({}_t\tilde{c}_T)))$$

for all ${}_0c_{t-1},{}_t\tilde{c}_T$ and $P_{t-k}(\theta) \equiv P_t(\theta)$.

(By $(P_{t-k},({}_{t-k}c_{t-1},{}_t\tilde{c}_T)$ we shall mean $(P_{t-k},({}_{t-k}\tilde{c}_{t-1},{}_t\tilde{c}_T))$ where $c_s(\theta) = c_s$ constant for $s = t-k,\ldots,t-1$. The condition $P_{t-k}(\theta) \equiv P_t(\theta)$ states that P_{t-k} and P_t are identical measures differing only in the time of resolution.)

Assumption 5:

The preordering $\preccurlyeq$ exhibits t.r.i.

Remark. Completely apart from the effect of early resolution of uncertainty on the set of feasible consumption trees discussed in section 4, one might reasonably argue on purely psychological grounds that the consumer prefers early (or late) resolution in consumption trees. We consider in [33] the question of representing preferences over $\mathcal{T}$ in the absence of (A.5). However, when one drops the t.r.i. assumption, a kind of "intergenerational" inconsistency results which is akin to that discussed in the certainty setting by Strotz [34], Peleg and Yaari [19] and Blackorby, Nissen, Primont and Russell

[2].

As we saw above, by restricting the consumer's preference ordering on trees to the subspace $\overline{Y}_t({}_oc_{t-1},{}_{t+1}c_T)$, one induces the set of orderings on one-period lotteries $\{\overset{t}{\leqslant}\}$--termed his "risk preferences". Analogously, the restriction of $\leqslant$ to the set of strings $\mathcal{J}^* \subset \mathcal{J}$ induces an ordering on C, denoted $\overset{C}{\leqslant}$ and referred to as the DM's (decision maker's) "time preferences".

Assumption 6:

The preference ordering $\overset{C}{\leqslant}$ is representable by the continuous (ordinal) utility function $U: C \to R$.

As we shall show over the ensuing sections, the collection of time preference and risk preference functions U, $\{V_1, \ldots, V_T\}$ constitute (under the assumptions spelled out above) the basic data for representing choices over trees.

Finally in order to establish the preference and representation results presented in section 3, it will be necessary to invoke the following technical restriction on the class of admissible consumption trees.

Integrability Property:

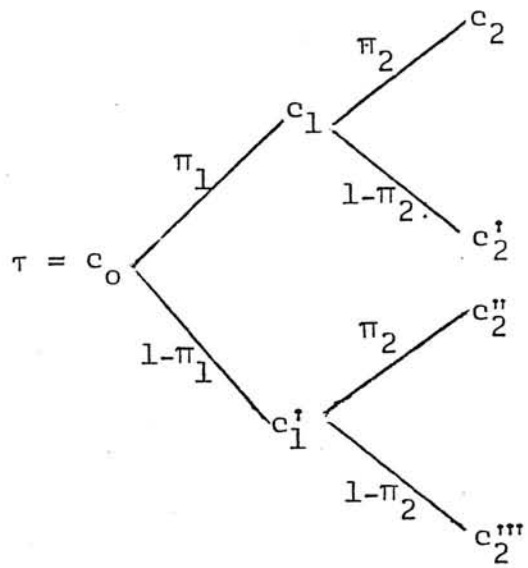
A given consumption tree $\tau \in \mathcal{J}$ will be said to possess property (I) if for each $1 \leq t \leq T$, the multiple (Lebesgue-Stieltjes) integrals[3]

$$\int \cdots \int\int V_t(c_t(\theta;\theta^{(1,t-1)})dP_t(\theta;\theta^{(1,t-1)})dP_{t-1}(\theta;\theta^{(1,t-2)})\ldots dP_1(\theta)$$

exist and are finite.[4]

An Example

To help clarify the relationship between the assumptions set forth above and the notion of a consumption tree, we next consider a simple example based on the following:



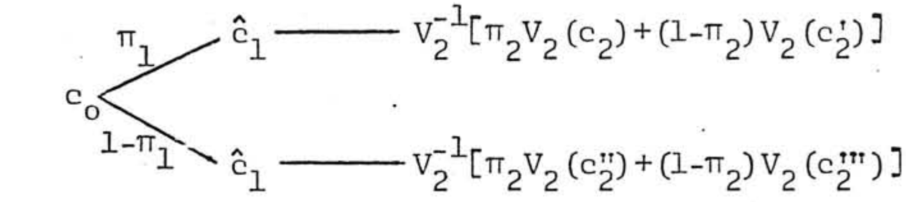
Under (A.1)-(A.5), we can convert τ into an indifferent consumption stream (string) by a simple series of steps. Let the "NM indices" V_1 and V_2 be given. By Assumptions 3 and 4,

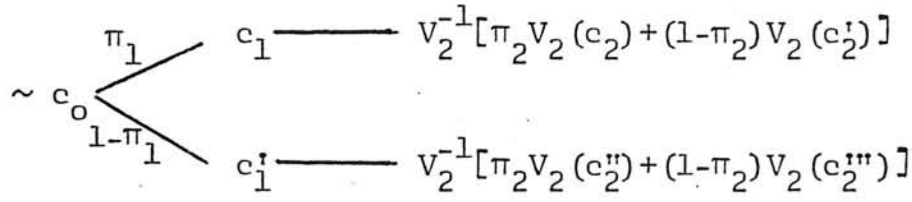
$$\tau \sim c_o \begin{array}{l} \xrightarrow{\pi_1} c_1 \text{ ——— } V_2^{-1}[\pi_2 V_2(c_2)+(1-\pi_2)V_2(c_2')] \\ \xrightarrow{1-\pi_1} c_1' \text{ ——— } V_2^{-1}[\pi_2 V_2(c_2'')+(1-\pi_2)V_2(c_2''')] \end{array} \tag{2.2}$$

Defining $\hat{c}_1 = V_1^{-1}[\pi_1 V_1(c_1)+(1-\pi_1)V_1(c_1')]$, we have by Assumptions 3 and 4 and the first Remark following (A.3) that for any c_2

$$c_o \begin{array}{l} \xrightarrow{\pi_1} \hat{c}_1 \text{ ——— } c_2 \\ \xrightarrow{1-\pi_1} \hat{c}_1 \text{ ——— } c_2 \end{array} \sim c_o \begin{array}{l} \xrightarrow{\pi_1} c_1 \text{ ——— } c_2 \\ \xrightarrow{1-\pi_1} c_1' \text{ ——— } c_2 \end{array} \tag{2.3}$$

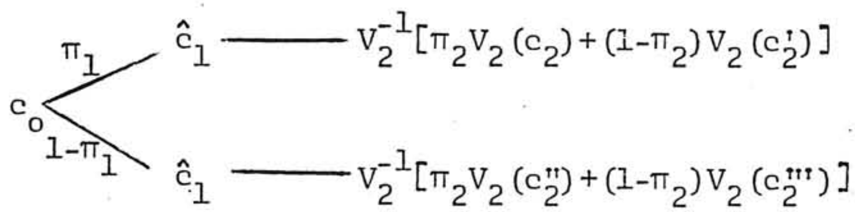
But by r.p.i., (2.3) implies that

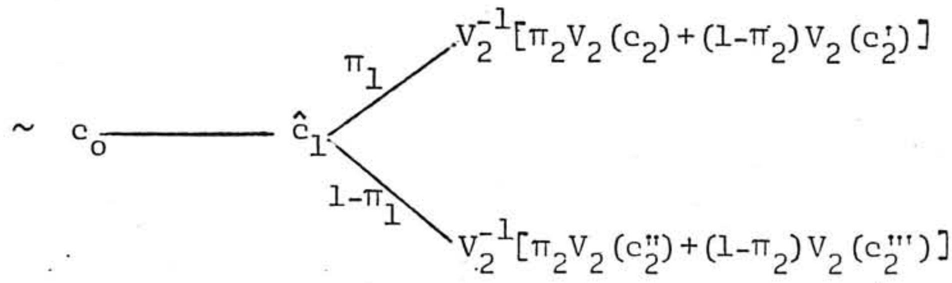




$\sim \tau.$

By the temporal resolution indifference assumption





Finally, defining

$$\hat{c}_2 = V_2^{-1}\{\pi_1 V_2(V_2^{-1}[\pi_2 V_2(c_2)+(1-\pi_2)V_2(c_2')])+(1-\pi_1)V_2(V_2^{-1}[\pi_2 V_2(c_2'')\div(1-\pi_2)V_2(c_2''')])\}$$

$$= V_2^{-1}[\pi_1\pi_2V_2(c_2) + \pi_1(1-\pi_2)V_2(c_2') + (1-\pi_1)\pi_2V_2(c_2'') + (1-\pi_1)(1-\pi_2)V_2(c_2''')] \quad (2.4)$$

and using r.p.i., we obtain

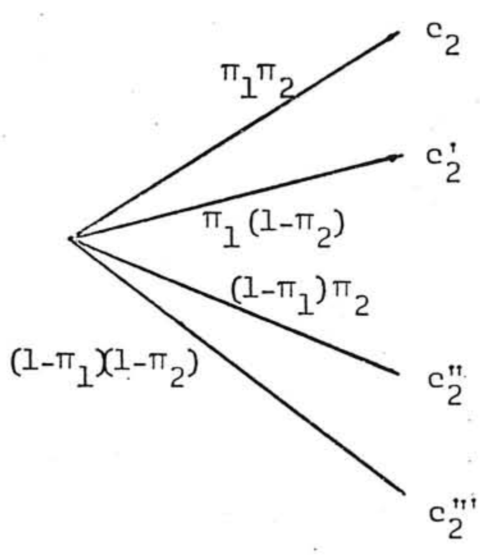
$$\tau \sim c_o \text{ ——— } \hat{c}_1 \begin{cases} \xrightarrow{\pi_1} V_2^{-1}[\pi_2V_2(c_2) + (1-\pi_2)V_2(c_2')] \\ \xrightarrow{1-\pi_1} V_2^{-1}[\pi_2V_2(c_2'') + (1-\pi_2)V_2(c_2''')] \end{cases}$$

$$\sim (c_o, \hat{c}_1, \hat{c}_2).$$

Remarks. (1) Note that in this example (under (A.1)-(A.5)), risk preferences (V_1 and V_2) yield a unique $(c_o, \hat{c}_1, \hat{c}_2)$ indifferent to the consumption tree τ. The fact that in this case (as well as in the proof of Lemma 1) one does not make recourse to time preferences in the replacement of a tree (or "one-period lotteries" contained therein) by an indifferent string may at first seem surprising. However, once the certainty equivalent stream is obtained, say $(c_o', \hat{c}_1', \hat{c}_2')$, time preferences can enter in identifying an, in general, infinite set of indifferent consumption triples $\{(c_o, c_1, c_2) \in R_+^3 \mid U(c_o, c_1, c_2) = U(c_o', \hat{c}_1', \hat{c}_2')\}$. This procedure differs significantly from that of the multiperiod expected utility approach. If one adds that extra axiomatic structure which implies the existence of a multiperiod expected utility function, then the vector payoff from each (T+1)-period branch is evaluated by a multiperiod NM utility function which simultaneously embodies risk and time preferences. (Cf., Rossman and Selden [24] for a discussion of the two-period case.) Then, for a given tree, an indifference set of certainty equivalent consumption streams can be determined in a single step. The basic point is that under the OCE approach we do not add that axiomatic structure which, by forcing a specific interdependence between risk and time preferences, requires both factors to enter into the evaluation simultaneously (Cf., Selden [30]).

(2) In deriving the string $(c_o, \hat{c}_1, \hat{c}_2)$, the r.p.i. assumption allows us to view the period-two consumption possiblities c_2, c_2', c_2'' and c_2''' independently from consumption in periods zero and one. We can partially relax this assumption and still obtain a tree paralleling (2.2) which is indifferent to τ. (The pair of "conditional" period-two certainty equivalents would simply be based on $V_{c_1}(c_2)$ and $V_{c_1'}(c_2)$, respectively.) However, if one then tries to replace the period-one "lottery" by $\hat{c}_1$, difficulties arise in knowing how to condition the period-two NM index so as to obtain a "joint" certainty equivalent as in (2.4).

(3) Whereas periods zero and one consumption do not enter into the determination of $\hat{c}_2$, period-one (state) uncertainty does. The total uncertainty for period-two consumption, when viewed from time zero, is the joint uncertainty attributable to (1) the "lottery" resolving in period one and (2) the "lotteries" resolving in period two, which can be cxpressed as



The t.r.i. assumption, roughly speaking, justifies "pushing" the period-one marginal probabilities through to period-two so that the joint distribution becomes the basis for determining $\hat{c}_2$ (Cf., eqn. (2.4)).

3. PREFERENCE AND REPRESENTATION THEOREMS

Let us first of all show that corresponding to each consumption tree, the DM's conditional risk preferences determine, under appropriate conditions, a unique indifferent string or certainty equivalent consumption stream.

Lemma 1:

Suppose (A.1)-(A.5) hold. Let $\mathcal{T}(I)$ denote the set of consumption trees in $\mathcal{T}$ satisfying Property I. Then for any $\tau \in \mathcal{T}(I)$, there exists a unique $\tau^* \in \mathcal{T}^*(\equiv C)$ such that $\tau \sim \tau^*$ where τ^* is essentially equivalent to a certain consumption amount in time-period zero followed by a certainty equivalent stream, $(c_o, {}_1\hat{c}_T) \equiv (c_o, \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_T)$.

(The proof is given in Appendix A.)

Remark. It should be noted that in the statement of the Lemma, $\hat{c}_t$ refers to the time-period t certainty equivalent with respect to the joint probability measure on $(\theta^{(1)}, \ldots, \theta^{(t)})$ or equivalently on $(\theta^{(1)}, \ldots, \theta^{(T)})$ since $c_t(\theta; \theta^{(1,t-1)})$ is independent of $\theta^{(t+1,T)}$. As we show in Appendix A, $\hat{c}_t$ can readily be computed according to

$$\hat{c}_t = V_t^{-1} \int V_t(c_t(\theta^{(1)}, \ldots, \theta^{(t)}))\, dJ_T(\theta^{(1)}, \ldots, \theta^{(T)}) \qquad (3.1)$$

where

$$dJ_T(\theta^{(1)}, \ldots, \theta^{(T)}) =_{\text{def}} dP_1(\theta)\, dP_2(\theta; \theta^{(1)}) \ldots dP_T(\theta; \theta^{(1,T-1)}).$$ [5]

It is clear from Lemma 1 that, as was suggested earlier, time preferences do not in any way enter into the determination of $(c_o, {}_1\hat{c}_T)$. Rather, this unique "consumption stream" is determined solely by the set $\{V_1,\ldots,V_T\}$.

However, by adding U to the collection of NM indices, $\{V_1,\ldots,V_T\}$, we are next able to show that the given preordering $\preccurlyeq$ on $\mathcal{T}(I)$ is uniquely determined.

Theorem 1:

Under (A.1)-(A.6), the collection of time and risk preference functions U, $\{V_1,\ldots,V_T\}$ determines the preordering on $\mathcal{T}(I)$ uniquely.

(The proof is given in Appendix B.)

The two-period OCE (Ordinal Certainty Equivalent) representation result, Corollary 2 in [30], can now be extended to the case of (T+1) time-periods.

Corollary 1 (OCE Representation):

Under (A.1)-(A.6), the collection U, $\{V_1,\ldots,V_T\}$ represents the preordering $\preccurlyeq$ on $\mathcal{T}(I)$ in that, for any $\tau_1, \tau_2 \in \mathcal{T}(I)$

$$\tau_1 \preccurlyeq \tau_2 \Leftrightarrow U((c_o,{}_1\hat{c}_T)_1) \leq U((c_o,{}_1\hat{c}_T)_2)$$

where $(c_o,{}_1\hat{c}_T)_i$ is the certainty equivalent consumption stream ("string") for τ_i (i = 1,2) obtained by using $\{V_1,\ldots,V_T\}$.

It is straightforward to show that this OCE representation is unique up to an increasing monotonic transform of U and positive affine transforms of $V_1,\ldots,V_T$.

Note that Theorem 1 does not rule out the possibility that two different U, $\{V_1,\ldots,V_T\}$-collections might each uniquely determine the same $\preccurlyeq$ on $\mathcal{T}(I)$. We next establish that under (A.1)-(A.6) this cannot be the case --i.e., there exists but a single decomposition of $\preccurlyeq$ into time and risk preferences.[6]

Theorem 2:

If at least one pair of utility functions from U', $\{V_1',\ldots,V_T'\}$ and U", $\{V_1'',\ldots,V_T''\}$ differ (by more than an increasing monotone or positive affine transform, respectively), then under (A.1)-(A.6) these two sets of representations produce different orderings on $\mathcal{T}(I)$.

(See Appendix C for the proof.)

Theorems 1 and 2 can also be interpreted in the following "constructive" way:

Corollary 2:

Suppose that we are given $U, \{V_1,\ldots,V_T\}$ which represent the preorderings $\overset{C}{\leqslant}$, $\{\overset{1}{\leqslant},\ldots,\overset{T}{\leqslant}\}$ on C, $\{\underline{\overline{X}}_1,\ldots,\underline{\overline{X}}_T\}$, respectively. Then there exists a unique, complete preordering $\leqslant$ on $\mathcal{T}(I)$ which (i) satisfies (A.3)-(A.6) and (ii) induces preorderings on C, $\{\underline{\overline{X}}_1,\ldots,\underline{\overline{X}}_T\}$ that are identical to $\overset{C}{\leqslant}$, $\{\overset{1}{\leqslant},\ldots,\overset{T}{\leqslant}\}$.

The proof follows from a straightforward application of (L.1) and (T.1) and hence is omitted.

We conclude this section with a brief comparison of our multiperiod OCE representation and the corresponding (T+1)-period expected utility model. For any pair of consumption trees $\tau', \tau'' \in \mathcal{T}$, let c_o' and c_o'' denote the corresponding period-zero consumption values and G' and G" the corresponding joint c.d.f.'s defined on the set of T-vectors $\{(c_1,\ldots,c_T)\}$. Then $\leqslant$ will be said to be "NM representable" if there exists a continuous $W: C \to R$ such that $\int W(c_o,c_1,\ldots,c_T)\,dG(c_1,\ldots,c_T)$ exists for all $c_o \in R_+$, is finite and satisfies for all $\tau', \tau'' \in \mathcal{T}$,

$$\tau' \leqslant \tau'' \Leftrightarrow \int W(c_o',c_1,\ldots,c_T)\,dG'(c_1,\ldots,c_T) \leq \int W(c_o'',c_1,\ldots,c_T)\,dG''(c_1,\ldots,c_T).$$

If we continue to assume that $\preccurlyeq$ exhibits risk preference independence, then W will take the following form

$$W(c_o,c_1,\ldots,c_T) = \alpha(c_o) + \beta(c_o)\sum_{t=1}^{T} k_t w_t(c_t) \qquad (3.2)$$

(see initial discussion in Appendix D).

We next show that under comparable assumptions the multiperiod expected utility model is but a special case of the OCE representation.

Theorem 3:

Every (T+1)-period expected utility representation of $\preccurlyeq$ satisfying (A.3) can be transformed into an OCE representation, but the converse is not true.

(See Appendix D for the proof.)

4. CONSUMPTION/SAVINGS DECISION

We next apply the multiperiod OCE representation to a consumption/savings allocation problem. For simplicity it will be assumed that T = 2, although our analysis extends in a straightforward way to any finite T. Imagine a consumer situated in time-period zero with an initial wealth of y_o. He must determine what portion of his initial wealth to consume and what portion to save, denoted respectively c_o and s_o. That is, he is constrained to satisfy $c_o + s_o = y_o$. The only means for investment is a single asset yielding a random (gross) rate of return $X_1(\theta)$. At time zero our agent will also need to formulate an "ex ante" allocation of his period-one wealth, $y_1(\theta) = s_o X_1(\theta)$, between consumption and savings, $c_1(\theta)$ and $s_1(\theta)$. Again only a single risky asset will be available in time-period one. Denote its (gross) rate of return by $X_2(\theta;\theta^{(1)})$ where the random return in period-two can depend on the $\theta^{(1)}$-outcome. Let both $X_1(\theta)$ and $X_2(\theta;\theta^{(1)})$ be strictly positive and finite for all $\theta^{(1)}$, $\theta^{(2)}$. The probability measures $P_1(\theta)$ and $P_2(\theta;\theta^{(1)})$ are defined as in section 2. However, for the present discussion it will be assumed that there are only a finite number of states $i = 1,\ldots,N$. Thus, $\theta_i^{(t)}$ will denote the i^{th} state in time-period t.

For the special case of two states in period one and two states in period two conditional on each $\theta^{(1)}$-outcome, the consumption/savings allocation problem is summarized schematically in Figure (1a).

Thus in determining his optimal allocation of y_o, the consumer will make one decision at $t = 0$ and formulate an ex ante allocation for $t = 1$. We express this decision problem in terms of the savings allocations

$$A_o = s_o/y_o \tag{4.1}$$

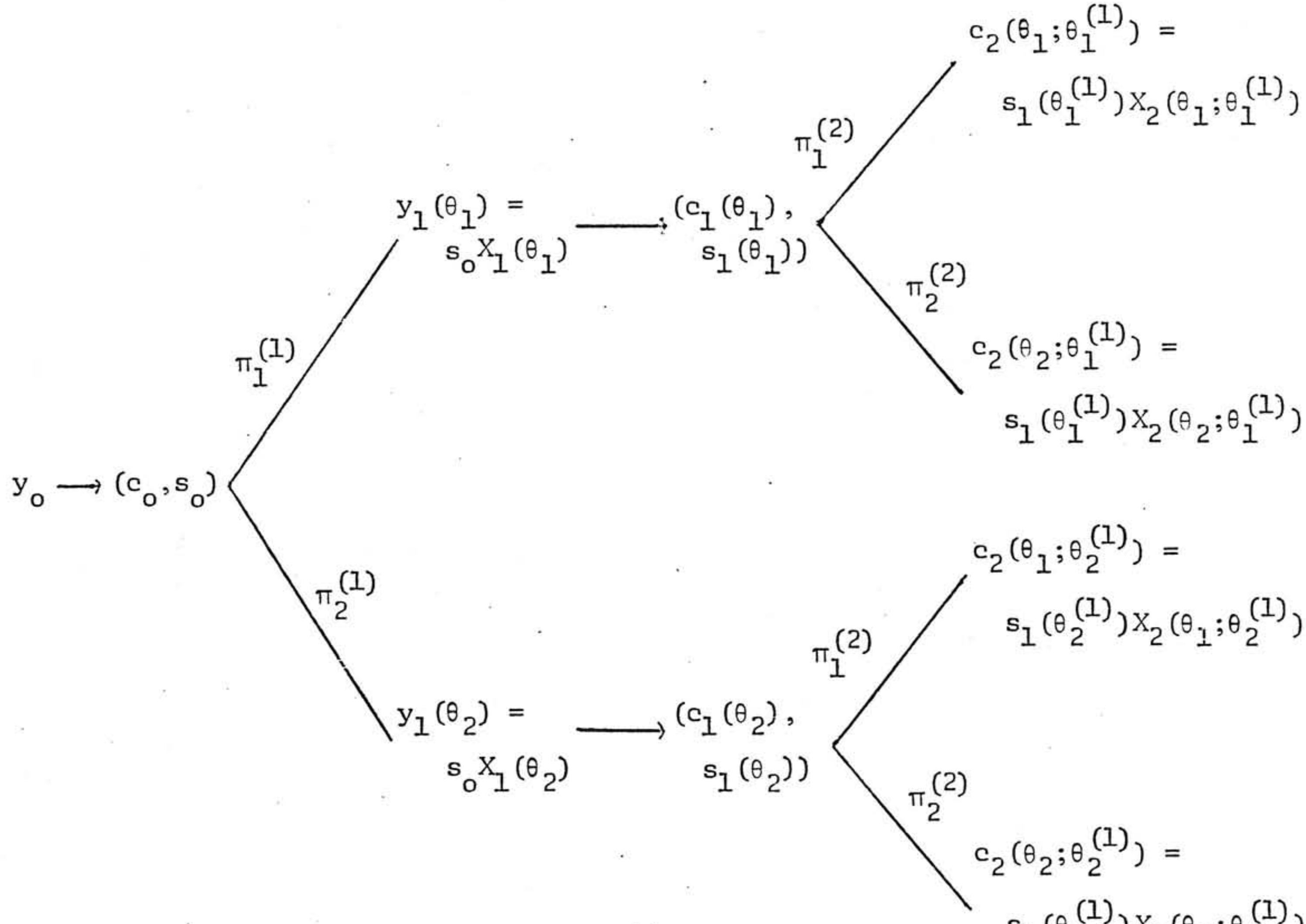
$$A_1(\theta_i) = s_1(\theta_i)/y_1(\theta_i) \quad i = 1,\ldots,N, \tag{4.2}$$

where $A_o \in [0,1]$ and the function $A_1(\theta_i) \in [0,1]$, $\forall\, \theta_i \in \Theta$.

For each vector $(A_o, A_1(\theta_1),\ldots,A_1(\theta_N))$ there will be a 3-period consumption tree. Thus, for instance, corresponding to the consumption/savings problem summarized in Figure (1a) there will be a tree of the form given in Figure (1b). More generally, one can express τ's dependence on A_o and $A_1(\theta)$ as

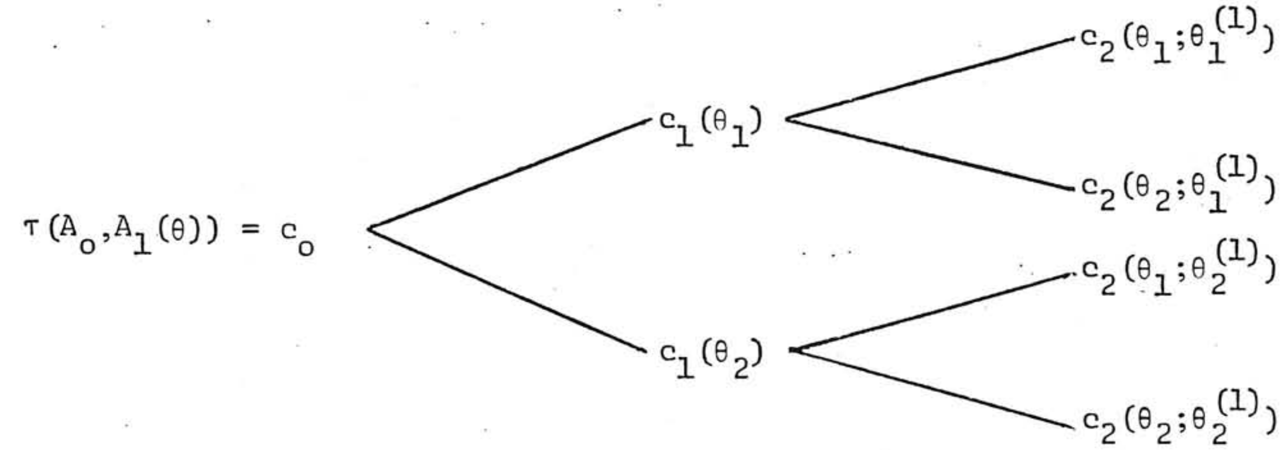
$$\begin{aligned}\tau(A_o,A_1(\theta)) &= (c_o,\{P_1(\theta),c_1(\theta)\},\{P_2(\theta;\theta^{(1)}),c_2(\theta;\theta^{(1)})\}) \\ &= (([1-A_o]y_o),\{P_1(\theta),(A_oy_oX_1(\theta)[1-A_1(\theta)])\},\{P_2(\theta; \\ &\qquad \theta^{(1)}),(A_oy_oX_1(\theta^{(1)})A_1(\theta^{(1)})X_2(\theta;\theta^{(1)}))\})\end{aligned} \tag{4.3}$$

Hence the consumer's problem is one of picking that current allocation and ex ante plan $(A_o^o, A_1(\theta_1)^o,\ldots,A_1(\theta_N)^o)$ which in time-period zero produces the most preferred consumption tree τ^o under $\preccurlyeq$ or, in terms of the OCE representation, maximizes $U(c_o, {}_1\hat{c}_2)$. We next show that such a maximizing allocation exists.[7]



$y_o \longrightarrow (c_o, s_o)$
$\pi_1^{(1)}$
$\pi_2^{(1)}$
$y_1(\theta_1) = s_o X_1(\theta_1)$
$(c_1(\theta_1), s_1(\theta_1))$
$y_1(\theta_2) = s_o X_1(\theta_2)$
$(c_1(\theta_2), s_1(\theta_2))$
$\pi_1^{(2)}$
$\pi_2^{(2)}$
$\pi_1^{(2)}$
$\pi_2^{(2)}$
$c_2(\theta_1;\theta_1^{(1)}) = s_1(\theta_1^{(1)}) X_2(\theta_1;\theta_1^{(1)})$
$c_2(\theta_2;\theta_1^{(1)}) = s_1(\theta_1^{(1)}) X_2(\theta_2;\theta_1^{(1)})$
$c_2(\theta_1;\theta_2^{(1)}) = s_1(\theta_2^{(1)}) X_2(\theta_1;\theta_2^{(1)})$
$c_2(\theta_2;\theta_2^{(1)}) = s_1(\theta_2^{(1)}) X_2(\theta_2;\theta_2^{(1)})$

(a)



$\tau(A_o, A_1(\theta)) = c_o$
$c_1(\theta_1)$
$c_1(\theta_2)$
$c_2(\theta_1;\theta_1^{(1)})$
$c_2(\theta_2;\theta_1^{(1)})$
$c_2(\theta_1;\theta_2^{(1)})$
$c_2(\theta_2;\theta_2^{(1)})$

(b)

Figure 1

Proposition 1:

Suppose that (A.1) - (A.6) hold so that $\preccurlyeq$ is "OCE representable". Restrict each $A_o, A_1(\theta_i) \epsilon [0,1]$. Then an optimal time-period zero $(A_o^o, A_1(\theta_1)^o, \ldots, A_1(\theta_N)^o)$ exists.

(The proof is given in Appendix E.)

Remark. It should be stressed that in this paper we are only addressing the question of an optimal current allocation. The $A_1(\theta)^o$ is thus a type of "instrumental policy" which is optimal in the sense that it (together with the current action A^o) yields the most preferred consumption tree at time zero. Once the period-one uncertainty is resolved and the consumer considers anew the question of how much to save in the then current period, he faces an allocation problem which is different in two respects: (i) state $\theta_i^{(1)}$ has occurred and a known wealth of $y_1(\theta_i)$ is available for allocation and (ii) the uncertainty regarding period-two consumption (as well as $c_3, \ldots, c_T$ in the more general case) has changed in that the period-one marginal probabilities have "dropped off". Suppose that in period-one, the consumer, paralleling the previous period's analysis, determines what portion of his (then known) wealth $y_1(\theta_i)$ to save. The resulting allocation, denoted $B_1(\theta_i)^o$, which produces the most preferred consumption tree in period one will, in general, be different from the "ex ante" $A_1(\theta_1)^o$ obtained in conjunction with A_o^o in period zero. The reason for this is quite simple: whereas tastes haven't changed (the V_2 and $U_{c_o}(c_1, c_2)$ used in the period one analysis are equivalent to the corresponding representations used in period zero), the allocation problem has. While the $A_1(\theta)^o$ takes into account the period-one state uncertainty $P_1(\theta)$, the optimal allocation $B_1(\theta)^o$ does not because the uncertainty surrounding $\theta^{(1)}$ has been resolved. This and other related issues are considered at length for the general (T+1)-period case in Selden and Stux [32].

To go further and characterize key properties of the optimal current allocation, we shall follow the standard practice of specializing the assumed form of utility. Thus, let

$$U(c_o,c_1,c_2) = \sum_{t=0}^{2} -\alpha_t c_t^{-\delta}/\delta \tag{4.4}$$

$$V(c_t) = -c_t^{-\gamma_t}/\gamma_t \qquad t = 1,2 \tag{4.5}$$

where $0 < \alpha_t < 1$, $\sum_t \alpha_t = 1$ and $-1 < \delta, \gamma_t < \infty$. Then it follows from Corollary 1 (and eqn. (3.1)) that the OCE utility for any 3-period consumption tree is given by

$$-\alpha_o\frac{c_o^{-\delta}}{\delta} - \frac{\alpha_1}{\delta}\left[\int[c_1(\theta)]^{-\gamma_1}dP_1(\theta)\right]^{\delta/\gamma_1} - \frac{\alpha_2}{\delta}\left[\iint[c_2(\theta;\theta^{(1)})]^{-\gamma_2}dP_2(\theta;\theta^{(1)}) \cdot dP_1(\theta^{(1)})\right]^{\delta/\gamma_2} \tag{4.6}$$

Following Selden [31], we shall interpret the constant elasticity of substitution for (4.4)

$$\eta = 1/(\delta+1)$$

as a measure of intuitive intertemporal <u>complementarity</u>. The common sense everyday usage of "complementarity" refers to the property of "belonging to" or "going with". The stronger or more intense the (preference) association between a pair of commodities, the greater their complementarity. Irving Fisher [5] referred to a pair of goods as perfect complements if they cannot be used separately but only in a fixed ratio and perfect substitutes if they can be substituted for one another in a constant ratio. It is in this spirit that we shall interpret c_o, c_1 and c_2 as being {complements, independents, substitutes} as η { <,=,> } 1. (See Katzner [10].)

The specification of risk preferences (4.5) exhibits constant relative risk aversion (Arrow [1], Pratt [22])

$$\rho_t =_{\text{def}} -c_t V_t''(c_t)/V_t'(c_t) = \gamma_t + 1.$$

Now it is easy to show that the only multiperiod expected utility representation of $\preccurlyeq$ consistent with the above specification of time and conditional risk preferences is the standard form defined by the 3-period "NM index"

$$W(c_o, c_1, c_2) = \sum_{t=0}^{2} - \alpha_t c_t^{-\gamma}/\gamma, \tag{4.7}$$

where $0 < \alpha_t < 1$, $\sum_t \alpha_t = 1$ and $-1 < \gamma < \infty$. This form of utility has been employed extensively in the study of consumption/savings (portfolio) problems (e.g., Phelps [20], Levhari and Srinivasan [14], Hakansson [9], Samuelson [28], Merton [15], Rothschild and Stiglitz [25], Mirrlees [17] and Kihlstrom and Mirman [12]). In terms of the OCE representation (4.6), <u>assuming $\preccurlyeq$ to be "NM representable" implies that the Arrow-Pratt measure of relative risk aversion must be the same in periods one and two and must equal the reciprocal of the measure of complementarity</u>, i.e.,

$$\rho_1 = \rho_2 = 1/\eta. \tag{4.8}$$

Under the more general OCE representation ρ_1, ρ_2 and η can be prescribed independently (restrictions are, of course, indirectly implied by the reasonableness of the corresponding consumption and savings behavior).

It is well-known that if the consumer is a multiperiod expected utility maximizer with W defined by (4.7), then both optimal initial consumption and savings are linear in y_o. The following establishes that this result generalizes to the OCE representation defined by CES time preferences and constant relative risk aversion risk preferences. We further show that the optimal <u>initial</u> allocation A_o^o is unique and lies between zero and one.

<u>Proposition 2</u>:

Let $\preccurlyeq$ be OCE representable where U is defined by eqn. (4.4) and V_1 and V_2 by (4.5). Then the optimal $(A_o^o, A_1(\theta_1)^o, \ldots, A_1(\theta_N)^o)$

(i) can be obtained by a two-stage optimization and

(ii) is independent of y_o.

Furthermore, A_o^o

(iii) is unique and an interior solution.

(See Appendix F for the proof.)

Before investigating the effects of changes in certain of the preference parameters and the degree of uncertainty in the single asset's rate of return on optimal initial savings (consumption), let us consider whether assuming that the preordering over <u>consumption trees</u> exhibits temporal resolution indifference necessarily implies that the consumer is indifferent to the time of resolution in an <u>allocation problem</u>. In the context of the consumption/savings problem portrayed in Figure (1a), we shall interpret "early resolution" to mean that the $\theta^{(2)}$-outcome becomes known <u>at t = 1</u> and consequently, the period-one savings decision will not be made under uncertainty. A_1 will thus depend on both $\theta^{(1)}$ and $\theta^{(2)}$ and will be denoted $A_1(\theta^{(1)},\theta^{(2)})$. Under early resolution there will be a 3-period consumption tree τ corresponding to each $(A_o, A_1(\theta_1^{(1)},\theta_1^{(2)}), \ldots, A_1(\theta_i^{(1)},\theta_j^{(2)}), \ldots, A_1(\theta_N^{(1)},\theta_N^{(2)}))$. Paralleling the non-early resolution expression (4.3) we can write

$$\tau(A_o,A_1(\theta^{(1)},\theta^{(2)})) = (c_o,\{P_2(\theta^{(2)};\theta^{(1)})P_1(\theta^{(1)}),(c_1(\theta^{(1)},$$

$$\theta^{(2)}),c_2(\theta^{(1)},\theta^{(2)}))\})$$

$$= ([1-A_o]y_o, \{P_2(\theta^{(2)};\theta^{(1)})P_1(\theta^{(1)}),$$

$$(A_o y_o X_1(\theta^{(1)})[1-A_1(\theta^{(1)},\theta^{(2)})],$$

$$A_o y_o X_1(\theta^{(1)})A_1(\theta^{(1)},\theta^{(2)})$$

$$\cdot X_2(\theta^{(2)};\theta^{(1)}))\}) \qquad (4.9)$$

Thus under early resolution $\tau(A_o, A_1(\theta^{(1)},\theta^{(2)}))$ takes the form of a vector lottery. Following Proposition 1, maximization of OCE utility will yield a time-zero optimal vector denoted $(A_o^{oo}, A_1(\theta^{(1)},\theta^{(2)})^{oo})$ and a corresponding optimal consumption tree τ^{oo}.

Suppose that the trees defined by (4.3) and (4.9) are identical in every way except for the time at which $\theta^{(2)}$ is known. Also suppose that the consumer's preordering $\preccurlyeq$ is OCE representable according to (4.6). Then the following gives sufficient conditions for the early resolution optimum τ^{oo} to always be preferred to the non-early resolution optimum τ^{o} corresponding to $(A^o, A_1(\theta)^o)$.[8]

Result A:

If $\rho_1, \rho_2 > 0$ and $\eta \leq 1$, then $\tau^o \preccurlyeq \tau^{oo}$.

(The proof is given in Appendix G.)

Remark. The above result shows that if the consumer's OCE preferences exhibit risk aversion and intertemporal complementarity, early resolution of uncertainty is, in general, preferred (τ^{oo} is strictly preferred to τ^o except for a set of quite special cases --e.g., when $\tilde{X}_2$ is independent of $\theta^{(2)}$). Thus, as indicated in section 1, while the temporal resolution indifference assumption eliminates the purely psychological preference for early resolution in consumption trees, it nevertheless is consistent with a preference for early resolution in allocation problems. As suggested from the discussion preceding Result A (and the proof), the value of early resolution in an allocation context arises from the use of this added information by the consumer to achieve a better consumption/savings decision. Kreps and Porteus [13] make a significant contribution in demonstrating explicitly the important role played by time of resolution in dynamic choice theory. What we do here is distinguish its role in choices among consumption trees from its role in allocation problems.

Increased Risk Aversion and Risk

Virtually all previous efforts at studying the effects of both increased risk aversion and risk on optimal savings have been cast in terms of a multi-period expected utility model. Most often, the "NM index" is assumed to be a two-period version of (4.7). But in this case since, as noted above in connection with (4.8), the consumer's degree of relative risk aversion (ρ_1 or ρ_2) cannot be altered independently of his measure of intertemporal complementarity, the question of how the optimal s_o^o varies with increased risk aversion is not well-posed. (Cf., the two-period discussions of Selden [31] and Kihlstrom and Mirman [12].) However under the OCE representation (4.6), ρ_1 and ρ_2 represent natural and unambiguous risk aversion shift paramters. We next show that whether an increase in (either period-one or period-two) risk aversion (in general) produces increased, unchanged or decreased initial savings depends on whether c_o,c_1 and c_2 are complements, independent or substitutes.

Result B:

Assume that ρ_t $(t = 1,2)$ is strictly positive and not equal to unity. Then in response to a small increase in ρ_t, optimal initial saving will, in general {increase, remain constant, decrease} as η {<,=,>} 1.[9]

(The proof is provided in Appendix H.)

One would expect that increases in risk aversion and increases in risk, appropriately defined, should have similar qualitative effects on the solutions to multi-period resource allocation problems (Diamond and Stiglitz [4]). We next show that this is, in fact, the case for the (CES time preference, constant relative risk aversion) OCE representation. To simplify matters, the Arrow-Sandmo ([1],[29]) notion of a mean preserving increase in capital risk is employed.[10] That is, write the (net) rate of return on investment as $\lambda_t\tilde{x}_t + \epsilon_t$ (t = 1,2) where λ_t and ϵ_t are, respectively, multiplicative and additive shift parameters and $x_t \equiv X_t - 1$. In order for a multiplicative shift around zero to keep the mean constant, we must have

$$d\epsilon_t/d\lambda_t = -E(\tilde{x}_t) \tag{4.10}$$

<u>Result C</u>:

Assume that $\rho_t > 0$ (t = 1,2), the period-two (gross) rate of return $\tilde{X}_2$ is independent of $\theta^{(1)}$ and the optimal $A_1(\theta_i)^o \in (0,1)$, $\forall\ \theta_i \in \Theta$. Then a mean preserving increase in period t capital risk has the same qualitative effect on optimal initial saving as a small increase in period t risk aversion.

(See Appendix I for the proof.)

Thus, whether optimal initial saving will {increase, remain constant, decrease} in response to a mean preserving increase in capital risk will depend on whether a (risk averse) consumer views c_o, c_1 and c_2 as intertemporal {complements, independents, substitutes}. Despite the apparent similarity between Results B and C, one caveat is in order. It will be observed from their respective proofs that increases in period-one risk aversion and capital risk differ in that the effect of the former is restricted to just period-one whereas the latter will, in general, impact both on $\hat{c}_1$ and $\hat{c}_2$. Thus for multi-period problems, some care should be taken in drawing parallels between the effects of increases in risk aversion and in risk.

APPENDIX

A. Proof of Lemma 1

Let τ be expressed schematically once again as

$$(c_o,\{P_1,\tilde{c}_1\},\{P_2,\tilde{c}_2\},\ldots,\{P_T,\tilde{c}_T\}) \tag{a.1}$$

where

$$\{P_t,\tilde{c}_t\} =_{\text{def}} \{(P_t(\theta;\theta^{(1,t-1)}),c_t(\theta;\theta^{(1,t-1)}))\}.$$

Consider the collection $\{P_T,\tilde{c}_T\}$. Corresponding to each specific sequence of parameters $\theta^{(1,T-1)}$ there will be a T-period consumption stream and a single one-period lottery, $(c_o,c_1,\ldots,c_{T-1},(P_T,\tilde{c}_T))$. Given Assumption 4, we can compute a T-period certainty equivalent for the pair $(P_T(\theta),\ c_T(\theta))$:

$$\hat{c}_T(\theta^{(1,T-1)}) = V_T^{-1}\int V_T(c_T(\theta;\theta^{(1,T-1)}))\,dP_T(\theta;\theta^{(1,T-1)}).$$

Now it follows from the continuity and strict monotonicity of V_T and the second Mean Value Theorem (for integrals) that for each $\theta^{(1,T-1)}$ a $\hat{c}_T$ exists and is unique.

By Property I, $\hat{c}_T(\theta^{(1,T-1)})$, viewed as a <u>function</u> of $\theta^{(T-1)}$, will be a measurable random variable with respect to the measure $P_{T-1}(\theta;\theta^{(1,T-2)})$. It follows from r.p.i. (risk preference independence) that

$$\tau \sim (c_o,\{P_1,\tilde{c}_1\},\ldots,\{(P_{T-1},(\tilde{c}_{T-1},\tilde{\hat{c}}_T))\}) \tag{a.2}$$

where $(P_{T-1},(\tilde{c}_{T-1},\tilde{\hat{c}}_T))$ is a vector lottery characterized by all of the uncertainty resolving at the beginning of time-period T-1 when $\theta^{(T-1)}$ becomes known. Here again, assume a specific sequence of parameters $\theta^{(1,T-2)}$. Corresponding thereto will be the branch

$$(c_o,c_1,\ldots,c_{T-2},(P_{T-1},(\tilde{c}_{T-1},\tilde{\hat{c}}_T))), \tag{a.3}$$

which we shall "resolve" in a series of steps paralleling the Example in section 2. First, using V_{T-1}, given by (A.4), and applying (A.3) we have that

$$({}_oc_{T-2},(P_{T-1},(\tilde{c}_{T-1},c_T))) \sim ({}_oc_{T-2},(P_{T-1},(\hat{c}_{T-1},c_T)))$$

for any choice of c_T and where

$$\hat{c}_{T-1}(\theta^{(1,T-2)}) = V_{T-1}^{-1}\int V_{T-1}(c_{T-1}(\theta;\theta^{(1,T-2)}))dP_{T-1}(\theta;\theta^{(1,T-2)}).$$

Now r.p.i. implies that the following is indifferent to the expression (a.3)

$$({}_oc_{T-2},(P_{T-1},(\hat{c}_{T-1},\tilde{\hat{c}}_T))).$$

Since $\hat{c}_{T-1}$ is the same for all values of $\theta^{(T-1)}$, we can next apply t.r.i. (temporal resolution indifference) to change the time of resolution of P_{T-1} from T-1 to T. This yields the following which is indifferent to the expression (a.4)

$$({}_oc_{T-2},\hat{c}_{T-1},\ (P_{T-1},\tilde{\hat{c}}_T))$$

(note that to avoid possible notational confusion, we have <u>not</u> altered the subscript on the measure from T-1 to T as perfect consistency with the definition of t.r.i. would require). The third step in "resolving" (a.3) is to use V_T and r.p.i. again to convert the above to the indifferent string

$$({}_oc_{T-2},\hat{c}_{T-1},\hat{\hat{c}}_T) \tag{a.5}$$

where

$$\hat{\hat{c}}_T = V_T^{-1}\int V_T(\hat{c}_T(\theta;\theta^{(1,T-2)}))dP_{T-1}(\theta;\theta^{(1,T-2)}).$$

Note that under (A.4), V_{T-1} and V_T are strictly monotonically increasing and hence (a.5) is uniquely determined.

Proceeding as above for each possible $\theta^{(1,T-3)}$ yields

$$\tau \sim (c_o,\{P_1,\tilde{c}_1\},\ldots,\{(P_{T-2},(\tilde{c}_{T-2},\tilde{\hat{c}}_{T-1},\tilde{\hat{\hat{c}}}_T)\}$$

where Property I ensures that $\hat{c}_{T-1}$ and $\hat{\hat{c}}_T$, when viewed as functions of $\theta^{(T-2)}$, are measurable random vairables with respect to the measure $P_{T-2}(\theta;\theta^{(1,T-3)})$. Now we can apply the above argument for each possible $\theta^{(1,T-3)}$ sequence. Doing so yields

$$({}_oc_{T-3},(P_{T-2},(c_{T-2},\tilde{\hat{c}}_{T-1},\tilde{\hat{\hat{c}}}_T))) \sim ({}_oc_{T-3},\hat{c}_{T-2},\hat{\hat{c}}_{T-1},\hat{\hat{\hat{c}}}_T)$$

where

$$\hat{c}_{T-2} = V_{T-2}^{-1}\int V_{T-2}(c_{T-2}(\theta;\theta^{(1,T-3)}))dP_{T-2}(\theta;\theta^{(1,T-3)})$$

$$\hat{\hat{c}}_{T-1} = V_{T-1}^{-1}\int V_{T-1}(\hat{c}_{T-1}(\theta;\theta^{(1,T-3)}))dP_{T-2}(\theta;\theta^{(1,T-3)})$$

$$\hat{\hat{\hat{c}}}_T = V_T^{-1}\int V_T(\hat{\hat{c}}_T(\theta;\theta^{(1,T-3)}))dP_{T-2}(\theta;\theta^{(1,T-3)}).$$

(A.4) ensures that each of these certainty equivalents is uniquely determined.

Finally, repeating this process for each of the T time-periods characterized by uncertain consumption yields

$$(c_o,\hat{c}_1,\hat{\hat{c}}_2,\ldots,\overset{\hat{\vdots}}{c}_T) \equiv \tau^* \sim \tau. \qquad \text{(a.6)}$$

(A.4) ensures the uniqueness of the τ^* obtained. $\boxtimes$

Now to verify our claim concerning eqn. (3.1) in the Remark, simply observe that (for $t = 2$)

$$\begin{aligned}\hat{\hat{c}}_2 &= V_2^{-1}\int V_2\Big(V_2^{-1}\int V_2(c_2(\theta;\theta^{(1)}))dP_2(\theta;\theta^{(1)})\Big)dP_1(\theta) \\ &= V_2^{-1}\iint V_2(c_2(\theta^{(1)},\theta^{(2)}))dJ_2(\theta^{(1)},\theta^{(2)})\end{aligned}$$

or more generally

$$\overset{\hat{\vdots}}{c}_t = V_t^{-1}\int V_t(c_t(\theta^{(1)},\ldots,\theta^{(t)})dJ_t(\theta^{(1)},\ldots,\theta^{(t)}) \qquad \text{(a.7)}$$

where integration is over the $\theta^{(1,t)}$ space.

To avoid introducing the quite unaesthetic notation $\overset{\hat{\vdots}}{c}_t$ in the text, we instead use simply $\hat{c}_t$ (see note (5)). Thus in the statement of the Lemma, eqn. (a.6) is expressed as $(c_o,{}_1\hat{c}_T) \equiv (c_o,\hat{c}_1,\ldots,\hat{c}_T) \equiv \tau^*$. Q.E.D.

B. Proof of Theorem 1

Consider any pair of consumption trees, $\tau_1,\tau_2 \in \mathcal{T}(I)$. We have from (L.1) that there exists a unique $(c_o,{}_1\hat{c}_T)_1 \equiv \tau_1^* \sim \tau_1$ and a unique $(c_o,{}_1\hat{c}_T)_2 \equiv \tau_2^* \sim \tau_2$. But clearly given that U represents $\overset{C}{\leqslant}$,

$$U((c_o,{}_1\hat{c}_T)_1) \leq U((c_o,{}_1\hat{c}_T)_2) \;\Leftrightarrow\; (c_o,{}_1\hat{c}_T)_1 \overset{C}{\leqslant} (c_o,{}_1\hat{c}_T)_2.$$

It follows immediately that

$$(c_o,{}_1\hat{c}_T)_1 \overset{C}{\leqslant} (c_o,{}_1\hat{c}_T)_2 \;\Leftrightarrow\; \tau_1^* \leqslant \tau_2^* \;\Leftrightarrow\; \tau_1 \leqslant \tau_2.$$

That $\leqslant$ is uniquely determined is an immediate consequence of the uniqueness of the string, $(c_o,{}_1\hat{c}_T)$. Q.E.D.

C. Proof of Theorem 2

Let $\leqslant'$ and $\leqslant''$ denote, respectively, the orderings corresponding to the sets U', $\{V_1',\ldots,V_T'\}$ and U", $\{V_1'',\ldots,V_T''\}$. Clearly for $\preccurlyeq'$ and $\preccurlyeq''$ to be different they will have to disagree over some subset of $\mathcal{T}$. The thrust of the proof will be to show that this must, in fact, be the case.

Case 1. Suppose U' and U" are essentially different. Then there will exist some ${}_oc_T' \equiv \tau_1^*$ and ${}_oc_T'' \equiv \tau_2^*$ such that

$$U'({}_oc_T') \leq U'({}_oc_T'') \quad \underline{\text{and}} \quad U''({}_oc_T') > U''({}_oc_T'').$$

But given (A.6) and the fact that $\overset{C}{\preccurlyeq}$ is obtained by restricting $\preccurlyeq$ to $\mathcal{T}^*$, this implies that

$$\tau_1^* \preccurlyeq' \tau_2^* \quad \underline{\text{and}} \quad \tau_1^* \succ'' \tau_2^*.$$

Hence $\preccurlyeq'$ and $\preccurlyeq''$ are different -- this is true without regard to $\{V_1',\ldots,V_T'\}$ and $\{V_1'',\ldots,V_T''\}$.

Case 2. Suppose U' and U" are essentially the same, but V_t' and V_t'' differ for at least some $t \in \{1,\ldots,T\}$. That is, $\exists\ F_t, G_t \in \underline{X}_t$ such that

$$F_t \overset{t}{\preccurlyeq'} G_t \quad \underline{\text{and}} \quad F_t \overset{t}{\succ''} G_t$$

where $\overset{t}{\preccurlyeq'}$ and $\overset{t}{\preccurlyeq''}$ are the preorderings corresponding respectively, under (A.4), to V_t' and V_t''. But as noted in the Remark (1) following (A.3), F_t and G_t correspond respectively to the one-period lotteries $(P_t, \tilde{c}_t')$ and $(Q_t, \tilde{c}_t'')$. Thus, if we consider the following specific pair of trees in $\mathcal{T}$

$$\tau_1 \equiv ({}_oc_{t-1}; P_t, (\tilde{c}_t', {}_{t+1}c_T))$$

$$\tau_2 \equiv ({}_oc_{t-1}; Q_t, (\tilde{c}_t'', {}_{t+1}c_T)),$$

we have (via risk preference independence) that

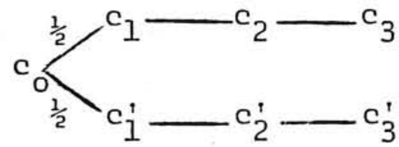
$$\tau_1 \preccurlyeq' \tau_2 \quad \underline{\text{and}} \quad \tau_1 \succ'' \tau_2$$

where $\preccurlyeq'$ and $\preccurlyeq''$ are, respectively, the preorderings on $\mathcal{T}(I)$ corresponding to U', $\{V_1',\ldots,V_t',\ldots,V_T'\}$ and U", $\{V_1'',\ldots,V_t'',\ldots,V_T''\}$. Hence $\preccurlyeq'$ and $\preccurlyeq''$ are different --this being true without regard to U', $\{V_1',\ldots,V_{t-1}',V_{t+1}',\ldots,V_T'\}$ and U", $\{V_1'',\ldots,V_{t-1}'',V_{t+1}'',\ldots,V_T''\}$. Q.E.D.

D. Proof of Theorem 3

To verify the first claim, let us begin by showing that if $\preccurlyeq$ is "NM representable" and exhibits risk preference independence then the multiperiod NM index W will be given by eqn. (3.2). (As noted in the text, Assumption 5 will hold automatically for multiperiod NM preferences.)

To establish that there exists a continuous additively separable NM index defined on $C_1 \times \ldots\ldots \times C_T$, we have only to show that our assumptions imply the hypothesis of Theorem 11.1 in Fishburn ([6], p.149). We shall do this in terms of the following generic example



for some fixed c_o. The key hypothesis in Fishburn's theorem is essentially that the pair (c_t, c_t') for any t can be "flipped" and the new tree (or joint distribution) will be indifferent to the original one. Now clearly, $\forall\ c_2^*, c_3^*$

(d.1)

and then by r.p.i. (condition i)

(d.2)

Next, we want to show that (c_2, c_2') [or (c_3, c_3')] can also be "flipped". Using r.p.i. (condition ii), (A.4) and (A.5) yields

(d.3)

Then proceeding as in (d.1) and (d.2) yields

 (d.4)

But clearly (as in (d.3))

and using (d.3) and (d.4), we have that

This implies (via Fishburn's result) that there exists a continuous additively separable NM index defined on $C_1 \times \ldots\ldots \times C_T$. Finally, the r.p.i. property also implies that the ordering over lotteries involving consumption for periods one through T does not depend on the level of c_o and hence the additively separable NM index on $C_1 \times \ldots \times C_T$ can depend on c_o only up to a positive affine transform. Thus, if an NM utility defined on $C_o \times C_1 \times \ldots \times C_T$ exists and exhibits r.p.i., it will take the form (3.2).

We next construct an OCE representation by defining $U({}_oc_T) = W({}_oc_T)$ and $V_t(c_t) = w_t(c_t)$, for $t = 1,\ldots,T$. Then showing that for any $\tau \in \mathcal{J}(I)$, $U(c_o, {}_1\hat{c}_T) = EW(c_o,\tilde{c}_1,\ldots,\tilde{c}_T)$ will clearly verify our first claim.

Let $P_t(\theta;\theta^{(1,t-1)})$ for all $t \in \{1,\ldots,T\}$ be defined as before. Denote the joint probability measure by $J_t(\theta^{(1,t)}) = P_t(\theta;\theta^{(1,t-1)})P_{t-1}(\theta;\theta^{(1,t-2)})\ldots \ldots P_1(\theta)$. Using the w_t's defined above and the observation following eqn. (a.6) (in Appendix A)

$$\hat{c}_T = w_T^{-1}\int w_T(c_T(\theta;\theta^{(1,T-1)}))dP_T(\theta;\theta^{(1,T-1)})$$

$$\hat{\hat{c}}_T = w_T^{-1} \iint w_T(c_T(\theta^{(T-1)}, \theta^{(T)}; \theta^{(1,T-2)})) dP_T(\theta; \theta^{(1,T-1)}) dP_{T-1}(\theta; \theta^{(1,T-2)})$$

or, more generally for any time-period t, we obtain

$$\hat{\hat{c}}_t = w_t^{-1} \int w_t(c_t(\theta^{(1)}, \ldots, \theta^{(t)})) dJ_t(\theta^{(1)}, \ldots, \theta^{(t)}),$$

where J_t is the joint measure. Furthermore, since under risk preference independence $w_t(c_t)$ does not depend on $\theta^{(t+1)}, \ldots, \theta^{(T)}$, we can integrate it as a constant with respect to the measures $P_T(\theta; \theta^{(1,T-1)}), \ldots\ldots, P_{t+1}(\theta; \theta^{(1,t)})$ and obtain

$$\hat{\hat{c}}_t = w_t^{-1} \int w_t(c_t(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(t)})) dJ_T(\theta^{(1,T)}).$$

Following Corollary 1, if $\preccurlyeq$ is OCE representable then

$$\tau_1 \preccurlyeq \tau_2 \Leftrightarrow U((c_o, {}_1\hat{c}_T)_1) \leq U((c_o, {}_1\hat{c}_T)_2),$$

where, as in the text, we follow the notational simplification of using ${}_1\hat{c}_T$ for $(\hat{c}_1, \hat{\hat{c}}_2, \ldots, \hat{\hat{c}}_T)$. In the present case we have

$$\begin{aligned} U(c_o, {}_1\hat{c}_T) &= U(c_o, \hat{c}_1, \ldots, \hat{\hat{c}}_T) \\ &= \alpha(c_o) + \beta(c_o) \sum_{t=1}^{T} k_t w_t(\hat{c}_t) \\ &= \alpha(c_o) + \beta(c_o) \sum_{t=1}^{T} k_t w_t [w_t^{-1} \int w_t\ (c_t(\theta^{(1,t)})) dJ_T(\theta^{(1,T)})] \\ &= \int [\alpha(c_o) + \beta(c_o) \sum_{t=1}^{T} k_t w_t (c_t(\theta^{(1,t)}))] dJ_T(\theta^{(1,T)}) \\ &= EW(c_o, {}_1\tilde{c}_T). \end{aligned}$$

To see that the converse is false, consider the case in which $U(c_o, \ldots, c_T) = \sum_{t=1}^{T} -c_t^{-\delta}/\delta$ $(-1 < \delta < \infty)$ and $V_t(c_t) = -c_t^{-\gamma_t}/\gamma_t$ $(-1 < \gamma_t < \infty)$ with $\gamma_t \neq \delta$ --Cf., Theorem 2 in Selden [30]. Q.E.D.

E. Proof of Proposition 1

Each feasible allocation $\{A_o, A_1(\theta_1), \ldots, A_1(\theta_N)\}$ results in a consumption tree τ in which

$$c_o = (1-A_o) y_o$$

$$c_1(\theta_i) = A_o y_o X_1(\theta_i)[1-A_1(\theta_i)] \qquad i=1,\dots,N$$

$$c_2(\theta_j;\theta_i^{(1)}) = A_o y_o X_1(\theta_i^{(1)}) A_1(\theta_i^{(1)}) X_2(\theta_j;\theta_i^{(1)}) \qquad \begin{matrix} j=1,\dots,N \\ i=1,\dots,N \end{matrix}$$

Following (L.1) and eqn. (3.1), we have that

$$\hat{c}_1 = V_1^{-1}\int V_1(c_1(\theta))\,dP_1(\theta) = V_1^{-1}\sum_{i=1}^{N} V_1(c_1(\theta_i))P_1(\theta_i)$$

and

$$\hat{c}_2 = V_2^{-1}\int V_2(c_2(\theta;\theta^{(1)}))\,dP_2(\theta;\theta^{(1)})\,dP_1(\theta)$$

$$= V_2^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N} V_2(c_2(\theta_j;\theta_i^{(1)}))P_2(\theta_j;\theta_i^{(1)})\;P_1(\theta_i^{(1)})$$

where $c_1(\theta)$ and $c_2(\theta;\theta^{(1)})$ are defined above and Stieltjes integrals are employed to accomodate the discrete probability measures P_2 and P_1 produced by the finite state assumption. Given that $\preccurlyeq$ on $\mathcal{T}(I)$ is OCE representable, the utility of any tree τ corresponding to a feasible $\{A_o, A_1(\theta_1),\dots,A_1(\theta_N)\}$ can be written as

$$U((1-A_o)y_o, V_1^{-1}\sum_{i=1}^{N} V_1(A_o y_o X_1(\theta_i)[1-A_1(\theta_i)])P_1(\theta_i), V_2^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N} V_2(A_o y_o X_1(\theta_i^{(1)})$$

$$\cdot A_1(\theta_i^{(1)}) X_2(\theta_j;\theta_i^{(1)}))P_2(\theta_j;\theta_i^{(1)})\;P_1(\theta_i^{(1)})).$$

Following (A.4) and (A.6), $V_1, V_1^{-1}, V_2^{-1}, V_2$ and U are continuous. But then U will be continuous in $A_o, A_1(\theta_1),\dots,$ and $A_1(\theta_N)$ since the composition of continuous functions is continuous. Furthermore, $(A_o, A_1(\theta_1),\dots,A_1(\theta_N)) \in [0,1]^{N+1}$ by assumption. Finally since we have a continuous function on a compact subset of $\mathbb{R}^{N+1}$, the Weierstrass Theorem assures us that a maximum exists. Q.E.D.

F. Proof of Proposition 2

It follows by straightforward calculation from the expressions in Appendix E and the assumed forms of V_1 and V_2 (eqn. (4.5)) that

$$\hat{c}_1 = A_o y_o \Gamma$$

$$\hat{c}_2 = A_o y_o \Delta$$

where

$$\Gamma =_{\text{def}} \left[\sum_{i=1}^{N} ([1-A_1(\theta_i)]X_1(\theta_i))^{-\gamma_1} P_1(\theta_i) \right]^{-1/\gamma_1}$$

$$\Delta =_{\text{def}} \left[\sum_{i=1}^{N} \sum_{j=1}^{N} (X_1(\theta_i^{(1)})A_1(\theta_i^{(1)})X_2(\theta_j;\theta_i^{(1)}))^{-\gamma_2} P_2(\theta_j;\theta_i^{(1)})P_1(\theta_i^{(1)}) \right]^{-1/\gamma_2}$$

Defining

$$M =_{\text{def}} \alpha_1 \Gamma^{-\delta} + \alpha_2 \Delta^{-\delta} \qquad \delta \neq 0$$

$$L =_{\text{def}} \alpha_1 \ell n\Gamma + \alpha_2 \ell n\Delta \qquad \delta = 0$$

we have that for any feasible consumption tree τ

$$U(c_o,\hat{c}_1,\hat{c}_2) = -\frac{1}{\delta}\left[\alpha_o(1-A_o)^{-\delta}y_o^{-\delta} + A_o^{-\delta}y_o^{-\delta}M\right] \qquad \delta \neq 0$$

$$U(c_o,\hat{c}_1,\hat{c}_2) = \alpha_o \ell n(1-A_o)y_o + (\alpha_1+\alpha_2)\ell nA_o y_o + L \qquad \delta = 0$$

The consumer's maximization problem can then be expressed for the case $\delta \neq 0$ as

$$\max_{\substack{0 \le A_o \le 1 \\ 0 \le \{A_1(\theta_i)\} \le 1}} \left\{ -\frac{1}{\delta}\left[\alpha_o(1-A_o)^{-\delta}y_o^{-\delta} + A_o^{-\delta}y_o^{-\delta}M\right]\right\}$$

$$= y_o^{-\delta} \max_{0 \le A_o \le 1} \left\{ -\frac{\alpha_o}{\delta}(1-A_o)^{-\delta} - A_o^{-\delta} \min_{0 \le \{A_1(\theta_i)\} \le 1} \left(\frac{M}{\delta}\right)\right\} \qquad \text{(f.1)}$$

and for the case $\delta = 0$ as

$$\max_{\substack{0 \le A_o \le 1 \\ 0 \le \{A_1(\theta_i)\} \le 1}} \left\{(\alpha_o+\alpha_1+\alpha_2)\ell ny_o + \alpha_o \ell n(1-A_o) + (\alpha_1+\alpha_2)\ell nA_o + L\right\} \qquad \text{(f.2)}$$

$$= \ell ny_o + \max_{0 \le A_o \le 1}\left\{\alpha_o \ell n(1-A_o) + (\alpha_1+\alpha_2)\ell nA_o + \max_{0 \le \{A_1(\theta_i)\} \le 1} L\right\}.$$

Clearly, in both instances the optimal allocation $\{A_o^o, A_1(\theta_1)^o, \ldots, A_1(\theta_N)^o\}$ does not depend on y_o. Also claim (i) is satisfied, since in the case of the problem (f.1) one can first $\min_{0 \le \{A_1(\theta_i)\} \le 1}\left(\frac{M}{\delta}\right)$ and <u>then</u> $\max_{0 \le A_o \le 1}\left\{-\frac{\alpha_o}{\delta}(1-A_o)^{-\delta} - A_o^{-\delta}\left(\frac{M^o}{\delta}\right)\right\}$ where M^o is optimal (as in Proposition 1, the existence of M^o follows from the Weierstrass Theorem). A similar argument applies for the log additive problem (f.2).

To prove condition (iii) of the Proposition, we start by assuming that M^o

has been optimally valued by the appropriate choice of the vector $\{A_1(\theta_1)^o, \ldots, A_1(\theta_N)^o\}$. Now it clearly follows from our assumptions that $0 < M^o < \infty$ and $0 < \alpha_o, \alpha_1, \alpha_2$. We have two cases.

<u>Case (1): $\delta = 0$</u>. Here it follows from (f.2) that we need to

$$\max_{0 \leq A_o \leq 1} \left\{ \alpha_o \ell n(1-A_o) + (\alpha_1 + \alpha_2)\ell n A_o \right\}.$$

Differentiating and solving for A_o yields

$$A_o^o = \frac{\alpha_1 + \alpha_2}{\alpha_o + \alpha_1 + \alpha_2} .$$

Thus in this case, A_o^o is interior and unique.

<u>Case (2): $-1 < \delta < 0$ and $\delta > 0$</u>. Here we need to solve

$$\max_{0 \leq A_o \leq 1} \left\{ -\frac{\alpha_o}{\delta} (1-A_o)^{-\delta} - A_o^{-\delta} \frac{M^o}{\delta} \right\}.$$

Calling the expression to be maximized $f(A_o)$, we have

$$f'(A_o) = -\alpha_o(1-A_o)^{-\delta-1} + M^o A_o^{-\delta-1}$$

and thus

$$f'(\epsilon) = -\alpha_o(1-\epsilon)^{-(\delta+1)} + M^o(1/\epsilon)^{\delta+1}$$

$$f'(1-\epsilon) = -\alpha_o(1/\epsilon)^{\delta+1} + M^o(1-\epsilon)^{-(\delta+1)} .$$

But the latter imply, since $\delta+1 > 0$, that for $\epsilon > 0$ but <u>small enough</u>, $f'(\epsilon) > 0$ while $f'(1-\epsilon) < 0$. Hence the max $f(A_o)$ <u>must</u> occur in the interior, i.e., $0 < A^o < 1$. Solving the above expression $f'(A^o) = 0$ yields the following unique solution

$$A_o^o = \left[(\alpha_o/M^o)^{1/(\delta+1)} + 1 \right]^{-1} . \tag{f.3}$$

Q.E.D.

G. <u>Proof of Result A</u>

Following our development in Appendices E and F and using eqn. (4.9) in the text, we have that the "early resolution" certainty equivalents, $\hat{c}_1^e$ and $\hat{c}_2^e$, are given by

$$\hat{c}_1^e = V_1^{-1} \sum_{i=1}^{N} \sum_{j=1}^{N} V_1(c_1(\theta_i^{(1)}, \theta_j^{(2)}))P_2(\theta_j^{(2)};\theta_i^{(1)})P_1(\theta_i^{(1)}) = A_o y_o \Gamma^e$$

$$\hat{c}_2^e = V_2^{-1} \sum_{i=1}^{N} \sum_{j=1}^{N} V_2(c_2(\theta_i^{(1)}, \theta_j^{(2)}))P_2(\theta_j^{(2)};\theta_i^{(1)})P_1(\theta_i^{(1)}) = A_o y_o \Delta^e$$

where

$$\Gamma^e =_{\text{def}} \left[\sum_{i=1}^{N}\sum_{j=1}^{N} \left([1-A_1(\theta_i^{(1)},\theta_j^{(2)})]X_1(\theta_i^{(1)})\right)^{-\gamma_1} P_2(\theta_j^{(2)};\theta_i^{(1)})P_1(\theta_i^{(1)})\right]^{-1/\gamma_1} \quad \text{(g.1)}$$

$$\Delta^e =_{\text{def}} \left[\sum_{i=1}^{N}\sum_{j=1}^{N} (A_1(\theta_i^{(1)},\theta_j^{(2)})X_1(\theta_i^{(1)})X_2(\theta_j^{(2)};\theta_i^{(1)}))^{-\gamma_2} P_2(\theta_j^{(2)};\theta_i^{(1)}) \cdot P_1(\theta_i^{(1)})\right]^{-1/\gamma_2} \quad \text{(g.2)}$$

Next define

$$M^e =_{\text{def}} \alpha_1(\Gamma^e)^{-\delta} + \alpha_2(\Delta^e)^{-\delta} \qquad \delta \neq 0.$$

Given the same U, V_1 and V_2, the only difference between the early and non-early resolution cases is in the definition of Γ and Δ. Rewriting the non-early resolution expressions from Appendix F, we have

$$\Gamma = \left[\sum_{i=1}^{N}\sum_{j=1}^{N} ([1-A_1(\theta_i^{(1)})]X_1(\theta_i^{(1)}))^{-\gamma_1} P_2(\theta_j;\theta_i^{(1)})P_1(\theta_i^{(1)})\right]^{-1/\gamma_1} \quad \text{(g.3)}$$

(since $\sum_{j=1}^{N} P_2(\theta_j;\theta^{(1)}) = 1$)

$$\Delta = \left[\sum_{i=1}^{N}\sum_{j=1}^{N} (A_1(\theta_i^{(1)})X_1(\theta_i^{(1)})X_2(\theta_j;\theta_i^{(1)}))^{-\gamma_2} P_2(\theta_j;\theta_i^{(1)})\ P_1(\theta_i^{(1)})\right]^{-1/\gamma_2} \quad \text{(g.4)}$$

$$M = \alpha_1\Gamma^{-\delta} + \alpha_2\Delta^{-\delta} \qquad \delta \neq 0.$$

Restricting ourselves to the case $\delta > 0$ ($\eta < 1$), the early-resolution optimization problem becomes one of

$$\max_{0 \le A_o \le 1} \left\{-\frac{\alpha_o}{\delta}(1-A_o)^{-\delta} - A_o^{-\delta} \min_{0 \le A_1(\theta_i^{(1)},\theta_j^{(2)}) \le 1} (M^e/\delta)\right\}.$$

It is clear from eqns. (g.1)-(g.4) that the class of functions $A_1(\theta^{(1)},\theta^{(2)})$ over which M^e is optimized contains the class $A_1(\theta^{(1)})$ over which the non-early resolution M is optimized; furthermore, for any fixed $A_1(\theta^{(1)})$, the resulting M^e and M will be equal. From this we can immediately conclude that

$$\text{optimal } (M^e/\delta) \le \text{optimal } (M/\delta). \quad \text{(g.5)}$$

But we have from Appendix F that the OCE utility conditional on the optimal $A_1(\theta^{(1)})^o$ is given by

$$-\frac{\alpha_o}{\delta}\, y_o^{-\delta}[1-A_o]^{-\delta} \; -A_o^{-\delta} y_o^{-\delta} (M^o/\delta)$$

where

$$A_o^o = \left[(\alpha_o/M^o)^{\frac{1}{\delta+1}} + 1\right]^{-1}. \qquad (g.6)$$

Substituting and then rearranging yields

$$\frac{-y_o^{-\delta}}{\delta}\left\{\alpha_o\left[1+(M^o/\alpha_o)^{\frac{1}{\delta+1}}\right]^{\delta} + \left[\alpha_o^{\frac{1}{\delta+1}}(M^o)^{\frac{1}{\delta(\delta+1)}} + (M^o)^{\frac{1}{\delta}}\right]^{\delta}\right\}$$

which is clearly decreasing in M^o assuming $\delta > 0$. But then combining this result with (g.5), we have that for $\delta > 0$ ($\eta < 1$)

$$\tau^o \lessdot \tau^{oo}$$

where τ^{oo} and τ^o denote, respectively, the early and non-early resolution optimal consumption trees.

A similar argument applies for the $\delta = 0$ case. Q.E.D.

H. Proof of Result B

We shall require the following result.

Lemma:

If $\alpha \leq \beta < 0$ or $0 < \alpha \leq \beta < \infty$, $\int_\Theta dP(\theta) = 1$,

$f(\theta) \geq 0 \;\forall\; \theta \in \Theta$ and $\int_\Theta f(\theta)dP(\theta) \neq 0$, then

$$\left\{\int_\Theta [f(\theta)]^\alpha dP(\theta)\right\}^{1/\alpha} \leq \left\{\int_\Theta [f(\theta)]^\beta dP(\theta)\right\}^{1/\beta}.$$

Proof. See Rudin [27], problem 5, p. 70, for the case of $0 < \alpha \leq \beta < \infty$. Assume now that $\alpha \leq \beta < 0$. But this implies that $0 < -\beta \leq -\alpha$. Hence

$$\left\{\int_\Theta [g(\theta)]^{-\beta} dP(\theta)\right\}^{-1/\beta} \leq \left\{\int_\Theta [g(\theta)]^{-\alpha} dP(\theta)\right\}^{-1/\alpha}$$

where this is in the form of the case covered by Rudin; setting $g(\theta) = 1/f(\theta)$ yields

$$\left\{\int_{\Theta}[f(\theta)]^{\alpha}dP(\theta)\right\}^{1/\alpha} \le \left\{\int_{\Theta}[f(\theta)]^{\beta}dP(\theta)\right\}^{1/\beta}$$

where $\alpha \le \beta < 0$. ☒

<u>Case 1.</u> Let $\eta < 1$ $(\delta > 0)$ and suppose that ρ_1 increases to $\rho_1' > \rho_1$. By definition $\rho_1 = \gamma_1 + 1$, and so γ_1 increases to $\gamma_1' > \gamma_1$. By the above Lemma and the definition of Γ in Appendix F, we have that

$$\Gamma(A_1(\theta),\gamma_1') \le \Gamma(A_1(\theta),\gamma_1) \qquad \forall A_1(\theta).$$

Thus, from the definition of M $(\delta > 0)$

$$M(A_1(\theta),\gamma_1') \ge M(A_1(\theta),\gamma_1) \qquad \forall A_1(\theta). \qquad \text{(h.1)}$$

But if $A_1(\theta)^o$ denotes the <u>minimizing</u> $(\delta > 0)$ function,

$$M(A_1(\theta),\gamma_1) \ge M(A_1(\theta)^o,\gamma_1) \qquad \forall A_1(\theta)$$

and hence

$$M(A_1(\theta),\gamma_1') \ge M(A_1(\theta)^o,\gamma_1) \qquad \forall A_1(\theta).$$

But since this holds for any $A_1(\theta)$, $M(A_1(\theta)^+,\gamma_1') \ge M(A_1(\theta)^o,\gamma_1)$ where $A_1(\theta)^+$ denotes the optimizing function for $M(A_1(\theta),\gamma_1')$. However this implies, given eqn. (f.3) in Appendix F, that $A_o^+ \ge A_o^o$. That is, the optimal A_o corresponding to ρ' will exceed that corresponding to the lower initial level of risk aversion, ρ (see the qualifications in note (9)).

A similar argument applies when ρ_2 increases to $\rho_2' > \rho_2$; the only difference being that Δ is altered instead of Γ.

<u>Case 2.</u> Let $\eta > 1$ $(\delta < 0)$ and again suppose that ρ_1 increases to $\rho_1' > \rho_1$. Since $\delta < 0$, (h.1) becomes

$$M(A_1(\theta),\gamma_1') \le M(A_1(\theta),\gamma_1) \qquad \forall A_1(\theta).$$

Letting $A_1(\theta)^o$ and $A_1(\theta)^+$ denote, respectively, the <u>maximizing</u> $(\delta < 0)$ functions for $M(A_1(\theta),\gamma_1)$ and $M(A_1(\theta),\gamma_1')$, and arguing as in Case 1 we have $M(A_1(\theta)^o,\gamma_1) \ge M(A_1(\theta)^+,\gamma_1')$. But this implies via eqn. (f.3) that $A_o^o \ge A_o^+$. A similar argument holds for a small increase in ρ_2.

Case 3. Let $\eta = 1$ ($\delta = 0$) and again suppose that ρ_1 increases to $\rho_1' > \rho_1$. But this can have no effect on the optimal initial allocation since, as was shown in Appendix F, $A_o^o = \alpha_1 + \alpha_2$. Q.E.D.

I. Proof of Result C

We shall only consider the case of a mean preserving increase in period one capital risk--leaving the simpler proof for a period-two increase to the interested reader. Result C is implied by the following Lemma:

Lemma:

Let M be defined as in Appendix F and suppose that the assumptions of Result C hold. Then the effect of a mean preserving increase in period-one risk on the optimal $(-M^o/\delta)$, defined in terms of $A_1(\theta_1)^o, \ldots, A_1(\theta_N)^o$, can be expressed as

$$\left.\frac{\partial}{\partial\lambda_1}\left(\frac{-M^o}{\delta}\right)\right|_{d\epsilon_1/d\lambda_1 = -E(\tilde{x}_1)} \le 0.$$

Proof. First of all, performing the maximization (f.1) in Appendix F, noting the assumption that $\tilde{x}_2$ is independent of $\theta^{(1)}$, yields for the optimal $A_1(\theta_1)^o, \ldots, A_1(\theta_N)^o$

$$\alpha_1\left[\sum_{i=1}^{N}([1-A_1(\theta_i)](1+x_1(\theta_i)))^{-\gamma_1}P_1(\theta_i)\right]^{\frac{\delta}{\gamma_1}-1}\left[1-A_1(\theta_k)\right]^{-\gamma_1-1}$$
$$\cdot[1+x_1(\theta_k)]^{-\gamma_1} = \alpha_2\left[\sum_{i=1}^{N}\sum_{j=1}^{N}[(1+x_1(\theta_i)]A_1(\theta_i)[1+x_2(\theta_j)])^{-\gamma_2}\right.$$
$$\left.\cdot P_2(\theta_j)P_1(\theta_i)\right]^{\frac{\delta}{\gamma_2}-1}\sum_{j=1}^{N}A_1(\theta_k)^{-\gamma_2-1}[1+x_1(\theta_k)]^{-\gamma_2}[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j) \qquad \text{(i.1)}$$
$$k = 1, \ldots, N.$$

Define

$$K_1 \equiv \alpha_1\left[\sum_{i=1}^{N}([1-A_1(\theta_i)^o]\ (1+x_1(\theta_i))^{-\gamma_1}P_1(\theta_i)\right]^{\frac{\delta}{\gamma_1}-1}$$

$$K_2 \equiv \alpha_2\Big[\sum_{i=1}^{N}\sum_{j=1}^{N}([1+x_1(\theta_i)]A_1(\theta_i)[1+x_2(\theta_j)])^{-\gamma_2}P_2(\theta_j)P_1(\theta_i)\Big]^{\frac{\delta}{\gamma_2}-1}.$$

Next, write $(-M^o/\delta)$ in terms of $\lambda_1\tilde{x}_1 + \epsilon_1$ and then differentiate with respect to λ_1 (evaluating all derivatives at $\lambda_1 = 1$, $\epsilon_1 = 0$ and setting $d\epsilon_1/d\lambda_1 = -E(\tilde{x}_1)$)

$$\frac{\partial}{\partial\lambda_1}\left(\frac{-M^o}{\delta}\right)\Bigg|_{d\epsilon_1/d\lambda_1 = -E(\tilde{x}_1)}$$

$$= K_1\sum_{i=1}^{N}[1-A_1(\theta_i)]^{-\gamma_1-1}[1+x_1(\theta_i)]^{-\gamma_1}P_1(\theta_i)\frac{-\partial A_1(\theta_i)}{\partial\lambda_1}$$

$$+ K_1\sum_{i=1}^{N}[1-A_1(\theta_i)]^{-\gamma_1}[1+x_1(\theta_i)]^{-\gamma_1-1}P_1(\theta_i)[x_1(\theta_i)- E(\tilde{x}_1)]$$

$$+ K_2\sum_{i=1}^{N}\sum_{j=1}^{N}[A_1(\theta_i)]^{-\gamma_2-1}([1+x_1(\theta_i)][1+x_2(\theta_j)])^{-\gamma_2}P_2(\theta_j)P_1(\theta_i)\frac{\partial A_1(\theta_i)}{\partial\lambda_1}$$

$$+ K_2\sum_{i=1}^{N}\sum_{j=1}^{N}(A_1(\theta_i)[1+x_2(\theta_j)])^{-\gamma_2}[1+x_1(\theta_i)]^{-\gamma_2-1}P_2(\theta_j)P_1(\theta_i)[x_1(\theta_i)-E(\tilde{x}_1)]$$

(or using the first-order conditions (i.1) since M^o is optimal)

$$= K_2\sum_{i=1}^{N}\sum_{j=1}^{N}(A_1(\theta_i)[1+x_1(\theta_i)])^{-\gamma_2-1}[1+x_2(\theta_j)]^{-\gamma_2}[x_1(\theta_i)-E(\tilde{x}_1)]P_2(\theta_j)\ P_1(\theta_i).$$

Thus to verify the assertion of the Lemma we need to show that

$$\sum_{i=1}^{N}\sum_{j=1}^{N}(A_1(\theta_i)[1+x_1(\theta_i)])^{-\gamma_2-1}[1+x_2(\theta_j)]^{-\gamma_2}[x_1(\theta_i)-E(\tilde{x}_1)]P_2(\theta_j)P_1(\theta_i) \le 0 \qquad \text{(i.2)}$$

since $K_2 \ge 0$.

<u>Step 1</u>. In order to establish (i.2), we first shall require the result that

$$(A_1(\theta_i)[1+x_1(\theta_i)])^{-\gamma_2-1}\sum_{j=1}^{N}[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j) \qquad \text{(i.3)}$$

is a <u>decreasing</u> function of $x_1(\theta_i)$ (where the $A_1(\theta_i)$ were obtained using the first-order conditions). This will be accomplished by proving that the more general expression

$$S^{-\gamma_2-1}(1+Z)^{-\gamma_2-1}\sum_{j=1}^{N}[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j) \quad (i.4)$$

is a <u>decreasing function of Z</u> where Z lies in the closed interval $[\min_i x_1(\theta_i), \max_i x_1(\theta_i)]$ and S is implicitly defined as a function of Z by

$$K_1(1+Z)^{-\gamma_1}(1-S)^{-\gamma_1-1}-K_2(1+Z)^{-\gamma_2}S^{-\gamma_2-1}\sum_{j=1}^{N}[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j)=0 \quad (i.5)$$

Note that K_1 and K_2 are defined as above and eqns. (i.4) and (i.5) are, respectively, functional extensions of the N-state expression (i.3) and the first-order conditions (i.1) (i.e., eqns. (i.4) and (i.5) include (i.3) and (i.1) as special cases when $Z = x_1(\theta_i)$ and $S = A_1(\theta_i)$). Further observe with respect to (i.5) that as $S \to 0$ one term in the expression goes to zero while the other is strictly positive and vice versa as $S \to 1$. This implies the existence of an $S \in (0,1)$ which satisfies (i.5).

First, observe that the function defined by (i.5) is continuously differentiable in Z and S. Also the partial derivative with respect to S is

$$(\gamma_1+1)K_1(1+Z)^{-\gamma_1}(1-S)^{-\gamma_1-2}+(\gamma_2+1)K_2(1+Z)^{-\gamma_2}S^{-\gamma_2-2}\sum_{j=1}^{N}[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j)\neq 0.$$

It then follows from the implicit function theorem that S is a continuously differentiable function of Z, denoted S(Z). Next, to show that (i.4) is decreasing in Z, we need to establish that

$$-(\gamma_2+1)\sum_{j=1}^{N}[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j)(1+Z)^{-\gamma_2-2}S^{-\gamma_2-2}[S+(1+Z)\frac{dS}{dZ}]\leq 0.$$

But since $S > 0$ and $\gamma_2+1 = \rho_2 > 0$, we only need to show that $[S+(1+Z)\frac{dS}{dZ}] \geq 0$. Implicitly differentiating (i.5) with respect to Z yields

$$(1+Z)\frac{dS}{dZ} = \Big[\gamma_1K_1(1-S)^{-\gamma_1-1}(1+Z)^{-\gamma_1}-\gamma_2K_2S^{-\gamma_2-1}(1+Z)^{-\gamma_2}\sum_j[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j)\Big]$$

$$\left[(\gamma_1+1)K_1(1-S)^{-\gamma_1-2}(1+Z)^{-\gamma_1}+(\gamma_2+1)K_2S^{-\gamma_2-2}(1+Z)^{-\gamma_2}\sum_j[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j)\right]^{-1}$$

(or substituting from (i.5) into the numerator)

$$=\frac{(\gamma_1-\gamma_2)K_2S^{-\gamma_2-1}(1+Z)^{-\gamma_2}\sum_j[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j)}{(\gamma_1+1)K_1(1-S)^{-\gamma_1-2}(1+Z)^{-\gamma_1}+(\gamma_2+1)K_2S^{-\gamma_2-2}(1+Z)^{-\gamma_2}\sum_j[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j)}.$$

If $\gamma_1 = \gamma_2$, then $(1+Z)\frac{dS}{dZ} = 0$ and $[S + (1+Z)\frac{dS}{dZ}] > 0$. Assuming $\gamma_1 \neq \gamma_2$, rearranging the above expression yields

$$(1+Z)\frac{dS}{dZ} = \frac{1}{\frac{(\gamma_2+1)}{(\gamma_1-\gamma_2)}\frac{1}{S}+\frac{(\gamma_1+1)}{(\gamma_1-\gamma_2)}\frac{K_1}{K_2}\frac{(1-S)^{-\gamma_1-2}}{S^{-\gamma_2-1}}\frac{(1+Z)^{-\gamma_1+\gamma_2}}{\sum_j[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j)}} \qquad \text{(i.6)}$$

If $\gamma_1 > \gamma_2$, then $(1+Z)\frac{dS}{dZ} > 0$ and $[S+(1+Z)\frac{dS}{dZ}] > 0$. On the other hand if $\gamma_1 < \gamma_2$, since $\rho_t > 0$, we have $-1 < \gamma_1 < \gamma_2$ and thus (i.6) becomes

$$(1+Z)\frac{dS}{dZ} = \left[\frac{-1}{\frac{(\gamma_2+1)}{(\gamma_2-\gamma_1)}+\frac{(\gamma_1+1)}{(\gamma_2-\gamma_1)}\frac{K_1}{K_2}\frac{(1-S)^{-\gamma_1-2}}{S^{-\gamma_2-2}}\frac{(1+Z)^{-\gamma_1+\gamma_2}}{\sum_j[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j)}}\right]S \quad .$$

But now $\frac{\gamma_2+1}{\gamma_2-\gamma_1} > 1$, which implies that the denominator of the bracket > 1 and thus $|(1+Z)\frac{dS}{dZ}| < S$ or $S + (1+Z)\frac{dS}{dZ} > 0$. Eqn. (i.4) is thus a decreasing function of Z; therefore the special case (i.3), where $Z = x_1(\theta_i)$ and $S = A_1(\theta_i)$, must also be decreasing in $x_1(\theta_i)$. This completes step 1.

<u>Step 2</u>. We are now ready to prove the inequality (i.2). Define $\bar{A}_1 = S(\bar{x}_1)$, i.e., $\bar{A}_1$ is the value of S when Z is set equal to $\bar{x}_1 \equiv E(\tilde{x}_1)$. It follows from (i.4) being decreasing in Z [or (i.3) being decreasing in x_1] that

$$Z <(>) \bar{x}_1 \Rightarrow [\bar{A}_1(1+\bar{x}_1)]^{-\gamma_2-1}\sum_{j=1}^{N}[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j) <(>) [S(1+Z)]^{-\gamma_2-1}$$
$$\cdot\sum_{j=1}^{N}[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j).$$

But then

$$[A_1(\theta_i)(1+x_1(\theta_i))]^{-\gamma_2-1}[x_1(\theta_i)-\bar{x}_1] \sum_{j=1}^{N}[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j) \leq$$

$$[\bar{A}_1(1+\bar{x}_1)]^{-\gamma_2-1}[x_1(\theta_i)-\bar{x}_1] \sum_{j=1}^{N}[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j)$$

$$i = 1,\ldots,N$$

and hence

$$\sum_{i=1}^{N}[A_1(\theta_i)(1+x_1(\theta_i))]^{-\gamma_2-1}[x_1(\theta_i)-\bar{x}_1]P_1(\theta_i)\sum_{j=1}^{N}[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j) \leq$$

$$[\bar{A}_1(1+\bar{x}_1)]^{-\gamma_2-1}\sum_{i=1}^{N}[x_1(\theta_i)-\bar{x}_1]P_1(\theta_i)\sum_{j=1}^{N}[1+x_2(\theta_j)]^{-\gamma_2}P_2(\theta_j) = 0.$$

$\boxtimes$

Now to prove Result C, note that

$$\left.\frac{\partial M^o}{\partial \lambda_1}\right|_{\frac{d\epsilon_1}{d\lambda_1} = -E(\tilde{x}_1)} = \left.(-\delta)\frac{\partial}{\partial \lambda_1}\left(\frac{-M^o}{\delta}\right)\right|_{\frac{d\epsilon_1}{d\lambda_1} = -E(\tilde{x}_1)}$$

and thus

$$\left.\frac{\partial M^o}{\partial \lambda_1}\right|_{\frac{d\epsilon_1}{d\lambda_1} = -E(\tilde{x}_1)} \{\geq, =, \leq\}\ 0$$

<u>as</u> $\delta\{>, =, <\}\ 0$, or

as $\eta\{<, =, >\}\ 1$.

Finally, since A^o is increasing in M^o (Cf., eqn. (f.3)) in Appendix F), the assertions in Result C follow immediately. Q.E.D.

NOTES

1. One exception is Pye's paper [23] in which the NM representation is assumed to be affinely multiplicative.

2. Note that under risk preference independence, the set of V_t's are well-defined in that each is independent, up to a positive affine transform, of consumption payoffs occuring before period t and of lotteries after t.

3. Here, after the first integration, we view the parameter $\theta^{(t-1)}$ as a new variable and integrate with respect to it. We continue in this fashion for $\theta^{(t-2)},\ldots,\theta^{(1)}$. Thus, in the T = 2 case, we first integrate with respect to $\theta^{(2)}$, yielding $f(\theta^1) = \int V_2(c_2(\theta;\theta^{(1)}))dP_2(\theta;\theta^{(1)})$, and then with respect to $\theta^{(1)}$, i.e., $\int f(\theta)dP_1(\theta)$.

4. When T = 1, this condition reduces to the existence and finiteness of period-one expected utility,

$$\int V_1(c_1(\theta))dP_1(\theta) = \int V_1(c_1)dF(c_1) < \infty.$$

If the random variable $\tilde{c}_1$ has a finite mean, then a sufficient condition for this to hold is that $\exists$ some constant K such that $|V_1(c_1)| \leq Kc_1$, $\forall c_1$. Thus essentially, Property I will be satisfied if $c_t(\theta;\theta^{(1,t-1)})$, viewed as a function of each its parameters sequentially, is measurable and has a finite mean with respect to each of the corresponding measures $P_1,\ldots,P_t$.

5. In order to simplify notation, the same symbol $\hat{c}_t$ is used in eqns. (3.1) and (2.1) even though (within the context of a consumption tree) it has two different meanings. That is, in eqn. (2.1) $\hat{c}_t$ denotes the certainty equivalent of a one-period lottery $(P_t,\tilde{c}_t)$ resolved at time t given that $\theta^{(1)},\ldots,\theta^{(t-1)}$ has occurred, while in (3.1) it refers to the certainty equivalent with respect to the joint measure J_t (or J_T).

6. We wish to thank Andy Postlewaite for raising the question of a unique decomposition.

7. We are indebted to Arjun Ray for pointing out, in an earlier version of this paper, an error in our analysis of the optimal allocation.

8. In [33], we address this same question where the preordering $\preccurlyeq$ is permitted to exhibit temporal resolution dependence. However, this increase in generality greatly complicates the analysis of optimal behavior in allocation problems.

9. Two technical points should be noted. First to simplify the proof, we assume that in the process of undergoing a small increase, ρ_t never equals unity (the log case). Secondly, some clarification of the phrase "in general" is required. As can be seen from the proof, if $\eta < (>) 1$ the change in s_o^o will be, strictly speaking, $\geq (\leq) 0$. But since the set of cases in which actual equality holds is both quite special and uninteresting, we ignore them.

10. More general definitions of increased risk can be found in Rothschild and Stiglitz [25] and Diamond and Stiglitz [4].

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