Time Consistency, Temporal Resolution Indifference and the Separation of Time and Risk

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Abstract

No existing dynamic preference model can simultaneously satisfy time consistency, temporal resolution of risk indifference and the separation of time and risk preferences. In the context of the consumption-portfolio optimization problem, we derive necessary and sufficient conditions such that all three of these properties are satisfied by the dynamic ordinal certainty equivalent (DOCE) preference structure axiomatized in Selden and Stux (1978). These conditions ensure that DOCE resolute, naive and sophisticated consumption and asset demands are (i) identical and (ii) the same as the demands generated by Kreps and Porteus (1978) (KP) preferences. When the conditions are violated, the elasticity of intertemporal substitution can play a key role in determining whether the differences between resolute, naive and sophisticated demands are material and the axiomatic differences between the DOCE and KP preference models are important.

KEYWORDS. Kreps-Porteus-Selden preferences, time consistency, separation of time and risk, temporal resolution indifference, consumption-portfolio problem

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1 Introduction

For economic models where consumers solve dynamic optimization problems under risk, assumptions on preferences play a key role in the resulting solutions and their comparative statics. At the level of preferences, the following three properties are often mentioned as being desirable: (i) time consistency (TC), (ii) the ability to separate time and risk preferences (SEP) and (iii) the ability to accommodate temporal resolution of risk indifference (TRI). Currently, no single dynamic preference model can simultaneously accommodate all three properties.

This paper makes two primary contributions. First in the context of a dynamic consumption-portfolio problem, it provides conditions under which the three properties can be satisfied. We show that there exists a preference model for which TC, SEP and TRI can hold on a meaningful subset of the choice space rather than on the full space as is typically considered and we give conditions ensuring that optimal demands always lie in this set. Second, when the conditions do not hold, this can have a surprisingly minimal effect on optimal demands when the consumer's time preferences exhibit sufficient aversion to intertemporal substitution.

Although KP (Kreps and Porteus 1978) motivate the introduction of their recursive preference structure on the basis of being able to accommodate a preference for early or late resolution of uncertainty, Epstein, Farhi and Strzalecki (2014) among others argue that while early resolution can be beneficial in decision making, it may not be desirable to require this property at the pure preference level. In fact in the EZ (Epstein and Zin 1989) homothetic version of KP utility, two parameters govern the seemingly distinct time preferences, risk preferences and a preference for the resolution of uncertainty. However, when setting the parameters at different values to achieve a preference for early or late resolution of risk, an analyst loses her ability to satisfy SEP. This limitation has been recognized from the start in EZ and in part motivates us to explore the use the DOCE (dynamic ordinal certainty equivalent) preference model axiomatized by Selden and Stux (1978). As a natural generalization of Selden (1978), DOCE preferences are based on independent risk and time preference building blocks which, respectively, are used to replace risky consumption in each period by certainty equivalent consumption and evaluate the resulting vector of certain and certainty equivalent consumption. Thus, DOCE preferences exhibit SEP. By assumption, they also exhibit TRI. In contrast to KP preferences, this attitude toward the resolution of risk is independent of the form of time and risk preferences as well as their interrelationship. However as suggested by Johnsen and Donaldson (1985), DOCE preferences in general violate time consistency. It should be noted that although the KP and DOCE utilities in general differ, they become ordinally equivalent in a two period setting where the first period is certain. The common representation is typically referred to as the KPS (Kreps-Porteus-Selden) utility.

Given that in general, neither KP nor DOCE preferences can simultaneously satisfy TC, SEP and TRI, are there any special circumstances under which either model can satisfy the three properties? We show that in a consumption-portfolio setting if a consumer has DOCE preferences corresponding to her underlying time and risk preference building blocks being homothetic and the distribution of asset returns is independent over time, consumption and asset demands will be time consistent. In this case, consumption along branches exhibits a special proportionality and the DOCE model satisfies TC as well as SEP and TRI. While the restriction that asset returns be independent over time is clearly a special case, the stronger assumption that asset returns are i.i.d. (identically and independently distributed) has been made in a number of important papers such as Levhari and Srinivasan (1969), Samuelson (1969), Weil (1993), Campbell and Cochrane (1999) and Barro (2009). Also, the assumption that the representations of time and risk preferences are homothetic has been widely used as, for instance, in the EZ special case of KP preferences. The intuition for our result is that the combination of independent returns and homotheticity permits the transformation of the choice over a multi-date event branch consumption tree into the choice over an equivalent single branch tree analogous to what the consumer confronts in a pure certainty time consistent setting. Moreover, at any given node of the dynamic consumption tree, since (i) each branch is characterized by the same asset return distribution and (ii) the assumed CRRA (constant relative risk aversion) risk preferences imply one fund portfolio separation, the optimal asset mix will be the same for each branch. Thus, the consumer's portfolio composition will be the same irrespective of the state outcome at the node and she will have no reason to revise her plans as risk is resolved.

Given the assumption of homothetic time and risk preferences, although independent asset returns is sufficient it is not necessary for DOCE preferences to satisfy TC. In general, optimal asset demands in any period prior to the last period will depend on both time and risk preferences. However given homothetic time and risk preferences, it is possible for asset returns to imply that (i) the optimal portfolio composition only depends on risk preferences and (ii) the certainty equivalent asset return for the optimal portfolio is the same on each branch. If asset returns satisfy this independent certainty equivalent portfolio return property [ICER], then asset demands are independent of time preferences and the consumer's consumption and asset demands satisfy TC. In this case, although certainty equivalent returns are constant across branches they can vary across time periods.

It would clearly be desirable to weaken the restriction that time and risk preferences must be homothetic. In fact, it is possible to extend our result to the class of HARA (hyperbolic absolute risk aversion)[1] time and risk preferences in a consumption-portfolio setting if one adds to [ICER] the assumption that one of the available assets is risk free. The quasihomothetic members of the HARA class other than the CRRA case can be viewed as being homothetic to translated origins (see Pollak 1971).[2] The risk free asset assumption in our result is crucial in dealing with the translations. A particularly striking result is obtained for the CARA (constant absolute risk aversion) member of the HARA class. In this case, the corresponding DOCE consumption and asset demands satisfy TC when the risk free rate is non-stochastic. No restriction need be made on risky asset returns.[3] If the HARA conditions are satisfied, DOCE preferences will exhibit TC, SEP and TRI on a restricted domain corresponding to the specific choice problem. Moreover as discussed in Section 3 below, satisfaction of the DOCE conditions for TC greatly simplifies the complex T-period dynamic consumption-portfolio problem.[4] One can transform the problem into $T-1$ single period portfolio optimizations and then solve an elementary time consistent consumption-saving problem based on certainty time preferences and a budget constraint involving certainty equivalent portfolio rates of return.

It should be emphasized that in choice problems such as we are considering, TC does not just depend on preferences but it also depends on prices or asset returns. It is standard in discussions of time consistent preference models to impose restrictions solely on preferences and assume that the conditions hold for all prices (e.g., Johnsen and Donaldson 1985). Our key conditions of independent asset returns or [ICER] can be viewed as effectively restrictions on contingent claim prices and probabilities.

Given these results, it is natural to wonder how the time consistent DOCE and KP demands relate to one another assuming the corresponding dynamic prefer-

[1]See Gollier (2001) for a characterization of HARA preferences and their properties.

[2]These include the translated origin CRRA utility used for instance by Campbell and Cochrane (1999) and the constant absolute risk aversion form.

[3]The dual assumptions of negative exponential EU preferences and non-stochastic interest rates are sometimes made in models of asset pricing under asymmetric information and in microstructure analyses (e.g., Wang 1993 and 1994).

[4]The first part of the discussion illustrates how the key assumptions that (i) risk preferences in each time period are represented by an EU representation with the same NM index and (ii) TRI work together to facilitate the very simple computation of certainty equivalent consumption in each period.

ences are based on the same time and risk preference building blocks and asset returns satisfy [ICER]. Although the utilities are not ordinally equivalent, the DOCE and KP demand functions are identical. This result can be understood, once it is realized that on the restricted set of consumption trees corresponding to the consumption-portfolio problem, HARA preferences, the existence of a risk free asset and asset returns satisfying [ICER], the DOCE and KP utilities converge.[5] As a result, under the assumptions outlined above a number of key consumption, saving and asset demand properties present in two period KPS applications[6] extend to the dynamic setting. For instance for the homothetic case, the ratio of risk free to risky asset demands depends only on risk preferences and not on time preferences.

Given that empirical evidence suggests that asset returns can deviate from being independent over time and likely violate [ICER] as well, DOCE preferences become time inconsistent. It then becomes necessary to consider the standard (Strotz-Pollak) resolute, naive and sophisticated solution techniques for the consumption-portfolio problem. In order to analyze this case, we assume a three period setting where DOCE and KP preferences share the same CES (constant elasticity of substitution) time and CRRA risk preference building blocks. We focus on differences in demand for the resolute, naive and sophisticated DOCE and KP cases, often based on numerical simulations. Our analysis suggests that two quite different sets of conclusions can be obtained depending on the value of the EIS (elasticity of intertemporal substitution). First, when the EIS is in the range of roughly 0.20 to 0.40, as estimated in a number of certainty empirical studies, we find that DOCE resolute, naive and sophisticated and KP period 1 consumption and asset demands exhibit the same qualitative properties and can be surprisingly close in absolute value. This suggests that axiomatic differences in the two preference models may not be critical. Second, if the EIS is considerably larger in the range of 1.5 to 2.0, as suggested by calibrations of some finance and macro models, then the DOCE resolute and sophisticated demands can differ quite substantially.[7] The KP and sophisticated DOCE demands also diverge. These differences can be explained by the impact of a strong preference for intertemporal substitution paralleling behavior in the simple two period certainty consumption-saving problem. Our results suggest the critical importance of developing empirical evidence (perhaps via experimental laboratory tests) on

[5] As discussed in Subsection 4.3, on this restricted domain early resolution consumption trees do not exist and the issue of potential violations of TRI does not arise for KP preferences.

[6] See, for example, Selden (1979), Barsky (1989) and Kimball and Weil (2009).

[7] For a review of the literature on the size of the EIS, see for instance Attanasio and Weber (2010), Havranek (2015) and Thimme (2017) and the references cited in these papers.

whether in a risky consumption-portfolio problem, consumer behavior can best explained by EIS values less than 0.40 or greater than 1.5.

The rest of the paper is organized as follows. In the next section, we introduce notation, definitions and the optimization problem. Section 3 illustrates (i) the intuitive appeal of DOCE utility and (ii) the significant simplification of complex dynamic consumption-portfolio problems when our conditions for time consistent DOCE demands hold. In Section 4, we provide our main theorems for DOCE preferences to satisfy TC and provide results relating DOCE and KP demands. Section 5 provides comparisons of consumption and asset demands for the DOCE resolute, naive, sophisticated and KP cases when asset returns deviate from satisfying [ICER]. Section 6 contains concluding comments. Proofs are given in Appendix A and supporting materials are provided in Supplemental Appendix B.

2 Preliminaries

2.1 Notation and Definitions

Assume time is indexed by $t = 1, \ldots, T$. Exogenous shocks s_t realize in a finite set S. A history of shocks up to some date t is denoted by $s^t = (s_1, s_2, \ldots, s_t)$ and called a date event. Since each chance node in a tree can be reached only through one historical path, we also use s^t to denote a chance node. The notation $s^{t+1} \succ s^t$ refers to the node s^{t+1} following node s^t. Let $\mathcal{S}$ denote the set of all nodes, s^t, of the tree. We consider an agent's choices over T periods, $t = 1, \ldots, T$. For simplicity, we often focus on the $T = 3$ case where we use a different notation and denote nodes at $t = 2$ by (21), (22),... and at $t = 3$ by (31), (32), ...

We next briefly describe the DOCE utility axiomatized in Selden and Stux (1978). (Their paper, although unpublished, is available on the website of Larry Selden, Columbia University Graduate Business School.) Assume a T period setting, where consumption in period $t = 1$ is certain and risky in periods $t = 2, ..., T$. In period t, the consumer's certainty time preferences over degenerate consumption streams $(c_t, ..., c_T)$ $(t \in \{1, ..., T\})$ are represented by the following additively separable utility

$$U_t(c_t, ..., c_T) = u(c_t) + \sum_{i=t+1}^{T} \beta^{i-t} u(c_i), \tag{1}$$

where $0 < \beta < 1$ is the standard discount function. The consumer's risk preferences in each period $t \in \{2, ..., T\}$ are identical and represented by the EU (Expected Utility) function

$$\sum_{s^t} \pi(s^t) V(c(s^t)),$$

where $\pi(s^t)$ is the probability of the date event (node) s^t and V is the NM (von Neumann-Morgenstern) index. DOCE preferences are assumed to be independent of when risk is resolved. This preference axiom, referred to as TRI, is one important difference from KP preferences described below. The stationary time preference u and NM index V will be assumed to satisfy $u' > 0, u'' < 0, V' > 0$ and $V'' < 0$ unless stated otherwise. In what follows, we use preferences over current and future consumption conditional on the current date event node being s^τ.

The period t certainty equivalent evaluated at node s^τ is defined by

$$(\widehat{c}_t|s^\tau) = V^{-1}\left(\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau)V(c(s^t))\right),$$

where $\pi(s^t|s^\tau)$ is the probability of date event s^t conditional on being at node s^τ. Throughout this paper we denote by $\mathbf{c}$ the consumption vector for the full tree. Thus, for a given s^τ, the DOCE representation is given by

$$\mathcal{U}\left(\mathbf{c}|s^\tau\right) = u(c(s^\tau)) + \sum_{t=\tau+1}^{T} \beta^{t-\tau} u(\widehat{c}_t|s^\tau).$$

Note that $\mathcal{U}(\cdot|s^\tau)$ is a function of $\mathbf{c}$ but only varies with $c(s^\tau)$ and $c(s^{\tau+i})$, $i = 1, ..., T-\tau$, where $s^{\tau+i} \succ s^\tau$. In period 1, the utility is given by

$$\mathcal{U}\left(\mathbf{c}\right) = u(c_1) + \sum_{t=2}^{T} \beta^{t-1} u(\widehat{c}_t|s^1).$$

For the DOCE preference model, (i) risk preferences are constant over time, (ii) there is a complete separation of time and risk preferences corresponding to u and V [8] and (iii) the consumer is psychically indifferent to when risk is resolved (see the TRI Definition 2 below).

Kreps and Porteus (1978) derived the recursive representation

$$\mathcal{U}(\mathbf{c}|s^\tau) = U\left(c(s^\tau), \sum_{s^{\tau+1} \succ s^\tau} \pi(s^{\tau+1}|s^\tau)\mathcal{U}(\mathbf{c}|s^{\tau+1})\right),$$

where U is continuous and strictly increasing.[9] Note that if U is linear in the second argument, the KP representation converges to the EU special case. The

[8] As discussed in Selden (1978), time and risk preferences satisfy SEP in the sense that (i) the OCE utility is constructed from the independent building blocks (U, V) and (ii) if a given general continuous and monotone Bernoulli utility satisfies the OCE axioms, then it is always possible to derive the unique, up to appropriate transformations, separate U and V indices.

[9] Unlike the DOCE and EZ cases, the KP preference building blocks are U and $\mathcal{U}$. An EU index V can be induced from the KP utility for the final time period T.

EZ representation is a special case of the KP utility,[10] where

$$U(c_t, x) = -\frac{\left(c_t^{-\delta_1} + \beta(-\delta_2 x)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \quad \text{and} \quad V_T(x) = -\frac{x^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1),$$

with V_T being induced from $\mathcal{U}$. If $\delta_1 = \delta_2 = \delta$, the EZ representation converges to the EU function

$$\mathcal{U}(\mathbf{c}|s^\tau) = -\frac{(c(s^\tau))^{-\delta}}{\delta} - E\left[\sum_{t=\tau+1}^{T} \beta^{t-\tau} \frac{(c_t(s^\tau))^{-\delta}}{\delta}\right].$$

Both the KP and EZ recursive preference structures can accommodate a preference for early or late resolution of risk. However as was mentioned in the prior section, this temporal resolution preference cannot be varied independently from time and risk preferences.

For our two key theorems derived in Section 4, it will be convenient to first give the following collection of homothetic time and risk preference building block utilities

$$u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_1 > -1, \delta_2 > -1, \delta_1, \delta_2 \neq 0), \tag{2}$$

$$u(c) = \ln c \quad \text{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_2 > -1, \delta_2 \neq 0), \tag{3}$$

$$u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = \ln c \quad (\delta_1 > -1, \delta_1 \neq 0), \tag{4}$$

and

$$u(c) = \ln c \quad \text{and} \quad V(c) = \ln c. \tag{5}$$

Note that eqn. (5) corresponds to a time consistent EU special case of DOCE preferences. The quasihomothetic time and risk preference utilities are given by[11]

$$u(c) = -\frac{(c-b)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{(c-b)^{-\delta_2}}{\delta_2}, \tag{6}$$

where $(\delta_1, \delta_2 > -1, \delta_1, \delta_2 \neq 0, b \in \mathbb{R}, c > \max(0, b))$,

$$u(c) = \ln(c-b) \quad \text{and} \quad V(c) = -\frac{(c-b)^{-\delta_2}}{\delta_2}, \tag{7}$$

where $(\delta_2 > -1, \delta_2 \neq 0, b \in \mathbb{R}, c > \max(0, b))$,

$$u(c) = -\frac{(c-b)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = \ln(c-b), \tag{8}$$

[10] Weil (1990) derives an alternative specialization of the KP preference model.

[11] To avoid corner solutions for the consumption-portfolio problems considered below, we do not include the $\delta_2 = -1$ case.

where $(\delta_1 > -1, \delta_1 \neq 0, b \in \mathbb{R}, c > \max(0, b))$,

$$u(c) = \ln(c - b) \quad \text{and} \quad V(c) = \ln(c - b) \quad (b \in \mathbb{R}, c > b), \tag{9}$$

$$u(c) = -\frac{\exp(-\kappa_1 c)}{\kappa_1} \quad \text{and} \quad V(c) = -\frac{\exp(-\kappa_2 c)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0) \tag{10}$$

and

$$u(c) = \frac{(b - c)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = \frac{(b - c)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 < -1, b > c > 0). \tag{11}$$

For the NM indices in (6), (10) and (11), respectively, the risk preferences exhibit decreasing, constant and increasing absolute risk aversion. This collection of NM indices is typically referred to as the HARA class. The corresponding certainty utilities are frequently referred to as the Modified Bergson family.[12] One important special case of (6) is the CES time and CRRA risk preference utilities (2) used in the EZ special case of KP preferences where the EIS (elasticity of intertemporal substitution) and Arrow-Pratt relative risk aversion measures are given by, respectively,

$$EIS = \frac{1}{1 + \delta_1} \quad \text{and} \quad -c_t \frac{V''(c_t)}{V'(c_t)} = 1 + \delta_2. \tag{12}$$

For the popular DARA (decreasing absolute risk aversion) case (6), it is standard to interpret $b > 0$ as a certain subsistence requirement.[13]

As discussed in Section 1, one key objective of this paper is to show that in addition to satisfying SEP and TRI, DOCE preferences can also satisfy TC. Before concluding this subsection, we formally define TC and TRI and then review what is known about their simultaneous satisfaction for the DOCE and KP preference models. The following can be shown to be equivalent in our setting to the definition of time consistency in Johnsen and Donaldson (1985).

Definition 1 *The consumer's preferences satisfy TC if and only if at any time t and all $\mathbf{c}$, $\mathbf{c}'$ with some payoff history s^t,*

$$\mathcal{U}\left(\mathbf{c}|s^{t+1}\right) \geq \mathcal{U}\left(\mathbf{c}'|s^{t+1}\right) \quad \left(\forall s^{t+1} \succ s^t\right) \Rightarrow \mathcal{U}\left(\mathbf{c}|s^t\right) \geq \mathcal{U}\left(\mathbf{c}'|s^t\right),$$

where $c(s^t) = c'(s^t)$.

[12] See Pollak (1971) for a description of the Modified Bergson class.

[13] For the DARA case we can have $b < 0$, but then the subsistence interpretation does not make sense (see Pollak 1970, p. 748). For the IARA (increasing absolute risk aversion) case (11), b can be interpreted as a bliss point.

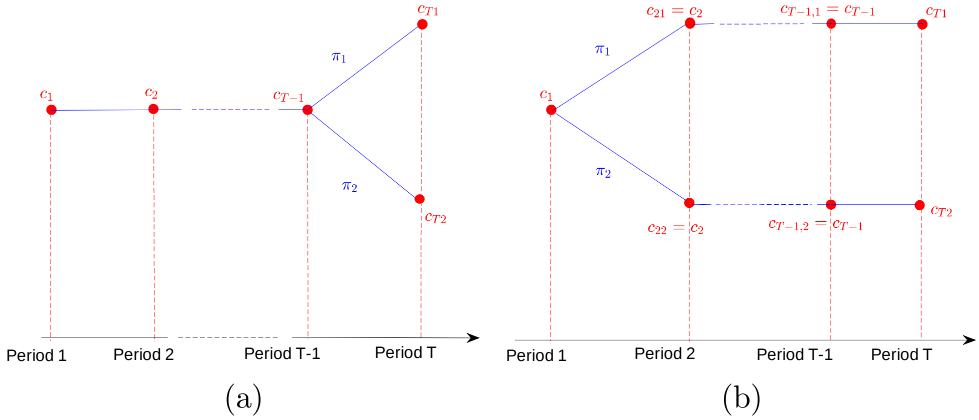


Figure 1:

Next, we define TRI. For this definition, note that for consumption $\mathbf{c}$ over an arbitrary tree $\mathcal{S}$ we can uniquely identify a node s^t by the unique s^T that succeeds it and by the time t. We write for any s^t,

$$c(t|s^T) = c(s^t) \quad \text{for} \quad s^T \succ s^t.$$

Definition 2 *The consumer's preferences satisfy TRI if for any* $\mathbf{c}$ *on a tree* $\mathcal{S}$ *with probabilities* $\boldsymbol{\pi}$ *the individual is indifferent between* $\mathbf{c}$ *and consumption* $\overline{\mathbf{c}}$ *over a tree* $\overline{\mathcal{S}}$ *with probabilities* $\overline{\boldsymbol{\pi}}$*, where on the tree* $\overline{\mathcal{S}}$ *all uncertainty realizes in the first period, i.e., for each* $s^T \in \mathcal{S}$ *there is a* $\overline{s^2} \in \overline{\mathcal{S}}$ *with* $\pi(s^T) = \pi(\overline{s^2})$*, each* $\overline{s^t}$*,* $t = 2, \ldots, T-1$ *has a unique successor, and where*

$$c(t|s^T) = \overline{c}(t|\overline{s^T}) \quad \textit{for all } s^T, t = 2, \ldots, T.$$

The tree structures in Figures 1(a) and (b) respectively illustrate the late resolution case where the risk is resolved in period T, and the early resolution case where risk is resolved in period 2. The early and late resolution cases correspond respectively to the trees $\overline{\mathcal{S}}$ and $\mathcal{S}$ in Definition 2. Property TRI states that a consumer is indifferent between these two trees.

It is clear that by construction DOCE and KP preferences are based on the two independent building blocks (u, V) and satisfy SEP.[14] The axiomatization in Selden and Stux (1978) explicitly assumes TRI. In Section 4, we provide conditions under which DOCE preferences also satisfies TC. Moreover, these conditions in no way preclude an independent variation of time and risk preferences and thus SEP

[14] The EU representation is a special case of both the DOCE and KP preference models where u and V are positive affine transforms of each other and hence SEP is violated.

holding. Next, consider the case of KP preferences. By construction, its recursive structure ensures TC. Also by construction, it always violates TRI except for the special case of EU preferences where SEP is also violated. Although formally KP preferences satisfy SEP, the fact that it also allows for a preference for early or late resolution is intertwined with the SEP property. To see this most clearly, consider the case where the building blocks (u, V) correspond to CES time and CRRA risk preference utilities (2). In this case, the consumer has a preference for the early (late) resolution tree depending on whether her risk preference parameter $\delta_2 > (<)$ her time preference parameter δ_1. Thus if a KP consumer has a preference for the early resolution tree, then her u and V building blocks cannot be prescribed independently as they must satisfy $\delta_2 > \delta_1$ and property SEP is violated.[15]

2.2 Optimization Problems

In this subsection, we formally define the consumption-portfolio problem and characterize time consistency at the demand level in terms of the standard resolute, naive and sophisticated solution techniques that are typically considered when preferences fail to be time consistent.[16]

At the beginning of each period $t = 1, \ldots, T-1$ there are J assets available for trade with returns $\mathbf{R}\left(s^{t+1}\right) = \left(R_j\left(s^{t+1}\right)\right)_{j=1}^{J} \geq 0$ being realized at node s^{t+1}. We assume that asset returns preclude arbitrage in that there exist $\rho(s^t) > 0$ for all s^t such that

$$\sum_{s^{t+1} \succ s^t} \rho(s^{t+1}) R_j(s^{t+1}) = \rho(s^t) \quad \forall s^t, j. \tag{13}$$

Suppose the special case of complete asset markets holds where the number of assets is the same as the number of states, or more formally at each s^t, $t < T$, the matrix $(R(s^{t+1}))_{\{s^{t+1} \succ s^t\}}$ has rank S. Then $\rho(s^{t+1})$ in eqn. (13) can be interpreted as the contingent claim price for $c(s^{t+1})$.

[15] This point is noted in Strzalecki (2013, pp. 1056 - 1057) where he comments that "[in the KP preference model] three features – intertemporal elasticity of substitution, elasticity of substitution between states, and preference for timing of resolution of uncertainty – are interdependent; roughly speaking, knowing two of them is sufficient to determine the third. For this reason, the Kreps–Porteus model may be seen as restrictive because it does not allow enough freedom to specify the three parameters independently". Epstein, Farhi and Strzalecki (2014, p. 2688), observe that allowing for a non-indifference to temporal resolution results in "a partial separation between EIS and RRA [relative risk aversion]".

[16] Phelps and Pollak (1968) and Peleg and Yaari (1973) argue that one should think of the time inconsistent choice problem as being equivalent to a game between divergent individuals – myself today and my selves in future periods. Caplin and Leahy (2006) argue that the sophisticated and game theoretic approaches result in equivalent solutions.

A much weaker assumption which plays a prominent role in our analysis is that there exists a one period risk free asset at each date event.

Assumption [RF] For each s^t, $t = 1, \ldots, T-1$, there exists an $\boldsymbol{\omega}(s^t)$, where $\boldsymbol{\omega} = \omega_1, ..., \omega_J$, such that

$$\sum_j \omega_j \left(s^t\right) R_j \left(s^{t+1}\right) = 1 \ \ \forall s^{t+1} \succ s^t.$$

Note that this assumption is automatically satisfied when markets are complete.

An individual is assumed to choose consumption and assets in periods $t = 1, \ldots, T-1$ so as to maximize utility. We assume throughout that the individual has rational expectations in that she knows future asset returns contingent on the nodes.

In period $t \in \{1, ..., T-1\}$, at the node s^t, denote the demand for asset $j \in \{1, ..., J\}$ by $n_j\left(s^t\right)$ and the vector of asset holdings by $\mathbf{n} = \left(\mathbf{n}\left(s^1\right), ... \left\{\mathbf{n}\left(s^t\right)\right\}, ... \left\{\mathbf{n}\left(s^{T-1}\right)\right\}\right)$, where $\mathbf{n}\left(s^t\right) = \left(n_1\left(s^t\right), ..., n_j\left(s^t\right)\right)$. Define the stream of time $t = 1, ..., T$ consumption and contingent claim quantities $\mathbf{c} = \left(c\left(s^1\right), ..., \left\{c(s^t)\right\}, ..., \left\{c(s^T)\right\}\right)$.

Let I and $I(s^t)$ denote, respectively, initial income and the income received from investment in period $t-1$ at the beginning of period $t > 1$ at the node s^t and $I\left(s^t\right) = I$ when $t = 1$.

The period 1 consumption-portfolio problem is defined as follows

$$\max_{\mathbf{c},\mathbf{n}} \mathcal{U}\left(\mathbf{c}\right) \quad S.T. \tag{14}$$

$$c(s^t) = I - \sum_j n_j(s^t), \quad t = 1, \tag{15}$$

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t) - \sum_j n_j(s^t), \quad 2 < t < T, \tag{16}$$

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t), \quad t = T. \tag{17}$$

It is assumed that in any period t, the consumer can only purchase assets with maturity of one time period.

In addition to defining time consistency at the preference level as in Definition 1, one can also define it at the demand level. To simplify notation, we use $(\mathbf{c}^\circ, \mathbf{n}^\circ)$, $(\mathbf{c}^*, \mathbf{n}^*)$ and $(\mathbf{c}^{**}, \mathbf{n}^{**})$ to denote resolute, naive and sophisticated demands, respectively. We follow the standard definitions of these solution techniques as in Selden and Wei (2016, p. 1916). (Formal definitions in the context of the consumption-portfolio optimization (14) - (17) are given in Appendix A.1.) To facilitate the comparison with KP preferences below, $\left(\mathbf{c}^{KP}, \mathbf{n}^{KP}\right)$ denotes the optimal demands corresponding to KP preferences. Consistent with the certainty analysis of Strotz

(1956) and Pollak (1968), DOCE demands are said to be time consistent if and only if $(\mathbf{c}^\circ, \mathbf{n}^\circ) = (\mathbf{c}^*, \mathbf{n}^*) = (\mathbf{c}^{**}, \mathbf{n}^{**})$, for a subset of possible prices.

In the certainty case, Blackorby, et al. (1973) prove that demands are time consistent if and only if each period $t+1$ utility can be embedded into the period t utility for all $t \in \{1, ..., T-1\}$ utilities. Johnsen and Donaldson (1985) extend this notion to the risky case, where time consistency holds if and only if the future utility function in each state can be embedded into the utility function of prior periods. Following Blackorby et al. (1973), in Section 4 we link the demand and preference definitions of time consistency in our setting.

3 Intuition

In this section, we first illustrate the very simple and intuitive calculation of DOCE utility. Second, we argue that if the necessary and sufficient conditions derived in the next section for DOCE preferences to be time consistent are satisfied, the consumer's dynamic consumption-portfolio problem can be reformulated as a sequence of single period portfolio optimizations and a certainty consumption-saving optimization where the former optimizations are based on risk preferences and the latter is based on time preferences.

Consider the three period consumption tree in Figure 2. The assumed TRI property implies that the tree on the left hand side of the figure can be decomposed into the two single period trees on the right hand side. The upper tree corresponds to the distribution of period 2 consumption. The bottom tree can be thought of as a single stage compound lottery paying off period 3 consumption. It further follows from TRI that the two subtrees can be evaluated independently using the assumed risk preferences. As shown in the figure, it is straightforward to compute the corresponding certainty equivalents $\widehat{c}_2$ and $\widehat{c}_3$. Then the utility of the three period tree on the left hand side of Figure 2 can be viewed as being equivalent to the utility of the degenerate tree $(c_1, \widehat{c}_2, \widehat{c}_3)$ computed by $U(c_1, \widehat{c}_2, \widehat{c}_3)$.

Next consider the three period consumption-portfolio problem (14) - (17) corresponding to Figure 3, where time and risk preferences are respectively represented by the CES and CRRA utilities in (2). The consumer can be viewed as solving single period portfolio optimizations at the period 1 node and at the two period 2 nodes. Denote the (gross) returns on a period 1 risky asset in the two states by R_{21} and R_{22} and the (gross) return on the risk free asset R_{f2}. With slight abuse of our general notation, period 1 risky and risk free asset holdings are respectively denoted by n_1 and n_{f1}. Thus in period 1, the consumer can be viewed

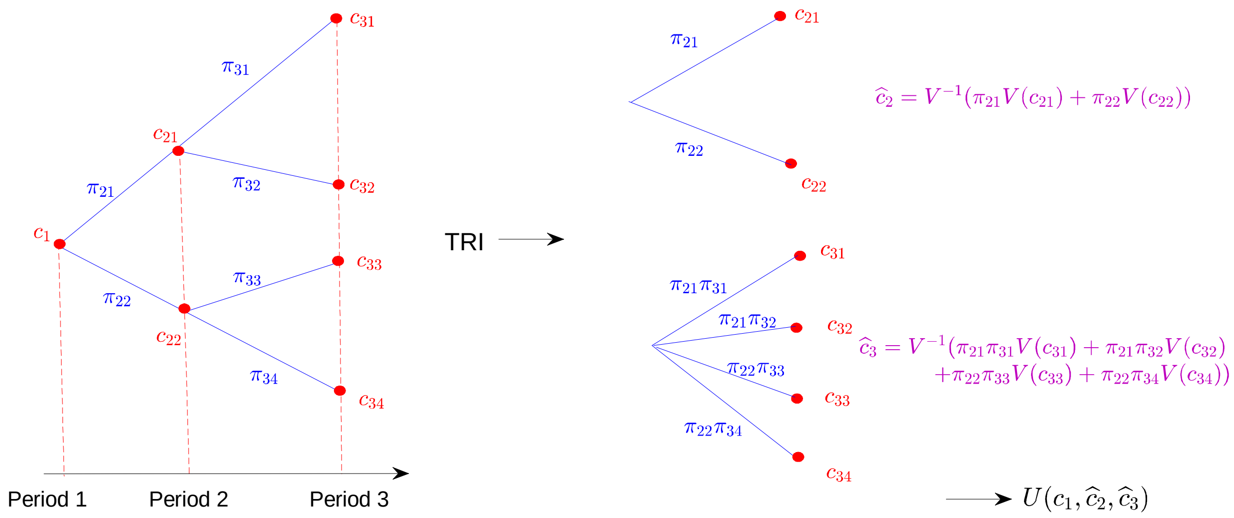


Figure 2:

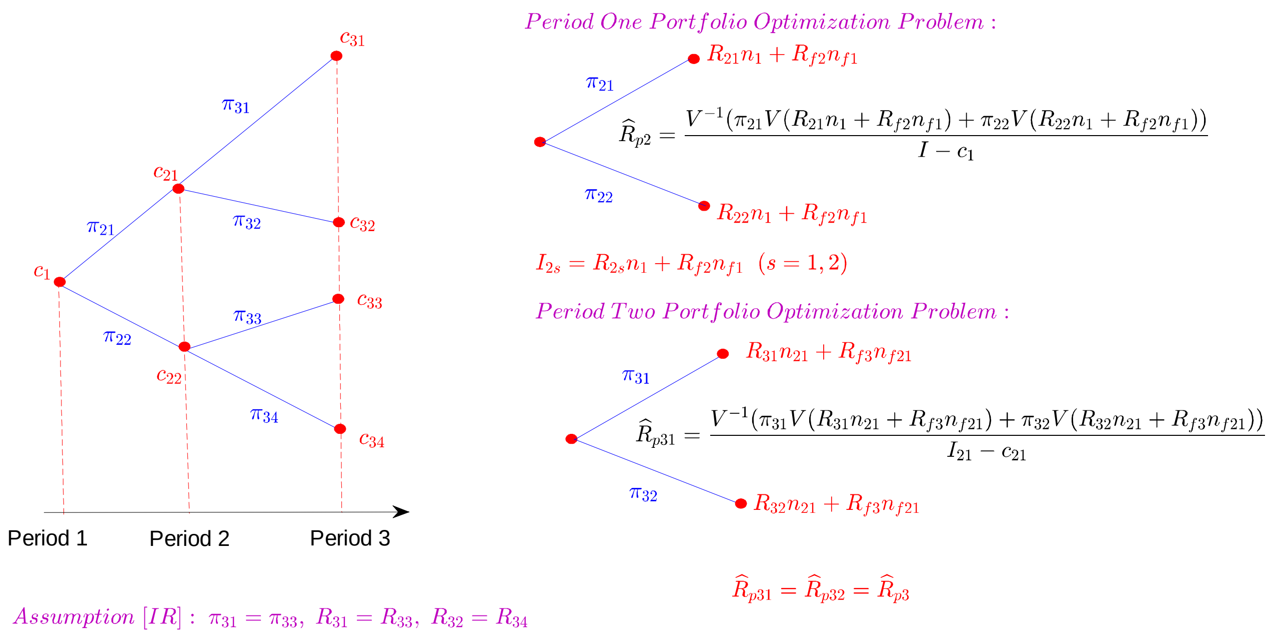


Figure 3:

as maximizing the following conditional on her level of period 1 saving $(I - c_1)$

$$\max_{n_1, n_{f1}} \pi_{21} V(R_{21} n_1 + R_{f2} n_{f1}) + \pi_{22} V(R_{22} n_1 + R_{f2} n_{f1}) \quad S.T.\ I - c_1 = n_1 + n_{f1}. \quad (18)$$

Conditional on her optimal period 1 asset demands, period 2 income for the two branches is given by

$$I_{21} = R_{21} n_1 + R_{f2} n_{f1} \quad \text{and} \quad I_{22} = R_{22} n_1 + R_{f2} n_{f1}.$$

Then in period 2, conditional on the upper node being realized and income equalling I_{21}, the consumer faces the following portfolio problem

$$\max_{n_{21}, n_{f21}} \pi_{31} V(R_{31} n_{21} + R_{f3} n_{f21}) + \pi_{32} V(R_{32} n_{21} + R_{f3} n_{f21}) \quad S.T.\ I_{21} - c_{21} = n_{21} + n_{f21}. \quad (19)$$

The period 3 risky and risk free asset returns are the same on the upper and lower branches of the consumption tree implying that asset returns are independent over time. This is referred to as Assumption [IR] and formally defined in Subsection 4.1.2 below. Given the homothetic form of the CRRA NM index V, certainty equivalent consumption in the upper and lower branches can be expressed as

$$\widehat{c}_{31} = (I_{21} - c_{21}) \widehat{R}_{p31} \quad \text{and} \quad \widehat{c}_{32} = (I_{22} - c_{22}) \widehat{R}_{p32},$$

where $\widehat{R}_{p31}$ and $\widehat{R}_{p32}$ denote respectively the portfolio certainty equivalent returns on the upper and lower branches and are calculated as indicated on the right hand side of Figure 3. Asset returns satisfying [IR] implies that $\widehat{R}_{p31} = \widehat{R}_{p32} = \widehat{R}_{p3}$. However, as we argue in Subsection 4.1.2, it is enough to assume that $\widehat{R}_{p31} = \widehat{R}_{p32}$ or that certainty equivalent returns are the same across branches [ICER].

It follows from Theorem 1 below that if Assumption [ICER] holds and the consumer's time and risk preferences take the homothetic form (2), resolute, naive and sophisticated consumption and asset demands are the same. Moreover, these assumption imply that the consumer's DOCE preferences exhibit time consistency on a restricted domain of consumption trees. Having solved for the optimal conditional asset demands for periods 1 and 2, the consumer's remaining consumption-saving problem takes the following very simple certainty form

$$\max_{c_1, \widehat{c}_2, \widehat{c}_3} \left(c_1^{-\delta_1} + \beta \widehat{c}_2^{\,-\delta_1} + \beta^2 \widehat{c}_3^{\,-\delta_1} \right)^{-\frac{1}{\delta_1}} \quad S.T.\ I = c_1 + \frac{\widehat{c}_2}{\widehat{R}_{p2}} + \frac{\widehat{c}_3}{\widehat{R}_{p2} \widehat{R}_{p3}},$$

where

$$\widehat{c}_2 = V^{-1} \left(\pi_{21} V(c_{21}) + \pi_{22} V(c_{22}) \right) \quad \text{and} \quad \widehat{c}_3 = V^{-1} \left(\pi_{21} V(\widehat{c}_{31}) + \pi_{22} V(\widehat{c}_{32}) \right)$$

and

$$\widehat{c}_{31} = V^{-1}\left(\pi_{31}V(c_{31}) + \pi_{32}V(c_{32})\right) \quad \text{and} \quad \widehat{c}_{32} = V^{-1}\left(\pi_{33}V(c_{33}) + \pi_{34}V(c_{34})\right).$$

Solving this problem yields

$$c_1 = \frac{I}{1 + \beta^{\frac{1}{1+\delta_1}} \widehat{R}_{p2}^{-\frac{\delta_1}{1+\delta_1}} + \beta^{\frac{2}{1+\delta_1}} \widehat{R}_{p2}^{-\frac{\delta_1}{1+\delta_1}} \widehat{R}_{p3}^{-\frac{\delta_1}{1+\delta_1}}}. \tag{20}$$

Optimal unconditional asset demands can be computed substituting period 1 consumption (20) into the conditional demands obtained from (18) and (19).

4 Time Consistent DOCE Demand

In this section, we derive conditions for when DOCE preferences satisfy property TC. When DOCE demands are time consistent, the same conditions imply that (i) the DOCE demands can also be rationalized by time consistent KP preferences based on the same assumed building blocks utilities (u, V) and (ii) key properties relating to asset demand behavior derived for two period KPS preferences also hold for T-period DOCE and KP preferences.

4.1 Homothetic Preferences

In the first part of this subsection, we provide necessary and sufficient conditions on consumption sets such that DOCE preferences are time consistent. In the second part, we state necessary and sufficient conditions on asset returns that ensure choices always (no matter whether resolute, naive or sophisticated) lie in this set.

4.1.1 Time Consistent Preferences over Restricted Domains

It is well known that in general, DOCE preferences violate time consistency as defined in Definition 1 above. We will give examples later in the paper demonstrating that there can be significant differences between sophisticated and resolute choice. On the other hand, we argue in this subsection that if one restricts the domain of preference (i.e., assumes that possible choices have to lie in a subset of all possible consumption choices on the event tree) time consistency can be restored.

Before stating the general result, Proposition 1, characterizing restrictions on the domain of preferences that ensure that DOCE preferences are time consistent, we first provide a transparent example for why this can happen. For simplicity assume the simple three period consumption tree in Figure 4. Denote the nodes by

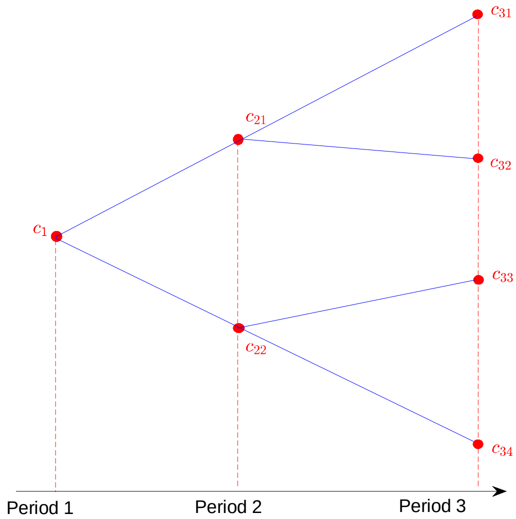


Figure 4:

the sequence of numbers $1, 21, 22, 31, 32, 33$ and 34 corresponding naturally to the subscripts for consumption at each node. Given the fixed tree structure in Figure 4 and set of probabilities, a given consumption tree can be fully characterized by the consumption vector

$$\mathbf{c} = (c_1, c_{21}, c_{22}, c_{31}, c_{32}, c_{33}, c_{34}) \in \mathbb{R}^7_+.$$

The vectors $(c_{21}, c_{31}, c_{32}) \in \mathbb{R}^3_+$ and $(c_{22}, c_{33}, c_{34}) \in \mathbb{R}^3_+$, respectively, characterize in a natural way the upper and lower subtrees. Let $\mathcal{U}(\mathbf{c}|(21))$ and $\mathcal{U}(\mathbf{c}|(22))$ represent, respectively, the DOCE preferences over the consumption on the subtrees corresponding to (c_{21}, c_{31}, c_{32}) and (c_{22}, c_{33}, c_{34}). In the current setting, the TC Definition 1 then simplifies to the following. For all $\mathbf{c}$, $\mathbf{c}'$ with $c_1 = c_1'$,

$$\mathcal{U}(\mathbf{c}|\,(21)) \geq \mathcal{U}(\mathbf{c}'|\,(21)) \quad \text{and} \quad \mathcal{U}(\mathbf{c}|\,(22)) \geq \mathcal{U}(\mathbf{c}'|\,(22)) \Longrightarrow \mathcal{U}(\mathbf{c}) \geq \mathcal{U}(\mathbf{c}')\,.$$

While DOCE preferences do not satisfy TC over $\mathbb{R}^7_+$, it is easy to see that for any $\bar{\mathbf{c}} \in \mathbb{R}^7_+$ they are time consistent over $\{\mathbf{c} \in \mathbb{R}^7_+ : \mathbf{c} = \alpha\bar{\mathbf{c}}, \alpha > 0\}$. This trivial example illustrates that time consistency is a joint property of preferences and the domain over which they are defined.

It turns out to be more interesting to assume that preferences are homothetic[17] and consider the following set as the domain of preferences.

$$\mathcal{I}_{\delta_2,\pi} = \left\{ \begin{array}{c} \mathbf{c} \in \mathbb{R}^7_+ : \mathbf{c} = (c_1, c_{21}, c_{22}, \alpha_1 c_{21}, \alpha_2 c_{21}, \alpha_3 c_{22}, \alpha_4 c_{22}), \\ (\alpha_1, \ldots, \alpha_4) \in \mathbb{R}^4_+, \pi_{31}\alpha_1^{-\delta_2} + \pi_{32}\alpha_2^{-\delta_2} = \pi_{33}\alpha_3^{-\delta_2} + \pi_{34}\alpha_4^{-\delta_2} \end{array} \right\}.$$

[17]It is easy to see that DOCE preferences will be homothetic if and only if the building block time and risk preference representations take the CES time and CRRA risk preference form (2).

We next argue that homothetic DOCE preferences are TC over the domain $\mathcal{I}_{\delta_2,\pi}$. Define

$$K = \pi_{33}\alpha_3^{-\delta_2} + \pi_{34}\alpha_4^{-\delta_2}$$

and rewrite the period 1 utility function as follows

$$\begin{aligned}
\mathcal{U}(\mathbf{c}) &= u(c_1) + \beta u \circ V^{-1}\left(\sum_{s=1}^{2} \pi_{2s} V(c_{2s})\right) + \beta^2 u \circ V^{-1}\left(\sum_{s=1}^{2} \pi_{2s} \sum_{s'} \pi_{3s'} V(\alpha_{s'} c_{2s})\right) \\
&= u(c_1) + \beta u \circ V^{-1}\left(\sum_{s} \pi_{2s} V(c_{2s})\right) + \beta^2 \left(u \circ V^{-1}\left(\sum_{s} \pi_{2s} V(c_{2s})\right) u \circ V^{-1}(K)\right) \\
&= u(c_1) + \beta u \circ V^{-1}\left(\sum_{s} \pi_{2s} V(c_{2s})\right) \left(1 + \beta u \circ V^{-1}(K)\right)
\end{aligned}$$

and depending on whether the upper or lower state is realized

$$\mathcal{U}(\mathbf{c}|\,(2s)) = u(c_{2s}) + \beta u \circ V^{-1}\left(\sum_{s'} \pi_{3s'} V(\alpha_{s'} c_{2s})\right) = u(c_{2s})\left(1 + \beta u \circ V^{-1}(K)\right) \quad (s = 1, 2).$$

It is now easy to see that homothetic DOCE preferences are time consistent over the domain $\mathcal{I}_{\delta_2,\pi}$. The following proposition generalizes this result to arbitrary date event consumption trees, where without loss of generality we denote the domain as simply $\mathcal{I}$.[18]

Proposition 1 *Suppose DOCE preferences are homothetic and are defined over consumption on a date event consumption tree with M nodes. Then the preferences are time consistent if and only if consumption is restricted to the set*

$$\mathcal{I} = \{\mathbf{c} \in \mathbb{R}^M : \exists V_t, t = 2, ..., T-1, \text{ such that } \forall s^t, t < T, \frac{-1}{\delta_2} \sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) \left(\frac{c(s^{t+1})}{c(s^t)}\right)^{-\delta_2} = V_t\}. \tag{21}$$

4.1.2 Main Result

To operationalize Proposition 1, we next provide simple conditions on asset returns that ensure that an individual's consumption $\mathbf{c}$ lies in the set $\mathcal{I}$. In order to do so, it is useful to define for each s^t, $\widetilde{\mathbf{n}}(s^t) \in \mathbb{R}^J$ to be the unique solution to the J equations

$$\sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) R(s^{t+1}) \left(R(s^{t+1}) \cdot \widetilde{\mathbf{n}}(s^t)\right)^{-\delta_2 - 1} = 1, \tag{22}$$

where the homothetic risk preference NM index takes the CRRA form in (2).

[18] It should be noted that $\mathcal{I}$ requires both u and V to be homothetic, but only depends on the risk preference parameter δ_2.

The following assumption then plays a key role in the time consistency of DOCE choice.

Assumption [ICER] Assume that for all s^t, $t < T$,

$$\sum_{s^{t+1}\succ s^t} \pi(s^{t+1}|s^t)\left(R(s^{t+1})\cdot \widetilde{\mathbf{n}}(s^t)\right)^{-\delta_2} = \left(\hat{R}_{pt}\sum_j \widetilde{\mathbf{n}}_j(s^t)\right)^{-\delta_2},$$

where $\hat{R}_{pt}$ only depends on t.

When markets are incomplete this assumption can be difficult to verify. However, a simple sufficient condition for Assumption [ICER] to hold is that the asset return distributions are identical across all nodes in a given period. Formally, this is stated as follows.

Assumption [IR] Assume that for each s^t, $R(s^t) = \bar{R}_{t+1}(s_t)$ for some functions $\bar{R}_{t+1}(.)$ and $\pi(s^t|s^{t-1}) = \bar{\pi}_t(s_t)$ for some function $\bar{\pi}_t(s_t)$.

When markets are complete, Assumption [ICER] can be directly translated into an assumption on asset returns as the following example shows. Based on the assumed CRRA representation of risk preferences, corresponding to different values of δ_2, different sets of asset return distributions will satisfy [ICER]. Each branch in period t has an asset return distribution from the same set parameterized by $\widehat{R}_{pt}$.

Example 1 *Consider the tree structure in Figure 4. Assume DOCE preferences corresponding to (2). Since the markets are complete, we have the following relationship between asset returns and contingent claim prices*

$$\begin{aligned}\rho_{31} &= \frac{R_{f31}-R_{32}}{(R_{31}-R_{32})R_{f31}} \quad \text{and} \quad \rho_{32} = \frac{R_{31}-R_{f31}}{(R_{31}-R_{32})R_{f31}},\\ \rho_{33} &= \frac{R_{f32}-R_{34}}{(R_{33}-R_{34})R_{f32}} \quad \text{and} \quad \rho_{34} = \frac{R_{33}-R_{f32}}{(R_{33}-R_{34})R_{f32}}.\end{aligned} \tag{23}$$

It follows from the first order conditions for the consumption-portfolio problem (14) - (17) that

$$c_{32} = \left(\frac{\pi_{32}\rho_{31}}{\pi_{31}\rho_{32}}\right)^{\frac{1}{1+\delta_2}} c_{31},$$

implying

$$\begin{aligned}\widehat{c}_{31} &= \left(\pi_{31}c_{31}^{-\delta_2}+\pi_{32}c_{32}^{-\delta_2}\right)^{-\frac{1}{\delta_2}} = \left(\pi_{31}+\pi_{32}\left(\frac{\pi_{32}\rho_{31}}{\pi_{31}\rho_{32}}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{-\frac{1}{\delta_2}} c_{31}\\ &= \frac{\left(\pi_{31}+\pi_{32}\left(\frac{\pi_{32}\rho_{31}}{\pi_{31}\rho_{32}}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{-\frac{1}{\delta_2}}(I_{21}-c_{21})}{\rho_{31}+\left(\frac{\pi_{32}\rho_{31}}{\pi_{31}\rho_{32}}\right)^{\frac{1}{1+\delta_2}}\rho_{32}}.\end{aligned}$$

Therefore, $\widehat{R}_{p31} = \widehat{R}_{p32} = \widehat{R}_{p3}$ is equivalent to

$$\frac{\left(\pi_{33} + \pi_{34}\left(\frac{\pi_{34}\rho_{33}}{\pi_{33}\rho_{34}}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{-\frac{1}{\delta_2}}}{\rho_{33} + \left(\frac{\pi_{34}\rho_{33}}{\pi_{33}\rho_{34}}\right)^{\frac{1}{1+\delta_2}}\rho_{34}} = \frac{\left(\pi_{31} + \pi_{32}\left(\frac{\pi_{32}\rho_{31}}{\pi_{31}\rho_{32}}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{-\frac{1}{\delta_2}}}{\rho_{31} + \left(\frac{\pi_{32}\rho_{31}}{\pi_{31}\rho_{32}}\right)^{\frac{1}{1+\delta_2}}\rho_{32}} \quad (\delta_2 \neq 0)\,. \tag{24}$$

Assuming

$$k_1 = \frac{\pi_{32}\rho_{31}}{\pi_{31}\rho_{32}} = 1$$

and using (23), the right hand side of (24) becomes

$$\frac{\left(\pi_{31} + \pi_{32}\left(\frac{\pi_{32}\rho_{31}}{\pi_{31}\rho_{32}}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{-\frac{1}{\delta_2}}}{\rho_{31} + \left(\frac{\pi_{32}\rho_{31}}{\pi_{31}\rho_{32}}\right)^{\frac{1}{1+\delta_2}}\rho_{32}} = R_{f31}.$$

If $\delta_2 = -\frac{1}{2}$, then

$$\pi_{33} = \frac{R_{f32} - R_{34} - \sqrt{R_{f32}\left(R_{f32} - R_{34}\right)\left(R_{f32} - R_{33}\right)\left(R_{f32} - R_{f31}\right)}/R_{f32}}{R_{33} - R_{34}}$$

Further assuming that $R_{f32} = R_{f31}$, it follows that

$$\pi_{33} = \frac{R_{f32} - R_{34}}{R_{33} - R_{34}} \in (0, 1)$$

defines a set of asset return distributions such $\widehat{R}_{p3}$ is the same for the upper and lower branches.

We have the following result.

Theorem 1 *Suppose the consumer solves the consumption-portfolio problem (14) - (17). Then the following hold.*

(i) *If the consumer's DOCE utility takes one of the forms in (2) - (5), her demands will be time consistent if and only if Assumption [ICER] holds.*

(ii) *If Assumption [ICER] holds, then the consumer's demands will be time consistent if and only if her time and risk preference utilities take one of the forms in (2) - (5).*

At first glance, the theorem seems very surprising: How can DOCE preferences be generally time inconsistent, but still generate time consistent demands when the portfolio certainty equivalent returns are independent over time? While our detailed proof of Theorem 1 gives a formal answer to this puzzle, the key insight is that Assumption [ICER] simplifies the first order conditions and implies the conditions in Proposition 1.

4.2 HARA Preferences

We next consider the case of DOCE preferences corresponding to the HARA (and modified Bergson) utilities (6) - (11). The key insight is that if u and V, respectively, take the shifted CES and CRRA forms with the same shift parameters, the problem can simply be viewed as consumers having homothetic preferences for consumption in excess of their subsistence consumption requirements and the result in Theorem 1 extends. Although Assumption [ICER] continues to play a key role for the DARA and IARA members of the HARA class, for the CARA member one can make the weaker assumption that the risk free rate R_f is "non-stochastic" or constant across branches. Then, we have the following result for DOCE consumption and asset demands to be time consistent.

Theorem 2 *Suppose the consumer solves the consumption-portfolio problem (14) - (17) and Assumption [RF] holds.*

(i) *Assumption [ICER] with $\widetilde{\mathbf{n}}(s^t)$ as defined in (22) holds. Then DOCE demands are time consistent if time and risk preference utilities take one of the forms in (6) - (9) or (11).*

(ii) *The risk free interest rate R_{ft} is non-stochastic. Then DOCE demands are time consistent if time and risk preference utilities take the form in (10).*

The proof of this result for the (6) case follows essentially from the (2) case in Theorem 1. In the presence of a risk free asset, the consumer's maximization problem based on utility (6) can just be viewed as that of another consumer who owns enough of the risk free asset to pay b in each period $t = 1, ..., T-1$ and maximizes (2).

One may wonder why for the CARA case, Theorem 2(ii), no restriction on risky asset returns such as [ICER] is assumed as in Theorem 1 and Theorem 2(i). The explanation follows immediately from the well-known property of CARA risk preferences that the demand for the risky asset n is independent of investment. As a result, one can show that the certainty equivalent portfolio return $\widehat{R}_{pt}$ equals the risk free rate R_{ft}. For any period t, an increase in $(I_{t-1} - c_{t-1})$ results only in an increase in the holdings of the risk free asset n_{ft-1} and an incremental increase in the portfolio return equal to R_{ft}.[19]

Remark 1 *It should be noted that (i) for both Theorems 1 and 2, the additive time preference U, eqn. (1), can have an arbitrary period* 1 *utility* $u_1(c_1)$ *which*

[19] For a more complete discussion of this phenomenon in the simple two period setting, see Selden and Wei (2018).

satisfies $u_1' > 0$ and $u_1'' < 0$ and (ii) for Theorem 2, the cases covered for risk preferences include the full HARA class.

4.3 Another Time Consistent Rationalization

We have shown that when appropriate restrictions are imposed on asset markets and DOCE time and risk preferences, demands are time consistent. Suppose that KP preferences are constructed from the same time and risk preference building block utilities (2) as in the time consistent DOCE case and one assumes that asset returns satisfy [ICER]. Quite surprisingly, we next show that the two preference relations which are not ordinally equivalent over the full choice space, nevertheless result in the same demands.

Proposition 2 *Suppose Assumption [ICER] holds and the consumer has DOCE utility corresponding to (2) and solves the consumption-portfolio problem (14) - (17). Then the optimal demands can also be rationalized by KP preferences, where*

$$U\left(c_t, x\right) = -\frac{\left(c_t^{-\delta_1} + \beta\left(-\delta_2 x\right)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \quad \text{and} \quad V_T\left(x\right) = -\frac{x^{-\delta_2}}{\delta_2}.$$

The proof of Proposition 2 shows that if restricted to the consumption set $\mathcal{I}$ as defined by (21), KP and DOCE preferences coincide. We then show that under Assumption [ICER], the optimal choice for KP utility lies in $\mathcal{I}$.

To see the intuition for why the two utilities are identical for consumption in $\mathcal{I}$, consider the three period case in Figure 4. Note that since the period 2 optimization problem is the same for DOCE sophisticated choice and KP and [ICER] holds, the period 2 portfolio certainty equivalent return $\widehat{R}_{p3}$ is the same across branches and we have

$$\widehat{c}_{31} = \alpha c_{21} \text{ and } \widehat{c}_{32} = \alpha c_{22},$$

where

$$\alpha = \beta^{\frac{1}{1+\delta_1}} \widehat{R}_{p3}^{\frac{1}{1+\delta_1}}.$$

Then for KP

$$\begin{aligned}
&\beta\left(\pi_1\left(c_{21}^{-\delta_1} + \beta\widehat{c}_{31}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}} + \pi_2\left(c_{22}^{-\delta_1} + \beta\widehat{c}_{32}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}\right)^{\frac{\delta_1}{\delta_2}} \\
= \;&\beta\left(\pi_1\left(1 + \beta\alpha^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}} c_{21}^{-\delta_2} + \pi_2\left(1 + \beta\alpha^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}} c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} \\
= \;&\beta\left(1 + \beta\alpha^{-\delta_1}\right)\left(\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}
\end{aligned}$$

and for DOCE sophisticated choice

$$\begin{aligned}
& \beta\left(\pi_1 c_{21}^{-\delta_2}+\pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}+\beta^2\left(\pi_1 \widehat{c}_{31}^{-\delta_2}+\pi_2 \widehat{c}_{32}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} \\
= & \beta\left(\pi_1 c_{21}^{-\delta_2}+\pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}+\beta^2\left(\pi_1 \alpha^{-\delta_2} c_{21}^{-\delta_2}+\pi_2 \alpha^{-\delta_2} c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} \\
= & \beta\left(\pi_1 c_{21}^{-\delta_2}+\pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}+\beta^2 \alpha^{-\delta_1}\left(\pi_1 c_{21}^{-\delta_2}+\pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} \\
= & \beta\left(1+\beta \alpha^{-\delta_1}\right)\left(\pi_1 c_{21}^{-\delta_2}+\pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}},
\end{aligned}$$

which are the same.

We next show that DOCE and KP demands are the same for the Theorem 2 case of HARA preferences.

Proposition 3 *Suppose the consumer solves the consumption-portfolio problem (14) - (17) and Assumption [RF] holds. For DOCE preferences,*

(i) *if Assumption [ICER] holds and we assume the time and risk preference building blocks (6), then the optimal demands can also be rationalized by KP preferences, where*

$$U\left(c_t, x\right)=-\frac{\left(\left(c_t-b\right)^{-\delta_1}+\beta\left(-\delta_2 x\right)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \quad \text{and} \quad V_T(x)=-\frac{(x-b)^{-\delta_2}}{\delta_2};$$

(ii) *if the risk free rate R_{ft} is non-stochastic and we assume the time and risk preference building blocks (10), then the optimal demands can also be rationalized by KP preferences, where*

$$U\left(c_t, x\right)=-\frac{\left(\exp\left(-\kappa_1 c_t\right)+\beta\left(-\kappa_2 x\right)^{\frac{\kappa_1}{\kappa_2}}\right)^{\frac{\kappa_2}{\kappa_1}}}{\kappa_2} \quad \text{and} \quad V_T(x)=-\frac{\exp\left(-\kappa_2 x\right)}{\kappa_2};$$

(iii) *if Assumption [ICER] holds and we assume the time and risk preference building blocks (11), then the optimal demands can also be rationalized by KP preferences, where*

$$U\left(c_t, x\right)=\frac{\left(\left(b-c_t\right)^{-\delta_1}+\beta\left(\delta_2 x\right)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \quad \text{and} \quad V_T(x)=\frac{(b-x)^{-\delta_2}}{\delta_2}.$$

The intuition for Propositions 2 and 3(i) and (iii) is that when Assumption [ICER] holds, effectively we do not receive any new information with the passage of the time. Thus the preference for early or late resolution for KP preferences cannot be distinguished from temporal resolution indifference for DOCE preferences.[20] In fact, Assumption [ICER] rules out the canonical early resolution consumption tree corresponding to the case in Figure 1(b).[21] Moreover, as proved in Proposition 2, over the domain $\mathcal{I}$, DOCE and KP preferences both satisfy time consistency. It is clear that property SEP holds for KP preferences. Moreover assuming homothetic preferences and Assumption [ICER], it can be seen from the computations following Proposition 2 that the KP utility takes the same form as the DOCE utility over the domain $\mathcal{I}$.

4.4 Extension of Two Period KPS Asset Demand Properties

One attractive feature of the complete separation of time and risk preferences implicit in the two period KPS utility corresponding to (2) is that in the classic consumption-portfolio problem, optimal asset ratios are determined by risk preferences and are independent of time preferences. In this subsection, we show that this result extends to the dynamic setting if the conditions in Theorem 2 are satisfied.

Proposition 4 *Suppose Assumptions [ICER] and [RF] hold and the consumer solves the consumption-portfolio problem (14) - (17). In each period* $t \in \{1, ..., T-1\}$, *given the node* s^t, *denote the return on the risk free asset on the branch starting from node* s^t *by* $R_f(s^t)$,[22] *the demands for risky and risk free assets by* $n_j(s^t)$ *and* $n_f(s^t)$, *respectively. If we further assume*

(i)

$$u(c) = -\frac{(c-b)^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = -\frac{(c-b)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1, c > \max(0, b)),$$

[20] Similarly, for the CARA case considered in Proposition 3(ii), the consumer receives no new information about the risk free rate of interest R_{ft} with the passage of time. The fact that the risky asset returns may change with the passage of time does not matter, since as indicated above $\widehat{R}_{pt} = R_{ft}$ and any change in beginning of period income $(I_{t-1} - c_{t-1})$ only effects the demand for the risk free asset.

[21] Assume $T = 3$ and a consumer prefers the early resolution tree to the late resolution consumption tree 1(a). Then following Kreps and Porteus (1978), she is said to have a preference for early resolution. Given three time periods and that $c_{21} = c_{22}$, Assumption [ICER] implies that $R_{f31} = R_{f32}$ and hence no matter how much is saved in period 1, period 2 income will be the same on the upper and lower branches. Since preferences are also the same on the upper and lower branches, optimal c_2 and c_3 will also be the same on the two branches. Thus the restricted domain will necessarily exclude early resolution consumption trees with different c_3-values.

[22] We use the notation $R_f(s^t)$ instead of $R_f(s^{t+1})$ since the risk free rate only depends on the starting node s^t and is the same for each $s^{t+1} \succ s^t$.

then in each period $t \in \{1, ..., T-1\}$,[23] *the optimal asset ratios* $\frac{n_f(s^t) - \frac{b}{R_f(s^t)}}{n_j(s^t)}$ *are the same for KP and DOCE preferences and independent of* δ_1 *and* β*;*

(ii)

$$u(c) = -\frac{\exp(-\kappa_1 c)}{\kappa_1} \quad \textit{and} \quad V(c) = -\frac{\exp(-\kappa_2 c)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0),$$

then in each period $t \in \{1, ..., T-1\}$*, the optimal risky asset demands* $n_j(s^t)$ *are the same for KP and DOCE preferences and independent of* κ_1 *and* β*; or*

(iii)

$$u(c) = \frac{(b-c)^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = \frac{(b-c)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1, b > c > 0),$$

then in each period $t \in \{1, ..., T-1\}$*, the optimal asset ratios* $\frac{\frac{b}{R_f(s^t)} - n_f(s^t)}{n_j(s^t)}$ *are the same for KP and DOCE preferences and independent of* δ_1 *and* β*.*

Remark 2 *There is a direct connection between Proposition 4 and a widely referenced result in Giovannini and Weil (1989, section 2.5). They prove that corresponding to the EZ special case of KP preferences associated with eqn. (2), if asset returns are i.i.d. the portfolio optimization is identical to that of a single period EU optimization and hence is independent of the consumer's time preference parameters* δ_1 *and* β*. Proposition 4 extends this result to a more general set of KP preferences and more general asset return condition [ICER] and establishes the connection to DOCE preferences. Also, Proposition 4 when combined with Theorem 2 can be viewed as providing necessary as well as sufficient conditions for asset ratios to be independent of time preferences since DOCE preferences are time consistent only under the indicated conditions.*

5 Departures from Asset Returns Satisfying [ICER]

In the prior section when Assumption [ICER] or the special case [IR] holds, the KP and DOCE models were shown to generate the same time consistent demands. In this section, we assume that DOCE and KP preferences have the same homothetic time and risk preference building blocks (u, V) corresponding to (2) and relax Assumption [ICER]. As a result, resolute and sophisticated DOCE and KP demands all diverge. As a short digression, the first subsection demonstrates in a certainty

[23] If $\delta_i = 0$, one can use $u(c) = \ln(c)$ instead of power utility as in Theorem 2. This statement also applies to subsequent results unless indicated otherwise.

setting, when CES time preferences exhibit time inconsistency, the sophisticated consumption-saving solution can diverge dramatically from the resolute and naive solutions when the consumer has a strong preference for intertemporal substitution. In Subsection 5.2, a similar divergence is shown to arise for the DOCE solutions to the consumption-portfolio problem. Conversely when the consumer exhibits an aversion to intertemporal substitution, period 1 consumption (and saving) and portfolio composition (as reflected by the ratio n_{f1}/n_1)[24] can exhibit very similar behavior for the KP and DOCE resolute and sophisticated cases.

5.1 Strong Preference for Intertemporal Substitution: Divergent Sophisticated Saving Behavior

In this subsection, a simple certainty consumption-saving analysis based on CES time preferences and quasi-hyperbolic discounting (Laibson 1997) is considered. The latter assumption enables us to focus on time inconsistency where resolute and sophisticated choice diverge as in the DOCE consumption-portfolio setting when Assumption [ICER] does not hold. We show that when consumers exhibit a strong preference for intertemporal substitution, sophisticated choice can result in seemingly counterintuitive consumption and saving behavior. A similar pattern will be exhibited in the next subsection when we consider risky asset returns and relax the [ICER] assumption.

Assume a three period consumption-saving problem, where the consumer can invest in periods 1 and 2 in a risk free asset with a (gross) return R_f in each period. The period 1 budget constraint is given by

$$c_1 + \frac{c_2}{R_f} + \frac{c_3}{R_f^2} = I.$$

Assume quasi-hyperbolic discounted time preferences corresponding to the following period 1 and 2 utilities, respectively,

$$U^{(1)}(c_1, c_2, c_3) = -\frac{c_1^{-\delta_1}}{\delta_1} - \gamma\beta\frac{c_2^{-\delta_1}}{\delta_1} - \gamma\beta^2\frac{c_3^{-\delta_1}}{\delta_1}$$

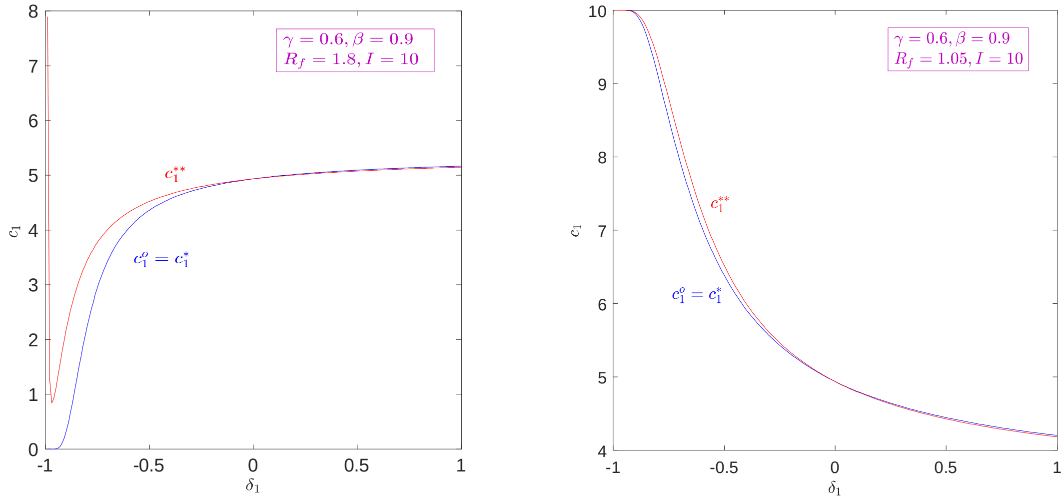
and

$$U^{(2)}(c_2, c_3) = -\frac{c_2^{-\delta_1}}{\delta_1} - \gamma\beta\frac{c_3^{-\delta_1}}{\delta_1},$$

where $\delta_1 > -1$ and $\delta_1 \neq 0$. Solving for $c_1^\circ = c_1^*$ and c_1^{**} and letting $\delta_1 \to -1$, one obtains

$$c_1^\circ = c_1^* \longrightarrow 0$$

[24] Since there is one node in period 1, we simplify the notation by denoting the risky and risk free asset holdings respectively by n_1 and n_{f1} instead of $n(s^1)$ and $n_f(s^1)$.



(a) $R_f = 1.8$ (b) $R_f = 1.05$

Figure 5:

if and only if

$$\gamma\beta R_f > 1 \qquad \text{or} \qquad \gamma\beta^2 R_f^2 > 1.$$

Based on numerical simulations, c_1^* and c_1^{**} can behave quite differently based on variations in the time preference parameter δ_1. For example, if[25]

$$\gamma = 0.6, \beta = 0.9, R_f = 1.8,$$

we have

$$\lim_{\delta_1 \to -1} c_1^* = 0 \qquad \text{and} \qquad \lim_{\delta_1 \to -1} c_1^{**} = I$$

as shown in Figure 5(a). To see the intuition for this result, notice that

$$\gamma\beta R_f = 0.918 \qquad \text{and} \qquad \gamma\beta^2 R_f^2 = 1.405. \tag{25}$$

Since $\gamma\beta^2 R_f^2 = 1.405 > 1$, for resolute (naive) choice, the return in the last period after taking into account the discount seems quite attractive. Assume the consumer is completely substitute oriented where $\delta_1 \to -1$ and the EIS, (12), goes to infinity. Then, the resolute (naive) consumer will not consume in periods 1 and 2, and will consume all of her initial income in period 3. The same substitute oriented consumer when following sophisticated choice behaves very differently. Again assuming $\delta_1 \to -1$, the sophisticated consumer, working

[25] The γ and β values assumed here are standard in the existing literature (see, for example, Laibson 1997, p. 456).

recursively, in period 2 views the return paying off in the third period $\gamma\beta R_f = 0.918 < 1$ as not good enough to merit saving. Hence she is inclined to consume everything in period 2. However, when viewed from the period 1 perspective, the period 2 return is the same unattractive $\gamma\beta R_f = 0.918 < 1$. Thus it seems best for the sophisticated consumer to do no saving and consume everything in period 1. Therefore, we have $c_1^{**} = I$ when $\delta_1 \to -1$. This behavior may seem counterintuitive since the sophisticated consumer is missing out on the attractive risk free return of $R_f = 1.8$ over two periods. In Figure 5(b), we illustrate the case where R_f takes the lower value of 1.05 and there is no opportunity for high two period returns after reflecting discounting. Resolute and naive choice no longer seek to postpone consumption to period 3 as $\delta_1 \to -1$ and all three solution approaches converge to consuming all of the initial income I in period 1.

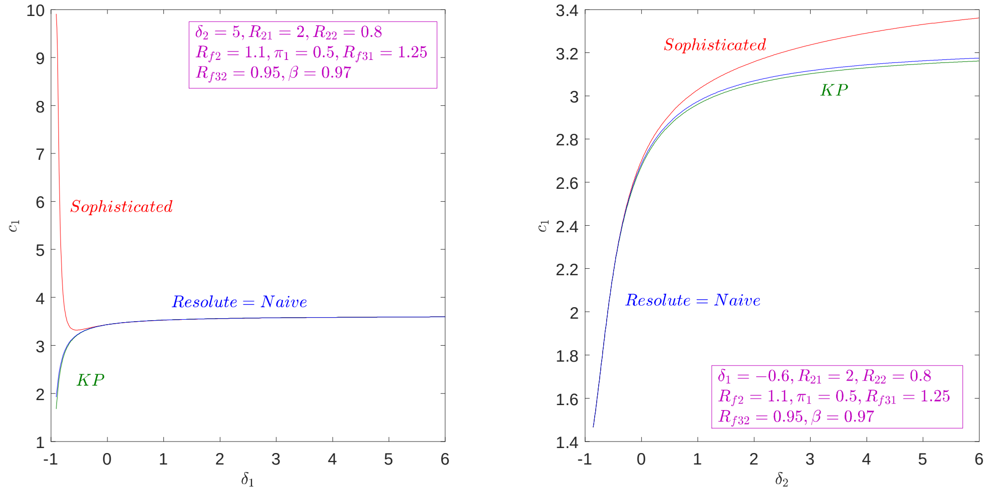
5.2 Disentangling the Effects of Time and Risk on Demand

In this subsection, the time and risk preference building blocks (u, V) take the special CES and CRRA forms in (2). Consistent with the analysis in the prior subsection, we show via a simple example that in a consumption-portfolio optimization where Assumption [ICER], or the special case [IR], does not hold, resolute, naive and sophisticated DOCE optimal demands can be quite similar so long as the time preference parameter δ_1 diverges sufficiently from -1 or the EIS (12) is sufficiently small. When δ_1 approaches -1, sophisticated DOCE demands can diverge significantly from resolute and naive demands. Also, sophisticated DOCE and KP demand can be quite similar or different depending on whether δ_1 diverges from or converges to -1. We illustrate these differences for optimal period 1 consumption and asset demands. With regard to the latter, it follows from Proposition 4 that when Assumption [ICER] holds, the portfolio composition as reflected in the asset ratio n_f/n is independent of the time preference parameters δ_1 and β. This independence fails to hold in the example below.

Assume a simplified version of the tree structure in Figure 4, where there are just two branches. In period 1, the consumer can buy one period risk free and risky assets, with respectively period 2 returns R_{f2} and R_{2s} and probability π_s $(s = 1, 2)$. In period 2, depending on which state is realized, there exists a risk free asset with return R_{f31} or R_{f32}. Period 1 asset holdings are denoted by n_1 and n_{f1}. Then period 2 income for the two branches is given by

$$I_{2s} = R_{2s}n_1 + R_{f2}n_{f1} \;\; (s = 1, 2) .$$

Optimal period 1 consumption and asset demands are derived for the DOCE



(a) (b)

Figure 6:

resolute, naive and sophisticated and KP cases. Their different responses to variations in the time and risk preference parameters δ_1 and δ_2 are illustrated.[26]

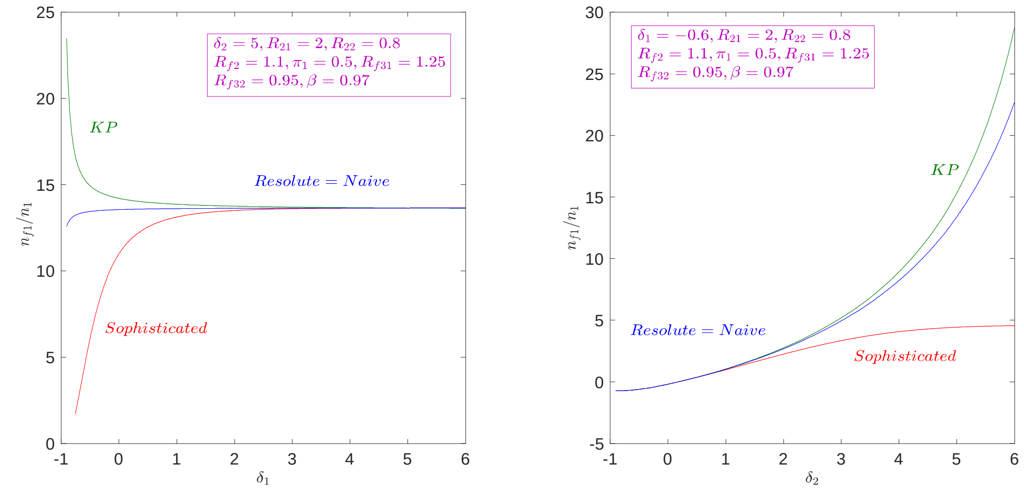
Example 2 *Assume the following parameter values*

$$R_{21} = 2, R_{22} = 0.8, R_{f2} = 1.1, R_{f31} = 1.25, R_{f32} = 0.95, \pi_1 = 0.5, \beta = 0.97, I = 10.$$

The results from numerical simulations of optimal c_1 as functions of δ_1 and δ_2 are plotted in Figures 6(a) and (b). Period 1 DOCE resolute and sophisticated and KP consumption values are generally quite close in value except when δ_1 is close to -1. The diverging pattern of DOCE sophisticated and resolute (naive) period 1 consumption in Figure 6(a) is similar to the quasi-hyperbolic discounting case in Figure 5(a). Interestingly, KP demand behaves similarly to the resolute (naive) case. Simulations of the optimal asset ratio n_{f1}/n_1 as functions of δ_1 and δ_2 are given in Figures 7(a) and (b). Based on the definitions of resolute and naive choice, $n^*_{f1}/n^*_1 = n^\circ_{f1}/n^\circ_1$. The KP and DOCE resolute and sophisticated asset ratios converge for the EU special case where $\delta_1 = \delta_2$. In contrast to Proposition 4 where Assumption [ICER] holds, in Figure 7(a), n_{f1}/n_1 varies with δ_1. In Figure 7(b) when $\delta_2 = 5$ and $\delta_1 = -0.6$, the KP and DOCE sophisticated asset ratios equal 15.28 and 4.43, respectively.[27] Clearly the differences in asset demand behavior reflects the presence of intertemporal risk in this example with asset returns being

[26] For details on these different optimizations, see Supplemental Appendix B.1.

[27] For comparison purposes, it should be noted that if one were to assume that [ICER] holds



(a) (b)

Figure 7:

positively correlated over time. The upper and lower branches in Figure 4 are associated respectively with (good, good) and (bad, bad) asset payoffs.

The results of this example suggest that when asset returns do not satisfy [ICER], if one follows much of the certainty empirical literature in assuming that the EIS is in the range of 0 and 0.4 (or using eqn. (12) $\delta_1 > 1.5$), then the DOCE and KP preference models generate qualitatively quite similar consumption and asset demand behavior. Alternatively, if one accepts the long-run risk and some macro EIS calibrations of 1.5 to 2.0 (or equivalently, $-0.5 < \delta_1 < -0.33$), then the demands differ significantly and differences in the respective preferences and underlying properties of TC, SEP and TRI become critical.

6 Concluding Comments

In this paper, we provide conditions such that DOCE preferences exhibit TC, SEP and TRI on a restricted domain of consumption trees corresponding to the consumption-portfolio problem. Under these same conditions, optimal consumption and asset demands for KP preferences are the same as the common DOCE resolute, naive and sophisticated demands. When the key Assumption [ICER] is

where $R_{f31} = R_{f32} = R_{f3}$ and the common risk free rate on both branches equals the value of the mean for this example, 1.10, then the asset ratio would be constant at the value $n_{f1}/n_1 = 4.70$ independent of δ_1.

relaxed, the demands for the KP and different DOCE solution techniques can be close but also can diverge significantly.

As mentioned in the Section 1, one key insight of our paper is that in intertemporal demand problems the presence of TC behavior does not depend just on preferences, but prices (or asset returns) can also play a crucial role. This differs from typical decision theoretic analyses such as in KP and Johnsen and Donaldson (1985) where the conditions for preferences to be TC are implicitly assumed to hold for all prices. Our key conditions [ICER] and [IR] have been shown to essentially be restrictions on return distributions or for complete markets on contingent claim prices. It would be interesting to consider more generally when such cases can arise. Consider the variation of HARA preferences in Theorem 2 where U takes the CES form and V takes the CARA form. Our result does not extend to this case. However, it can be verified that for a simplified tree structure and under additional restrictions including $\beta R_{f3} = 1$, a consumer with DOCE preferences becomes time consistent. It is interesting to note that this particular combination of time and risk preferences is assumed in Weil (1993) and more recently in variations of Hansen and Singleton (1995) preferences such as Tallarini (2000). Collectively, these results suggest the potential value of future research into the general question of joint restrictions on preferences and prices such that dynamic choice behavior is time consistent.

Another natural extension of our work would be to consider the critical role played by the value of the EIS measure for the case of KP and DOCE preferences based on the CRRA and translated origin CRRA preference models (2) and (6). Significant differences in both optimal consumption and asset demands can arise when the $EIS > 1$ (or $\delta_1 < 0$). Given that existing empirical research is inconclusive on whether the EIS measure is larger or smaller than unity, it would seem desirable to investigate this question particularly in the context of the simple dynamic structure in Example 2. Although a number of challenges exist in applying parametric or non-parametric tests to this setting, it would nevertheless seem to be an important area for future research.

Appendix

A Definitions and Proofs

A.1 Solution Techniques: Definitions

Definition 3 *The consumption-portfolio problem (14) - (17) is said to be solved via resolute choice if and only if the agent makes all choices at $t = 1$ and these* *choices are not revised over time as new choices become optimal. Given returns and initial income, we define resolute choice as*

$$(c^{\circ}(s^t), \mathbf{n}^{\circ}(s^t))_{s^t\in\mathcal{S}}\left((\mathbf{R}(s^t))_{s^t\in\mathcal{S}}, I\right) = \underset{c(s^t),\mathbf{n}(s^t)}{\arg\max}\ \mathcal{U}(\mathbf{c}|s^1) \quad S.T.$$

$$c(s^t) = I - \sum_j n_j(s^t), \quad t = 1,$$

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t) - \sum_j n_j(s^t), \quad 2 < t < T,$$

and

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t), \quad t = T.$$

Definition 4 *The consumption-portfolio problem (14) - (17) is said to be solved via naive choice if and only if the agent reoptimizes and revises her choices every period based on her current period preferences. Naive choice is defined sequentially for $\tau = 1, 2, ..., T$ as*

$$(c^*(s^\tau), \mathbf{n}^*(s^\tau))\left(I\left(s^\tau\right)\right) = (c^{\circ}(s^\tau), \mathbf{n}^{\circ}(s^\tau))\left((\mathbf{R}(s^t))_{s^t\in\mathcal{S}}, I\right)$$

where

$$(c^{\circ}(s^t), \mathbf{n}^{\circ}(s^t))_{s^t\succeq s^\tau}\left((\mathbf{R}(s^t))_{s^t\in\mathcal{S}}, I\right) = \underset{c(s^t),\mathbf{n}(s^t)}{\arg\max}\ \mathcal{U}(\mathbf{c}|s^\tau) \quad S.T.$$

$$c(s^t) = I - \sum_j n_j(s^t), \quad t = \tau,$$

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t) - \sum_j n_j(s^t), \quad \tau < t < T,$$

and

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t), \quad t = T.$$

Definition 5 *The consumption-portfolio problem (14) - (17) is said to be solved via sophisticated choice if and only if the agent takes into account her future period preferences when making her choices in earlier periods. The sophisticated choice can be defined recursively for $\tau = T, T-1, \ldots$ as*[28]

$$(c^{**}(s^\tau), \mathbf{n}^{**}(s^\tau))\left(I\left(s^\tau\right)\right) = \underset{c(s^\tau),\mathbf{n}(s^\tau)}{\arg\max}\ u\left(c(s^\tau)\right) + \sum_{t=\tau+1}^{T} \beta^{t-\tau} u(\widehat{c}_t|s^\tau) \quad S.T.$$

[28]It should be noted that in general a unique sophisticated choice may not exist in the recursive solution process. However for the utility functions we consider in this paper, a unique solution always exists since (quasi)homotheticity ensures concavity of the corresponding utility functions. Also, note that we have written $U(\mathbf{c}|s^\tau)$ as a separable form in order to highlight the role of $(\widehat{c}_t|s^\tau)$.

$$c(s^t) = I(s^t) - \sum_j n_j(s^t), \quad t = \tau,$$

and

$$(\widehat{c}_t|s^\tau) = V^{-1}\left(\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau)V(c^{**}(s^t)(\mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t)))\right).$$

A.2 Proof of Proposition 1

Generalizing the example in Subsection 4.1.1 and denoting by $\alpha(s^t) = \frac{c(s^t)}{c(s^{t-1})}$, we obtain

$$\begin{aligned}
\mathcal{U}(\alpha|s^\tau) &= u(c(s^\tau)) + \beta u \circ V^{-1}\left(\sum_{s^{\tau+1}} \pi(s^{\tau+1}|s^\tau)V(\alpha(s^{\tau+1})c(s^\tau))\right) + \ldots + \\
&\quad \beta^{T-1} u \circ V^{-1}\left(\sum_{s^{\tau+1} \succ s^\tau} \pi(s^{\tau+1}|s^\tau) \ldots \sum_{s^T \succ s^\tau} \pi(s^T|s^{T-1})V(\alpha(s^{\tau+1}) \cdot \ldots \cdot \alpha(s^T)c(s^\tau))\right) \\
&= u(c(s^\tau)) + \beta u \circ V^{-1}\left(\sum_{s^{\tau+1} \succ s^\tau} \pi(s^{\tau+1}|s^\tau)V(\alpha(s^{\tau+1})c(s^\tau))\right) K_{\tau+1},
\end{aligned}$$

where $K_{\tau+1}$ is recursively defined as

$$K_T = 1 + \beta u \circ V^{-1}\left(\sum_{s^T \succ s^{T-1}} \pi(s^T|s^{T-1})V(\alpha(s^T))\right)$$

and

$$K_t = 1 + \beta u \circ V^{-1}\left(\sum_{s^t \succ s^{t-1}} \pi(s^t|s^{t-1})V(\alpha(s^t))\right) K_{t+1}$$

for $t = \tau + 1, \ldots T - 1$. Note that $\mathbf{c} \in \mathcal{I}$ ensures that K_t does not depend on s^t.

By the same argument as in the example, it is now clear that if α is preferred to $\widetilde{\alpha}$ at τ it must be preferred at $\tau - 1$ and, by induction, preferred at any $\tau - i$, $i = 1, \ldots, \tau - 1$.

A.3 Proof of Theorem 1

In order to facilitate the proof, we first introduce Assumption [ICER*] and show that it is equivalent to [ICER]. Define recursively for each s^t, $t = T-1, T-2, \ldots$, $\widehat{\mathbf{n}}(s^t) \in \mathbb{R}^J$ to be the unique solution to the J equations

$$\sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t)R(s^{t+1})V'\left(\frac{R(s^{t+1}) \cdot \widehat{\mathbf{n}}(s^t)}{1 + \sum_j \widehat{\mathbf{n}}_j(s^{t+1})}\right) =$$

$$\frac{1}{\beta(u \circ V^{-1})'\left(\sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t)V\left(\frac{R(s^{t+1}) \cdot \widehat{\mathbf{n}}(s^t)}{1 + \sum_j \widehat{\mathbf{n}}_j(s^{t+1})}\right)\right)},$$

where $\widehat{\mathbf{n}}(s^t) = \frac{\mathbf{n}(s^t)}{c(s^t)}$ and $\widehat{\mathbf{n}}(s^T) = 0$ for all s^T.
Assumption [ICER*] Assume that for all s^t, $t < T$,

$$\sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) V \left(\frac{R(s^{t+1}) \cdot \widehat{\mathbf{n}}(s^t)}{1 + \sum_j \widehat{\mathbf{n}}_j(s^{t+1})} \right) = K_t,$$

where K_t only depends on t.

For this it suffices to show that under [ICER], $\widetilde{\mathbf{n}}_j(s^t)$ (as defined in eqn. (22)) is constant across all s^t for given t. By induction, we consider first $t = T - 1$. Homotheticity ensures that [ICER] can be written as

$$\sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) R(s^{t+1}) \cdot \widetilde{\mathbf{n}}(s^t) V' \left(R(s^{t+1}) \cdot \widetilde{\mathbf{n}}(s^t) \right) = \tilde{K}_t,$$

which is independent of s^t. Taking the J equations in (22) and weighting each j with $\widetilde{\mathbf{n}}_j(s^t)$ and summing up, this implies that $\sum_j \widetilde{\mathbf{n}}_j(s^t)$ must be independent of s^t, for $t = T - 1$. But then the same argument applies for each $t < T$ and [ICER] and [ICER*] are equivalent conditions.

Next, we prove that [ICER*] together with homothetic utility is sufficient. The following first order conditions are necessary and sufficient for naive choice at s^τ for consumption at some future node $\bar{s}^t \succeq s^\tau$

$$V'(c(\bar{s}^t)) \left(u \circ V^{-1}\right)' \left(\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau) V(c(s^t)) \right) =$$
$$\beta (u \circ V^{-1})' \left(\sum_{s^{t+1} \succ s^\tau} \pi(s^{t+1}|s^\tau) V\left(c(s^{t+1})\right) \right) \sum_{s^{t+1} \succ \bar{s}^t} R(s^{t+1}) \pi(s^{t+1}|\bar{s}^t) V'\left(c\left(s^{t+1}\right)\right),$$

for all $\bar{s}^t$, $t < T$. Since $u(.)$ and $V(.)$ are assumed to be homothetic, it is clear that these necessary and sufficient first order conditions will be satisfied for some $\alpha(s^t)$ that satisfy [ICER*], and that these $\alpha(s^t)$ do not change with τ. Therefore naive choice does not change with τ and choices are time consistent.

To prove necessity of homothetic utility given [ICER*], consider the simplified case of three periods, $t = 1, 2, 3$, based on a version of the event tree depicted in Figure 4 where there are just two branches. Suppose markets are complete. To satisfy Assumption [ICER*] suppose that the prices of the contingent claims are identical and denoted by $p(2)$.

The first order conditions for optimal naive choice at $t = 2$ are

$$p(2) u'(c_{2s}) = \beta u'(c_{3s}), \quad (s = 1, 2)$$

and, at $t = 1$, planning for $t = 2$, are

$$p(2)V'(c_{2s})(u \circ V^{-1})' \left(\sum_s \pi_s V(c_{2s}) \right) = \beta (u \circ V^{-1})' \left(\sum_s \pi_s V(c_{3s}) \right) V'(c_{3s}).$$

The first equation implies

$$c_{3s} = u'^{-1} \left(\frac{p(2)}{\beta} u'(c_{2s}) \right)$$

and substituting this into the second equation, we obtain

$$\begin{aligned} & p(1)V'(c_{2s})(u \circ V^{-1})' \left(\sum_s \pi_s V(c_{2s}) \right) = \\ & \beta (u \circ V^{-1})' \left(\sum_s \pi_s V \left(u'^{-1} \left(\frac{p(1)}{\beta} u'(c_{2s}) \right) \right) \right) \\ & V' \left(u'^{-1} \left(\frac{p(2)}{\beta} u'(c_{2s}) \right) \right). \end{aligned} \tag{A.1}$$

Denote the price $p(2)$ simply by p. Then we consider variations in $p(2) = p$ as well as first period prices $p(1)$ that keep second period consumption fixed. Taking the derivative with respect to p on both sides and then setting $p = \beta$, one obtains

$$\begin{aligned} 1 \;=\; & \frac{(u \circ V^{-1})'' \left(\sum_s \pi_s V(c_{2s}) \right) \sum_s \pi_s \left(V'(c_{2s}) u'^{-1\prime} \circ u'(c_{2s}) u'(c_{2s}) \right)}{(u \circ V^{-1})' \left(\sum_s \pi_s V(c_{2s}) \right)} \\ & + \frac{V''(u'^{-1\prime} u'(c_{2s})) u'(c_{2s})}{V'(c_{2s})}. \end{aligned}$$

Taking the derivatives with respect to c_{2s}, $s = 1, 2$, we obtain[29]

$$\frac{d}{dc} \frac{f'^{-1\prime}(g(c))g(c)}{f(c)} = 0,$$

where $f(c) = V'(c)$ and $g(c) = u'(c)$.

Since

$$g^{-1\prime}(g(c))g'(c) = 1,$$

we obtain

$$\frac{d}{dc} \frac{f'(c)g(c)}{g'(c)f(c)} = 0.$$

Consider the following ordinary differential equation

$$\frac{d}{dc} \left(\frac{f'(c)\, g(c)}{f(c)\, g'(c)} \right) = 0.$$

[29] This is possible since we can vary the prices of both Arrow securities at $t = 1$ independently.

We have

$$\frac{f'(c)\,g(c)}{f(c)\,g'(c)} = K_1,$$

where K_1 is a constant. Therefore,

$$\frac{f'(c)}{f(c)} = (\ln f(c))' = K_1 \frac{g'(c)}{g(c)} = K_1 (\ln g(c))',$$

implying that

$$\ln f(c) = K_1 \ln g(c) + K_2,$$

where K_2 is a constant. Thus we have

$$f(c) = K_3 (g(c))^{K_1},$$

where K_3 is a constant.

Assuming $K > 0$, we can write $V'(c) = u'^K$ and $V'^{-1}(x) = u'^{-1}(x^{\frac{1}{K}})$. Substituting this into (A.1), we obtain

$$p(u \circ V^{-1})' \left(\sum_s \pi_s V(c_{2s}) \right) = \beta(u \circ V^{-1})' \left(\sum_s \pi_s V \left(u'^{-1} \left(\frac{p}{\beta} u'(c_{2s}) \right) \right) \right) \left(\frac{p}{\beta} \right)^{\frac{1}{K}}.$$

Since $u \circ V^{-1}(x) = x^{\nu}$ for some ν, it follows that the above can only hold if $u\left((u')^{-1}(x)\right)$ is homothetic. In this case, we can write

$$u\left((u')^{-1}(x)\right) = ax^{\delta}.$$

Then we have

$$(u')^{-1}(x) = u^{-1}\left(ax^{\delta}\right).$$

Assuming

$$(u')^{-1}(x) = y,$$

then

$$u^{-1}\left(ax^{\delta}\right) = y \Leftrightarrow x = \left(\frac{u(y)}{a}\right)^{\frac{1}{\delta}}.$$

Therefore, we have

$$u'(x) = a\,(u(x))^{\delta}.$$

Thus if $\delta \neq 1$, we have

$$\frac{d\,(u(x))^{1-\delta}}{dx} = a(1-\delta) \Rightarrow u(x) = (a(1-\delta)x + c)^{-\frac{1}{1-\delta}}.$$

This corresponds to the DARA or IARA case of the HARA class. If $\delta = 1$,

$$\frac{d \ln u(x)}{dx} = a(1-\delta) \Rightarrow u(x) = \exp(a(1-\delta)x + c).$$

A simple numerical example can show that DARA and IARA utilities within the HARA class do not produce time consistent demand unless the conditions of Theorem 2 holds.

In the last step we prove that under homothetic utility, the assumption [ICER*] is necessary for time consistency. Suppose [ICER*] does not hold and consider the first order conditions for optimal resolute choice at some date τ of assets at some future $t > \tau$

$$V'(c(s^t))\left(u \circ V^{-1}\right)'\left(\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau)V(c(s^t))\right) =$$

$$\beta(u \circ V^{-1})'\left(\sum_{s^{t+1} \succ s^\tau} \pi(s^{t+1}|s^\tau)V\left(c(s^{t+1})\right)\right)\sum_{s^{t+1} \succ s^t} R(s^{t+1})\pi(s^{t+1}|s^t)V'\left(c\left(s^{t+1}\right)\right).$$

It is clear that they can only be satisfied for the same choices $c(s^t), c(s^{t+1})$, $s^{t+1} \succ s^t$, at two different dates τ, τ' if the ratio of the terms in the brackets are the same, i.e., if

$$\frac{\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau)V(c(s^t))}{\sum_{s^{t+1} \succ s^\tau} \pi(s^{t+1}|s^\tau)V\left(c(s^{t+1})\right)}$$

is independent of τ. But this implies $\mathbf{c} \in \mathcal{I}$ which can only hold if [ICER] holds. This completes the proof.

A.4 Proof of Theorem 2

For (i) note that the maximization problem of an individual with utilities given by (6) - (9) is identical to the maximization problem of an individual who has utilities given by (2) except that she needs to purchase b/R_f units of the risk free asset at each s^t, $t < T$, to fund her subsistence requirement b. For the case (11), one can apply a similar argument. Since this in turn is equivalent to a problem where the individual's utility is homothetic and her income is appropriately adjusted, [ICER] remains necessary and sufficient for TC given the homothetic utility function.

For (ii) consider the first order conditions for optimal resolute choice at some date-event s^τ of assets at some future $t > \tau$ for the risk free asset

$$V'(c(s^t))\left(u \circ V^{-1}\right)'\left(\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau)V(c(s^t))\right) =$$

$$\beta(u \circ V^{-1})'\left(\sum_{s^{t+1} \succ s^\tau} \pi(s^{t+1}|s^\tau)V\left(c(s^{t+1})\right)\right)\sum_{s^{t+1} \succ s^t} R_{t+1}\pi(s^{t+1}|s^t)V'\left(c\left(s^{t+1}\right)\right).$$

Adding over all $s^t \succ s^\tau$ weighted by $\pi(s^t|s^\tau)$, taking into account that $V'(c) =$

$-\frac{1}{\kappa_2}V(c)$ we obtain

$$\left(u \circ V^{-1}\right)'\left(\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau)V(c(s^t))\right) =$$
$$\beta(u \circ V^{-1})\left(\sum_{s^{t+1} \succ s^\tau} \pi(s^{t+1}|s^\tau)V\left(c(s^{t+1})\right)\right) R_{t+1}.$$

Since this holds for an s^τ it follows that the first order conditions for resolute choice do not change with τ and hence choice satisfies time consistency.

A.5 Proof of Proposition 2

The first key insight is that DOCE and KP preferences generate identical utility functions over I. To see this, let $\alpha(s^t) = \frac{c(s^t)}{c(s^{t-1})}$ and recall that DOCE utility can be written as follows

$$\begin{aligned}
\mathcal{U}(\alpha|s^\tau) &= u(c(s^\tau)) + \beta u \circ V^{-1}\left(\sum_{s^{\tau+1} \succ s^\tau} \pi\left(s^{\tau+1}|s^\tau\right) V(\alpha(s^{\tau+1})c(s^\tau))\right) + \ldots + \\
&\quad \beta^{T-1} u \circ V^{-1}\left(\sum_{s^{\tau+1} \succ s^\tau} \pi\left(s^{\tau+1}|s^\tau\right) \ldots \sum_{s^T \succ s^{T-1}} \pi\left(s^T|s^{T-1}\right) V(\alpha(s^{\tau+1}) \cdot \ldots \cdot \alpha(s^T)c(s^\tau))\right) \\
&= u(c(s^\tau)) + \beta u \circ V^{-1}\left(\sum_{s^{\tau+1} \succ s^\tau} \pi\left(s^{\tau+1}|s^\tau\right) V(\alpha(s^{\tau+1})c(s^\tau))\right) K_{\tau+1},
\end{aligned}$$

where $K_{\tau+1}$ is recursively defined as

$$K_T = 1 + \beta u \circ V^{-1}\left(\sum_{s^T \succ s^{T-1}} \pi\left(s^T|s^{T-1}\right) V(\alpha(s^T))\right)$$

and

$$K_t = 1 + \beta u \circ V^{-1}\left(\sum_{s^t \succ s^{t-1}} \pi\left(s^t|s^{t-1}\right) V(\alpha(s^t))\right) K_{t+1}.$$

Similarly KP utility can be written as

$$
\begin{aligned}
&\mathcal{U}^{KP}(\mathbf{c}|s^\tau) \\
= \quad & -\frac{c(s^\tau)^{-\delta_1}}{\delta_1} - \frac{\beta\left(\sum_{s^{\tau+1}\succ s^\tau}\pi(s^{\tau+1}|s^\tau)\mathcal{U}^{KP}(\mathbf{c}|s^{\tau+1})^{-\frac{\delta_2}{\delta_1}}\right)^{\frac{\delta_1}{\delta_2}}}{\delta_1} \\
= \quad & -\frac{c(s^\tau)^{-\delta_1}}{\delta_1} - \beta\frac{c(s^\tau)^{-\delta_1}}{\delta_1} \\
& \left(\begin{array}{c}\left(\sum_{s^{\tau+1}\succ s^\tau}\pi(s^{\tau+1}|s^\tau))\alpha(s^{\tau+1})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} \\ \left(1+\beta\left(\sum_{s^{\tau+2}\succ s^{\tau+1}}\pi(s^{\tau+2}|s^{\tau+1})\alpha(s^{\tau+2})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}(1+\ldots)\right)\end{array}\right),
\end{aligned}
$$

which, when multiplied out, is identical to DOCE utility. It remains to be shown that optimal choice under KP utility lies in $\mathcal{I}$. The necessary and sufficient conditions for optimal choice can be written as

$$
u'(c(s^t)) = \beta(u\circ V^{-1})'\left(\sum_{s^{t+1}\succ s^t} V\circ\mathcal{U}(\mathbf{c}|s^{t+1})\right)\sum_{s^{t+1}\succ s^t}\pi(s^{t+1}|s^t)R(s^{t+1})(V\circ u^{-1})'\mathcal{U}(\mathbf{c}|s^{t+1})u'(c(s^{t+1})).
$$

At $T-1$ KP and DOCE coincide, hence we can substitute for $\mathcal{U}$ and we obtain that $\sum_{s^{T-1}\succ s^{T-2}}\pi(s^{T-1}|s^{T-2})V(\alpha(s^{T-1}))$ is constant for all s^{T-2}. By induction this is then true for all t and hence KP and DOCE preferences generate the same demands.

A.6 Proof of Proposition 3

Part (i) follows from exactly the same argument as in the proof of Theorem 2. For both preference specifications we can rewrite the optimization problem as maximizing homothetic utility subject to an adjusted income and Proposition 2 then implies equivalence.

To show (ii), as in the Proof of Proposition 2, it is easy to see that for CARA utility functions utilities become identical whenever $\mathbf{c}\in\mathcal{I}$. We have shown above that non-stochastic risk free interest rates ensure that optimal choices for DOCE preferences lie in $\mathcal{I}$. To see that optimal choices for KP must also lie in $\mathcal{I}$, consider the first order conditions as in the proof of Proposition 2 – clearly under CRRA utility they are satisfied for $\mathbf{c}\in\mathcal{I}$. This completes the proof.

A.7 Proof of Proposition 4

It follows from Proposition 3 that DOCE demands are time consistent and the same as those for KP preferences. Therefore, it is enough to consider DOCE sophisticated choice. Consider case (i) with $b=0$. As in the proof of Proposition 2, assuming [ICER] holds, let $\alpha(s^t)=\frac{c(s^t)}{c(s^{t-1})}$ and the period $\tau-1$ DOCE utility can be written as follows

$$
\begin{aligned}
\mathcal{U}(\alpha|s^{\tau-1}) &= u(c(s^{\tau-1}))+\beta u\circ V^{-1}\left(\sum_{s^\tau\succ s^{\tau-1}}\pi\left(s^\tau|s^{\tau-1}\right)V(c(s^\tau))\right)+\\
&\quad \beta^2 u\circ V^{-1}\left(\sum_{s^{\tau+1}\succ s^\tau}\pi\left(s^{\tau+1}|s^\tau\right)V(\alpha(s^{\tau+1})c(s^\tau))\right)+\ldots+\\
&\quad \beta^{T-1}u\circ V^{-1}\left(\sum_{s^{\tau+1}\succ s^\tau}\pi\left(s^{\tau+1}|s^\tau\right)\ldots\sum_{s^T\succ s^{T-1}}\pi\left(s^T|s^{T-1}\right)V(\alpha(s^{\tau+1})\cdot\ldots\cdot\alpha(s^T)c(s^\tau))\right)\\
&= u(c(s^{\tau-1}))+\beta u\circ V^{-1}\left(\sum_{s^\tau\succ s^{\tau-1}}\pi\left(s^\tau|s^{\tau-1}\right)V(c(s^\tau))\right)K_{\tau+1},
\end{aligned}
$$

where $K_{\tau+1}$ is recursively defined as

$$K_T=1+\beta u\circ V^{-1}\left(\sum_{s^T\succ s^{T-1}}\pi\left(s^T|s^{T-1}\right)V(\alpha(s^T))\right)$$

and

$$K_t=1+\beta u\circ V^{-1}\left(\sum_{s^t\succ s^{t-1}}\pi\left(s^\tau|s^{\tau-1}\right)V(\alpha(s^t))\right)K_{t+1}.$$

Following sophisticated choice, the optimal asset ratios $n_f\left(s^{\tau-1}\right)/n_j\left(s^{\tau-1}\right)$ are determined by maximizing the EU function

$$\sum_{s^\tau\succ s^{\tau-1}}\pi\left(s^\tau|s^{\tau-1}\right)V\left(c(s^\tau)\right)=\frac{-1}{\delta_2}\sum_{s^\tau\succ s^{\tau-1}}\pi\left(s^\tau|s^{\tau-1}\right)c(s^\tau)^{-\delta_2},$$

which is independent of the time preference parameters δ_1 in u and β. This also implies that $\frac{n_f(s^\tau)}{n_j(s^\tau)}$ is independent of the time preference parameters δ_1 and β. For the other cases, the argument is the same as in the proof of Theorem 2.

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B For Online Publication: Supplemental Appendix

B.1 Supporting Materials for Section 5.2

To solve for the asset ratio for the DOCE resolute case, first note that from the constraints

$$c_{31} = R_{f31}\left(R_{21}n_1 + R_{f2}n_{f1} - c_{21}\right) \quad \text{and} \quad c_{32} = R_{f32}\left(R_{22}n_1 + R_{f2}n_{f1} - c_{22}\right).$$

It follows that

$$n_1 = \frac{\frac{c_{31}}{R_{f31}} + c_{21} - \frac{c_{32}}{R_{f32}} - c_{22}}{R_{21} - R_{22}} \quad \text{and} \quad n_{f1} = \frac{R_{21}\left(\frac{c_{32}}{R_{f32}} + c_{22}\right) - R_{22}\left(\frac{c_{31}}{R_{f31}} + c_{21}\right)}{\left(R_{21} - R_{22}\right)R_{f2}}.$$

Therefore, the period 1 budget constraint is

$$\begin{aligned} I &= c_1 + n_1 + n_{f1} \\ &= c_1 + \frac{\frac{c_{31}}{R_{f31}} + c_{21} - \frac{c_{32}}{R_{f32}} - c_{22}}{R_{21} - R_{22}} + \\ &\quad \frac{R_{21}\left(\frac{c_{32}}{R_{f32}} + c_{22}\right) - R_{22}\left(\frac{c_{31}}{R_{f31}} + c_{21}\right)}{\left(R_{21} - R_{22}\right)R_{f2}} \\ &= c_1 + \frac{R_{f2} - R_{22}}{\left(R_{21} - R_{22}\right)R_{f2}}c_{21} + \frac{R_{21} - R_{f2}}{\left(R_{21} - R_{22}\right)R_{f2}}c_{22} \\ &\quad + \frac{R_{f2} - R_{22}}{\left(R_{21} - R_{22}\right)R_{f31}R_{f2}}c_{31} + \frac{R_{21} - R_{f2}}{\left(R_{21} - R_{22}\right)R_{f32}R_{f2}}c_{32}. \end{aligned}$$

Thus the first order conditions for DOCE resolute choice are

$$\frac{\pi_1 c_{21}^{-1-\delta_2}}{\pi_2 c_{22}^{-1-\delta_2}} = \frac{R_{f2} - R_{22}}{R_{21} - R_{f2}} \Leftrightarrow c_{22} = \left(\frac{\pi_2\left(R_{f2} - R_{22}\right)}{\pi_1\left(R_{21} - R_{f2}\right)}\right)^{\frac{1}{1+\delta_2}} c_{21}$$

and

$$\frac{\pi_1 c_{31}^{-1-\delta_2}}{\pi_2 c_{32}^{-1-\delta_2}} = \frac{\left(R_{f2} - R_{22}\right)R_{f32}}{\left(R_{21} - R_{f2}\right)R_{f31}} \Leftrightarrow c_{32} = \left(\frac{\pi_2\left(R_{f2} - R_{22}\right)R_{f32}}{\pi_1\left(R_{21} - R_{f2}\right)R_{f31}}\right)^{\frac{1}{1+\delta_2}} c_{31}.$$

Therefore, the period 1 DOCE utility function can be transformed into a certainty utility of the single branch (c_1, c_{21}, c_{31}), which is

$$\left(\begin{array}{c} c_1^{-\delta_1} + \beta\left(\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} c_{21}^{-\delta_1} \\ +\beta^2\left(\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)R_{f32}}{\pi_1\left(R_{21}-R_{f2}\right)R_{f31}}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} c_{31}^{-\delta_1} \end{array}\right)^{-\frac{1}{\delta_1}} \tag{B.1}$$

and the budget constraint can be rewritten as

$$I = c_1 + \left(\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f2}} \left(\frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \right)^{\frac{1}{1+\delta_2}} \right) c_{21} + \left(\begin{array}{c} \frac{R_{f2}-R_{22}}{(R_{21}-R_{22})R_{f31}R_{f2}} + \\ \frac{R_{21}-R_{f2}}{(R_{21}-R_{22})R_{f32}R_{f2}} \left(\frac{\pi_2 \left(R_{f2}-R_{22}\right) R_{f32}}{\pi_1 \left(R_{21}-R_{f2}\right) R_{f31}} \right)^{\frac{1}{1+\delta_2}} \end{array} \right) c_{31}. \tag{B.2}$$

The first order condition is

$$\frac{\beta \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2}-R_{22}\right)}{\pi_1 \left(R_{21}-R_{f2}\right)} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}} c_{21}^{-1-\delta_1}}{\beta^2 \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2}-R_{22}\right) R_{f32}}{\pi_1 \left(R_{21}-R_{f2}\right) R_{f31}} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}} c_{31}^{-1-\delta_1}}$$

$$= \frac{\frac{R_{f2}-R_{22}}{(R_{21}-R_{22})R_{f2}} + \frac{R_{21}-R_{f2}}{(R_{21}-R_{22})R_{f2}} \left(\frac{\pi_2 \left(R_{f2}-R_{22}\right)}{\pi_1 \left(R_{21}-R_{f2}\right)} \right)^{\frac{1}{1+\delta_2}}}{\frac{R_{f2}-R_{22}}{(R_{21}-R_{22})R_{f31}R_{f2}} + \frac{R_{21}-R_{f2}}{(R_{21}-R_{22})R_{f32}R_{f2}} \left(\frac{\pi_2 \left(R_{f2}-R_{22}\right) R_{f32}}{\pi_1 \left(R_{21}-R_{f2}\right) R_{f31}} \right)^{\frac{1}{1+\delta_2}}},$$

or equivalently,

$$c_{31} = \kappa c_{21},$$

where

$$\kappa = \left(\frac{\beta \left(\frac{R_{f2}-R_{22}}{(R_{21}-R_{22})R_{f2}} + \frac{R_{21}-R_{f2}}{(R_{21}-R_{22})R_{f2}} \left(\frac{\pi_2 \left(R_{f2}-R_{22}\right)}{\pi_1 \left(R_{21}-R_{f2}\right)} \right)^{\frac{1}{1+\delta_2}} \right) \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2}-R_{22}\right) R_{f32}}{\pi_1 \left(R_{21}-R_{f2}\right) R_{f31}} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}}}{\left(\frac{R_{f2}-R_{22}}{(R_{21}-R_{22})R_{f31}R_{f2}} + \frac{R_{21}-R_{f2}}{(R_{21}-R_{22})R_{f32}R_{f2}} \left(\frac{\pi_2 \left(R_{f2}-R_{22}\right) R_{f32}}{\pi_1 \left(R_{21}-R_{f2}\right) R_{f31}} \right)^{\frac{1}{1+\delta_2}} \right) \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2}-R_{22}\right)}{\pi_1 \left(R_{21}-R_{f2}\right)} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}}} \right)^{\frac{1}{1+\delta_1}}.$$

Therefore, we have

$$
\begin{aligned}
\frac{n_{f1}^{\circ}}{n_1^{\circ}} &= \frac{R_{21}\left(\frac{c_{32}}{R_{f32}}+c_{22}\right)-R_{22}\left(\frac{c_{31}}{R_{f31}}+c_{21}\right)}{R_f\left(\frac{c_{31}}{R_{f31}}+c_{21}-\frac{c_{32}}{R_{f32}}-c_{22}\right)} \\
&= \frac{\left(\begin{array}{c}\frac{\kappa R_{21}}{R_{f32}}\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)R_{f32}}{\pi_1\left(R_{21}-R_{f2}\right)R_{f31}}\right)^{\frac{1}{1+\delta_2}}+R_{21}\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{\frac{1}{1+\delta_2}} \\ -\left(\frac{\kappa R_{22}}{R_{f31}}+R_{22}\right)\end{array}\right)}{R_{f2}\left(\frac{\kappa}{R_{f31}}+1-\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)R_{f32}}{\pi_1\left(R_{21}-R_{f2}\right)R_{f31}}\right)^{\frac{1}{1+\delta_2}}\frac{\kappa}{R_{f32}}-\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{\frac{1}{1+\delta_2}}\right)}.
\end{aligned}
$$

For DOCE sophisticated choice, the period 1 utility function is

$$
\begin{aligned}
\mathcal{U}(\mathbf{c}) &= \left(c_1^{-\delta_1}+\beta\left(\pi_1 c_{21}^{-\delta_2}+\pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}+\beta^2\left(\pi_1 c_{31}^{-\delta_2}+\pi_2 c_{32}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}} \\
&= \left(\begin{array}{c}
c_1^{-\delta_1}+\beta\left(\pi_1\frac{\left(R_{21}n_1+R_{f2}n_{f1}\right)^{-\delta_2}}{\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}}+\pi_2\frac{\left(R_{22}n_1+R_{f2}n_{f1}\right)^{-\delta_2}}{\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}}\right)^{\frac{\delta_1}{\delta_2}}+ \\
\beta^2\left(\begin{array}{c}\pi_1\frac{\beta^{-\frac{\delta_2}{1+\delta_1}}R_{f31}^{-\frac{\delta_2}{1+\delta_1}}\left(R_{21}n_1+R_{f2}n_{f1}\right)^{-\delta_2}}{\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}}+ \\ \pi_2\frac{\beta^{-\frac{\delta_2}{1+\delta_1}}R_{f32}^{-\frac{\delta_2}{1+\delta_1}}\left(R_{22}n_1+R_{f2}n_{f1}\right)^{-\delta_2}}{\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}}\end{array}\right)^{\frac{\delta_1}{\delta_2}}
\end{array}\right)^{-\frac{1}{\delta_1}}.
\end{aligned}
$$

The DOCE sophisticated case simulations in Example 2 can follow the first order conditions based on the above equation. For KP preferences, the period 1 utility function is

$$
\begin{aligned}
\mathcal{U}(\mathbf{c}) &= \left(c_1^{-\delta_1}+\beta\left(\pi_1\left(c_{21}^{-\delta_1}+\beta c_{31}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}+\pi_2\left(c_{22}^{-\delta_1}+\beta c_{32}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}} \\
&= \left(c_1^{-\delta_1}+\beta\left(\begin{array}{c}\pi_1\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}\left(R_{21}n_1+R_{f2}n_{f1}\right)^{-\delta_2}+ \\ \pi_2\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}\left(R_{22}n_1+R_{f2}n_{f1}\right)^{-\delta_2}\end{array}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}}.
\end{aligned}
$$

Defining

$$k_2 = \frac{\pi_2\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}(R_{f2}-R_{22})}{\pi_1\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}(R_{21}-R_{f2})},$$

we have the following conditional demands

$$n_{f1} = \frac{\left(R_{21}k_2^{\frac{1}{1+\delta_2}}-R_{22}\right)(I-c_1)}{R_{f2}-R_{22}+k_2^{\frac{1}{1+\delta_2}}(R_{21}-R_{f2})} \quad \text{and} \quad n_1 = \frac{\left(1-k_2^{\frac{1}{1+\delta_2}}\right)R_{f2}(I-c_1)}{R_{f2}-R_{22}+k_2^{\frac{1}{1+\delta_2}}(R_{21}-R_{f2})}.$$

The period 1 utility function can be rewritten as

$$\begin{aligned}
\mathcal{U}(\mathbf{c}) &= \left(c_1^{-\delta_1}+\beta\left(\pi_1\left(c_{21}^{-\delta_1}+\beta c_{31}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}+\pi_2\left(c_{22}^{-\delta_1}+\beta c_{32}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}} \\
&= \left(c_1^{-\delta_1}+\beta\left(\begin{array}{c}\pi_1\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}\left(\frac{(R_{21}-R_{22})R_{f2}(I-c_1)}{R_{f2}-R_{22}+k_2^{\frac{1}{1+\delta_2}}\left(R_{21}-R_{f2}\right)}\right)^{-\delta_2}+ \\ \pi_2\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}\left(\frac{(R_{21}-R_{22})R_{f2}k_2^{\frac{1}{1+\delta_2}}(I-c_1)}{R_{f2}-R_{22}+k_2^{\frac{1}{1+\delta_2}}\left(R_{21}-R_{f2}\right)}\right)^{-\delta_2}\end{array}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}}.
\end{aligned}$$

The first order condition is

$$c_1^{-\delta_1-1} = \beta\left(\begin{array}{c}\left(\frac{(R_{21}-R_{22})R_{f2}}{R_{f2}-R_{22}+k_2^{\frac{1}{1+\delta_2}}\left(R_{21}-R_{f2}\right)}\right)^{-\delta_2} \\ \left(\begin{array}{c}\pi_1\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}+ \\ \pi_2 k_2^{-\frac{\delta_2}{1+\delta_2}}\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}\end{array}\right)\end{array}\right)^{\frac{\delta_1}{\delta_2}}(I-c_1)^{-1-\delta_1},$$

implying that

$$c_1^{KP} = \frac{I}{1+\beta^{\frac{1}{1+\delta_1}}\left(\begin{array}{c}\left(\frac{(R_{21}-R_{22})R_{f2}}{R_{f2}-R_{22}+k_2^{\frac{1}{1+\delta_2}}\left(R_{21}-R_{f2}\right)}\right)^{-\delta_2} \\ \left(\begin{array}{c}\pi_1\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}+ \\ \pi_2 k_2^{-\frac{\delta_2}{1+\delta_2}}\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}\end{array}\right)\end{array}\right)^{\frac{\delta_1}{(1+\delta_1)\delta_2}}}.$$

Moreover, for this case, if

$$\pi_1 R_{21} + \pi_2 R_{22} > R_{f2},$$

we have $k_2 < 1$, implying that $n_1^{KP} > 0$. However, if $R_{f31} \neq R_{f32}$ then it is possible for $k_2 > 1$ and $n_1^{KP} < 0$. The general condition for determining the sign of n_1^{KP} is given by the following expression

$$n_1^{KP} \gtreqless 0 \Leftrightarrow \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \gtreqless \frac{\left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}}{\left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}}. \tag{B.3}$$

Note that when $n_1^{KP} < 0$, we require that period 2 income

$$I_{2s} = R_{2s} n_1 + R_{f2} n_{f1} > 0 \ (s = 1, 2)$$

in order for c_{21}, c_{22}, c_{31} and c_{32} to be positive and for the consumer's utility function to be well-defined. The increase in the risk free asset holdings financed by the shorting of the risky asset should not be viewed as reflecting increased intraperiod risk aversion. Instead, one can view the reduction in the positive quantity of n_1 as an attempt to decrease the period 2 portfolio intraperiod risk whereas the shift to shorting the risky asset can be thought of a move to decrease the intertemporal risk via dynamic hedging. For resolute choice, the period 1 DOCE utility function can be transformed into a certainty utility of the single branch (c_1, c_{21}, c_{31}) as in eqn. (B.1) and the budget constraint can be rewritten as eqn. (B.2). The first order conditions are

$$c_1^{-1-\delta_1} = \frac{\beta \left(\pi_1 + \pi_2 \left(\frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} c_{21}^{-\delta_1 - 1}}{\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f2}} \left(\frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})}\right)^{\frac{1}{1+\delta_2}}}$$

and

$$c_1^{-1-\delta_1} = \frac{\beta^2 \left(\pi_1 + \pi_2 \left(\frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} c_{31}^{-\delta_1 - 1}}{\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f31} R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f32} R_{f2}} \left(\frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}}\right)^{\frac{1}{1+\delta_2}}}.$$

Therefore

$$c_{21} = \left(\frac{\beta \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right)}{\pi_1 \left(R_{21} - R_{f2} \right)} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}}}{\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right)}{\pi_1 \left(R_{21} - R_{f2} \right)} \right)^{\frac{1}{1+\delta_2}}} \right)^{\frac{1}{1+\delta_1}} c_1$$

and

$$c_{31} = \left(\frac{\beta^2 \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}}}{\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f31} R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f32} R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{\frac{1}{1+\delta_2}}} \right)^{\frac{1}{1+\delta_1}} c_1.$$

It follows that

$$c_1^\circ = \frac{I}{\left(\begin{array}{c} 1 + \left(\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right)}{\pi_1 \left(R_{21} - R_{f2} \right)} \right)^{\frac{1}{1+\delta_2}} \right) \\ \times \left(\frac{\beta \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right)}{\pi_1 \left(R_{21} - R_{f2} \right)} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}}}{\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right)}{\pi_1 \left(R_{21} - R_{f2} \right)} \right)^{\frac{1}{1+\delta_2}}} \right)^{\frac{1}{1+\delta_1}} + \\ \left(\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f31} R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f32} R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{\frac{1}{1+\delta_2}} \right) \\ \times \left(\frac{\beta^2 \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}}}{\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f31} R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f32} R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{\frac{1}{1+\delta_2}}} \right)^{\frac{1}{1+\delta_1}} \end{array} \right)}.$$

The considerable variation in the n_{f1}/n_1 ratio in Figure 7(b) suggests a significant difference in risk attitudes. There are two dimensions of risk – the intraperiod portfolio risk in period 2 and the interperiod risk corresponding to the correlation pattern of the period 2 risky and period 3 risk free asset returns. The latter phenomenon can very clearly be observed if we switch the pattern of returns for R_{f31} and R_{f32} in Figure 7(b) to that shown in Figure 8. In the latter case, the period 2 risk can be viewed as being partially hedged by the period 3 risk as the intertem-

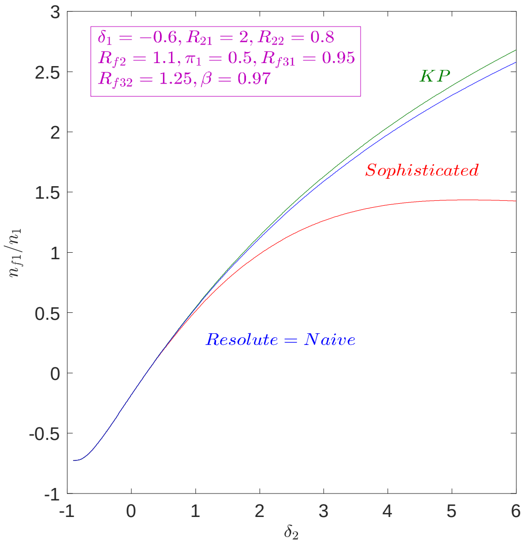

Figure 8:

poral correlation has gone from positive to negative. As a result, continuing to assume that $\delta_1 = -0.6$ and $\delta_2 = 5$, the n_{f1}/n_1 ratio for all four models drops substantially from the case in Figure 7(b), where the asset return intertemporal correlation is positive. Moreover, it is not surprising that the asset ratio for each of the models is the same when $R_{f31} = R_{f32}$ since there is no intertemporal risk. Also, the common ratio is intermediate between the positive correlation case of Figure 7(b) and the negative correlation case of Figure 8.