

The U.S. Government Risk Premium Puzzle*

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Abstract

The market value of outstanding government debt reflects the expected present discounted value of current and future primary surpluses. When the pricing kernel fits U.S. equity and Treasury prices and the government surpluses are consistent with U.S. post-war data, a government risk premium puzzle emerges. Since tax revenues are pro-cyclical while government spending is counter-cyclical, the tax revenue claim has a higher short-run discount rate and a lower value than the spending claim. Since revenue and spending are co-integrated with GDP, the long-run risk discount rates of both claims are much higher than the long Treasury yield. This implies a negative present value of U.S. government surpluses: the U.S. government should be a creditor rather than a debtor.

JEL codes: fiscal policy, term structure, debt maturity, convenience yield

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1 Introduction

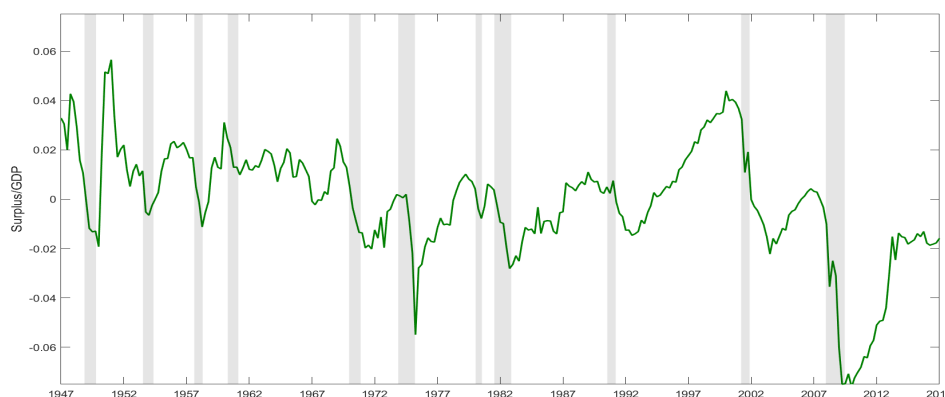
The U.S. Treasury is the largest borrower in the world. As of December 31 2018, the outstanding federal government debt held by the public was valued at \$16.1 trillion. Outstanding debt nearly doubled after the Great Recession to 76.4% of the U.S. annual GDP. Yet, some have argued that the U.S. has ample debt capacity to fund additional spending by rolling over its debt because T-bill rates are below GDP growth rates (Blanchard, 2019).

We show that, absent a bubble in government debt, the relevant “interest rate” on the portfolio of the entire outstanding debt is higher than Treasury bond rates and higher than GDP growth, reversing the former argument. To see why, note that the price of a stock is the expected present discount value of future dividends. Risk-free interest rates are below dividend growth rates, yet the price of the stock is finite. Why? As the stock’s dividend growth is pro-cyclical, cash flows are low when the investor’s marginal utility is high. The relevant “interest rate” for the stock contains a risk premium because of the risk exposures.

Analogously, we consider the portfolio strategy that buys all new government bonds issues and receives all bond repayments. This portfolio’s cash flow is the government’s primary surplus. As shown in Figure 1, the primary surpluses are strongly pro-cyclical just like the dividends. In recessions, when marginal utility is high, surpluses are negative and net bond issuance is high. In addition, revenue and spending are cointegrated with GDP and subject, as a result, to the same long run risk as GDP. Therefore, the claim to future government surpluses is risky, and the relevant “interest rate” for government

Figure 1: Government Cash Flows

The figure plots the U.S. federal government primary surplus as a fraction of GDP. The sample period is from 1947Q1 to 2017Q4.



surpluses also contains a risk premium.

More precisely, the value of a claim to current and future government primary surpluses, P_t^S , is the difference between the value of a claim to current and future federal tax revenues, P_t^T , and the value of a claim to current and future federal spending, P_t^G . Since tax revenues are pro-cyclical, the representative investor requires a high risk premium to hold the claim to future tax revenue. Put differently, P_t^T is low. Government spending is counter-cyclical, so that the claim which pays out government spending is a great recession hedge. It commands a lower risk premium. Put differently, P_t^G is high. If the average primary surplus is about zero, as it has been over the past 70 years, the claim to current and future government surpluses, $P_t^S = P_t^T - P_t^G$, should have a negative present discounted value. However, by the government's dynamic budget constraint and in the absence of bubbles, the value of the surplus claim must equal the market value of outstanding debt, which is positive rather than negative. We refer to this difference between the positive valuation of outstanding debt and the negative valuation of the surplus claim as the *government debt valuation puzzle*, which has been 196% of GDP on average since 1947.

In addition, both claims are subject to the same long-run GDP risk. Hence, we expect the long position in the tax claim and short position in the spending claim to earn an additional long-run risk premium, because the long position is larger than the short position and the market value of outstanding debt is positive. However, the return on the U.S. government debt portfolio is only 0.93% in excess of three-month Tbill rate. We refer to this as *government debt risk premium puzzle*.

Put in terms of interest rates rather than valuations, the U.S. government's promised payments on its outstanding debt, future surpluses, are a risky cash-flow stream and risk averse investors demand a risk premium to compensate for this risk. Thus, the relevant "interest rate" or discount rate for the government bond portfolio's cash flows is high. Yet, Treasury investors seem willing to purchase government debt at low yields. Government bond yields in the U.S. and other developed bond markets are puzzlingly low.

The above argument relies on a realistic model of risk and asset pricing. First, adequately capturing the dynamics of government spending and tax revenue is crucial. Only when we correctly specify the risk characteristics of the cash flows to the G-claim and T-claim, will we be able to correctly value that claim. As emphasized above, spending and revenue growth covary with GDP growth. We additionally allow them to covary with inflation, interest rate levels, the slope of the term structure, dividend growth, the price-

dividend ratio, and to be predictable by their own lags and the lags of these real and financial variables.

Importantly, we impose that government tax revenues and spending are co-integrated with GDP, and that revenues, spending, and GDP adjust when revenue-to-GDP or spending-to-GDP are away from their long-run relationship. This imposes a form of long-run automatic stabilization. When spending has been higher than usual for a long period, as in the aftermath of the Great Recession, spending growth will be lower than average in the future to return the spending-GDP ratio back to its long-run average. The same mean-reversion is present for tax revenues. Without the assumption of co-integration, all shocks to spending and revenue would permanently affect the levels of spending-to-GDP and revenue-to-GDP.

Co-integration has important implications for the risk of the T- and G-claims. For example, a deep recession not only raises current government spending and lowers current tax revenue as a fraction of GDP, but also lowers future spending and raises future revenue, as a fraction of future GDP. Both the spending and the revenue claim are exposed to the same long-run risk as GDP. As a result, the long-run discount rate for government debt is the same as the rate that investors use when pricing a claim to GDP.

Another way of stating the puzzle is to point out that the surplus claim simply cannot be risk-free, because the federal government's surpluses trend with GDP: GDP innovations permanently alter all future surpluses. As a result, Blanchard's argument cannot hold. The risk-free rate is not the right discount rate.

Second, to adequately capture risk aversion, we posit a state-of-the-art stochastic discount factor (SDF) model. Rather than committing to a specific utility function, we use a reduced form SDF that accurately prices the term structure of Treasury bond yields of various maturities in each quarter since 1947. Inflation and GDP growth risk are two key macro-economic sources of risk that affect the price of government bonds. As such, it is by construction consistent with the history of safe interest rates and GDP growth rates. The model matches also the time series of bond risk premia. To obtain a realistic SDF model, we further insist that the model prices a claim to aggregate stock market dividends correctly. Having extracted the market prices of risk associated with the aggregate sources of risk, we have a realistic SDF that can be used to price a claim to future tax revenues and to future government spending. The SDF model's rich implications for the term structure of risk allow it to adequately price not only short-run but also long-run risk to spending and revenue.

One potential explanation of the puzzle is based on the finding that the U.S. government debt earns a convenience yield, which lowers the yield investors require for holding government bonds. Convenience yields are another source of revenue for the U.S. government, which we need to add to the primary surpluses and properly value. Convenience yields also lower equilibrium returns on government debt. The seignorage revenue increases the debt capacity of the Treasury substantially, but not by enough. To close the gap, seignorage revenue accruing to the U.S. Treasury should account for at least 12.66% of the federal government's tax revenue, while our estimates of the actual seignorage are closer to 1.74%.

We also explore the possibility of a future large fiscal correction that is absent from our sample, but priced into the bond market. We back out from the time series of government debt that bond investors would have to assign a large probability of about 50% (and even 90% at the end of our sample) to a spending cut equivalent to 8% of GDP. Such a high probability seems *prima facie* implausible and inconsistent with a peso event.

Missing government assets or market segmentation cannot resolve the puzzle either. One final "resolution" to the puzzle is to argue that there is a bubble in U.S. government debt. Indeed, our approach quantifies the bubble as the difference between the value of outstanding government debt and the value of the surplus claim. Over the post-war period, the average size of the bubble is 196% of GDP. Since 2000, the size of the bubble has tripled from 65% of GDP in 2000 to 235% of GDP in 2017. This is both because the outstanding value of government debt has doubled from about 35% to 75% of GDP and because the value of the surplus claim has fallen from -30% to -160% of GDP. The Treasury markets do not seem to enforce the transversality condition. The bond market vigilantes seem to have vanished after the 1990s.

We then use our model to study the optimal maturity structure of government debt. We are guided by the insight of [Bhandari, Evans, Golosov, and Sargent \(2017\)](#) that the optimal debt portfolio choice minimizes the variance of the government's funding needs. We can compute the variance of the government's funding needs using the estimated model and gauge how far the Treasury is from the optimal debt portfolio. This minimization amounts to equalizing the sensitivities of the value of the outstanding bond portfolio to each of the shocks that hit the economy to the sensitivities of the value of the surplus claim to the same shocks. Matching duration, the sensitivity to interest rates, is the simplest example in a world where the only risk is to the level of the term structure. On average, the typical maturity of the Treasury portfolio seems too short. More generally,

the positions in bonds of the various maturities can be chosen to fully immunize the government debt portfolio to all shocks. We show that the actual bond portfolio displays substantial deviations from full immunization.

The rest of the paper is organized as follows. We discuss the related literature next. Section 2 introduces the government budget constraint and characterizes the relationship between government surpluses and government debt. Section 3 sets up and solves the quantitative model. Section 4 formulates the government risk premium puzzle. Section 5 revisits the puzzle in a world with convenience yields on Treasury debt. Section 6 discusses potential resolutions for this puzzle, including a fiscal austerity peso event. Section 7 evaluates the government's optimal government debt maturity choice. Section 8 concludes. The appendix presents the details of model derivation and estimation.

Related Literature Our paper thus contributes to the literature on the optimal maturity structure of government debt. One view is that the government ought to minimize its expected funding cost by issuing short-term debt when the slope of the yield curve is steep, thus exploiting the failure of the expectations hypothesis. Conversely, the government should issue more long-term debt when the yield curve inverts (see for example [Campbell, 1995](#)). A normative literature on optimal government taxation and debt management offers a different prescription. In this class of dynamic models with distortionary taxation going back to [Lucas and Stokey \(1983\)](#), the government chooses the tax rate optimally to hedge shocks to government spending. If the government can issue state-contingent debt, the optimal tax rate inherits the serial correlation of government spending. To the extent that the government's debt securities do not span all the shocks that hit the economy, maturity choice plays an important role. In a model in which only spending shocks drive the term structure, [Angeletos \(2002\)](#) and [Buera and Nicolini \(2004\)](#) show how the government can choose the maturity of non-state-contingent government debt to mimic the complete markets allocations in [Lucas and Stokey \(1983\)](#), thus creating an explicit role for the maturity structure. In general, the government will not try to replicate the complete markets allocation if variation in interest rates is largely explained by non-spending shocks, as is the case in the data. [Lustig, Sleet, and Yeltekin \(2008\)](#) examine the optimal maturity structure when the government issues nominal non-state-contingent debt. Market incompleteness imputes more persistence to the optimal tax rates, as shown by [Aiyagari, Marcet, Sargent, and Seppälä \(2002\)](#). An important shortcoming of the conventional Ramsey analysis is that optimal debt management is derived in a setting that fails

to generate realistic asset prices. [Karantounias \(2018\)](#) shows that key optimal tax policy prescriptions change dramatically with [Epstein and Zin \(1989\)](#) preferences. Bond prices becomes more realistic under such preferences ([Piazzesi and Schneider, 2006](#); [Bansal and Shaliastovich, 2013](#)).

Motivated by this observation, we take a pragmatic approach and choose a flexible SDF model that prices the term structure of interest rates precisely. Our asset pricing model builds on [Lustig, Van Nieuwerburgh, and Verdelhan \(2013\)](#), who price a claim to aggregate consumption and study the properties of the price-dividend ratio of this claim, the wealth-consumption ratio. Here we focus on pricing claims to government revenue and spending growth instead. The asset pricing model combines a vector auto-regression model for the state variables as in [Campbell \(1991, 1993, 1996\)](#) with a no-arbitrage model for the (SDF) as in [Duffie and Kan \(1996\)](#); [Dai and Singleton \(2000\)](#); [Ang and Piazzesi \(2003\)](#). [Gupta and Van Nieuwerburgh \(2018\)](#) use a similar framework to evaluate the performance of private equity funds. For parsimony, our current work focuses on estimating the model with a single regime, whereas [Bianchi and Melosi \(2014, 2017, 2018\)](#) study different regimes of the fiscal policy and their real effects.

Our approach takes spending and tax policy as given, rather than being optimally determined. However, both policies are allowed to depend on a rich set of state variables and are estimated from the data. To keep the model tractable, we shut down feedback from tax and spending policy onto the SDF.¹

Our work also connects to the literature on the specialness of U.S. government bonds. [Longstaff \(2004\)](#); [Krishnamurthy and Vissing-Jorgensen \(2012, 2015\)](#); [Nagel \(2016\)](#) find that U.S. government bonds are traded at a premium relative to other risk-free bonds. [Greenwood, Hanson, and Stein \(2015\)](#) study the government debt's optimal maturity in the presence of such premium, and [Valchev \(2017\)](#); [Du, Im, and Schreger \(2018\)](#); [Jiang, Krishnamurthy, and Lustig \(2018\)](#) study this premium in international finance. Our approach takes this premium into account by incorporating Treasury yields in the pricing kernel, and tackles the fundamental question whether the U.S. government fiscal condition justifies such a premium.

Lastly, our work contributes to the fiscal theory of the price level, which requires a pos-

¹If the marginal investor in U.S. Treasuries is a foreign official institution or the domestic central bank, demand for Treasuries may not be materially influenced by fiscal policy considerations. Indeed, the share of U.S. Treasury debt held by foreigners has been rising steadily since the 1980s, and the Federal Reserve's holdings ballooned after the financial crisis. Foreign official institutions and the Fed combined have held about two-thirds of U.S. Treasuries over the past twenty years ([Kohn, 2016](#); [Favilukis, Kohn, Ludvigson, and Nieuwerburgh, 2013](#)) and their demand has been characterized as price inelastic.

itive present value of government surpluses to determine the price level and the exchange rate (Sargent and Wallace, 1984; Leeper, 1991; Woodford, 1994; Sims, 1994; Cochrane, 2001, 2005, 2019a,b; Jiang, 2019a,b). Our work quantitatively estimates this present value and discovers a valuation puzzle and a risk premium puzzle.

2 Theoretical Characterizations

2.1 Value Equivalence

Let G_t denote nominal government spending before interest expenses, T_t denote nominal government tax revenue, and $S_t = T_t - G_t$ denote the nominal primary government surplus. Let $P_t^\$(h)$ denote the price at time t of a nominal zero-coupon bond that pays \$1 at time $t + h$, where h is the maturity expressed in quarters. Let $Q_{t,h}^\$$ denote the outstanding face value at time t of government bond payments that are due at time $t + h$. Iterating on the one-period government budget constraint, we show two equivalences between the government debt portfolio and the government's primary surpluses.

Proposition 1 (Value equivalence). Today's market value of the outstanding government debt portfolio equals the expected present discounted value of current and all future primary surpluses:

$$D_t \equiv \sum_{h=0}^H P_t^\$(h) Q_{t-1,h+1}^\$ = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^\$ (T_{t+j} - G_{t+j}) \right] \equiv P_t^\tau - P_t^\$, \quad (1)$$

where the value of the tax claim and value of the spending claim are defined as:

$$P_t^\tau = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^\$ T_{t+j} \right], \quad P_t^\$ = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^\$ G_{t+j} \right].$$

The proof is given in Appendix A. The multi-period stochastic discount factor (SDF) $M_{t,t+h}^\$ = \prod_{k=0}^h M_{t+k}^\$$ is the product of the adjacent one-period SDFs, $M_{t+k}^\$$. Bond prices satisfy $P_t^\$(h) = \mathbb{E}_t \left[M_{t,t+h}^\$ \right] = \mathbb{E}_t \left[M_{t+1}^\$ P_{t+1}^\$(h-1) \right]$. By convention $P_t^\$(0) = M_{t,t}^\$ = M_t^\$ = 1$ and $M_{t,t+1}^\$ = M_{t+1}^\$$. The government bond portfolio is stripped into zero-coupon bond positions $Q_{t,h}^\$$. $Q_{t-1,1}^\$$ is the total amount of debt payments that is due today. The outstanding debt reflects all past bond issuance decisions, i.e., all past primary deficits. The proof relies only on the existence of a SDF, i.e., the absence of arbitrage opportuni-

ties, but not on complete markets. It imposes a transversality condition that rules out a government debt bubble.

Eq. (1) implies that when the government runs a deficit in a future date and state, it will need to issue new bonds to the investing public. If those dates and states are associated with a high value of the SDF for the representative bond investor, that debt issuance occurs at the “wrong” time. Investors will need to be induced by low prices (high yields) to absorb that new debt. To see this more clearly, we can write the right-hand side of eq. (1) as:

$$D_t = \sum_{j=0}^{\infty} P_t^{\$}(j) \mathbb{E}_t [S_{t+j}] + \sum_{j=0}^{\infty} \text{Cov}_t \left(M_{t,t+j}^{\$}, T_{t+j} \right) - \sum_{j=0}^{\infty} \text{Cov}_t \left(M_{t,t+j}^{\$}, G_{t+j} \right)$$

The first term on the right-hand side is the present discounted value of all expected future surpluses, using the term structure of risk-free bond prices. It is the PDV for a risk-neutral investor. If the SDF is constant, this is the only term on the right-hand side (Hansen, Roberds, and Sargent, 1991; Sargent, 2012). The government’s capacity to issue debt today is constrained by, or collateralized by its ability to generate current and future surpluses. The second and third terms encode the riskiness of the government debt portfolio, and only arise in the presence of time-varying discount rates. Since tax revenues tend to be high when times are good ($M_{t,t+j}$ is low), the second term is expected to be negative. Since government spending tends to be high when times are bad ($M_{t,t+j}$ is high), the third term is expected to be positive. Thus, the difference between the second and the third term is unambiguously negative. The covariance terms lower the government’s debt capacity. Put differently, to support a given, positive amount of government debt, D_t , the first term will need to be higher by an amount equal to the absolute value of the covariance terms. This paper quantifies that covariance term in a realistic model of risk and return, while most macro-economic models imply only small risk premia. The key finding of this paper is that this covariance term is large in absolute value, on the order of two times GDP.

2.2 Risk Premium Equivalence

Recall that P_t^{τ} denotes the cum-dividend value of claim to tax revenue, $P_t^{\$}$ denotes the cum-dividend value of a claim to government spending, and D_t denotes the market value of the outstanding government debt portfolio. Define the holding period returns on the

bond portfolio, the tax claim, and the spending claim as:

$$r_{t+1}^D = \frac{\sum_{h=1}^{\infty} P_{t+1}^{\$}(h-1)Q_{t,h}^{\$}}{\sum_{h=1}^{\infty} P_t^{\$}(h)Q_{t,h}^{\$}}, \quad r_{t+1}^{\tau} = \frac{P_{t+1}^{\tau}}{P_t^{\tau} - T_t}, \quad r_{t+1}^g = \frac{P_{t+1}^g}{P_t^g - G_t}.$$

We can further prove equivalence between the discount rate of the government surplus claim and that of the government debt portfolio.

Proposition 2 (Risk premium equivalence). Today's expected holding return $\mathbb{E}_t[r_{t+1}^D]$ on the government debt portfolio equals the expected holding return $\mathbb{E}_t[r_{t+1}^{\tau}]$ on the claim to future tax revenues minus the expected holding return $\mathbb{E}_t[r_{t+1}^g]$ on the claim to future government spending, weighted appropriately:

$$\mathbb{E}_t[r_{t+1}^D] = \frac{P_t^{\tau} - T_t}{D_t - S_t} \mathbb{E}_t[r_{t+1}^{\tau}] - \frac{P_t^g - G_t}{D_t - S_t} \mathbb{E}_t[r_{t+1}^g]. \quad (2)$$

where we have used $D_t - S_t = (P_t^{\tau} - T_t) - (P_t^g - G_t)$,

This second equivalence can be understood as the Modigliani-Miller theorem in the context of government finance. Absent frictions, the average discount rate on government liabilities is equal to the average discount rate on government assets, which are a claim to primary surpluses. Since the primary surpluses are tax revenues minus government spending, the discount rate on government debt equals the difference between the discount rates of tax revenues and government spending, appropriately weighted.

We can restate this expression in terms of expected excess returns:

$$\mathbb{E}[r_{t+1}^D - r_t^f] = \frac{P_t^{\tau} - T_t}{D_t - S_t} \mathbb{E}[r_{t+1}^{\tau} - r_t^f] - \frac{P_t^g - G_t}{D_t - S_t} \mathbb{E}[r_{t+1}^g - r_t^f].$$

To develop intuition, we consider a few simple scenarios. If the expected returns on both claims are identical, $\mathbb{E}[r_{t+1}^{\tau}] = \mathbb{E}[r_{t+1}^g]$, then the expected return on government debt is given by

$$\mathbb{E}[r_{t+1}^D - r_t^f] = \mathbb{E}[r_{t+1}^{\tau} - r_t^f] = \mathbb{E}[r_{t+1}^g - r_t^f].$$

However, if the tax revenue claim is riskier than the spending claim and hence earns a higher excess return, $\mathbb{E}[r_{t+1}^{\tau}] > \mathbb{E}[r_{t+1}^g]$, then the expected return on government debt exceeds the expected excess returns on the revenue and the spending claims:

$$\mathbb{E}[r_{t+1}^D - r_t^f] > \mathbb{E}[r_{t+1}^{\tau} - r_t^f] > \mathbb{E}[r_{t+1}^g - r_t^f].$$

Long-run Discount Rates We start by analyzing the long-run discount rates on the tax revenue and government spending claims. We consider a spending (revenue) strip that pays off G_{t+j} (T_{t+j}) at time $t + j$ and nothing at other times. Let $R_{t,t+j}^g$ ($R_{t,t+j}^\tau$) be the holding period return on such an j -period spending (revenue) strip. We analyze the limit of the log returns on these strips as $j \rightarrow \infty$ under two assumptions on the time-series properties of government spending and tax revenues.

Proposition 3. If the log of government spending G (tax revenue T) is stationary in levels (after removing a deterministic time trend), then the long-run expected log return on spending (revenue) strips equals the yield on a long-term government bond as the payoff date approaches maturity.

$$\lim_{j \rightarrow \infty} \mathbb{E}_t r_{t,t+j}^{g,j} = y_t^\infty, \quad \lim_{j \rightarrow \infty} \mathbb{E}_t r_{t,t+j}^{\tau,j} = y_t^\infty,$$

where y_t^∞ is the yield at time t on a nominal government bond of maturity $+\infty$. The proof is given in Appendix A. The result builds on work by [Alvarez and Jermann \(2005\)](#); [Hansen and Scheinkman \(2009\)](#); [Borovička, Hansen, and Scheinkman \(2016\)](#); [Backus, Boyarchenko, and Chernov \(2018\)](#), among others.

This result implies that the long-run strips can be discounted off the term-structure for zero coupon bonds. In this case, the long-run discount rate on government debt is the *yield on a long-term risk-free bond*. However, if there are no permanent shocks to T or G , then it is imperative to assume that GDP and aggregate consumption are not subject to permanent shocks either. If there are no permanent shocks to marginal utility, then the long bond is the riskiest asset in economy. That clearly seems counterfactual ([Alvarez and Jermann, 2005](#)). Put differently, the gap between the long-run discount rates on strips and the long yields is governed by the entropy of the permanent component of the pricing kernel. Explaining the high returns on risky assets such as stocks requires that entropy to be large, not zero (e.g., [Borovička, Hansen, and Scheinkman, 2016](#)). Next we consider a more realistic case.

Corollary 1. If the log of government spending/GDP ratio G/GDP (revenue/GDP T/GDP) is stationary in levels, then the long-run expected log excess return on long-dated spending (revenue) strips equals that on GDP strips:

$$\lim_{j \rightarrow \infty} \mathbb{E}_t r_{t,t+j}^{g,j} = \mathbb{E}_t r_{t,t+n}^{gdp,\infty} \gg y_t^\infty, \quad \lim_{j \rightarrow \infty} \mathbb{E}_t r_{t,t+j}^{\tau,j} = \mathbb{E}_t r_{t,t+n}^{gdp,\infty} \gg y_t^\infty.$$

This corollary implies that government bond investors have a net long position in a claim that is exposed to the same long-run risk as the GDP claim. It follows that the value of the long-run spending minus revenue strips will be smaller than what is predicted by the yields at the long end of the term structure:

$$\lim_{j \rightarrow \infty} (T_t \widehat{p}_t^j [T] - G_t \widehat{p}_t^j [G]) = \lim_{j \rightarrow \infty} (T_t - G_t) \widehat{p}_t^j [Y] \ll \lim_{n \rightarrow \infty} (T_t - G_t) \exp(-ny_t^n)$$

If spending/revenue are cointegrated with GDP, then the long-run discount rate is the *long-run discount rate on gdp*, which we can think of as *unlevered equity*. This return is much higher than the yield on long-term risk-free bonds because of permanent shocks to marginal utility.

Short-run Discount Rates Next, we turn our attention to cyclical risk which drives the expected returns on short maturity strips. Even though revenue and spending claims have the same long-run discount rates, short-run discount rates will likely be higher for the revenue claim because tax revenue is highly pro-cyclical while government spending is counter-cyclical. This property of short-run discount rates deepens the risk premium puzzle, because government debt investors have a net long position in a riskier claim than the short position.

Combining the properties of short-run and long-run discount rates together, theory predicts that $\mathbb{E}[r_{t+1}^D - r^f] > \mathbb{E}[r_{t+1}^\tau - r^f] > \mathbb{E}[r_{t+1}^g - r^f]$. To summarize, a model of asset prices will have to confront two forces that push up the equilibrium returns on government debt. First, the long-run discount rates are higher than the yield on a long-maturity bond, because of the long-run cash flow risk in the spending and revenue claims equals that of long-run GDP risk. Government debt investors have a net long position in a claim that is exposed to the same long-run cash flow risk as GDP. The excess returns on government debt will tend to be much higher than those on long-maturity bonds. Second, there is short-run cash flow risk that pushes the expected return on the revenue claim above the expected return on the spending claim. As a result, government debt investors earn a much larger risk premium on the long end than what they pay on the short end. This further increases the fair expected return on the debt claim. Discounting future surpluses at the short-run discount rate is inappropriate.²

²Section 4.5 below derives conditions under which future surpluses can be discounted at the risk-free interest rate and shows that these conditions are severely violated in the data.

3 Quantitative Results

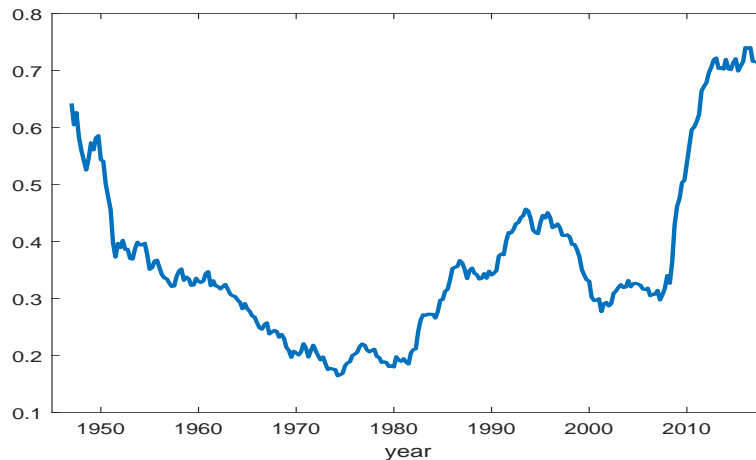
3.1 Data and State Variables

Following [Hall and Sargent \(2011\)](#) and extending their sample, we construct zero coupon bond (strip) positions from all coupon-bearing Treasury bonds (all cusips) issued in the past and outstanding in the current quarter. This is done separately for nominal and real bonds. Since zero-coupon bond prices are also observable, we can construct the left-hand side of Eq. (1) as the market value of outstanding U.S. government debt.³ Figure 2 plots its evolution over time, scaled by the U.S. GDP.

Table 1 reports the average returns and excess returns of various bond portfolios. On average, the government bond portfolio has a duration of 4.24 years, with a small average excess return of 0.93% per annum.

Figure 2: The Market Value of Outstanding Debt to GDP

The figure plots the ratio of the nominal market value of outstanding government debt divided by nominal GDP. GDP Data are from the Bureau of Economic Analysis. The market value of debt is constructed as follows. We multiply the nominal price (bid/ask average) of each cusip by its total amount outstanding (normalized by the face value), and then sum across all issuance (cusip). The series is quarterly from 1947.Q1 until 2017.Q4. Data Source: CRSP U.S. Treasury Database and BEA.



Next, we propose a simple no-arbitrage model for stocks, bonds, and government cash flows. We take a stance on the key sources of aggregate risk in the economy, and postulate that their dynamics follow a VAR. The goal is to estimate the market prices of these macro-economic risks, such that the model matches observed government bond yields and equity prices. With these market prices of risk in hand, we compute the expected present discounted value of future surpluses, the right-hand side of (1).

³Since the model fits nominal bond prices very well, as shown below, we can equivalently use model-implied bond prices. Similarly, we can use model-implied prices for real zero-coupon bonds.

Table 1: Summary Statistics for Government Bond Portfolio

Panel A reports summary statistics for the holding period return on the aggregate government bond portfolio: the mean and the standard deviation of the holding period return, r^D , the excess return, $r^D - r_f$, the three-month Tbill rate, r_f , and the weighted average Macaulay duration. Panel B reports the mean and the standard deviation of the holding period returns of three-month T-bill and T-bonds with time-to-maturity of one year, five years and ten years. All returns are expressed as annual percentage points. Duration is expressed in years. The sample period is from 1947.Q1 to 2017.Q4.

Panel A				
	r^D	$r^D - r_f$	r_f	Duration
Mean	5.08	0.93	4.16	4.24
Std.	3.92	3.87	3.16	0.70
Sharpe Ratio	0.24			
Panel B				
	1 Yr	5 Yr	10 Yr	20 Yr
Mean	4.74	4.79	5.56	5.75
Std.	1.04	1.72	4.80	6.80
Sharpe Ratio	0.42	0.29	0.23	0.23

We assume that the $N \times 1$ vector of state variables follows a Gaussian first-order VAR:

$$z_t = \Psi z_{t-1} + \Sigma^{\frac{1}{2}} \varepsilon_t, \quad (3)$$

with shocks $\varepsilon_t \sim i.i.d. \mathcal{N}(0, I)$ whose variance is the identity matrix. The companion matrix Ψ is a $N \times N$ matrix. The vector z is demeaned. The covariance matrix of the innovations to the state variables is Σ ; the model is homoscedastic. We use a Cholesky decomposition of the covariance matrix, $\Sigma = \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2} \prime}$, which has non-zero elements only on and below the diagonal. In this way, we interpret the shock to each state variable as a linear combination of structural shocks ε_t , each one of which is orthogonal to the shocks to the state variables that precede it in the VAR.

First, we consider state variables that govern the term structure of interest rates. We follow the empirical term structure literature and specify a term structure model that contains two key macro-economic sources of risk, inflation (π_t) and real GDP growth (x_t), as well as two interest rates, the nominal short rate ($y_t^{\$}(1)$) and the yield spread ($yspr_t^{\$}$) defined as the difference between the 5-year and 1-quarter nominal bond yields: $yspr_t^{\$} = y_t^{\$}(20) - y_t^{\$}(1)$. This is akin to a model with two observable macro-economic time series and two latent factors. It is well understood that two latent factors are needed to describe the term structure of interest rates since interest rates are not fully spanned by

macro-economic time series (Joslin, Priebisch, and Singleton, 2014).

Second, we include the log price-dividend ratio and the log real dividend growth on the aggregate stock market in the VAR system. Together they encode sufficient information for the time series of stock returns.

Third, to capture the government's cash flows, we include $\Delta \log \tau_t$ and $\Delta \log g_t$, the log change in government revenue to GDP and the log change in government spending to GDP. We denote the ratio of government spending to GDP by g_t , the ratio of tax revenues to GDP by τ_t , and the ratio of the primary surplus to GDP by s_t .

In addition, the levels of tax revenue and government spending could be mean-reverting. In Appendix D, we run the Johansen and Phillips-Ouliaris cointegration tests. The results support two cointegration relationships between log tax revenue and log GDP and between log spending and log GDP. The coefficients estimates of the cointegration relationships, however, tend to vary across sample periods. As a result, we take an a priori stance that the tax-to-GDP ratio, $\log \tau$, and the spending-to-GDP ratio, $\log g$, are stationary. That is, we assume cointegration coefficients of (1,-1) for both relationships. Cointegration introduces a notion of automatic stabilizers in fiscal policy in the long run. Spending (tax revenue) may be temporarily high (low) relative to GDP but must eventually mean revert.

Imposing cointegration requires us to include the levels of $\log \tau$ and $\log g$ in our vector of state variables. The VAR variables are:

$$\mathbf{z}_t = [\pi_t - \pi_0, x_t - x_0, y_t^\$(1) - y_0^\$(1), yspr_t^\$ - yspr_0^\$(1), pd_t - \overline{pd}, \Delta d_t - \mu_d, \Delta \log \tau_t - \mu_0^\tau, \Delta \log g_t - \mu_0^g, \log \tau_t - \log \tau_0, \log g_t - \log g_0]'$$

Our main data sources are NIPA and FRED. Tax revenue, government spending before interest expense, real GDP growth, inflation, and GDP are from NIPA. Constant maturity Treasury yields are from Fred. Stock price and dividend data are from CRSP; we use the CRSP value-weighted total market to represent the U.S. stock market. Dividends are seasonally adjusted.

We use selector vectors to pick out particular elements of the state vector. For example, we use e_π to denote the vector $[1, 0, \dots, 0]'$, which picks out the row of the VAR corresponding to π_t :

$$\pi_t = \pi_0 + e_\pi' \mathbf{z}_t.$$

Similarly, the one-month nominal bond yield is $y_t^\$(1) = y_0^\$(1) + e'_{yn} \mathbf{z}_t$, where $y_0^\$(1)$ is

the unconditional average yield and e_{yn} is a vector that selects the element of the state vector corresponding to the one-month yield. e_x picks out the row of the VAR corresponding to real GDP growth, x_t , $e_{\Delta\tau}$ picks out the row of the VAR corresponding to $\Delta \log \tau_{t+1}$, $e_{\Delta g}$ picks out the row of the VAR corresponding to $\Delta \log g_{t+1}$, and so on.

3.2 Cash Flow Dynamics

We estimate the VAR system $z_t = \Psi z_{t-1} + \Sigma^{\frac{1}{2}} \varepsilon_t$ using OLS, iteratively restricting the statistically insignificant elements in Ψ to 0. To reflect the relationship between GDP, tax revenue and government spending, we do not restrict the loadings of x , $\Delta \log \tau_t - \mu_0^\tau$, and $\Delta \log g_t - \mu_0^g$ on the last 4 state variables (i.e. $\Psi_{[2,7,8],[7,8,9,10]}$) to 0 even if their coefficients are statistically insignificant.

The estimates of Ψ are

$$\Psi = \begin{pmatrix} 0.536 & 0.065 & 0.241 & 0 & 0 & 0.028 & 0 & 0 & 0 & 0.005 \\ 0 & 0.346 & 0 & 0 & 0 & 0 & 0.013 & -0.002 & -0.010 & 0.002 \\ 0.043 & 0.043 & 0.958 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.018 & 0 & 0.863 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.978 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.453 & 0.071 & -0.092 & 0 & 0 \\ 0 & 1.199 & 0 & -4.871 & 0.024 & 0 & -0.141 & -0.114 & -0.142 & 0.053 \\ 0 & -1.275 & 0 & 0 & -0.018 & 0 & -0.098 & -0.060 & 0.019 & -0.064 \\ 0 & 1.199 & 0 & -4.871 & 0.024 & 0 & -0.141 & -0.114 & 0.858 & 0.053 \\ 0 & -1.275 & 0 & 0 & -0.018 & 0 & -0.098 & -0.060 & 0.019 & 0.936 \end{pmatrix}.$$

In the Ψ matrix, the first 4 rows govern the dynamics of bond market variables. It shows substantial diagonal elements (persistence) as well as several non-zero off-diagonal elements. For example, the lagged GDP growth and the lagged short rate predict the inflation rate, and the lagged inflation and the lagged GDP growth also predict the short rate.

The next two rows govern the dynamics of stock market variables. The pd ratio is highly persistent, but does not load on other lagged variables. The dividend growth has a quarterly persistence of 0.453, exceeding that of GDP growth of 0.346. Dividend growth is predicted by the tax revenue-to-GDP growth and the spending-to-GDP growth.

The last four rows govern the dynamics of government cash flows. Consistent with the cointegration relationships, the tax revenue-to-GDP growth loads negatively on the

lagged level $\log \tau_{t-1}$ ($\Psi_{[7,9]} = -0.142$), and the government spending-to-GDP growth $\Delta \log g_t$ loads negatively on the lagged level $\log g_{t-1}$ ($\Psi_{[8,10]} = -0.064$). These loadings imply long-run mean reversion of the tax-to-GDP ratio and the spending-to-GDP ratio. Moreover, a higher lagged GDP growth predicts a higher tax revenue-to-GDP growth ($\Psi_{[8,2]} > 0$) and a lower spending-to-GDP growth ($\Psi_{[9,2]} < 0$).

The estimates of $\Sigma^{\frac{1}{2}}$ are reported in Appendix C.1. The tax revenue-to-GDP growth loads positively on the GDP growth rate, while the spending-to-GDP growth loads negatively on the GDP growth rate. In other words, tax revenues are pro-cyclical and government spending is counter-cyclical. The government spending-to-GDP growth also loads negatively on the shock to stock dividend growth.

3.2.1 Cointegration and Long-run Predictability of Tax Revenue and Spending

Figure 3 plots the impulse responses of the tax revenue-to-GDP ratio ($\log \tau_t$) and the government spending-to-GDP ratio ($\log g_t$) to a x shock, a $\Delta \log \tau_t$ shock, and a $\Delta \log g$ shock. The $\Delta \log \tau_t$ shock is defined as the shock that increases $\Delta \log \tau_t$ by the standard deviation of its VAR residuals. By definition, it also raises the level $\log \tau_t$ by the same amount, but it does not affect the GDP growth rate x_t . Conversely, the x_t shock is defined as the shock that increases x_t by the standard deviation of its VAR residuals. As it does not affect the level of government tax and spending, it lowers the tax-to-GDP ratio $\log \tau_t$ and the spending-to-GDP ratio $\log g_t$ by the same amount.

The blue curves represent the results under the benchmark VAR system. For example, the $\Delta \log \tau_t$ shock raises the tax-to-GDP ratio $\log \tau_t$ on impact. Then, as the tax-to-GDP ratio is above the long-run average, its growth rate $\Delta \log \tau_t$ adjusts downward, leading to a reversion in the level.

For comparison, the red curves represent the results under a restricted VAR, in which the first 8 state variables do not load on the cointegration variables $\log \tau_t$ and $\log g_t$. In this case, the impact of the $\Delta \log \tau_t$ shock and the $\Delta \log g_t$ shock is permanent. For example, a positive $\Delta \log \tau_t$ shock raises the tax-to-GDP ratio $\log \tau_t$ permanently.

The impulse responses show that the VAR system with cointegration variables and the VAR system without cointegration variables imply very different dynamics in government cash flows. Which one is more consistent with the data? We regress the annual $\Delta \log \tau_{t+k}$ and $\Delta \log g_{t+k}$ in the following year $k = 1, \dots, 5$ on the current $\log \tau_t$ and $\log g_t$. Table 2 reports the regression result. In the data, a higher level of $\log \tau_t$ predicts a lower tax revenue-to-GDP growth in the next 3 years, and a higher level of $\log g_t$ predicts a

Figure 3: The impulse responses of $\log \tau_t$ and $\log g_t$ to $\Delta \log \tau_t$ shock, $\Delta \log g_t$ shock, and x_t shock. In percentage units.

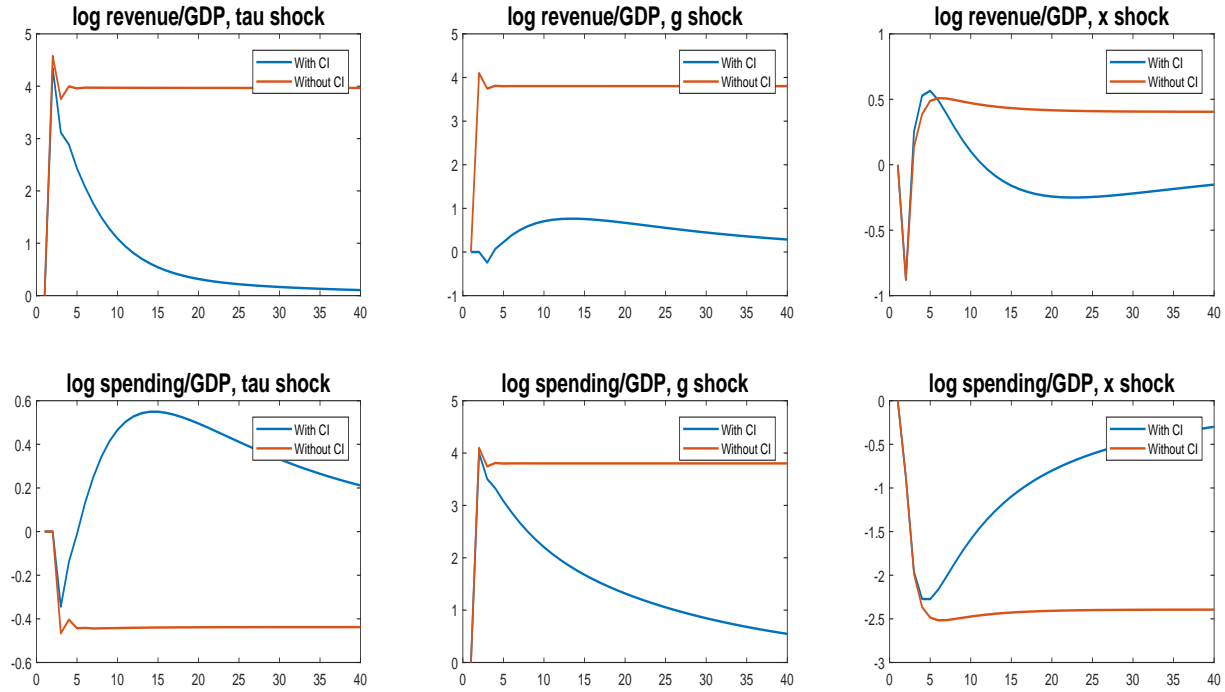


Table 2: The Predictability of Government Cash Flow Growth

This table reports how the levels of $\log \tau_t$ and $\log g_t$ predict the future tax revenue-to-GDP growth and the future government spending-to-GDP growth. The rows labeled by data report the coefficients from the regression of the annual $\Delta \log \tau_{t+k}$ and $\Delta \log g_{t+k}$ in the following year 1 through 5 on the current $\log \tau_t$ and $\log g_t$. Constants are omitted. Standard errors in parentheses are HAC-consistent. The rows labeled by model report the coefficients implied from the VAR system with cointegration variables.

<i>Dependent variable: $\Delta \log \tau_{t+k}$</i>					
k	1	2	3	4	5
$\log \tau_t$ - data	-0.37 (0.07)	-0.38 (0.08)	-0.21 (0.06)	-0.07 (0.07)	0.06 (0.10)
$\log \tau_t$ - model	-0.46	-0.25	-0.12	-0.06	-0.03
$\log g_t$ - data	0.08 (0.07)	0.05 (0.05)	0.04 (0.05)	0.04 (0.06)	-0.02 (0.06)
$\log g_t$ - model	0.17	0.06	0.00	-0.02	-0.03
<i>Dependent variable: $\Delta \log g_{t+k}$</i>					
k	1	2	3	4	5
$\log \tau_t$ - data	0.11 (0.10)	0.07 (0.08)	-0.04 (0.13)	0.01 (0.07)	0.03 (0.09)
$\log \tau_t$ - model	0.13	0.07	0.01	-0.02	-0.02
$\log g_t$ - data	-0.15 (0.07)	-0.15 (0.05)	-0.13 (0.05)	-0.10 (0.04)	-0.05 (0.04)
$\log g_t$ - model	-0.24	-0.17	-0.11	-0.08	-0.07

lower government spending-to-GDP growth in the next 4 years.

For comparison, Table 2 also reports the model counterparts implied by the VAR with

cointegration variables. The coefficients are the expected annual tax revenue-to-GDP growth and the expected annual government spending-to-GDP growth, conditional on an increase in $\log \tau_t$ by 1% or an increase in $\log g_t$ by 1%. The regression coefficients are quantitatively similar to the conditional expectations implied from our VAR model.

3.3 The Asset Pricing Model

Motivated by the no-arbitrage term structure literature (Ang and Piazzesi, 2003), we specify an exponentially affine stochastic discount factor (SDF). The nominal SDF $M_{t+1}^{\$} = \exp(m_{t+1}^{\$})$ is conditionally log-normal:

$$m_{t+1}^{\$} = -y_t^{\$(1)} - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1}, \quad (4)$$

The real SDF is $M_{t+1} = \exp(m_{t+1}) = \exp(m_{t+1}^{\$} + \pi_{t+1})$; it is also conditionally Gaussian. The innovations in the state vector ε_{t+1} from equation (3) are associated with a $N \times 1$ market price of risk vector Λ_t of the affine form:

$$\Lambda_t = \Lambda_0 + \Lambda_1 z_t,$$

The $N \times 1$ vector Λ_0 collects the average prices of risk while the $N \times N$ matrix Λ_1 governs the time variation in risk premia. We specify the restrictions on the market price of risk vector below. Asset pricing in this model amounts to estimating the market prices of risk in Λ_0 and Λ_1 .

3.3.1 Bond Pricing

This model offers a simple way to price nominal bonds. Nominal bond yields of maturity h are affine in the state vector:

$$y_t^{\$(h)} = -\frac{A^{\$(h)}}{h} - \frac{B^{\$(h)'} }{h} z_t,$$

the scalar $A^{\$(h)}$ and the vector $B^{\$(h)'}$ follow ordinary difference equations that depend on the properties of the state vector and of the market prices of risk.

Appendix B presents the proof and also shows a similar formula prices real bonds. We use this affine pricing equation to calculate the real interest rate, real bond risk premia, and inflation risk premia on bonds of various maturities.

Since both the nominal short rate ($y_t^\$(1)$) and the slope of the term structure ($y_t^\$(20) - y_t^\(1)) are included in the VAR, the SDF model must price the unconditional mean and the dynamics of the five-year bond yield:

$$-A^\$(20)/20 = y_0^\$(1) + yspr_0^\$ = y_0^\$(20) \quad (5)$$

$$-B^\$(20)/20 = e_{y1} + e_{yspr} \quad (6)$$

3.3.2 Equity Pricing

Let $PD_t^m(h)$ denote the price-dividend ratio of the dividend strip with maturity h (Wachter, 2005; van Binsbergen, Brandt, and Koijen, 2012). Then, the aggregate price-to-dividend ratio can be expressed as

$$PD_t^m = \sum_{h=0}^{\infty} PD_t^m(h). \quad (7)$$

Log price-dividend ratios on dividend strips are affine in the state vector:

$$pd_t^m(h) = \log(PD_t^m(h)) = A^m(h) + B^{m'}(h)z_t.$$

Since we include the log price-dividend ratio on the stock market in the state vector, it is affine in the state vector by assumption; see the left-hand side of (8):

$$\exp(\overline{pd} + e'_{pd}z_t) = \sum_{h=0}^{\infty} \exp(A^m(h) + B^{m'}(h)z_t), \quad (8)$$

Equation (8) rewrites the present-value relationship (7), and articulates that it implies a restriction on the coefficients $A^m(h)$ and $B^{m'}(h)$. We impose this restriction in the estimation.

3.4 Model Estimation

The state vector z_t is observed quarterly from 1947.Q1 until 2017.Q4 (284 observations). Under the VAR system, we estimate the constant market prices of risk Λ_0 and the time-varying market prices of risk Λ_1 that best fit the prices and expected returns on bonds of various maturities and on the aggregate stock market. Appendix C reports the point estimates as well as a detailed discussion of how the market price of risk parameters are identified.

3.4.1 Bonds

We use the following moments to estimate the 14 market price of risk parameters that govern the bond block. We include the distance between the observed and model-implied time-series of nominal bond yields for maturities of one quarter, one year, two years, five years, ten years, and thirty years.⁴ We also impose the 11 conditions implied by equations (5) and (6). Since it is part of the VAR, we insist on matching the 5-year bond yield precisely. This gives a total of $6T+11$ moments. The unconditional market prices of inflation, GDP growth, the level of interest rates, and the slope of the yield curve all have the expected sign. Figure 4 shows that the model matches the time series of bond yields in the data closely.

The top panels of Figure 5 show the model's implications for the average nominal (left panel) and real (right panel) yield curves at longer maturities. These yields are well behaved, with very long-run nominal (real) yields stabilizing at around 6.53% (3.15%) per year.⁵

The bottom left panel of Figure 5 shows that the model matches the dynamics of the nominal bond risk premium, defined as the expected excess return on the five-year nominal bond, quite well. Bond risk premia decline in the latter part of the sample, possibly reflecting the arrival of foreign investors who value U.S. Treasuries highly. The bottom right panel shows a decomposition of the nominal bond yield on a five-year bond into the five-year real bond yield, annual expected inflation over the next five years, and the five-year inflation risk premium. On average, the 5.1% nominal bond yield is comprised of a 1.9% real yield, a 3.2% expected inflation rate, and a 0.1% inflation risk premium. The graph shows that the importance of these components fluctuates over time.

⁴We use constant-maturity Treasury (CMT) yield data from FRED. For the 1-year, 5-year, and 10-year bonds, we supplement the time series with data from the Federal Reserve Board's FRASER archive for the period 1947.Q1-1953.Q1. The 2-year CMT yields are only available in 1976.Q3 and the 30-year CMT yields are available only for 1977.Q2-2002.Q1 and 2006.Q1-2017.Q3. Since our estimation is quarter by quarter, it can handle missing data points.

⁵We impose conditions that ensure that the nominal and real term structure are well behaved at very long maturities, for which we have no data. Specifically, we impose that average nominal (real) yields of bonds with maturities of 600, 800, 1000, 2000, 3000, and 4000 quarters remain above 6.24% (3.05%) per year, which is the long-run nominal (real) GDP growth rate $4x_0 + 4\pi_0$ ($4x_0$) observed in our sample. Second, we impose that nominal yields remain above real yields plus 3.19% expected inflation at those same maturities. This imposes that the inflation risk premium remain positive at very long maturities. Third, we impose that the nominal and real term structures of interest rates flatten out, with an average yield difference between 400 and 200 quarter yields that must not exceed 2% per year and between 1000 and 600 quarters that must not exceed 1% per year. These restrictions are satisfied at the optimum.

Figure 4: Dynamics of the Nominal Term Structure of Interest Rates

The figure plots the observed and model-implied 1-, 4-, 8-, 20-, 40-, and 120-quarter nominal bond yields. Data are from FRED and FRASER.

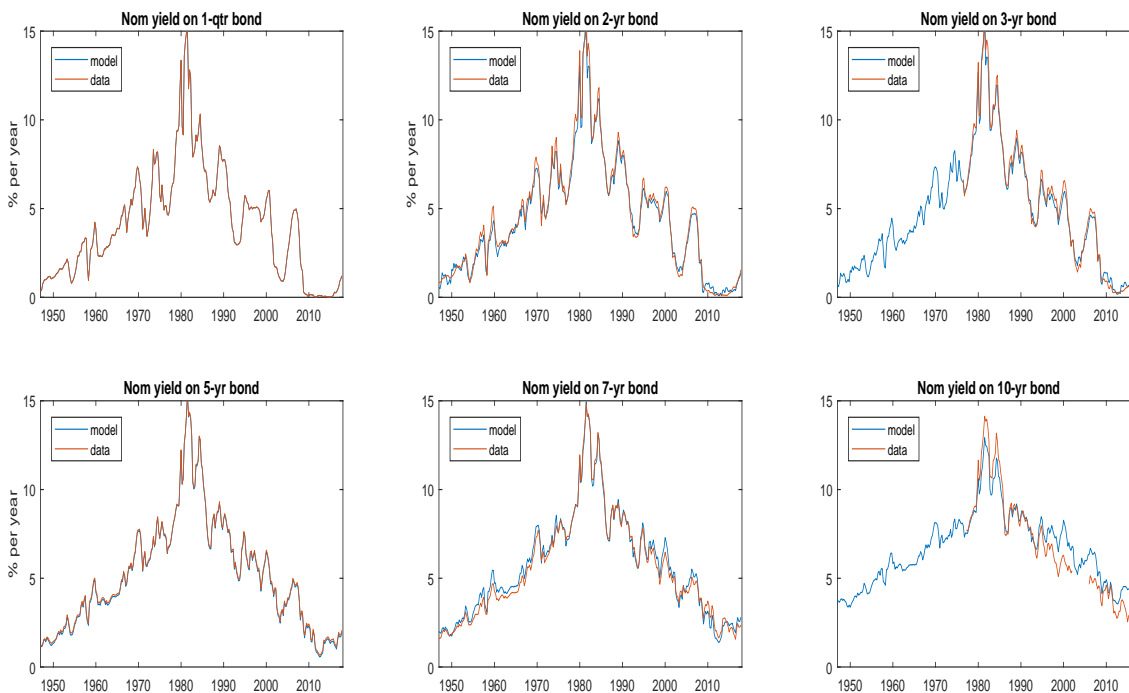
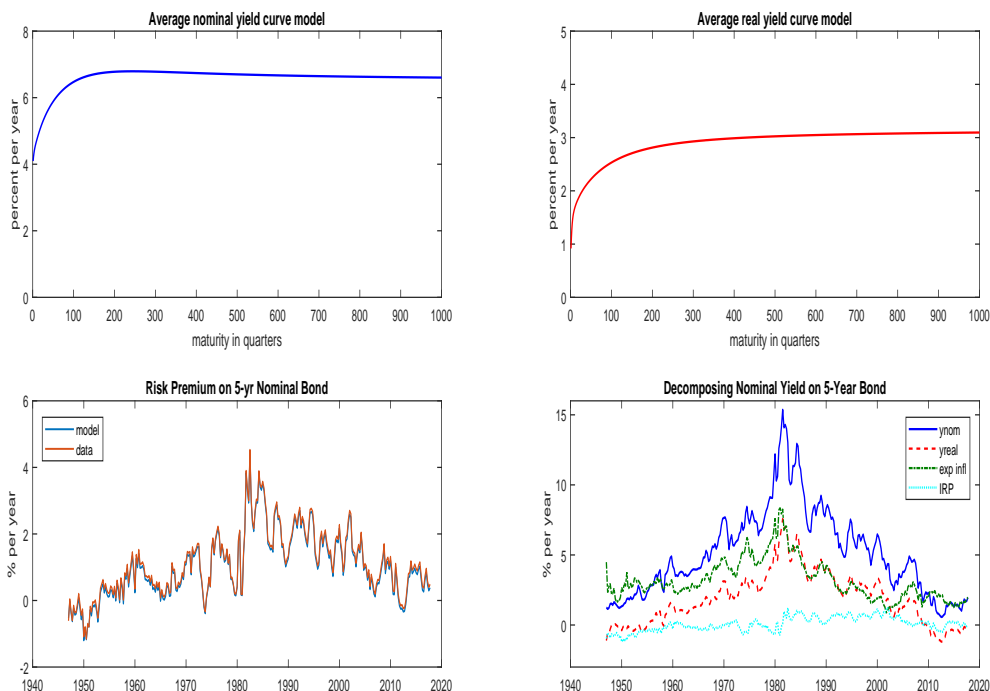


Figure 5: Long-term Yields and Bond Risk Premia

The top panels plot the average bond yield on nominal (left panel) and real (right panel) bonds for maturities ranging from 1 quarter to 1000 quarters. Yields are annualized. The bottom left panel plots the nominal bond risk premium on the five year bond in model and data. The bottom right panel decomposes the model's five-year nominal bond yield into the five-year real bond yield, the five-year inflation risk premium and the five-year real risk premium.



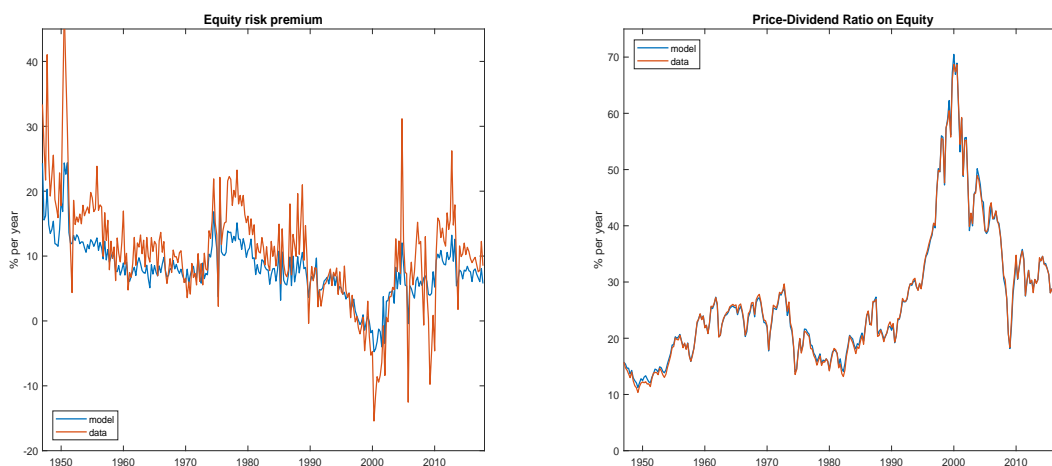
3.4.2 Stocks

We allow for non-zero market prices of risk in the sixth element of Λ_0 and the first six entries of the sixth row of Λ_1 ; the sixth element is the aggregate dividend growth rate on the U.S. stock market. We use the following moments to identify these parameters. First, we include the distance between the observed and model-implied time-series of the price-dividend ratio on the aggregate stock market in each quarter. The model-implied series is constructed from the dividend strips per (8). Second, we impose that the risk premium in the model matches that in the VAR, both in terms of its unconditional average and its dependence on the state variables. This gives a total of $T+11$ moments.

Figure 6 shows the equity risk premium, the expected excess return, in the left panel and the price-dividend ratio in the right panel. The risk premia in the data are the expected equity excess return predicted by the VAR. Their dynamics are sensible, with low risk premia in the dot-com boom of 1999-2000 and very high risk premia in the Great Financial Crisis of 2008-09.⁶ The figure's right panel shows a tight fit for equity price levels. We conclude that the model captures the observed prices and returns on stocks and bonds well.

Figure 6: Equity Risk Premium and Price-Dividend Ratio

The figure plots the observed and model-implied equity risk premium on the overall stock market in the left panel and the price-dividend ratio in the right panel. The quarterly equity risk premium in model and data is multiplied by 400 to express it as an annual percentage number. The price-dividend ratio is the price divided by the annualized dividend.



⁶Equity risk premia are multiplied by 4 to express them as annual quantities. The VAR-implied quarterly equity risk premium occasionally turns negative. The model-implied one rarely does. We could impose further restrictions on the variables that drive time-variation in expected excess stock returns to limit the in-sample presence of negative equity risk premia. We expect this will make little difference for our results.

3.4.3 Good deal bounds

Finally, when estimating the market prices of risk, we impose good deal bounds on the standard deviation of the log SDF in the spirit of [Cochrane and Saa-Requejo \(2000\)](#). Specifically, we impose a quadratic penalty for quarterly Sharpe ratios in excess of 1.5.

4 Valuing a Claim to Government Surpluses

4.1 Surplus Pricing Model

With the VAR dynamics and the SDF in hand, we can calculate the expected present discounted value of the primary surplus:

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$} S_{t+j} \right] = \sum_{j=0}^{\infty} \mathbb{E}_t \left[M_{t,t+j}^{\$} T_{t+j} \right] - \sum_{j=0}^{\infty} \mathbb{E}_t \left[M_{t,t+j}^{\$} G_{t+j} \right] = P_t^{\tau} - P_t^g, \quad (9)$$

where P_t^{τ} is the cum-dividend value of a claim to future nominal tax revenues and P_t^g is the cum-dividend value of a claim to future nominal government spending.

By construction, the nominal tax revenue growth is

$$\Delta \log T_{t+1} = \Delta \log \tau_{t+1} + x_{t+1} + \pi_{t+1} = \mu_0^{\tau} + x_0 + \pi_0 + (e_{\Delta\tau} + e_x + e_{\pi})' z_{t+1}, \quad (10)$$

where we recall that $\tau_t = T_t/GDP_t$ is the ratio of government revenue to GDP, and $e_{\Delta\tau}$ picks out the row of the VAR corresponding to $\Delta \log \tau_{t+1}$.

Similarly, the nominal government spending growth is

$$\Delta \log G_{t+1} = \Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} = \mu_0^g + x_0 + \pi_0 + (e_{\Delta g} + e_x + e_{\pi})' z_{t+1}, \quad (11)$$

where $g_t = G_t/GDP_t$ is the ratio of government spending to GDP, and $e_{\Delta g}$ picks out the row of the VAR corresponding to $\Delta \log g_{t+1}$.

Since we impose cointegration on the level of GDP, tax and spending, the unconditional growth rates of the tax-to-GDP ratio and the spending-to-GDP ratio (μ_0^{τ} and μ_0^g) have to be zero. On the other hand, the unconditional growth rate of the GDP x_0 is measured from the sample. The following proposition shows us how to price the government cash flows.

Proposition 4. (a) The price-dividend ratios on the tax claim and the spending claim are the sum of the price-dividend ratios of their strips, whose logs are affine in the state vector z_t :

$$PD_t^\tau = \frac{P_t^\tau}{T_t} = \sum_{h=0}^{\infty} \exp(A_\tau(h) + B'_\tau(h)z_t), \quad (12)$$

$$PD_t^g = \frac{P_t^g}{G_t} = \sum_{h=0}^{\infty} \exp(A_g(h) + B'_g(h)z_t). \quad (13)$$

(b) After we log-linearize the returns to the tax and spending claims, we can approximately express their risk premia (expected excess returns corrected for a Jensen term) as

$$\mathbb{E}_t [r_{t+1}^\tau] - y_t^\$(1) + Jensen = (e_{\Delta\tau} + e_x + e_\pi + \kappa_1^\tau \bar{B}_\tau)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t), \quad (14)$$

$$\mathbb{E}_t [r_{t+1}^g] - y_t^\$(1) + Jensen = (e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}_g)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t). \quad (15)$$

The proof is in Appendix B.4. In Part (b), the vectors \bar{B}_τ and \bar{B}_g describe the exposures of the “price-dividend ratios” of the revenue and spending claims to the state variables. The right-hand side denotes the covariance of the claims’ returns with the SDF. These covariances are crucially driven by the exposure vectors \bar{B}_g and \bar{B}_τ .

4.2 Results with No Cointegration

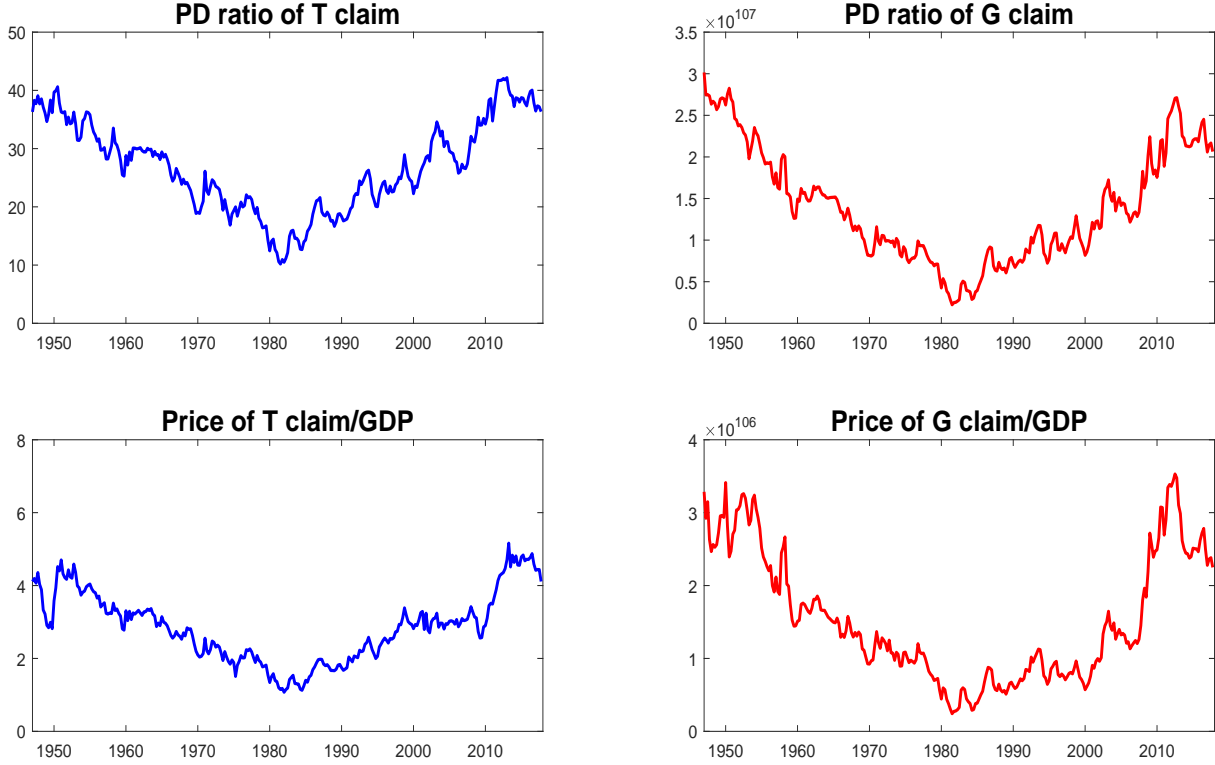
We first report the pricing results under the VAR in which the first 8 state variables do not load on the cointegration variables $\log \tau_t$ and $\log g_t$. As shown in Figure 3, the tax and spending shocks are permanent under this restricted VAR system.

Figure 7 reports the pricing results. The top left panel plots the (cum-dividend) price-dividend ratio on a claim to future tax revenue, PD_t^τ in (12). The time-series average of this ratio is 26.8, and the average risk premium is 10.1% per year. In other words, the representative agent who is pricing assets in this economy would be willing to pay 26.8 times annual tax revenues for the right to receive all current and future tax revenues. This valuation ratio reflects the risk of tax revenues. Since tax revenues accrue in good times, i.e., low marginal utility times, the tax revenue asset is risky and therefore has a low valuation ratio.

The top right panel plots the (cum-dividend) price-dividend ratio on a claim to future government spending, PD_t^g in (13). Under the estimated dynamics for spending-to-GDP

Figure 7: Government Cash Flows and Prices, No Cointegration

The top panels plot the (cum-dividend) price-dividend ratio on the claim to tax revenues (left) and government spending (middle). Both are annualized (divided by 4). The bottom left panel plots the value of a claim to future tax revenue, scaled by GDP. The middle panel plots the value of a claim to future government spending divided by GDP. The bottom right panel plots the value of future government surpluses scaled by GDP.



growth and market prices of risk, the time-series average of this ratio has an order of magnitude of 107. This very large valuation ratio translates into an average risk premium of 3.1% per year, lower than that of the revenue claim.

Intuitively, the lack of the cointegration variables makes an increase in the current $\Delta \log g_t$ not checked by the mean-reversion in the level of $\log g_t$. In this case, the increase in the future government spending becomes permanent, which tends to happen during recessions and makes the spending claim much safer. Similarly, the lack of the cointegration variables makes the tax claim much riskier, because a decline in the tax revenue during recessions also becomes permanent.

4.3 Main Results with Cointegration

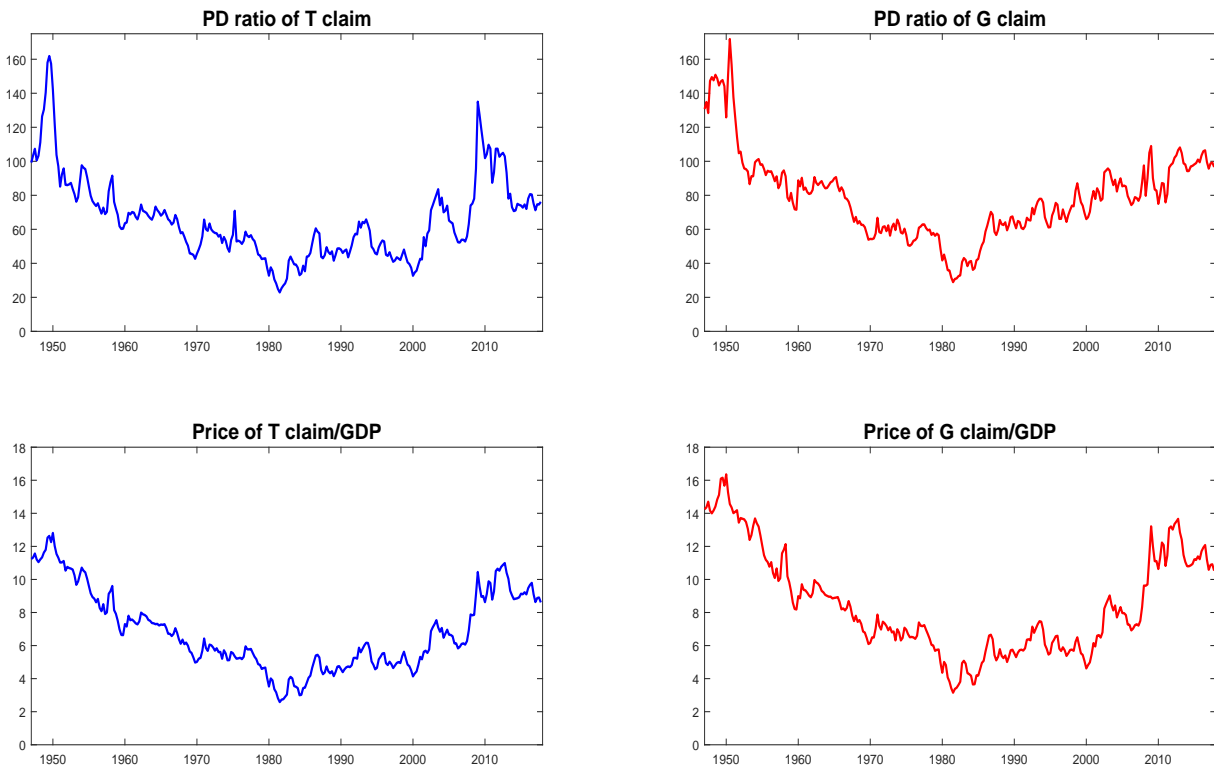
Now we report the pricing results under the benchmark VAR, in which the first 8 state variables can load on the cointegration variables $\log \tau_t$ and $\log g_t$. The top left panel of

Figure 8 plots the (cum-dividend) price-dividend ratio on a claim to future tax revenue, PD_t^T . The time-series average of this ratio is 65.58. In other words, the representative agent who is pricing assets in this economy would be willing to pay 65.58 times annual tax revenues for the right to receive all current and future tax revenues. This valuation ratio reflects the risk of tax revenues. Since tax revenues accrue in good times, i.e., low marginal utility times, the tax revenue asset is risky. The annual risk premium on the tax claim is 4.71% per year. The risk premium reflects mostly compensation for interest rate risk (12.82%), but also for GDP risk (2.31%), offset by stock market risk (-0.72%) and slope risk (-9.73%). The high risk premium translates into a low valuation ratio.

In addition, the price-dividend ratio of the tax claim displays substantial time-variation. A pronounced V-shape arises from the inverse V-shape of the long-term real interest rate, which, as manifest in the bottom right panel of Figure 5, is high in the middle of the sample and low at the beginning and end of the sample. Intuitively, discounting future tax revenues by a low (high) long-term real rate results in a high (low) valuation ratio.

Figure 8: Government Cash Flows and Prices

The top panels plot the (cum-dividend) price-dividend ratio on the claim to tax revenues (left) and government spending (middle). Both are annualized (divided by 4). The bottom left panel plots the value of a claim to future tax revenue, scaled by GDP. The middle panel plots the value of a claim to future government spending divided by GDP. The bottom right panel plots the value of future government surpluses scaled by GDP.



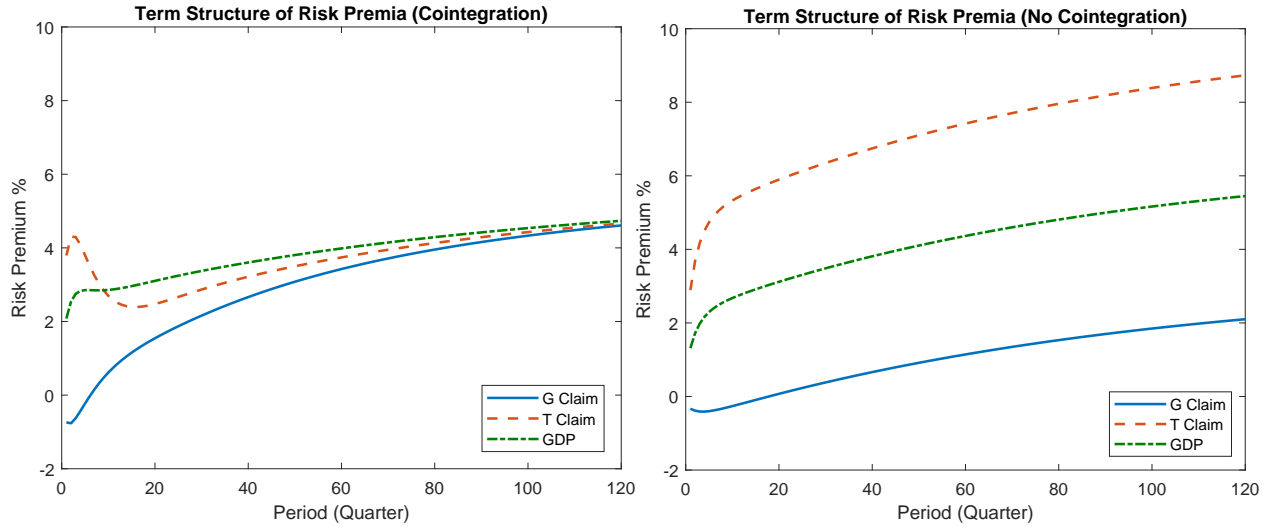
The top right panel of Figure 8 plots the (cum-dividend) price-dividend ratio on a claim to future government spending, PD_t^g . Under the estimated dynamics for spending-to-GDP growth and market prices of risk, the time-series average of this ratio is 78.87, and the average risk premium is 4.59% per year. These results are much less extreme than those from the VAR with no cointegration, because a higher government spending today lowers the growth rate of government spending in the future. In this case, the risk premium reflects mostly compensation for interest rate risk (13.61%), but also for GDP risk (2.18%), offset by stock market risk (-0.90%) and slope risk (-10.28%). The high risk premium translates into a low valuation ratio. The price-dividend ratio shows the same inverse V-shape dynamics of the price-dividend ratio on the revenue claim.

Although the gap between the unconditional risk premium on the tax claim and that on the spending claim seems small, the term structure of their risk premia behaves very differently, especially over the shorter horizon. Figure 9 presents the risk premia of government spending strips and tax strips over different horizons. The average risk premium of tax claim over the five-year horizon is 3.00%, much larger than that of the spending claim, 0.52%. A claim to government revenues is a safe asset. It pays out high “dividends” in bad economic times, i.e., high marginal utility states of the world. Therefore, agents are willing to pay a higher price/a lower risk premium for such an asset. The unconditional risk premia of two claims are both dominated by the risk premia of long term strips, which converge to the long-run risk premium on a GDP strip, given the cointegration restriction. The right-hand side shows that the assumption of cointegration is crucial for this result. Absent cointegration, there is no long-run risk premium convergence of T- and G-claims.

4.4 The Puzzle

Now we are in a position to evaluate the claim to future government surpluses as the tax claim minus the spending claim, the right-hand side of equation (9). Figure 10 plots the present value of government surpluses scaled by GDP as the dashed line. The value of the surplus claim is not enough to honor the market value of the US government debt, plotted as the solid line. The unconditional average present value of the government surplus is -1.59 times GDP, far below the average market value of outstanding government debt, 0.37 times GDP. The gap is 196% of GDP on average. In the time series, the present value of the government surplus does not match the dynamics of government debt value, either. We refer to this finding as the government debt valuation puzzle.

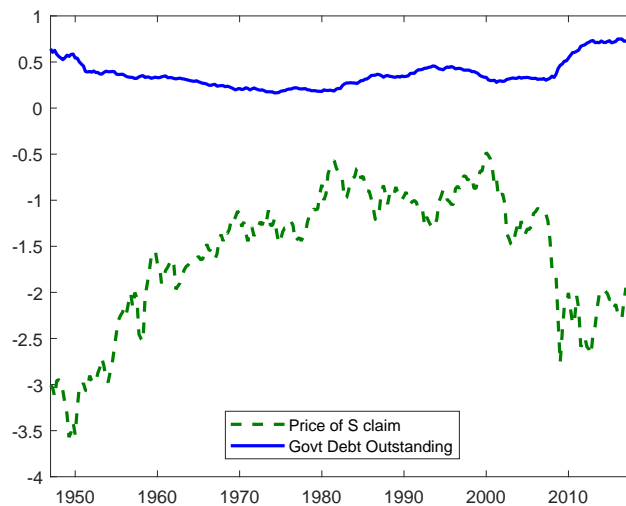
Figure 9: Term Structure of Risk Premia on the T-Claim and the G-Claim



This puzzle deepens further in the last 20 years, as the level of government debt rises to about 75% of the GDP, while the valuation of the government surplus claim goes down to about -200% of the GDP. In other words, the U.S. government has been issuing government debt while simultaneously reducing the expected government surpluses to back it up.

Figure 10: Present Value of Government Surpluses and Market Value of Government Debt

Both time series are scaled by the US GDP.



This puzzle is also deeper than merely the fact that the government does not generate

enough surplus to cover the debt payments. We can rewrite Eq. (1) as

$$\begin{aligned} \mathbb{E} \left[\sum_{h=0}^H P_t^\$(h) Q_{t-1,h+1}^\$ \right] &= \sum_{j=0}^{\infty} \mathbb{E} \left[M_{t,t+j}^\$ \right] \mathbb{E} [T_{t+j} - G_{t+j}] \\ &+ \text{cov} (M_{t,t+j}, T_{t+j}) - \text{cov} (M_{t,t+j}, G_{t+j}). \end{aligned}$$

On the right-hand side, as the average government surplus has been just about zero in our sample, $\mathbb{E} \left[M_{t,t+j}^\$ \right] \mathbb{E} [T_{t+j} - G_{t+j}]$ is approximately 0. Therefore, the nearly 200% of the GDP wedge between the left-hand side and the right-hand side stems from the differential riskiness of the revenue and the spending claims. Put differently, without the covariance terms, the government would need to generate about 75% of GDP in PDV of future surpluses to support 75% in debt relative to GDP. With the covariance terms present, about 275% of GDP in terms of future surpluses are needed to support the same debt.

This view of the government budget is different from [Blanchard \(2019\)](#), who argues that the US government has infinite debt capacity because the risk-free interest rate is often below the growth rate. In our estimation sample, the unconditional average 1-quarter log nominal interest rate is $y_0^\$(1)=1.02\%$ whereas the unconditional average 1-quarter log nominal GDP growth rate is $x_0 + \pi_0= 1.56\%$. The risk-free interest rate is on average below the growth rate. However, government tax and spending processes are sufficiently risky, so that their average nominal discount rates ($r_0^\tau = 0.0191$ and $r_0^\$ = 0.0178$) are above the average nominal GDP growth rate. We generate these discount rates while maintaining a good fit for the term structure of the Treasury yields. The claim to government surpluses reflects the governments' future debt issuance strategy. Future net debt issuances at inopportune (high SDF) times make the overall bond portfolio riskier than an individual Treasury bond. Therefore, even if risk-free interest rates are often below growth rates, the risk premia on government tax and spending processes are large enough to make these claims to have finite valuation. Our estimation result suggests that the government spending claim on average has a higher valuation than the tax revenue claim, which leads to the government risk premium puzzle.

4.5 Government's Measurability Constraints

In order for [Blanchard \(2019\)](#)'s argument to be valid, measurability constraints would need to hold which constrain the state-contingency of the surplus claim to mimic that

of risk-free debt. In general, the measurability constraints imply that the value of the surplus claim at the start of the period can only depend on shocks in the same way that the government bond portfolio does (Hansen, Roberds, and Sargent, 1991; Aiyagari, Marcet, Sargent, and Seppälä, 2002).

Proposition 5. Measurability Condition: The value of the surplus claim responds to all innovations in the same way as the bond portfolio. Exploiting the affine nature of the price-dividend ratio of tax revenue and spending strips and of zero-coupon bond prices, this produces the following system of N equations:

$$\frac{T_t}{GDP_t} \sum_{h=0}^{\infty} \exp(B'_\tau(h)) - \frac{G_t}{GDP_t} \sum_{h=0}^{\infty} \exp(B'_g(h)) = \sum_{h=0}^{\infty} \exp(B^\$(h)) \frac{Q_{t-1,h+1}^\$}{GDP_t}. \quad (16)$$

Corollary 2. Blanchard-Hansen-Roberds-Sargent condition: If the government only issues one-period risk-free debt, then the value of the previous period's bond portfolio at the start of the next period cannot depend on any shocks. The measurability conditions become:

$$\frac{T_t}{GDP_t} \sum_{h=0}^{\infty} \exp(B'_\tau(h)) - \frac{G_t}{GDP_t} \sum_{h=0}^{\infty} \exp(B'_g(h)) = \mathbf{0} \quad (17)$$

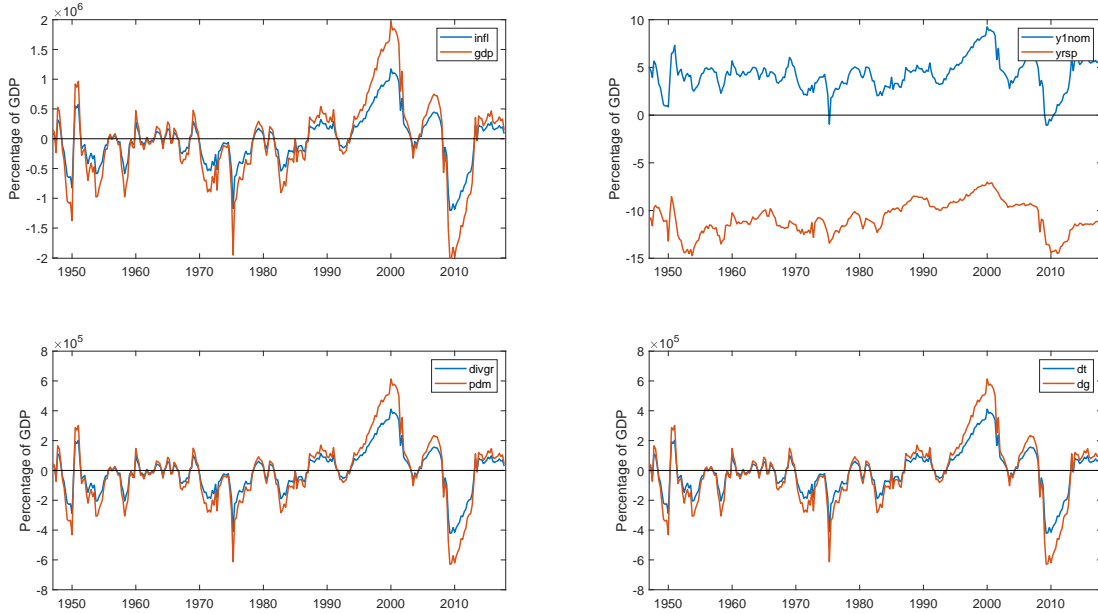
Only if condition (17) is satisfied can we discount future surpluses at the one-period risk-free bond rate, as Blanchard (2019) suggests one should do. Hansen, Roberds, and Sargent (1991) deliver a univariate version of this measurability condition.

This condition is severely violated in the data. Figure 11 plots the left hand side of this equation for our benchmark model estimates (case with cointegration) as well as the zero line. Deviations from zero are on the order of tens or even hundreds of times GDP. Given that the U.S. federal government's surpluses are clearly trending with GDP (see Figure 1), every innovation to GDP permanently alters the cash flows that accrue to investors in the surplus claim. The Blanchard-Hansen-Roberds-Sargent condition cannot hold: there is long-run GDP risk in the cash flows that simply cannot be replicated by a position in risk-free debt.

If the yield curve spans all the innovations, there exists a dynamic, highly levered long-short portfolio in government debt $Q_{t-1,h+1}^\$$ of various maturities that replicates the state-contingency of the surplus claim and satisfies Proposition 5. To do so, we select N bonds, we can stack N of the $\exp(B^\$(h))$ vectors in a matrix B , and back out the portfolio

Figure 11: Violations of the Measurability Condition With Only One-Period Debt

The figure shows the time series of $\frac{T_t}{GDP_t} \sum_{h=0}^{\infty} \exp(B'_t(h)) - \frac{G_t}{GDP_t} \sum_{h=0}^{\infty} \exp(B'_g(h))$ for each state variable. They are expressed as a



of government bonds $Q^{\$,*}$ that satisfies measurability for each t :

$$B^{-1} \left(\frac{T_t}{GDP_t} \sum_{h=0}^{\infty} \exp(B'_t(h)) - \frac{G_t}{GDP_t} \sum_{h=0}^{\infty} \exp(B'_g(h)) \right) = \frac{Q_{t-1}^{\$,*}}{GDP_t}.$$

However, we know that this dynamic portfolio $Q_{t-1}^{\$,*}$ is very different from the government's actual bond portfolio.⁷ The violations of the general measurability condition evaluated at the actual Treasury portfolio in Proposition 5 are equally large as those in shown Figure 11. This is not surprising: we need to construct a Treasury portfolio with long-run risk exposure to GDP equivalent to that of a claim to GDP.

5 Valuing Claim to Convenience Yield Seignorage Revenue

We define the convenience yield λ_t as the US government bonds' expected returns that investors are willing to forgo under the risk-neutral measure. We assume U.S. Treasury bonds carry a uniform convenience yield across the maturity spectrum. Then, the Euler

⁷Similar spanning arguments were explored by [Angeletos \(2002\)](#) and [Buera and Nicolini \(2004\)](#).

equation for a Treasury bond with maturity $k + 1$ is

$$e^{-\lambda_t} = \mathbb{E}_t \left[M_{t+1} \frac{P_{t+1}^\$(k)}{P_t^\$(k+1)} \right].$$

Proposition 6. If the transversality condition holds, the value of the government debt portfolio equals the value of future surpluses plus the value of future seigniorage revenue:

$$E_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^\$ \left(T_{t+j} - G_{t+j} + (1 - e^{-\lambda_{t+j}}) \sum_{h=1}^H Q_{t+j,h}^\$ P_{t+j}^\$(h) \right) \right] = \sum_{h=0}^H Q_{t-1,h+1}^\$ P_t^\$(h), \quad (18)$$

where $\sum_{h=0}^H Q_{t-1,h+1}^\$ P_t^\(h) on the right-hand side denotes the cum-dividend value of the government's debt portfolio at the start of period t , and $\sum_{h=1}^H Q_{t+j,h}^\$ P_{t+j}^\(h) on the left-hand side denotes the ex-dividend value of the government's debt portfolio at the end of period $t + j$.

When there is no convenience yield, we end up back in the standard case, Proposition 1. When the convenience yield is positive and the government debt outstanding is positive in the future, the convenience yields will always *increase* the value of government debt, acting as if the government has an additional source of income. We call this additional income the seigniorage revenue, which could potentially turn government deficits into surpluses when properly adjusted.

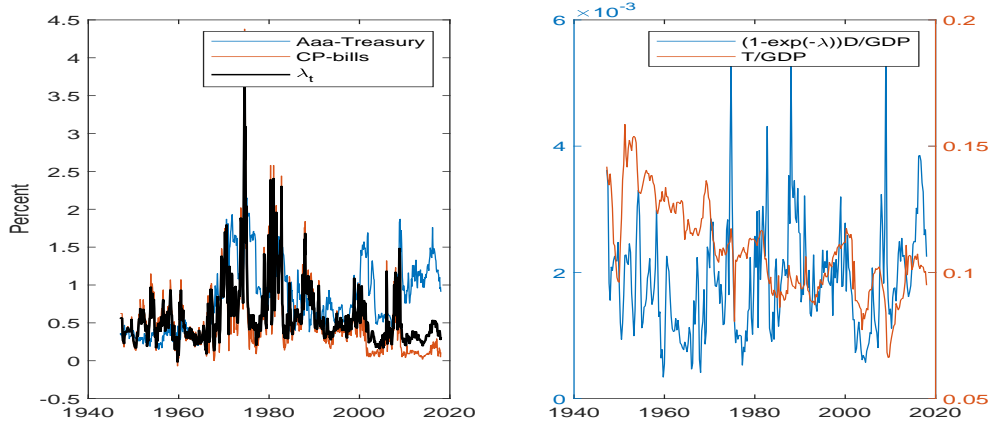
As an empirical strategy, we start by measuring the average convenience yield following [Krishnamurthy and Vissing-Jorgensen \(2012\)](#). We use the weighted average of the Aaa-Treasury yield spread and the high-grade commercial papers-bills yield spread to proxy λ_t , where the time series of weights are computed to match the duration of the government bond portfolio period by period. The left panel of Figure 12 shows the time series of the convenience yield (thick black line). Over the sample period from 1947 to 2017, the average convenience yield is 15 basis points per quarter or 0.57% per year, which implies an average seigniorage revenue 12.28 billions per year, or 0.19% of U.S. GDP as shown in the right panel of Figure 12.

Then, we rewrite equation (18) as:

$$E_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^\$ T_{t+j} K_{t+j} \right] - E_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^\$ G_{t+j} \right] = \sum_{h=0}^K Q_{t-1,h+1}^\$ P_t^\$(h),$$

Figure 12: Convenience Yield and Seigniorage Revenue

The left panel reports the time series of Aaa-Treasury yield spreads, high-grade commercial papers-bills yield spreads, and the proxy for λ_t . We measure λ_t using the weighted average of the Aaa-Treasury yield spread and the high-grade commercial papers-bills yield spread. All yields are expressed in percentage per annum. The right panel reports time series of $(1 - \exp(-\lambda_t))D_t/GDP$ and government tax revenue to GDP ratio at different scales. The sample period is from 1947-01 to 2017-04.



and model

$$K_{t+j} = \left(1 + \frac{(1 - e^{-\lambda_{t+j}}) \sum_{h=1}^H Q_{t+j,h}^{\$} P_{t+j}^{\$(h)}}{T_{t+j}} \right)$$

as a reduced-form variable. We preserve the structure of the state vectors and the SDF, and introduce its log growth rate $\Delta \log K_t$ as an additional state variable. Specifically, we define $\tilde{z}_t = [z_t, \Delta \log K_t]$. The seigniorage revenue $\log K_t$ follows the following process: $\Delta \log K_{t+1} = e'_k \tilde{z}_{t+1}$. $\Delta \log K_{t+1}$ has a mean of zero because $\log K_t$ is stationary.

We use the same method as in Proposition 4 to price the modified tax claim. The adapted pricing formula is:

$$E_t \left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$} T_{t+j} K_{t+j} \right] = T_t K_t PD_t^K,$$

where PD_t^K is a function of the state variables \tilde{z}_t . The left panel of Figure 13 reports the present value of the government surpluses under the modified model. The convenience yield always increases the present value of the government surpluses. On average, the seigniorage revenue has a present value of 93% of annual U.S. GDP. While substantial, this seigniorage revenue only bridges less than half of the gap between the present value of surpluses and the value of the government debt portfolio. The government debt valuation puzzle remains standing.

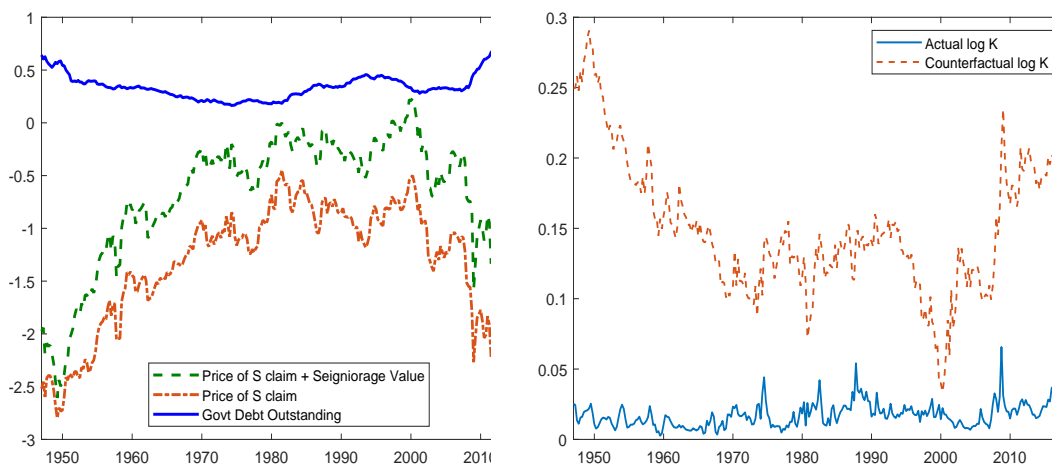
How large should the seigniorage revenue become to match the present value of the government surplus claim to the actual debt value? To answer this question, we fix the coefficients of the VAR system for pricing purposes, but change the seigniorage revenue term $\log K_t$ to $\log \tilde{K}_t$ at each point of time t so that

$$T_t \tilde{K}_t \widetilde{PD}_t^K - G_t P_t^g = \sum_{h=0}^H Q_{t-1, h+1}^{\$} P_t^{\$(h)}$$

Since the last element of \tilde{z}_t is $\Delta \log \tilde{K}_t$, $\log \tilde{K}_t$ enters this equation through both \tilde{K}_t and \widetilde{PD}_t^K , the latter of which is a function of \tilde{z}_t . We solve for variable $\log \tilde{K}_t$ in this equation, taking other variables as given. The right panel of Figure 13 reports the resulting \tilde{K}_t process. It requires seigniorage revenue that is on average 12.66% of tax revenue to match the present value of the government surplus claim to the actual debt value. Actual seigniorage revenue only averages 1.74% of tax revenue. In sum, the convenience yield would have to be implausibly large to bridge the gap.

Figure 13: Present Value of Government Surpluses and Seigniorage Revenues

The left panel plots the present value of government surpluses with and without seigniorage revenues, scaled by the US GDP. The right



6 Potential Resolutions of the Puzzle

6.1 Other Government Assets and Liabilities

The government owns various assets, including outstanding student loans and other credit transactions, cash balances, and various financial instruments. Based on Congressional Budget Office data, the total value of these government assets is 8.8% of the GDP as

of 2018. While these assets bring the net government debt held by the public from 77.8% to 69.1% of the GDP, the bulk of the government debt valuation puzzle remains.

Other significant sources of government revenues and outlays are those associated with the Social Security Administration (SSA). Based on the CBO data, the net flows from SSA are close to 0 as of 2018, but will turn into a deficit of 0.7% of GDP per annum from 2020 to 2029. SSA revenues and outlays are included in our definition of government tax revenue and spending data. That is, we merge the balance sheets of the Treasury and the SSA. Correspondingly, Treasuries bonds accumulated by the SSA trust fund are netted out against liabilities of the U.S Treasury. We consider only debt held by the public, which excludes intra-governmental holding. As the SSA turns from a net contributor of primary surpluses into a net contributor to the deficit in 2019, the government will need to issue additional debt to the public.

6.2 Market Segmentation

Can market segmentation resolve the government debt risk premium puzzle? U.S. Treasury securities are owned by both foreign and domestic investors.⁸ One natural question is whether foreign investors use a different SDF to price Treasury bonds. Whatever SDFs foreign investors use, the projections of their SDFs on the state space z must agree with those of the domestic investors to price bonds. Our benchmark exercise already identifies the SDFs that both foreign and domestic investors consistently use to price government bonds.

One could also argue that marginal investors in Treasury bonds don't necessarily overlap with investors in the U.S. equity market. We conduct the following analysis assuming that investors in the U.S. bond markets are not exposed to stock market risks. We write down the vector of state variables \hat{z}_t without including the log price-dividend ratio and the log real dividend growth on the aggregate stock market:

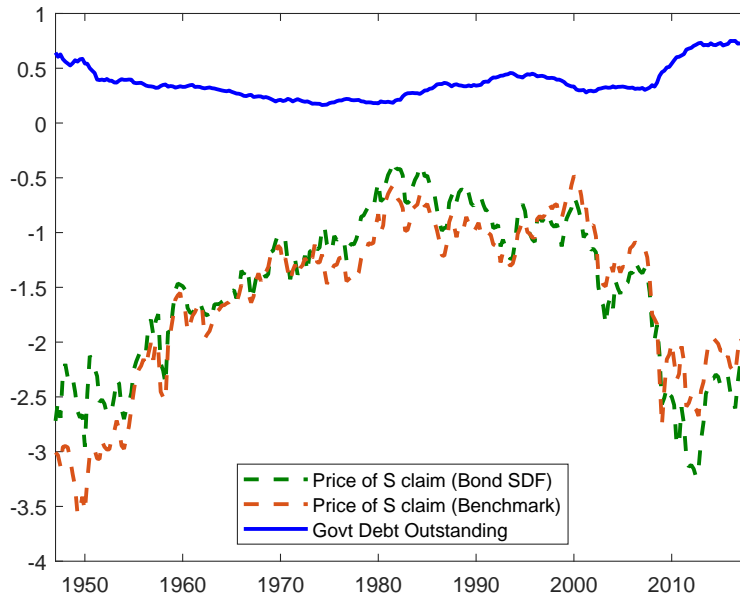
$$\hat{z}_t = [\pi_t - \pi_0, x_t - x_0, y_t^\$ - y_0^\$(1), yspr_t^\$ - yspr_0^\$(1), \Delta \log \tau_t - \mu_0^\tau, \Delta \log g_t - \mu_0^g, \log \tau_t - \log \tau_0, \log g_t - \log g_0]'$$

⁸At the end of our sample period, 43% of U.S. Treasury securities are owned by foreign investors and governments, while domestic investors excluding the Federal Reserve system (mutual funds, pension funds, banks, and insurance companies) own about 40% of the public debt. Foreign holdings accounted for 60% of Treasury debt held by the public in 2008 (see Favilukis, Kohn, Ludvigson, and Nieuwerburgh, 2013).

We estimate the VAR system $\hat{z}_t = \hat{\Psi}\hat{z}_{t-1} + \hat{\Sigma}^{\frac{1}{2}}\hat{\epsilon}_t$. Then we estimate the constant market prices of risk $\hat{\Lambda}_0$ and the time-varying market prices of risk $\hat{\Lambda}_1$ to fit the prices and expected returns on only bonds with different maturities. Both the average and time-varying prices of risk for inflation, real GDP growth, interest rate, and the slope of the term structure are similar to the estimates from our benchmark specification in Section 3.4. It is worth noting that zeroing out the stock market risk factors presents an extreme case of segmentation since government bond investors are almost certainly exposed to some U.S. stock market risks. Our estimates of $\hat{\Lambda}_0$ and $\hat{\Lambda}_1$ remain similar to Λ_0 and Λ_1 in our benchmark estimation.

Using the estimated VAR system and market prices of risk, we obtain an average price-dividend ratio for the tax revenue claim of 82.4 and for the spending claim of 94.4. Figure 14 shows that the present value of the government surpluses using the SDF that prices Treasury bonds only behaves similarly to the present value of the government surpluses under our benchmark specification. Market segmentation does not resolve the puzzle.

Figure 14: Present Value of Government Surpluses with the Bond Market SDF



6.3 Peso Problem

Lastly, we consider a model in which bond investors price in the possibility of a major government spending cut, but that such a spending cut never realizes in our 70-year sample. How large should the spending cut probability be in order to match the market valuation of the government debt to the present value of government surpluses?

We set the spending cut to be 2 times the standard deviation of the log spending-to-GDP shock. When it happens, the spending-to-GDP ratio decreases by $2 \times 3.85\% = 7.7\%$ of U.S. GDP.

Let $\bar{\phi}$ is the unconditional average of the spending cut probability. We augment the vector of demeaned state variables \mathbf{z}_t with the demeaned probability of a spending cut, ϕ_t : $\mathbf{w}_t = [\mathbf{z}_t; \phi_t]$, and we augment the vector of VAR shocks $\boldsymbol{\varepsilon}_t$ with an additional shock to ϕ_t : $\mathbf{u}_t = [\boldsymbol{\varepsilon}_t; \varepsilon_t^\phi]$. The new state vector \mathbf{w}_t follows:

$$\mathbf{w}_t = \tilde{\Psi} \mathbf{w}_{t-1} + \tilde{\Sigma}^{\frac{1}{2}} \mathbf{u}_t.$$

The time-varying probability ϕ_t can load on \mathbf{w}_{t-1} and \mathbf{u}_t , with loadings that are to be estimated. We extend Proposition 4, with proof in Appendix B.5, and show that the price-dividend ratios of the tax claim and the spending claim can be expressed in similar form as in the benchmark analysis.

Proposition 7. In the presence of the spending cut, the price-dividend ratios on the tax claim and the spending claim are the sum of the price-dividend ratios of their strips. The log price-dividend ratios on these strips are affine in the state vector \mathbf{z}_t :

$$\begin{aligned} PD_t^\tau &= \sum_{h=0}^{\infty} \exp(\tilde{A}_\tau(h) + \tilde{B}'_\tau(h) \mathbf{w}_t), \\ PD_t^g &= \sum_{h=0}^{\infty} \exp(\tilde{A}_g(h) + \tilde{B}'_g(h) \mathbf{w}_t). \end{aligned}$$

We estimate the process of the spending cut probability such that the present value of government surpluses is exactly equal to the market valuation of the government debt in every period. We do not change the market prices of risk Λ_t . That is, we assume the equity and bond investors still observe the same state variable processes, including the processes of tax and spending. The state variable dynamics are unaffected by the time-varying probability of the spending cut because it never realizes in our sample.

We estimate the unconditional average probability $\bar{\phi}$, the probability's loading on the lagged probability $e'_\phi \Psi e_\phi$, the probability's loading on the contemporaneous GDP shock $e_x \mathbf{u}_t$, and the probability's loading on the contemporaneous ϕ shock $e_\phi \mathbf{u}_t$. For the benchmark estimation, we set the other loadings of the probability to zero.

Our estimation algorithm proceeds as follows. We start with an initial guess for the aforementioned parameters. Under this set of parameter values, we solve for the ϕ_t pro-

cess that matches the market value of government debt with the present value of government surpluses. The market value of government debt is observed in the data, and the present value of government surpluses can be calculated following Proposition 7. Then, we calculate the empirical moments of the implied ϕ_t process. The moments are the average value of the probability and the loadings of the residual $\phi_t - e_\phi \tilde{\Psi} w_{t-1}$ on the 9 independent shocks u_t . We search for the parameter values that minimize the L_2 distance between the moments implied from the initial guess and these empirical moments. The resulting parameter estimates are reported in Table 3.

Table 3: Parameter Estimates and Corresponding Moments

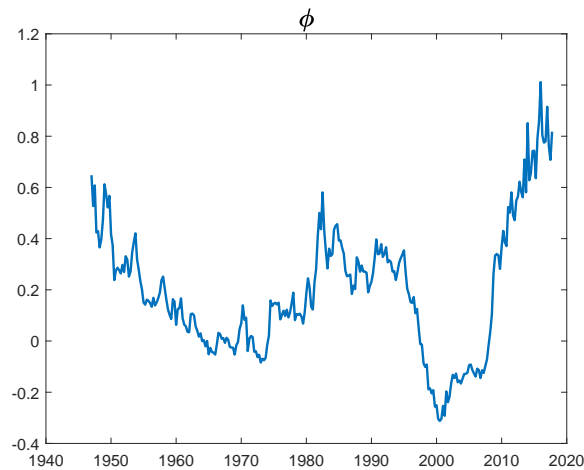
The table reports the estimated parameters in the extended model with spending cuts.

<i>Parameter Estimates</i>			
$\bar{\phi}$	$\bar{\Psi}_\phi$	$\bar{\Sigma}^{1/2}(11,2)$	$\bar{\Sigma}^{1/2}(11,11)$
0.1922	0.9884	-0.0117	0.0280

The estimated $\bar{\phi} + \phi_t$ process is shown in Figure 15. The gap between the market value of debt and the present value of surpluses under the benchmark model is nearly two hundred percent of GDP. To match such a large gap, the probability of the potential spending cut has to be large and have large fluctuations. The spending cut probability is around 50% on average and fluctuates strongly between -30% and 90%. It rises sharply after the year 2000. Such a large probability is at odds with the notion of a peso event that never happens in a 70-year sample. We interpret this result as a restatement of the puzzle.

Figure 15: Probabilities of Spending Cut Implied by Debt-to-GDP Ratio

This figure reports the time series of probabilities of spending cuts implied by the debt to GDP ratio, $\bar{\phi} + \phi_t$.



7 Government Risk Management

The peso problem formulation has the virtue that it reconciles the market value of outstanding government debt with the present value of future surpluses. With the government budget constraint holding with equality, we can consider the question of optimal government debt portfolio management. As shown by [Bhandari, Evans, Golosov, and Sargent \(2017\)](#), optimal maturity choice in a large class of models amounts to minimizing the variance of government funding needs.

Suppose there is only nominal debt, and let $Q_{t,h}^{\$}$ denote the outstanding nominal bond quantity at $t + 1$ chosen at time t . Then, the funding need at $t + 1$ is defined as the time $t + 1$ value of its time- t bond portfolio minus the expected discounted value of its future surpluses, holding fixed the government's tax and spending policy rules:

$$Need_{t+1} = \sum_{h=0}^{\infty} P_{t+1}^{\$}(h) Q_{t,h+1}^{\$} - \mathbb{E}_{t+1} \left[\sum_{j=0}^{\infty} M_{t+1,t+1+j}^{\$} S_{t+1+j} \right]. \quad (19)$$

A positive funding need means that the government faces a funding shortfall and has to issue additional debt at time $t + 1$.

Under this formulation, the government's risk management problem amounts to choosing its debt issuance policy along the maturity curve to minimize the variance of its funding needs:

$$\min_{\{Q_{t+1,h}^{\$}\}} Var_t [Need_{t+1}] \quad (20)$$

To develop some intuition, we start by considering a portfolio that is locally immunized against the interest rate shock to $y_{t+1}^{\$}(1)$. We define this shock as the realization of the shocks ε_{t+1} at time $t + 1$ such that $\Sigma^{\frac{1}{2}} \varepsilon_{t+1}$ is zero except the row corresponding to the 3-month interest rate. Recall that the state transition equation is $z_{t+1} = \Psi z_t + \Sigma^{\frac{1}{2}} \varepsilon_{t+1}$. So, if this interest rate shock is 1%, the 1-quarter interest rate becomes:

$$y_{t+1}^{\$}(1) = y_0^{\$}(1) + e'_{yn} \Psi z_t + 1\%,$$

while all of the other state variables evolve according to $z_{t+1} = \Psi z_t$.

Definition 1. A government bond portfolio $\{Q_{t+1,h}^*\}$ is locally hedged against a change in short-term interest rates if the dollar value of the surplus claim adjusts by the same amount as the dollar value of the outstanding bond portfolio in response to an interest

rate shock:

$$\sum_{h=0}^{\infty} Q_{t+1,h}^* \frac{-\partial P_{t+1}^{\$}(h)}{\partial y_{t+1}^{\$}(1)} = \frac{-\partial \mathbb{E}_{t+1} \left[\sum_{j=0}^{\infty} M_{t+1,t+1+j}^{\$} S_{t+1+j} \right]}{\partial y_{t+1}^{\$}(1)} \quad (21)$$

We refer to these derivatives above as dollar durations. Note they are the dollar value changes with respect to the 3-month interest rate, *holding the shocks to other state variables to be zero*. An increase in the nominal yield will decrease the value of the bond portfolio and the surplus claim. The negative signs turn the dollar durations into positive numbers. Rewritten in percentages, they become elasticities:

$$\frac{\sum_{h=0}^{\infty} P_t^{\$}(h) Q_{t+1,h}^* \frac{-\partial \log P_{t+1}^{\$}(h)}{\partial y_{t+1}^{\$}}}{\sum_{h=0}^{\infty} P_t^{\$}(h) Q_{t+1,h}^*} = \frac{\mathbb{E}_t \left[\sum_{j=1}^{\infty} M_{t,t+j}^{\$} S_{t+j} \right] \frac{-\partial \log \mathbb{E}_{t+1} \left[\sum_{j=0}^{\infty} M_{t+1,t+1+j}^{\$} S_{t+1+j} \right]}{\partial y_{t+1}^{\$}}}{\mathbb{E}_t \left[\sum_{j=1}^{\infty} M_{t,t+j}^{\$} S_{t+j} \right]}$$

where $-\partial \log P_{t+1}^{\$}(h) / \partial y_{t+1}^{\$}$ is the modified duration of the zero coupon bond.

The left panel of Figure 16 shows the dollar duration of the tax and spending claims, expressed as a percent of annual GDP. The right panel shows the dollar duration of government surpluses, the difference between the two lines in the left panel, and that of the actual government bond portfolio. The dollar duration of the outstanding government debt portfolio grows in recent years in part because the amount of debt outstanding grows. The modified durations in the middle panel take out this size-of-debt effect. The modified duration of the government surplus claim is positive, like that of the debt portfolio, but substantially larger at various times during the sample. For example, in the mid-2000s, the government debt portfolio has a modified duration of 3 years whereas the surplus claim has a modified duration of 8 years. Most of the time, the actual government bond portfolio insufficiently hedges the interest rate risk of the surplus claim.

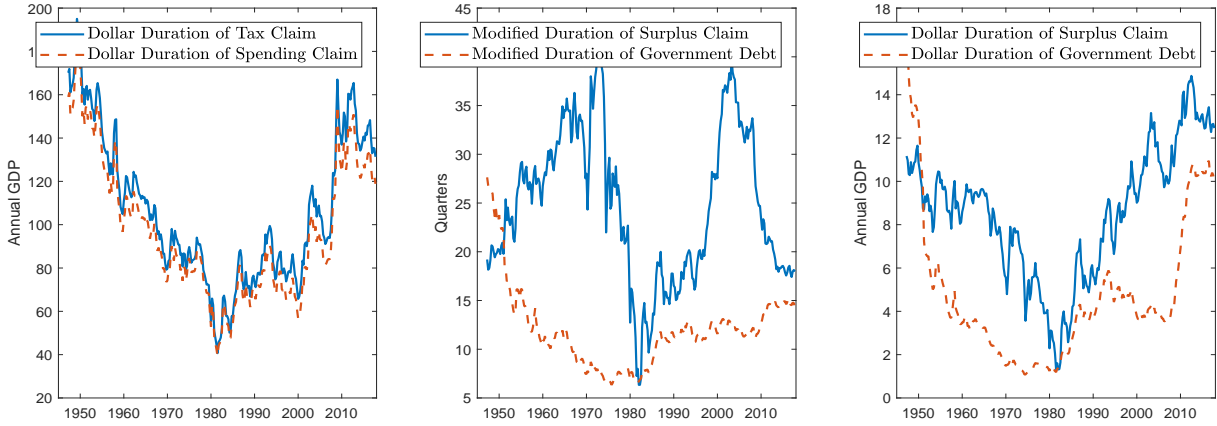
We can now generalize the notion of duration to other shocks. Minimize the variability of funding needs requires that the government's bond portfolio should also hedge those shocks.

Definition 2. A government bond portfolio $\{Q_{t+1,h}^*\}$ is fully immunized against any shock to the state of the economy provided that the changes in value of the outstanding bond portfolio equal the changes in the value of the surplus claim for each shock:

$$\sum_{h=0}^{\infty} Q_{t+1,h}^* \frac{\partial P_{t+1}^{\$}(h)}{\partial z_{t+1}} = \frac{\partial \mathbb{E}_{t+1} \left[\sum_{j=0}^{\infty} M_{t+1,t+1+j}^{\$} S_{t+1+j} \right]}{\partial z_{t+1}} \quad (22)$$

Figure 16: The Durations of the Government Bond Portfolio and Government Surpluses

The figure plots the duration of the government bond portfolio and that of government surpluses, weighted by annual GDP. The series is quarterly from 1940.Q1 until 2017.Q4. Data Source: CRSP U.S. Treasury Database and BEA.



Equation (22) describes a system of N equations, where N is the number of state variables (shocks) in the VAR. Appendix E contains the details. If the government can issue N different bond maturities, it is possible to fully immunize its debt portfolio against all economic shocks.⁹

Our objective is to evaluate how far the actual government debt portfolio (maturity structure) deviates from full immunization. To measure the deviation, we express the conditional standard deviation of the funding needs as:

$$\sqrt{\text{Var}_t [\text{Need}_{t+1}]} = \left(\text{Var}_t \left[\sum_{h=0}^{\infty} P_{t+1}^{\$}(h) Q_{t,h}^{\$} - P_{t+1}^{\tau} + P_{t+1}^{\$} \right] \right)^{1/2} = (U_t \Sigma U_t')^{1/2} \quad (23)$$

$$\text{where } U_t = \left(\sum_{h=0}^{\infty} Q_{t,h}^{\$} \frac{\partial P_{t+1}^{\$}(h)}{\partial \mathbf{z}_{t+1}} - \frac{\partial \mathbb{E}_{t+1} \left[\sum_{j=0}^{\infty} M_{t+1,t+1+j}^{\$} S_{t+1+j} \right]}{\partial \mathbf{z}_{t+1}} \right)$$

The left plot in Figure 17 reports this standard deviation. In 2017.Q3, the standard deviation of the government's funding need is around 3% of the annual GDP. This is a sizeable deviation from full immunization. The right plot decomposes this standard deviation to the component $e_i U_t \Sigma^{\frac{1}{2}}$ driven by each shock i . By Eq. (23), the square root of

⁹The immunization conditions look similar to the measurability conditions discussed in section 4.5. However, note that we are imposing the government budget constraint here through the peso model assumption.

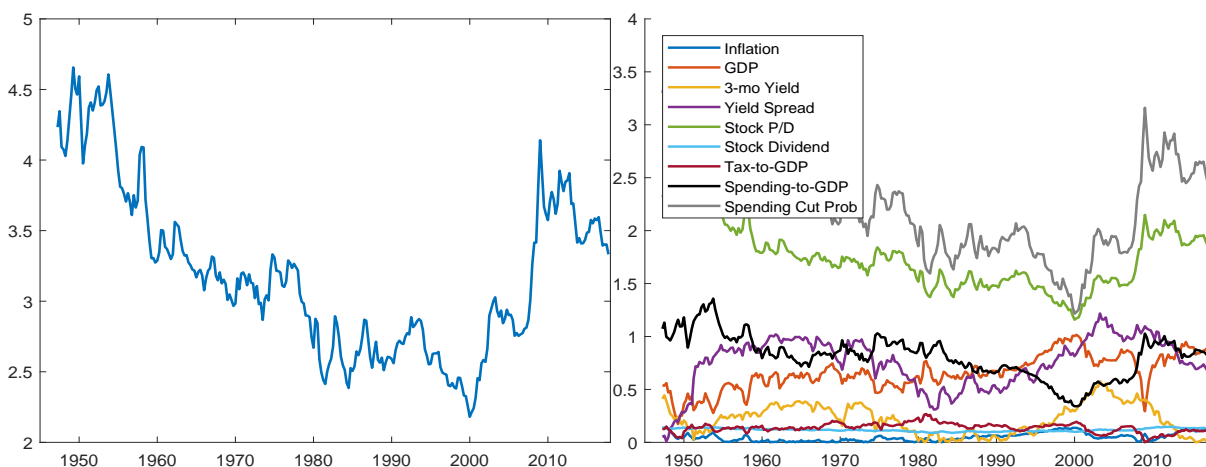
the sum of their squares is the total variation in the funding need:

$$\sqrt{\text{Var}_t[\text{Need}_{t+1}]} = \left(\sum_i (e_i U_t \Sigma^{\frac{1}{2}})^2 \right)^{1/2}$$

This calculation suggests that the deviations from full immunization arise mainly from failure to hedge exposure to the spending cut probability and the stock price-to-dividend ratio.

Figure 17: Conditional Standard Deviation of Funding Needs

The left figure plots the standard deviation of the funding needs, $(\text{Var}_t[\text{Need}_{t+1}])^{1/2}$, scaled by the annual GDP. The right figure decomposes the standard deviation to the variation due to each shock. Data Source: CRSP U.S. Treasury Database and BEA.



8 Conclusion

Because government deficits tend to occur in recessions, times when bond investors face high marginal utility, governments must tap debt markets at inopportune times. This consideration reduces the government's debt capacity by about 200% of GDP. If tax and spending policies remain on their current course, government debt capacity is negative. Put differently, government debt is a risky claim whose expected return far exceeds risk-free bond yields. We call this violation of the government budget constraint the government risk premium puzzle. We explore potential resolutions to this puzzle, but conclude they fall short of explaining it. Our framework can be used to study the optimal maturity structure of government debt. The current maturity structure leaves the government's fiscal position exposed to several macro-economic shocks.

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A Proof

Proposition 1

Proof. All objects in this appendix are in nominal terms but we drop the superscript $\$$ for ease of notation. The government faces the following one-period budget constraint:

$$G_t - T_t + Q_{t-1}^1 = \sum_{h=1}^H (Q_t^h - Q_{t-1}^{h+1}) P_t^h,$$

where G_t is total nominal government spending, T_t is total nominal government revenue, Q_t^h is the number of nominal zero-coupon bonds of maturity h outstanding in period t each promising to pay back \$1 at time $t + h$, and P_t^h is today's price for a h -period zero-coupon bond with \$1 face value. A unit of $h + 1$ -period bonds issued at $t - 1$ becomes a unit of h -period bonds in period t . That is, the stock of bonds evolves of each maturity evolves according to $Q_t^h = Q_{t-1}^{h+1} + \Delta Q_t^h$. Note that this notation can easily handle coupon-bearing bonds. For any bond with deterministic cash-flow sequence, we can write the price (present value) of the bond as the sum of the present values of each of its coupons.

The left-hand side of the budget constraint denotes new financing needs in the current period, due to primary deficit $G - T$ and one-period debt from last period that is now maturing. The right hand side shows that the money is raised by issuing new bonds of various maturities. Alternatively, we can write the budget constraint as total expenses equalling total income:

$$G_t + Q_{t-1}^1 + \sum_{h=1}^H Q_{t-1}^{h+1} P_t^h = T_t + \sum_{h=1}^H Q_t^h P_t^h,$$

We can now iterate the budget constraint forward. The period t constraint is given by:

$$\begin{aligned} T_t - G_t &= Q_{t-1}^1 - Q_t^1 P_t^1 + Q_{t-1}^2 P_t^1 - Q_t^2 P_t^2 + Q_{t-1}^3 P_t^2 - Q_t^3 P_t^3 \\ &+ \dots - Q_t^H P_t^H + Q_{t-1}^{H+1} P_t^H. \end{aligned}$$

Consider the period- $t + 1$ constraint,

$$\begin{aligned} T_{t+1} - G_{t+1} &= Q_t^1 - Q_{t+1}^1 P_{t+1}^1 + Q_t^2 P_{t+1}^1 - Q_{t+1}^2 P_{t+1}^2 + Q_t^3 P_{t+1}^2 - Q_{t+1}^3 P_{t+1}^3 \\ &+ \dots - Q_{t+1}^H P_{t+1}^H + Q_t^{H+1} P_{t+1}^H. \end{aligned}$$

multiply both sides by M_{t+1} , and take expectations conditional on time t :

$$\begin{aligned} \mathbb{E}_t [M_{t+1}(T_{t+1} - G_{t+1})] &= Q_t^1 P_t^1 - \mathbb{E}_t [Q_{t+1}^1 M_{t+1} P_{t+1}^1] + Q_t^2 P_t^2 - \mathbb{E}_t [Q_{t+1}^2 M_{t+1} P_{t+1}^2] + Q_t^3 P_t^3 \\ &- \mathbb{E}_t [Q_{t+1}^3 M_{t+1} P_{t+1}^3] + \dots + Q_t^H P_t^H \\ &- \mathbb{E}_t [Q_{t+1}^H M_{t+1} P_{t+1}^H] + Q_t^{H+1} P_t^{H+1}, \end{aligned}$$

where we use the asset pricing equations $\mathbb{E}_t [M_{t+1}] = P_t^1$, $\mathbb{E}_t [M_{t+1} P_{t+1}^1] = P_t^2$, \dots , $\mathbb{E}_t [M_{t+1} P_{t+1}^{H-1}] = P_t^H$, and $\mathbb{E}_t [M_{t+1} P_{t+1}^H] = P_t^{H+1}$.

Consider the period $t + 2$ constraint, multiplied by $M_{t+1} M_{t+2}$ and take time- t expectations:

$$\begin{aligned} \mathbb{E}_t [M_{t+1} M_{t+2} (T_{t+2} - G_{t+2})] &= \mathbb{E}_t [Q_{t+1}^1 M_{t+1} P_{t+1}^1] - \mathbb{E}_t [Q_{t+2}^1 M_{t+1} M_{t+2} P_{t+2}^1] + \mathbb{E}_t [Q_{t+1}^2 M_{t+1} P_{t+1}^2] \\ &- \mathbb{E}_t [Q_{t+2}^2 M_{t+1} M_{t+2} P_{t+2}^2] + \mathbb{E}_t [Q_{t+1}^3 M_{t+1} P_{t+1}^3] - \dots \\ &+ \mathbb{E}_t [Q_{t+1}^H M_{t+1} P_{t+1}^H] - \mathbb{E}_t [Q_{t+2}^H M_{t+1} M_{t+2} P_{t+2}^H] \\ &+ \mathbb{E}_t [Q_{t+1}^{H+1} M_{t+1} P_{t+1}^{H+1}], \end{aligned}$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1} [M_{t+2}] = P_{t+1}^1$, $\mathbb{E}_{t+1} [M_{t+2} P_{t+2}^1] = P_{t+1}^2$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected

discounted surpluses at t , $t + 1$, and $t + 2$ we get:

$$T_t - G_t + \mathbb{E}_t [M_{t+1}(T_{t+1} - G_{t+1})] + \mathbb{E}_t [M_{t+1}M_{t+2}(T_{t+2} - G_{t+2})] = \sum_{h=0}^H Q_{t-1}^{h+1} P_t^h + \\ -\mathbb{E}_t [Q_{t+2}^1 M_{t+1} M_{t+2} P_{t+2}^1] - \mathbb{E}_t [Q_{t+2}^2 M_{t+1} M_{t+2} P_{t+2}^2] - \dots - \mathbb{E}_t [Q_{t+2}^H M_{t+1} M_{t+2} P_{t+2}^H].$$

Similarly consider the one-period government budget constraints at times $t + 3$, $t + 4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t + J$, we get:

$$\sum_{h=0}^H Q_{t-1}^{h+1} P_t^h = \mathbb{E}_t \left[\sum_{j=0}^J M_{t,t+j} (T_{t+j} - G_{t+j}) \right] + \mathbb{E}_t \left[M_{t,t+J} \sum_{h=1}^H Q_{t+J}^h P_{t+J}^h \right]$$

where we used the cumulate SDF notation $M_{t,t+j} = \prod_{i=0}^j M_{t+i}$ and by convention $M_{t,t} = M_t = 1$ and $P_t^0 = 1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next J years plus the present value of the government bond portfolio that will be outstanding at time $t + J$. The latter is the cost the government will face at time $t + J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J \rightarrow \infty$:

$$\sum_{h=0}^H Q_{t-1}^{h+1} P_t^h = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} (T_{t+j} - G_{t+j}) \right] + \lim_{J \rightarrow \infty} \mathbb{E}_t \left[M_{t,t+J} \sum_{h=1}^H Q_{t+J}^h P_{t+J}^h \right].$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j} - G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the transversality condition:

$$\lim_{J \rightarrow \infty} \mathbb{E}_t \left[M_{t,t+J} \sum_{h=1}^H Q_{t+J}^h P_{t+J}^h \right] = 0.$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise there is a government debt bubble.

If the transversality condition is satisfied, the outstanding debt today, D_t , reflects the expected present-discounted value of the current and all future primary surpluses:

$$D_t = \sum_{h=0}^H Q_{t-1}^{h+1} P_t^h = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} (T_{t+j} - G_{t+j}) \right].$$

This is equation (1) in the main text.

Proposition 2 From the time- t budget constraint, we get that the primary surplus

$$-S_t = -Q_{t-1}^1 + \sum_{h=1}^H (Q_t^h - Q_{t-1}^{h+1}) P_t^h.$$

It follows that

$$D_t - S_t = \sum_{h=0}^H Q_{t-1}^{h+1} P_t^h - Q_{t-1}^1 + \sum_{h=1}^H (Q_t^h - Q_{t-1}^{h+1}) P_t^h = \sum_{h=1}^H Q_t^h P_t^h.$$

We obtain equation (2) in the main text.

$$r_{t+1}^D (D_t - S_t) = \sum_{h=0}^{\infty} P_{t+1}^{\$}(h) Q_{t,h+1}^{\$} = D_{t+1} = P_{t+1}^{\tau} - P_{t+1}^{\$} \\ = (P_t^{\tau} - T_t) r_{t+1}^{\tau} - (P_{t+1}^{\$} - G_t) r_{t+1}^{\$}.$$

□

Proposition 3

Proof. We follow the proof in the working paper version of [Backus, Boyarchenko, and Chernov \(2018\)](#) on page 16 (Example 5). [Hansen and Scheinkman \(2009\)](#) consider the following equation:

$$\mathbb{E}_t[M_{t,t+1}v_{t+1}] = \nu v_t, \quad (\text{A.1})$$

where ν is the dominant eigenvalue and v_t is the eigenfunction. Claims to stationary cash flows earn a return equal to the yield on the long bond. We consider the following decomposition of the pricing kernel:

$$M_{t,t+1}^1 = M_{t,t+1}v_{t+1}/\nu v_t, \quad (\text{A.2})$$

$$M_{t,t+1}^2 = \nu v_t/v_{t+1}. \quad (\text{A.3})$$

By construction, $\mathbb{E}_t[M_{t,t+1}^1] = 1$. The long yields converge to $-\log \nu$. The long-run bond return converges to $\lim_{n \rightarrow \infty} R_{t,t+1}^n = \frac{1}{M_{t,t+1}^2} = v_{t+1}/\nu v_t$. This implies that $\mathbb{E}[\log R_{t,t+1}^\infty] = -\log \nu$.

To value claims to uncertain cash flows with one-period growth rate $g_{t,t+1}$, we define \hat{p}_t^n to denote the price of a strip that pays off $d_{t,t+n}$, n periods from now.

$$\hat{p}_t^n = \mathbb{E}_t[M_{t,t+1}g_{t,t+1}\hat{p}_{t+1}^{n-1}] = \mathbb{E}_t[\hat{M}_{t,t+1}\hat{p}_{t+1}^{n-1}],$$

where $\hat{M}_{t,t+1} = M_{t,t+1}g_{t,t+1}$. Consider the problem of finding the dominant eigenvalue:

$$\mathbb{E}_t[\hat{M}_{t,t+1}\hat{v}_{t+1}] = \nu \hat{v}_t. \quad (\text{A.4})$$

If the cash flows are stationary, then the same ν that solves this equation for $M_{t,t+1}$ in eqn. [A.1](#) solves the one for $\hat{M}_{t,t+1}$. Hence, if (ν, v_t) solves eqn. [A.1](#), then $(\nu, v_t/d_t)$ solves the hat equation eqn. [A.4](#). □

B Asset Pricing Model

B.1 Risk-free rate

The real short yield $y_t(1)$, or risk-free rate, satisfies $E_t[\exp\{m_{t+1} + y_t(1)\}] = 1$. Solving out this Euler equation, we get:

$$\begin{aligned} y_t(1) &= y_t^{\$}(1) - E_t[\pi_{t+1}] - \frac{1}{2}e'_\pi \Sigma e_\pi + e'_\pi \Sigma^{\frac{1}{2}} \Lambda_t \\ &= y_0(1) + [e'_{y\pi} - e'_\pi \Psi + e'_\pi \Sigma^{\frac{1}{2}} \Lambda_1] z_t. \end{aligned} \quad (\text{A.5})$$

$$y_0(1) \equiv y_0^{\$}(1) - \pi_0 - \frac{1}{2}e'_\pi \Sigma e_\pi + e'_\pi \Sigma^{\frac{1}{2}} \Lambda_0. \quad (\text{A.6})$$

where we used the expression for the real SDF

$$\begin{aligned} m_{t+1} &= m_{t+1}^{\$} + \pi_{t+1} \\ &= -y_t^{\$}(1) - \frac{1}{2}\Lambda'_t \Lambda_t - \Lambda'_t \varepsilon_{t+1} + \pi_0 + e'_\pi \Psi z_t + e'_\pi \Sigma^{\frac{1}{2}} \varepsilon_{t+1} \\ &= -y_t(1) - \frac{1}{2}e'_\pi \Sigma e_\pi + e'_\pi \Sigma^{\frac{1}{2}} \Lambda_t - \frac{1}{2}\Lambda'_t \Lambda_t - (\Lambda'_t - e'_\pi \Sigma^{\frac{1}{2}}) \varepsilon_{t+1} \end{aligned}$$

The real short yield is the nominal short yield minus expected inflation minus a Jensen adjustment minus the inflation risk premium.

B.2 Nominal and real term structure

Proposition 8. Nominal bond yields are affine in the state vector:

$$y_t^{\$}(h) = -\frac{A^{\$}(h)}{h} - \frac{B^{\$}(h)'}{h} z_t,$$

where the coefficients $A^{\$}(h)$ and $B^{\$}(h)$ satisfy the following recursions:

$$A^{\$}(h+1) = -y_0^{\$(1)} + A^{\$}(h) + \frac{1}{2} (B^{\$}(h))' \Sigma (B^{\$}(h)) - (B^{\$}(h))' \Sigma^{\frac{1}{2}} \Lambda_0, \quad (\text{A.7})$$

$$(B^{\$}(h+1))' = (B^{\$}(h))' \Psi - e'_{yn} - (B^{\$}(h))' \Sigma^{\frac{1}{2}} \Lambda_1, \quad (\text{A.8})$$

initialized at $A^{\$}(0) = 0$ and $B^{\$}(0) = \mathbf{0}$.

Proof. We conjecture that the $t+1$ -price of a τ -period bond is exponentially affine in the state:

$$\log(P_{t+1}^{\$(h)}) = A^{\$}(h) + (B^{\$}(h))' z_{t+1}$$

and solve for the coefficients $A^{\$}(h+1)$ and $B^{\$}(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned} P_t^{\$(h+1)} &= E_t[\exp\{m_{t+1}^{\$} + \log(P_{t+1}^{\$(h)})\}] \\ &= E_t[\exp\{-y_t^{\$(1)} - \frac{1}{2} \Lambda_t' \Lambda_t - \Lambda_t' \varepsilon_{t+1} + A^{\$}(h) + (B^{\$}(h))' z_{t+1}\}] \\ &= \exp\{-y_0^{\$(1)} - e'_{yn} z_t - \frac{1}{2} \Lambda_t' \Lambda_t + A^{\$}(h) + (B^{\$}(h))' \Psi z_t\} \times \\ &\quad E_t \left[\exp\{-\Lambda_t' \varepsilon_{t+1} + (B^{\$}(h))' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\} \right]. \end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned} P_t^{\$(h+1)} &= \exp \left\{ -y_0^{\$(1)} - e'_{yn} z_t + A^{\$}(h) + (B^{\$}(h))' \Psi z_t + \frac{1}{2} (B^{\$}(h))' \Sigma (B^{\$}(h)) \right. \\ &\quad \left. - (B^{\$}(h))' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t) \right\}. \end{aligned}$$

Taking logs and collecting terms, we obtain a linear equation for $\log(P_t(h+1))$:

$$\log(P_t^{\$(h+1)}) = A^{\$}(h+1) + (B^{\$}(h+1))' z_t,$$

where $A^{\$}(h+1)$ satisfies (A.7) and $B^{\$}(h+1)$ satisfies (A.8). The relationship between log bond prices and bond yields is given by $-\log(P_t^{\$(h)}) / \tau = y_t^{\$(h)}$. \square

Define the one-period return on a nominal zero-coupon bond as:

$$r_{t+1}^{b,\$}(h) = \log(P_{t+1}^{\$(h)}) - \log(P_t^{\$(h+1)})$$

The nominal bond risk premium on a bond of maturity τ is the expected excess return corrected for a Jensen term, and equals

negative the conditional covariance between that bond return and the nominal SDF:

$$\begin{aligned} E_t \left[r_{t+1}^{b,\$}(h) \right] - y_t^\$(1) + \frac{1}{2} V_t \left[r_{t+1}^{b,\$}(h) \right] &= -\text{Cov}_t \left[m_{t+1}^\$, r_{t+1}^{b,\$}(h) \right] \\ &= \left(B^\$(h) \right)' \Sigma^{\frac{1}{2}} \Lambda_t \end{aligned}$$

Real bond yields, $y_t(h)$, denoted without the \$ superscript, are affine as well with coefficients that follow similar recursions:

$$A(h+1) = -y_0(1) + A(h) + \frac{1}{2} (B(h))' \Sigma (B(h)) - (B(h))' \Sigma^{\frac{1}{2}} \left(\Lambda_0 - \Sigma^{\frac{1}{2}} e_\pi \right), \quad (\text{A.9})$$

$$(B(h+1))' = -e'_{ym} + (e_\pi + B(h))' \left(\Psi - \Sigma^{\frac{1}{2}} \Lambda_1 \right). \quad (\text{A.10})$$

For $\tau = 1$, we recover the expression for the risk-free rate in (A.5)-(A.6).

B.3 Stocks

B.3.1 Aggregate Stock Market

We define the real return on the aggregate stock market as $R_{t+1}^m = \frac{P_{t+1}^m + D_{t+1}^m}{P_t^m}$, where P_t^m is the ex-dividend price on the equity market. A log-linearization delivers:

$$r_{t+1}^m = \kappa_0^m + \Delta d_{t+1}^m + \kappa_1^m p d_{t+1}^m - p d_t^m. \quad (\text{A.11})$$

The unconditional mean log real stock return is $r_0^m = E[r_t^m]$, the unconditional mean real dividend growth rate is $\mu^m = E[\Delta d_{t+1}^m]$, and $\overline{p d^m} = E[p d_t^m]$ is the unconditional average log price-dividend ratio on equity. The linearization constants κ_0^m and κ_1^m are defined as:

$$\kappa_1^m = \frac{\overline{e^{p d^m}}}{e^{p d^m} + 1} < 1 \quad \text{and} \quad \kappa_0^m = \log \left(e^{\overline{p d^m}} + 1 \right) - \frac{\overline{e^{p d^m}}}{e^{p d^m} + 1} \overline{p d^m}. \quad (\text{A.12})$$

Our state vector z contains the (demeaned) log real dividend growth rate on the stock market, $\Delta d_{t+1}^m - \mu^m$, and the (demeaned) log price-dividend ratio $p d_t^m - \overline{p d^m}$.

$$\begin{aligned} p d_t^m(h) &= \overline{p d^m} + e'_{pd} z_t, \\ \Delta d_t^m &= \mu^m + e'_{divm} z_t, \end{aligned}$$

where e'_{pd} (e'_{divm}) is a selector vector that has a one in the fifth (sixth) entry, since the log pd ratio (log dividend growth rate) is the fifth (sixth) element of the VAR.

We define the log return on the stock market so that the log return equation holds exactly, given the time series for $\{\Delta d_t^m, p d_t^m\}$. Rewriting (A.11):

$$\begin{aligned} r_{t+1}^m - r_0^m &= \left[(e'_{divm} + \kappa_1^m e'_{pd})' \Psi - e'_{pd} \right] z_t + (e'_{divm} + \kappa_1^m e'_{pd})' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}. \\ r_0^m &= \mu^m + \kappa_0^m - \overline{p d^m} (1 - \kappa_1^m). \end{aligned}$$

The equity risk premium is the expected excess return on the stock market corrected for a Jensen term. By the Euler equation, it equals minus the conditional covariance between the log SDF and the log return:

$$\begin{aligned} 1 &= E_t \left[M_{t+1} \frac{P_{t+1}^m + D_{t+1}^m}{P_t^m} \right] = E_t \left[\exp \{ m_{t+1}^\$ + \pi_{t+1} + r_{t+1}^m \} \right] \\ &= E_t \left[\exp \left\{ -y_{t+1}^\$ - \frac{1}{2} \Lambda_t' \Lambda_t - \Lambda_t' \varepsilon_{t+1} + \pi_0 + e'_{\pi} z_{t+1} + r_0^m + (e'_{divm} + \kappa_1^m e'_{pd})' z_{t+1} - e'_{pd} z_t \right\} \right] \end{aligned}$$

$$\begin{aligned}
&= \exp \left\{ -y_0^{\$}(1) - \frac{1}{2} \Lambda_t' \Lambda_t + \pi_0 + r_0^m + \left[(e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Psi - e'_{pd} - e'_{yn} \right] z_t \right\} \\
&\quad \times E_t \left[\exp \left\{ -\Lambda_t' \varepsilon_{t+1} + (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Sigma^{\frac{1}{2}} \varepsilon_{t+1} \right\} \right] \\
&= \exp \left\{ r_0^m + \pi_0 - y_0^{\$}(1) + \left[(e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Psi - e'_{pd} - e'_{yn} \right] z_t \right\} \\
&\quad \times \exp \left\{ \frac{1}{2} (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Sigma (e_{divm} + \kappa_1^m e_{pd} + e_{\pi}) - (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Sigma^{\frac{1}{2}} \Lambda_t \right\}
\end{aligned}$$

Taking logs on both sides delivers:

$$\begin{aligned}
r_0^m + \pi_0 - y_0^{\$}(1) + \left[(e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Psi - e'_{pd} - e'_{yn} \right] z_t & \quad (A.13) \\
+ \frac{1}{2} (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Sigma (e_{divm} + \kappa_1^m e_{pd} + e_{\pi}) &= (e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Sigma^{\frac{1}{2}} \Lambda_t \\
E_t \left[r_{t+1}^{m,\$} \right] - y_{t,1}^{\$} + \frac{1}{2} V_t \left[r_{t+1}^{m,\$} \right] &= -Cov_t \left[m_{t+1}^{\$,} r_{t+1}^{m,\$} \right]
\end{aligned}$$

The left-hand side is the expected excess return on the stock market, corrected for a Jensen term, while the right-hand side is the negative of the conditional covariance between the (nominal) log stock return and the nominal log SDF. We refer to the left-hand side as the equity risk premium in the data, since it is implied directly by the VAR. We refer to the right-hand side as the equity risk premium in the model, since it requires knowledge of the market prices of risk.

Note that we can obtain the same expression using the log real SDF and log real stock return:

$$\begin{aligned}
E_t \left[r_{t+1}^m \right] - y_{t,1} + \frac{1}{2} V_t \left[r_{t+1}^m \right] &= -Cov_t \left[m_{t+1}, r_{t+1}^m \right] \\
r_0^m - y_0(1) + \left[(e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Psi - e'_{pd} - e'_{yn} - e'_{\pi} \Sigma^{1/2} \Lambda_1 \right] z_t & \\
+ \frac{1}{2} (e_{divm} + \kappa_1^m e_{pd})' \Sigma (e_{divm} + \kappa_1^m e_{pd}) &= (e_{divm} + \kappa_1^m e_{pd})' \Sigma^{1/2} (\Lambda_t - (\Sigma^{1/2})' e_{\pi})
\end{aligned}$$

We combine the terms in Λ_0 and Λ_1 on the right-hand side and plug in for $y_0(1)$ from (A.6) to get:

$$\begin{aligned}
r_0^m + \pi_0 - y_{0,1}^{\$} + \frac{1}{2} e'_{\pi} \Sigma e_{\pi} & \\
+ \frac{1}{2} (e_{divm} + \kappa_1^m e_{pd})' \Sigma (e_{divm} + \kappa_1^m e_{pd}) + e'_{\pi} \Sigma (e_{divm} + \kappa_1^m e_{pd}) & \\
+ \left[(e_{divm} + \kappa_1^m e_{pd} + e_{\pi})' \Psi - e'_{pd} - e'_{yn} \right] z_t & \\
= (e_{divm} + \kappa_1^m e_{pd})' \Sigma^{1/2} \Lambda_t + e'_{\pi} \Sigma^{\frac{1}{2}} \Lambda_0 + e'_{\pi} \Sigma^{1/2} \Lambda_1 z_t &
\end{aligned}$$

This recovers equation (A.13).

B.3.2 Dividend Strips

Proposition 9. Log price-dividend ratios on dividend strips are affine in the state vector:

$$p_t^d(h) = \log \left(P_t^d(h) \right) = A^m(h) + B^m(h) z_t,$$

where the coefficients $A^m(h)$ and $B^m(h)$ follow recursions:

$$\begin{aligned}
A^m(h+1) &= A^m(h) + \mu_m - y_0(1) + \frac{1}{2} (e_{divm} + B^m(h))' \Sigma (e_{divm} + B^m(h)) \\
&\quad - (e_{divm} + B^m(h))' \Sigma^{\frac{1}{2}} \left(\Lambda_0 - \Sigma^{\frac{1}{2}} e_{\pi} \right), & (A.14)
\end{aligned}$$

$$B^m(h+1) = (e_{divm} + e_{\pi} + B^m(h))' \Psi - e'_{yn} - (e_{divm} + e_{\pi} + B^m(h))' \Sigma^{\frac{1}{2}} \Lambda_1, \quad (A.15)$$

initialized at $A_0^m = 0$ and $B_0^m = 0$.

Proof. We conjecture the affine structure and solve for the coefficients $A^m(h+1)$ and $B^m(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned}
P_t^d(h+1) &= \mathbb{E}_t \left[M_{t+1} P_{t+1}^d(h) \frac{D_{t+1}^m}{D_t^m} \right] = E_t \left[\exp\{m_{t+1}^{\$} + \pi_{t+1} + \Delta d_{t+1}^m + p_{t+1}^d(h)\} \right] \\
&= \mathbb{E}_t \left[\exp\left\{-y_{t,1}^{\$} - \frac{1}{2} \Lambda_t' \Lambda_t - \Lambda_t' \varepsilon_{t+1} + \pi_0 + e_{\pi}' z_{t+1} + \mu^m + e_{divm}' z_{t+1} + A^m(h) + B(h)^m z_{t+1}\right\} \right] \\
&= \exp\left\{-y_0^{\$}(1) - e_{yn}' z_t - \frac{1}{2} \Lambda_t' \Lambda_t + \pi_0 + e_{\pi}' \Psi z_t + \mu_m + e_{divm}' \Psi z_t + A^m(h) + B(h)^m \Psi z_t\right\} \\
&\quad \times \mathbb{E}_t \left[\exp\left\{-\Lambda_t' \varepsilon_{t+1} + (e_{divm} + e_{\pi} + B^m(h))' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\right\} \right].
\end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned}
P_t^d(h+1) &= \exp\left\{-y_0^{\$}(1) + \pi_0 + \mu_m + A^m(h) + \left[(e_{divm} + e_{\pi} + B^m(h))' \Psi - e_{yn}'\right] z_t\right. \\
&\quad \left. + \frac{1}{2} (e_{divm} + e_{\pi} + B^m(h))' \Sigma (e_{divm} + e_{\pi} + B^m(h))\right. \\
&\quad \left. - (e_{divm} + e_{\pi} + B^m(h))' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t)\right\}
\end{aligned}$$

Taking logs and collecting terms, we obtain a log-linear expression for $p_t^d(h+1)$:

$$p_t^d(h+1) = A^m(h+1) + B^m(h+1) z_t,$$

where:

$$\begin{aligned}
A^m(h+1) &= A^m(h) + \mu_m - y_0^{\$}(1) + \pi_0 + \frac{1}{2} (e_{divm} + e_{\pi} + B^m(h))' \Sigma (e_{divm} + e_{\pi} + B^m(h)) \\
&\quad - (e_{divm} + e_{\pi} + B^m(h))' \Sigma^{\frac{1}{2}} \Lambda_0, \\
B^m(h+1) &= (e_{divm} + e_{\pi} + B^m(h))' \Psi - e_{yn}' - (e_{divm} + e_{\pi} + B^m(h))' \Sigma^{\frac{1}{2}} \Lambda_1.
\end{aligned}$$

We recover the recursions in (A.14) and (A.15) after using equation (A.6). □

We define the dividend strip risk premium as:

$$\begin{aligned}
\mathbb{E}_t \left[r_{t+1}^{d,\$} \right] - y_{t,1}^{\$} + \frac{1}{2} V_t \left[r_{t+1}^{d,\$} \right] &= -Cov_t \left[m_{t+1}^{\$}, r_{t+1}^{d,\$} \right] \\
&= (e_{divm} + e_{\pi} + B^m(h))' \Sigma^{\frac{1}{2}} \Lambda_t
\end{aligned}$$

B.4 Claim to Future Government Spending and Tax Revenues

This appendix computes P_t^r , the value of a claim to future tax revenues, and P_t^g , the value of a claim to future government spending.

B.4.1 Spending Claim

Nominal government spending growth equals

$$\Delta \log G_{t+1} = \Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \mu_0^g + (e_{\Delta g} + e_x + e_{\pi})' z_{t+1}. \quad (\text{A.16})$$

We conjecture the log price-dividend ratios on spending strips are affine in the state vector:

$$p_t^s(h) = \log(P_t^s(h)) = A^s(h) + B^{s'}(h)z_t.$$

We solve for the coefficients $A^s(h+1)$ and $B^s(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned} P_t^s(h+1) &= \mathbb{E}_t \left[M_{t+1} P_{t+1}^s(h) \frac{G_{t+1}}{G_t} \right] = E_t \left[\exp\{m_{t+1}^s + \Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} + P_{t+1}^s(h)\} \right] \\ &= \exp\{-y_0^s(1) - e'_{yn} z_t - \frac{1}{2} \Lambda_t' \Lambda_t + \mu^s + x_0 + \pi_0 + (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Psi z_t + A^s(h)\} \\ &\quad \times \mathbb{E}_t \left[\exp\{-\Lambda_t' \varepsilon_{t+1} + (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\} \right]. \end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned} P_t^s(h+1) &= \exp\{-y_0^s(1) + \mu^s + x_0 + \pi_0 + ((e_{\Delta g} + e_x + e_\pi + B^s(h))' \Psi - e'_{yn}) z_t + A^s(h) \\ &\quad + \frac{1}{2} (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma (e_{\Delta g} + e_x + e_\pi + B^s(h)) \\ &\quad - (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t)\} \end{aligned}$$

Taking logs and collecting terms, we obtain

$$\begin{aligned} A^s(h+1) &= -y_0^s(1) + \mu^s + x_0 + \pi_0 + A^s(h) + \frac{1}{2} (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma (e_{\Delta g} + e_x + e_\pi + B^s(h)) \\ &\quad - (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma^{\frac{1}{2}} \Lambda_0, \\ B^s(h+1)' &= (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Psi - e'_{yn} - (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma^{\frac{1}{2}} \Lambda_1, \end{aligned}$$

and the price-dividend ratio of the cum-dividend spending claim is

$$\sum_{h=0}^{\infty} \exp(A^s(h+1) + B^s(h+1)' z_t)$$

Next, we define the (nominal) return on the claim as $R_{t+1}^s = \frac{P_{t+1}^s}{P_t^s - G_t} = \frac{P_{t+1}^{s,ex} + G_{t+1}}{P_t^{s,ex}}$, where P_t^s is the cum-dividend price on the spending claim and $P_t^{s,ex}$ is the ex-dividend price. We log-linearize the return around $z_t = 0$:

$$r_{t+1}^s = \kappa_0^s + \Delta \log G_{t+1} + \kappa_1^s p g_{t+1} - p g_t. \quad (\text{A.17})$$

where $p g_t \equiv \log\left(\frac{P_t^{s,ex}}{G_t}\right) = \log\left(\frac{P_t^s}{G_t} - 1\right)$. The unconditional mean log real stock return is $r_0^s = E[r_t^s]$.

We obtain $\bar{p}\bar{g}$ from the precise valuation formula Eq. (13) at $z_t = 0$. The linearization constants κ_0^s and κ_1^s are defined as:

$$\kappa_1^s = \frac{e^{\bar{p}\bar{g}}}{e^{\bar{p}\bar{g}} + 1} < 1 \quad \text{and} \quad \kappa_0^s = \log\left(e^{\bar{p}\bar{g}} + 1\right) - \frac{e^{\bar{p}\bar{g}}}{e^{\bar{p}\bar{g}} + 1} \bar{p}\bar{g}. \quad (\text{A.18})$$

Then, the unconditional expected return is:

$$r_0^s = x_0 + \pi_0 + \kappa_0^s - \bar{p}\bar{g}(1 - \kappa_1^s).$$

We conjecture that the log ex-dividend price-dividend ratio on the spending claim is affine in the state vector and verify the

conjecture by solving the Euler equation for the claim.

$$pg_t = \overline{p\bar{g}} + \bar{B}'_g z_t \quad (\text{A.19})$$

This allows us to write the return as:

$$r_{t+1}^s = r_0^s + (e_{\Delta g} + e_x + e_\pi + \kappa_1^s \bar{B}_g)' z_{t+1} - \bar{B}'_g z_t. \quad (\text{A.20})$$

Proof. Starting from the Euler equation:

$$\begin{aligned} 1 &= E_t \left[\exp\{m_{t+1}^s + r_{t+1}^s\} \right] \\ &= \exp\{-y_0^s(1) - e'_{yn} z_t - \frac{1}{2} \Lambda'_t \Lambda_t + r_0^s + [(e_{\Delta g} + e_x + e_\pi + \kappa_1^s \bar{B}_g)' \Psi - \bar{B}'_g] z_t\} \\ &\quad \times E_t \left[\exp\{-\Lambda'_t \varepsilon_{t+1} + (e_{\Delta g} + e_x + e_\pi + \kappa_1^s \bar{B}_g)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\} \right]. \end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned} 1 &= \exp\{r_0^s - y_0^s(1) + [(e_{\Delta g} + e_x + e_\pi + \kappa_1^s \bar{B}_g)' \Psi - \bar{B}'_g - e'_{yn}] z_t \\ &\quad + \underbrace{\frac{1}{2} (e_{\Delta g} + e_x + e_\pi + \kappa_1^s \bar{B}_g)' \Sigma (e_{\Delta g} + e_x + e_\pi + \kappa_1^s \bar{B}_g)}_{Jensen} \\ &\quad - (e_{\Delta g} + e_x + e_\pi + \kappa_1^s \bar{B}_g)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t)\} \end{aligned}$$

Taking logs and collecting terms, we obtain the following system of equations:

$$r_0^s - y_0^s(1) + Jensen = (e_{\Delta g} + e_x + e_\pi + \kappa_1^s \bar{B}_g)' \Sigma^{\frac{1}{2}} \Lambda_0 \quad (\text{A.21})$$

and

$$(e_{\Delta g} + e_x + e_\pi + \kappa_1^s \bar{B}_g)' \Psi - \bar{B}'_g - e'_{yn} = (e_{\Delta g} + e_x + e_\pi + \kappa_1^s \bar{B}_g)' \Sigma^{\frac{1}{2}} \Lambda_1 \quad (\text{A.22})$$

The left-hand side of this equation is the unconditional expected excess log return with Jensen adjustment. The right hand side is the unconditional covariance of the log SDF with the log return. This equation describes the unconditional risk premium on the claim to government spending. Equation (A.22) describes the time-varying component of the government spending risk premium. Given Λ_1 , the system of N equations (A.22) can be solved for the vector \bar{B}_g :

$$\bar{B}_g = \left(I - \kappa_1^s (\Psi - \Sigma^{\frac{1}{2}} \Lambda_1)' \right)^{-1} \left[(\Psi - \Sigma^{\frac{1}{2}} \Lambda_1)' (e_{\Delta g} + e_x + e_\pi) - e_{yn} \right]. \quad (\text{A.23})$$

□

B.4.2 Revenue Claim

Nominal government revenue growth equals

$$\Delta \log T_{t+1} = \Delta \log \tau_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \mu_0^\tau + (e_{\Delta \tau} + e_x + e_\pi)' z_{t+1}. \quad (\text{A.24})$$

where $\tau_t = T_t / GDP_t$ is the ratio of government revenue to GDP. Note that this ratio is assumed to have a long-run growth rate of zero. This imposes cointegration between government revenue and GDP. The growth ratio in this ratio can only temporarily deviate from zero.

The remaining proof exactly mirrors the proof for government spending, with

$$p\tau_t \equiv \log\left(\frac{P_t^{\tau,ex}}{T_t}\right) = \log\left(\frac{P_t^\tau}{T_t} - 1\right) = \bar{p}\tau + B'_\tau z_t \quad (\text{A.25})$$

$$r_{t+1}^\tau = r_0^\tau + (e_{\Delta\tau} + e_x + e_\pi + \kappa_1^\tau B_\tau)' z_{t+1} - B'_\tau z_t, \quad (\text{A.26})$$

and

$$\begin{aligned} r_0^\tau &= x_0 + \pi_0 + \kappa_0^\tau - \bar{p}\tau(1 - \kappa_1^\tau). \\ r_0^\tau - y_0^\tau(1) + Jensen &= (e_{\Delta\tau} + e_x + e_\pi + \kappa_1^\tau B_\tau)' \Sigma^{\frac{1}{2}} \Lambda_0. \end{aligned} \quad (\text{A.27})$$

B.5 Pricing in the Presence of Spending Cut

The original state space is

$$z_t = \Psi z_{t-1} + \Sigma^{\frac{1}{2}} \varepsilon_t, \quad (\text{A.28})$$

$$\begin{aligned} z_t = & [\pi_t - \pi_0, x_t - x_0, y_t^\tau(1) - y_0^\tau(1), y_s p r_t^\tau - y_s p r_0^\tau(1), p d_t - \bar{p}d, \Delta d_t - \mu_d, \\ & \Delta \log \tau_t - \mu_0^\tau, \Delta \log g_t - \mu_0^g, \log \tau_t - \log \tau_0, \log g_t - \log g_0]'. \end{aligned}$$

We augment it with the probability of a spending cut

$$w_t = [z_t; \phi_t], \quad (\text{A.29})$$

and ϕ_t can load on z_{t-1} and ε_t , and the loadings are to be estimated. ϕ_t describes the demeaned probability of a spending cut in time $t+1$, and $\bar{\phi}$ is the mean. The spending cut is i.i.d. When it happens, the shock to the growth rate of the tax-GDP ratio increases by c times standard deviation. This increase is separate from the ordinary shocks ε_{t+1} . We denote $\tilde{\varepsilon}_{t+1} = \varepsilon_{t+1} - c e_{\Delta g}$, and $\tilde{w}_{t+1} = w_{t+1} - \Sigma^{\frac{1}{2}} c e_{\Delta g}$.

We conjecture

$$p_t^\tau(h) = \log(P_t^\tau(h)) = A^g(h) + B^g(h)' w_t,$$

and solve for the coefficients $A^g(h+1)$ and $B^g(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned} P_t^\tau(h+1) &= \mathbb{E}_t \left[M_{t+1} P_{t+1}^\tau(h) \frac{G_{t+1}}{G_t} \right] = E_t \left[\exp\{m_{t+1}^\tau + \Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} + p_{t+1}^\tau(h)\} \right] \\ &= \mathbb{E}_t \left[\exp\{-y_{t,1}^\tau - \frac{1}{2} \Lambda_t' \Lambda_t - \Lambda_t' \varepsilon_{t+1} + \mu^\tau + x_0 + \pi_0 + A^g(h) + (e'_{\Delta g} + e'_x + e'_\pi + B^g(h)') w_{t+1}\} \right] \\ &= \mathbb{E}_t \left[(1 - \phi_t) \exp\{-y_{t,1}^\tau - \frac{1}{2} \Lambda_t' \Lambda_t - \Lambda_t' \varepsilon_{t+1} + \mu^\tau + x_0 + \pi_0 + A^g(h) + (e'_{\Delta g} + e'_x + e'_\pi + B^g(h)') w_{t+1}\} \right. \\ &\quad \left. + \phi_t \exp\{-y_{t,1}^\tau - \frac{1}{2} \Lambda_t' \Lambda_t - \Lambda_t' \tilde{\varepsilon}_{t+1} + \mu^\tau + x_0 + \pi_0 + A^g(h) + (e'_{\Delta g} + e'_x + e'_\pi + B^g(h)') \tilde{w}_{t+1}\} \right] \\ &= \exp\{-y_0^\tau(1) - e'_{y_m} w_t - \frac{1}{2} \Lambda_t' \Lambda_t + \mu^\tau + x_0 + \pi_0 + (e_{\Delta g} + e_x + e_\pi + B^g(h))' \Psi w_t + A^g(h)\} \\ &\quad \times \mathbb{E}_t \left[\exp\{(-\Lambda_t' + (e_{\Delta g} + e_x + e_\pi + B^g(h))' \Sigma^{\frac{1}{2}}) \varepsilon_{t+1}\} \right] \\ &\quad (1 - \phi_t + \phi_t \exp\{((e_{\Delta g} + e_x + e_\pi + B^g(h))' \Sigma^{\frac{1}{2}}) (-c e_{\Delta g})\}) \end{aligned}$$

For small ϕ_t ,

$$\begin{aligned} \log(1 - \phi_t + \phi_t \exp\{(e_{\Delta g} + e_x + e_\pi + B^g(h))' \Sigma^{\frac{1}{2}} (-c e_{\Delta g})\}) &\approx \\ -\phi_t \left(1 - \exp\{(e_{\Delta g} + e_x + e_\pi + B^g(h))' \Sigma^{\frac{1}{2}} (-c e_{\Delta g})\}\right) & \end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned} p_t^s(h+1) &= -y_0^s(1) + \mu^s + x_0 + \pi_0 + ((e_{\Delta g} + e_x + e_\pi + B^s(h))' \Psi - e'_{ym}) w_t + A^s(h) \\ &\quad + \frac{1}{2} (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma (e_{\Delta g} + e_x + e_\pi + B^s(h)) \\ &\quad - (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 w_t) \\ &\quad - \left(1 - \exp\{-(e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma^{\frac{1}{2}} c e_{\Delta g}\} \right) (\bar{\phi} + e'_\phi w_t) \end{aligned}$$

Taking logs and collecting terms, we obtain a log-linear expression for $p_t^s(h+1)$:

$$p_t^s(h+1) = A^s(h+1) + B^s(h+1)' w_t,$$

where:

$$\begin{aligned} A^s(h+1) &= -y_0^s(1) + \mu^s + x_0 + \pi_0 + A^s(h) + \frac{1}{2} (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma (e_{\Delta g} + e_x + e_\pi + B^s(h)) \\ &\quad - (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma^{\frac{1}{2}} \Lambda_0 - \left(1 - \exp\{-(e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma^{\frac{1}{2}} c e_{\Delta g}\} \right) \bar{\phi}, \\ B^s(h+1)' &= (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Psi - e'_{ym} - (e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma^{\frac{1}{2}} \Lambda_1 \\ &\quad - \left(1 - \exp\{-(e_{\Delta g} + e_x + e_\pi + B^s(h))' \Sigma^{\frac{1}{2}} c e_{\Delta g}\} \right) e'_\phi. \end{aligned}$$

Then the price of the cum-dividend spending claim is

$$G_t \sum_{h=0}^{\infty} \exp(p_t^s(h))$$

We also obtain the cum-dividend tax claim from our previous formula

$$T_t \sum_{h=0}^{\infty} \exp(p_t^\tau(h)) = T_t \sum_{h=0}^{\infty} \exp(A^\tau(h) + B^\tau(h)' w_t)$$

with

$$\begin{aligned} A^\tau(h+1) &= -y_0^\tau(1) + \mu^\tau + x_0 + \pi_0 + A^\tau(h) + \frac{1}{2} (e_{\Delta \tau} + e_x + e_\pi + B^\tau(h))' \Sigma (e_{\Delta \tau} + e_x + e_\pi + B^\tau(h)) \\ &\quad - (e_{\Delta \tau} + e_x + e_\pi + B^\tau(h))' \Sigma^{\frac{1}{2}} \Lambda_0 - \left(1 - \exp\{-(e_{\Delta \tau} + e_x + e_\pi + B^\tau(h))' \Sigma^{\frac{1}{2}} c e_{\Delta g}\} \right) \bar{\phi}, \\ B^\tau(h+1)' &= (e_{\Delta \tau} + e_x + e_\pi + B^\tau(h))' \Psi - e'_{ym} - (e_{\Delta \tau} + e_x + e_\pi + B^\tau(h))' \Sigma^{\frac{1}{2}} \Lambda_1 \\ &\quad - \left(1 - \exp\{-(e_{\Delta \tau} + e_x + e_\pi + B^\tau(h))' \Sigma^{\frac{1}{2}} e_{\Delta g}\} \right) c e'_\phi. \end{aligned}$$

Then we can back out ϕ by setting the different in the value of tax and spending claim to the valuation of the government debt. Under this framework, we can run a Kalman filter to find the best-fitting parameters that govern the dynamics of ϕ .

C Coefficient Estimates

C.1 The VAR System

The Cholesky decomposition of the residual variance-covariance matrix, $\Sigma^{\frac{1}{2}}$, multiplied by 100 for readability is given by:

$$100 \times \Sigma^{\frac{1}{2}} = \begin{pmatrix} 0.36 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.03 & 0.88 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.02 & 0.04 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.00 & -0.01 & -0.07 & 0.09 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.75 & 0.84 & -1.14 & -0.06 & 8.04 & 0 & 0 & 0 & 0 & 0 \\ 0.13 & 0.08 & 0.03 & -0.16 & -0.33 & 1.94 & 0 & 0 & 0 & 0 \\ 0.77 & 1.13 & 0.18 & -0.22 & 0.25 & 0.03 & 4.10 & 0 & 0 & 0 \\ -0.26 & -0.86 & -0.34 & 0.17 & 0.24 & -0.39 & -0.16 & 3.85 & 0 & 0 \\ 0.77 & 1.13 & 0.18 & -0.22 & 0.25 & 0.03 & 4.10 & 0.00 & 0.00 & 0 \\ -0.26 & -0.86 & -0.34 & 0.17 & 0.24 & -0.39 & -0.16 & 3.85 & 0.00 & 0.00 \end{pmatrix}$$

In this matrix, the last two columns are all zero. This is because the dependent variables $\log \tau_t - \log \tau_0$ and $\log g_t - \log g_0$ do not have independent shocks. For example, $\log \tau_t - \log \tau_0$ can be expressed as

$$\begin{aligned} \log \tau_t - \log \tau_0 &= \Delta \log \tau_t + (\log \tau_{t-1} - \log \tau_0) \\ &= (e'_{\Delta\tau} \Psi + e'_t) z_{t-1} + e'_{\Delta\tau} \Sigma^{\frac{1}{2}} \varepsilon_t, \end{aligned}$$

which loads on the first 8 shocks in the same way as $\Delta \log \tau_t - \mu_0^g$.

C.2 Market Prices of Risk

C.2.1 Parameter Estimates

The constant market price of risk vector is estimated at:

$$\Lambda'_0 = [0.01, 0.33, -0.53, 0.11, 0, 0.60, 0, 0, 0]$$

The time-varying market price of risk matrix is estimated at:

$$\Lambda_1 = \begin{bmatrix} 33.47 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -10.53 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -27.45 & -171.15 & 0 & 0 & 0 & 0 & 0 & 0 \\ 37.22 & 34.51 & -15.86 & -65.40 & 0 & 0 & 0 & 0 & -0.11 & 0.13 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 36.14 & 12.03 & -45.44 & -103.91 & -0.97 & 6.77 & 3.61 & 0.28 & -0.17 & 0.33 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C.2.2 Identification

We allow for first four elements of Λ_0 to be non-zero. The market price of the first shock, the inflation shock, will partly capture a shock to expected inflation given the persistence of inflation. Movements in expected inflation are a key determinants of parallel shifts in the term structure of interest rates. i.e., they are the main driver of level of the term structure. Since inflation is usually bad news to the representative agent, we expect a negative price of risk for this shock. Shocks to GDP growth affect the slope of the term structure. They affect long rates more than short rates. We expect a positive price of risk since positive innovations to GDP growth are good news. The third risk price $\Lambda_0(3)$ is the price of risk for a shock to the interest rate that is orthogonal to inflation and GDP growth shocks. As in the classic term structure models of Vasicek and Cox, Ingersoll, and Ross, we expect this risk price to be negative. We expect the shock to the yield spread that is orthogonal to the preceding three shocks to carry a positive risk price $\Lambda_0(4)$, as positive slopes indicate improving economic conditions. This risk price helps the model match the average slope of the term structure.

We allow for ten non-zero elements in the first four rows (term structure block) of Λ_1 , which describes the dynamics in the risk prices. We let the price of inflation risk depend on the level of inflation to capture that periods like the late 1970s, early 1980s may have had elevated inflation risk. We let the price of GDP growth risk depend on the level of GDP growth. We let the price of short rate risk depend on the short rate as well as the term spread. The first dependence is a feature of the Vasicek and Cox, Ingersoll model, for example. The second dependence captures that the slope of the term structure predicts higher future returns on bonds (Capmbell and Shiller). We also need six non-zero elements in the fourth row of Λ_1 in order to allow the model to closely match the dynamics of the slope of the term structure, which is one of the variables included in the VAR. The dynamics of the five-year bond yield must satisfy (6). Given the first three rows of Λ_1 , satisfying these conditions requires that the first four elements of the fourth row of Λ_1 all be no-zero.

Lastly, we set all elements in the sixth row of Λ_1 to be non-zero, so that we have enough degrees of freedom to fit (A.13).

D Cointegration Tests

We run the Johansen cointegration test with the auxiliary specification

$$\Delta w_t = A(B'w_{t-1} + c) + D\Delta w_{t-1} + \varepsilon_t, \quad \text{where } w_t = \begin{pmatrix} \log T_t \\ \log G_t \\ \log GDP_t \end{pmatrix}.$$

Both trace and max eigenvalue tests do not reject the null of cointegration rank 1 or 2, but reject the null of cointegration rank 0. In other words, there are at least one cointegration relationship between variables in w_t .

We also conduct the Phillips-Ouliaris cointegration test and reject the null hypothesis that w is not cointegrated with a p value of 0.030 when the truncation lag parameter is 2, or a p value of 0.011 when the truncation lag parameter is 9.

E Immunization

To immunize against all shocks, we construct a replicating bond portfolio for the surplus claim. For the G-claim, we can approximate the change in the valuation of the spending claim as:

$$\begin{aligned} P_{t+1}^g - P_t^g &= G_{t+1} \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'z_{t+1}) - G_t \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'z_t) \\ &= G_t \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'z_{t+1} + \Delta \log G_{t+1}) - G_t \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'z_t) \\ &= G_t \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'(\Psi z_t + \Sigma^{\frac{1}{2}} \varepsilon_{t+1}) + x_0 + \pi_0 + \mu^g) \\ &\quad + (e_g + e_x + e_\pi)' (\Psi z_t + \Sigma^{\frac{1}{2}} \varepsilon_{t+1}) - G_t \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)'z_t) \\ &\approx G_t \sum_{h=0}^{\infty} \exp(A_g(h) + x_0 + \pi_0 + \mu^g + (B_g(h) + e_g + e_x + e_\pi)' \Psi z_t) \end{aligned}$$

$$\begin{aligned}
& \cdot (1 + (B_g(h) + e_g + e_x + e_\pi)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}) - G_t \sum_{h=0}^{\infty} \exp(A_g(h) + B_g(h)' z_t) \\
& = G_t \sum_{h=0}^{\infty} \{\exp(A_g(h) + x_0 + \pi_0 + \mu^g + (B_g(h) + e_g + e_x + e_\pi)' \Psi z_t) \\
& - \exp(A_g(h) + B_g(h)' z_t)\} + \{G_t \sum_{h=0}^{\infty} \exp(A_g(h) + x_0 + \pi_0 + \mu^g \\
& + (B_g(h) + e_g + e_x + e_\pi)' \Psi z_t) (B_g(h) + e_g + e_x + e_\pi)' \Sigma^{\frac{1}{2}}\} \varepsilon_{t+1}
\end{aligned}$$

Similarly, we can approximate the change in the valuation of the T-claim, assuming a constant tax revenue-to-GDP ratio, as:

$$\begin{aligned}
P_{t+1}^T - P_t^T & \approx T_t \sum_{h=0}^{\infty} \{\exp(A_\tau(h) + x_0 + \pi_0 + \mu^T + (B_\tau(h) + e_\tau + e_x + e_\pi)' \Psi z_t) \\
& - \exp(A_\tau(h) + B_\tau(h)' z_t)\} + \{T_t \sum_{h=0}^{\infty} \exp(A_\tau(h) + x_0 + \pi_0 + \mu^T \\
& + (B_\tau(h) + e_\tau + e_x + e_\pi)' \Psi z_t) (B_\tau(h) + e_\tau + e_x + e_\pi)' \Sigma^{\frac{1}{2}}\} \varepsilon_{t+1}
\end{aligned}$$

After collecting terms, we can state the change in the valuation of the surplus claim as:

$$(P_{t+1}^T - P_t^T) - (P_{t+1}^S - P_t^S) \approx a_t^S + b_t^S \Sigma^{\frac{1}{2}} \varepsilon_{t+1}$$

Next, we compute the sensitivity of the nominal bond price to the state variables, for a generic bond of maturity h quarters:

$$\log P_{t+1}^S(h) - \log P_t^S(h+1) = A^S(h) - A^S(h+1) + [(B^S(h))' \Psi - (B^S(h+1))'] z_t + (B^S(h))' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}.$$

Hence, we can approximate the change in the price of the bond as:

$$\begin{aligned}
(P_{t+1}^S(h) - P_t^S(h+1)) & \approx P_t^S(h+1) (A^S(h) - A^S(h+1)) + P_t^S(h+1) [((B^S(h))' \Psi - (B^S(h+1))') \\
& + P_t^S(h+1) (B^S(h))' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}.
\end{aligned}$$

We can state the latter, collecting terms, as:

$$(P_{t+1}^S(h) - P_t^S(h+1)) = P_t^S(h+1) a_t^S(h) + P_t^S(h+1) B^S(h) \Sigma^{\frac{1}{2}} \varepsilon_{t+1}$$

where

$$a_t^S(h) = (A^S(h) - A^S(h+1)) + [((B^S(h))' \Psi - (B^S(h+1))')] z_t$$

The sensitivity of real bonds takes the exact same expression, except without the dollar superscripts:

$$(P_{t+1}(h) - P_t(h+1)) = P_t(h+1) a_t(h) + P_t(h+1) B(h) \Sigma^{\frac{1}{2}} \varepsilon_{t+1}$$

where

$$a_t(h) = (A(h) - A(h+1)) + [((B(h))' \Psi - (B(h+1))')] z_t$$

Let $Q_{t,h}^S$ ($Q_{t,h}$) denote the observed position in the h -quarter nominal (real) zero coupon bond in the data. If the government is immunizing the risk exposure of its funding shock according to [Bhandari, Evans, Golosov, and Sargent \(2017\)](#), the dollar exposure of the government bond portfolio to each shock should equal the dollar exposure of the surplus claim:

$$\sum_{h=0}^H Q_{t,h+1}^S P_t^S(h+1) a_t^S(h) + \sum_{h=0}^H Q_{t,h+1} P_t(h+1) a_t(h) = a_t^S$$

$$\sum_{h=0}^H Q_{t,h+1}^S P_t^S(h+1)B^S(h) + \sum_{h=0}^H Q_{t,h+1} P_t(h+1)B(h) = b_t^S$$

We quantify the differences between the left-hand side and the right-hand side.

F Other Measures of the Convenience Yield

In this section, we compare our measure of the convenience yield with the implied convenience yields from [van Binsbergen, Diamond, and Grotteria \(2019\)](#). Figure A.1 shows the 6-month, 12-month, and 18-month convenience yields from [van Binsbergen, Diamond, and Grotteria \(2019\)](#), which are spreads between the SPX option implied interest rates and government bond rates with corresponding maturities. All measures of the convenience yield exhibit similar time-series patterns over the sample period from 2004-01 to 2017-04.

Figure A.1: Measures of the Convenience Yield

The figure shows the time series of different measures of the convenience yield. The dashed blue line is the spread of 6-month zero coupon interest rates implied from SPX options with 6-month Treasury bill rate. The dotted red line is the spread of 12-month zero coupon interest rates implied from SPX options with 12-month Treasury bill rate. The dashed yellow line is the spread of 18-month zero coupon interest rates implied from SPX options with 18-month Treasury bond rate. The data is from [van Binsbergen, Diamond, and Grotteria \(2019\)](#). The spread between the implied rate of the 6-month Treasury bill and the high-grade commercial papers-bills yield spread. A

