Incomplete Market Demand Tests for Kreps-Porteus-Selden Preferences

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Abstract

What does utility maximization subject to a budget constraint imply for intertemporal choice under uncertainty? Assuming consumers face a two period consumption-portfolio problem where asset markets are incomplete, we address this question following both the standard local infinitesimal and finite data approaches. To focus on the separate roles of time and risk preferences, individuals maximize KPS (Kreps-Porteus-Selden) preferences. The consumption-portfolio problem is decomposed into a one period portfolio problem and a two period certainty consumption-saving problem. We derive demand restrictions which are necessary and sufficient, for portfolio choices and certainty intertemporal consumption to have been generated by maximization, respectively, of a one period expected utility representation and a certainty representation of time preferences. Conditions are provided for recovering the building block time and risk preference utilities. For the finite data case, we derive a set of linear inequalities that are necessary and sufficient for observations to be consistent with the maximization of KPS utility.

KEYWORDS. Kreps-Porteus-Selden preferences, expected utility, contingent claims, integrability, incomplete markets.

JEL CLASSIFICATION. D01, D11, D80.

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1 Introduction

The neoclassical certainty model of consumer behavior postulates that a consumer's demand can be described as having been derived from utility maximization subject to a budget constraint. One is then naturally led to ask what this model implies about demand behavior. This question has been addressed using two quite distinct approaches. The first, typically referred to as *integrability*, originating in the infinitesimal analysis of Slutsky (1915) and Antonelli (1886), derives necessary and sufficient conditions such that a given demand function arises from the maximization of utility. The second approach known as "revealed preference", following the classic work of Samuelson (1938), provides necessary and sufficient restrictions on a finite set of demand-price pairs such that the demand behavior of a consumer is consistent with utility maximization. Among many others, Hurwicz and Uzawa (1971), Mas-Colell (1978) and Afriat (1967) have provided complete answers to these questions for the case of demand for commodities under certainty. While the analysis carries over directly to a static uncertainty setting with complete asset markets, the case of incomplete markets is not fully understood. Moreover for the case of the intertemporal demand for assets and first period consumption in incomplete markets, the answer is far less clear, in particular for the integrability approach. The question of what the hypothesis of utility maximization implies for intertemporal choice under uncertainty has received very little attention in the literature. This is surprising given the well-recognized importance of understanding the separate roles of risk and time preferences in saving and portfolio decisions.

In this paper we address the question of intertemporal choice where asset markets are incomplete utilizing both the integrability and revealed preference approaches. We assume the classic two period consumption-portfolio problem, where period 1 consumption is certain and period 2 consumption is risky. In the first period the consumer chooses a level of period 1 consumption and a portfolio of financial assets, where the market for assets is incomplete. In order to distinguish the separate roles of time and risk, we assume that consumers have preferences of the form axiomatized by Kreps and Porteus (1978) and Selden (1978) which include two period expected utility preferences as a special case. These KPS (Kreps-Porteus-Selden) preferences are fully characterized by a representation of **time preferences** defined over certain periods 1 and 2 consumption and **conditional risk preferences**, where the latter are parameterized by period 1 consumption and are defined over risky period 2 consumption. This separation of time and risk preferences is well known and has been widely used in the analysis of

saving behavior and asset pricing.[1] In addition to giving necessary and sufficient conditions such that consumption and asset demands are consistent with the maximization of KPS preferences, we also provide conditions under which the building block time and risk preference utilities can be recovered from the demands.

Our first observation is that, under mild conditions, an agent's utility maximization problem can be decomposed into a two stage problem. First conditional on a given value of period one consumption, one solves the single period portfolio problem resulting in optimal second period consumption. The solution is referred to as the conditional asset demand function. Then a second stage consumption-saving optimization for optimal first period consumption is solved. It will prove convenient to base our demand test for and identification of conditional risk preferences, on the solution to the conditional asset optimization problem. Then for the existence and identification of time preferences we utilize the solution to the second problem.

The solution to the conditional asset demand problem is formally identical to the solution of a one period asset choice problem. We argue that when financial markets are incomplete, it is generally impossible to extend the Hurwicz and Uzawa (1971) integrability result to the most direct case, a utility for assets. But what if preferences are defined over contingent claims and are representable by an EU (expected utility) function? Assuming one can vary probabilities as well as prices and income, we derive an asset demand test which verifies the existence of a unique EU representation that rationalizes the given demand. Moreover, we provide a means for recovering the EU function.

The assumption of varying probabilities is different from the traditional Arrow-Debreu setting, where probabilities are assumed to be given and fixed. This key difference enables us to define an implicit relationship between probabilities and asset prices from the given asset demand functions. It follows from McLennan (1979) that without this assumption the EU preferences cannot be uniquely recovered - in this sense it is a necessary condition for our analysis. In reality, investor beliefs over asset returns obviously vary over time but it is not clear how they can be observed. In laboratory experiments where subjects are given the probabilities they can naturally be varied across observations.

Within the KPS framework, establishing the existence of a well behaved time preference utility turns out to be more difficult. One of the challenges is that

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[1] For dynamic extensions (i.e., more than two periods) of these preferences such as the widely used Epstein and Zin (1989) model, it is not possible in general to achieve a complete separation of time and risk preferences (over consumption). See Epstein, Farhi and Strzalecki (2014, p. 2687).

the second stage consumption-saving problem can in general have a nonlinear budget constraint. Building on an insight in Polemarchakis and Selden (1984), we derive a local demand test for the existence of a unique (up to an increasing transformation) representation of time preferences that can rationalize the solution to the second stage consumption-saving problem. A key input into this test is the EU function recovered from the conditional asset demands. In addition to integrability results for the first and second stage optimizations, we also provide local conditions on demand which are necessary and sufficient for it to be derived from a KPS utility function.

Thus, together our theorems extend the integrability results of Hurwicz and Uzawa (1971) and Mas-Colell (1978) to the consumption-portfolio problem where asset markets are incomplete for the case of KPS preferences. To illustrate the application of our key results, we include a sequence of examples in which given demands are shown to satisfy the necessary and sufficient conditions for the existence of both a representation of conditional risk preferences and a representation of time preferences. Moreover, we recover the specific representations of risk and time preferences generating the demands.

We also translate our ideas to a revealed preference setting, with finitely many observations on prices, probabilities and asset demands. Kubler (2004) considered a special case of this setting but was unable to give a tractable characterization of necessary and sufficient conditions. In this setting we provide a set of linear inequalities that are necessary and sufficient for a finite set of observations to be consistent with the maximization of KPS utility. To obtain this result, we assume that in addition to observing the utility maximizing choices, one also observes the certainty equivalents of risky consumption corresponding to these choices. In the recent literature on contingent claim demand tests of different preference models (e.g., Choi, et al. 2007), it is standard to assume that prices, income and probabilities are known. Also, the required certainty equivalents could in principle be solicited from the experimental subjects. Our revealed preference extension would seem to facilitate addressing the interesting questions of whether (i) consumer demands are consistent with KPS preferences, (ii) time and risk preferences are independent and (iii) they perform better in complete versus incomplete markets.

The desire to separately identify risk and time preferences from given consumption and asset demands is a clear motivation for why we have chosen to focus on the consumption-portfolio optimization rather than just the portfolio problem. With regard to potential applications of the theoretical results in this paper, recent laboratory experimental work investigating the separate roles of risk and time preferences would seem quite complementary. Numerous studies have

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been conducted in this area (see, for example, Andreoni and Sprenger 2012, 2015, Wölbert and Riedl 2013, Cheung 2015, Epper and Fehr-Duda 2015 and Miao and Zhong 2015).

For the case of revealed preferences, Varian (1983) shows how to extend Afirat's (1967) analysis to a portfolio problem with possibly incomplete asset markets. Our contribution is to extend his analysis to an intertemporal setting. There is a substantial literature on the question whether EU preferences can be uniquely identified from asset demand and how to recover utility (see e.g., Dybvig and Polemarchakis 1981, Polemarchakis and Selden 1984). Clearly, if one can recover a candidate EU function, one can plug it into the first order conditions for optimal demand and test whether demand is generated by this utility. There are several limitations to this approach. First, although it allows one to uniquely identify a candidate EU representation, it may be difficult to analytically derive the utility function. Second, McLennan (1979) shows that locally the same incomplete market asset demand can be generated by two different EU representations. To overcome this problem, one must assume the existence of a risk free asset and apply the approach globally or assume that the utility function is analytic. Our approach proves existence of a unique EU function without having to recover the utility and avoids the requirement to have a risk free asset and apply the approach globally as well as the assumption that utility is analytic. Kubler and Polemarchaks (2017) examine a complementary problem. They also work in a two period setting with incomplete markets but assume the existence of stationary expected utility, and directly recover the NM (von Neuman-Morgenstern) index. The main contribution of that paper is to give conditions that allow for an identification of (fixed) beliefs from the observed asset demand as a function of asset prices. We consider the opposite problem: beliefs are observable (objective) and vary but no assumption is made on preferences. We give necessary and sufficient conditions for the existence of a KPS representation and show that the utility function can be recovered from observations on prices, probabilities and demands.

The rest of the paper is organized as follows. In the next section, we introduce the setup and define notation. In Section 3, we provide several examples illustrating a number of specific obstacles in an incomplete market setting to directly solving the integrability problem for a utility over assets rather than contingent claims. In Section 4, we first consider integrability for the case where conditional risk preferences are representable by expected utility and then provide necessary and sufficient conditions for the existence of a utility representing time preferences over certain periods 1 and 2 consumption and a means for identifying the utility. Section 5 gives a revealed preference test to verify that discrete data is consis-

tent with a KPS representation which can be conducted in a lab setting. Proofs are given in Appendix A and supporting materials are provided in Supplemental Appendix B.

2 Preliminaries

In the first subsection, we describe the consumption-portfolio setting and then review the structure and properties of KPS preferences. One of the motivations for assuming these preferences is to be able to identify, based on consumption and asset demands, the specific underlying risk and time preferences. To achieve this, it will prove useful to utilize a two stage process for solving the consumption-portfolio optimization, which is discussed in the second subsection.

2.1 Notation and Definitions

At the beginning of period 1, the consumer chooses a level of certain first period consumption c_1 and a set of asset holdings, where the returns on the latter fund consumption in period 2. The asset market can be incomplete with $J \geq 2$ independent assets and S states, where $J \leq S$. Denote the payoff for asset j ($j \in \{1, ..., J\}$) in state s ($s \in \{1, ..., S\}$) by $\xi_{js} \geq 0$, where for each j, there exists at least one $s \in \{1, ..., S\}$ such that $\xi_{js} > 0$. The quantities of assets and contingent claims are denoted, respectively, by z_j and c_{2s}, with $\mathbf{z}$ and $\mathbf{c}_2$ being the corresponding vectors. Random period 2 consumption can thus be expressed as

$$c_{2s} = \sum_{j=1}^{J} z_j \xi_{js} \quad (s = 1, ..., S). \tag{1}$$

The prices of period 1 consumption, c_1 and asset z_j are given by, respectively, p_1 and q_j. The vector of state probabilities is denoted $\boldsymbol{\pi} \in \Delta_{++}^{S-1} = \{\boldsymbol{\pi} \in \mathbb{R}_{++}^{S} | \sum_{s=1}^{S} \pi_s = 1\}$. Both asset prices and state probabilities are allowed to vary. We assume throughout that the payoffs of the J assets across states, $(\xi_{j1}, \ldots, \xi_{jS})$, are linearly independent for all $j = 1, \ldots, J$. Asset prices preclude arbitrage in that there are $p_{2s} > 0$, $s = 1, \ldots, S$ such that

$$q_j = \sum_{s=1}^{S} \xi_{js} p_{2s} \quad (j = 1, 2, , ..., J). \tag{2}$$

The consumer's preferences over the consumption vectors $(c_1, c_{21}, ..., c_{2S})$ are

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assumed to be representable by the KPS form[2]

$$\mathcal{U}\left(c_1, c_{21}, ..., c_{2S}\right) = U\left(c_1, V_{c_1}^{-1}\left(\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})\right)\right) = U(c, \widehat{c}_2), \qquad (3)$$

where $\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})$ is the standard single period state independent EU representation over risky period 2 consumption, V_{c_1} is the NM index conditional on period 1 consumption.[3] The NM index V_{c_1} is strictly increasing in c_{2s} and twice continuously differentiable in c_1 and c_{2s}. The assumption that V_{c_1} is strictly increasing in c_{2s} ensures the existence of a unique certainty equivalent $\widehat{c}_2$. The time preference representation U is twice continuously differentiable and strictly quasi-concave and V_{c_1} is concave in c_2 for each c_1-value. The second argument of U in (3) is the period 2 certainty equivalent associated with $(c_{21}, ..., c_{2S})$

$$\widehat{c}_2 = V_{c_1}^{-1}\left(\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})\right).$$

If V_{c_1} takes the form

$$V_{c_1}\left(\cdot\right) = a\left(c_1\right) V\left(\cdot\right) + b\left(c_1\right),$$

where $a\left(c_1\right) > 0$ and $b\left(c_1\right)$ are functions of c_1 and V is independent of c_1, it will be said to exhibit RPI (risk preference independence). Otherwise, it will be said to exhibit RPD (risk preference dependence).[4] Clearly for the case of an RPD conditional NM index, $\widehat{c}_2$ will depend not only on $(c_{21}, ..., c_{2S})$ but also on c_1. Thus, the KPS utility (3) is defined by the indices $(U, \{V_{c_1}\})$.

Given the dual contingent claim structure assumed, the consumer's consumption-portfolio optimization problem is given by

$$\max_{c_1, \mathbf{z}} U\left(c_1, \widehat{c}_2\right) \qquad (4)$$

$$S.T. \quad \widehat{c}_2\left(\mathbf{c}_2; \boldsymbol{\pi}\right) = V_{c_1}^{-1}\left(\sum_{s=1}^{S} \pi_s V_{c_1}\left(c_{2s}\right)\right),$$

$$c_{2s} = \sum_{j=1}^{J} \xi_{js} z_j \quad \text{and} \quad p_1 c_1 + \sum_{j=1}^{J} q_j z_j = I. \qquad (5)$$

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[2]The particular form (3) is axiomatized in Selden (1978). As is well-known, this utility is equivalent to the two period Kreps and Porteus (1978) form if one embeds $V_{c_1}^{-1}$ in the outside aggregator.

[3]In general, the representation (3) is not linear in probabilities and diverges from the two period EU $\sum_{s=1}^{S} \pi_s W(c_1, c_{2s})$.

[4]One familiar example of such a dependence is the internal habit formation formulation in Constantinides (1990).

Since these assumptions do not imply that the first order conditions for the problem (4) - (5) are sufficient for a unique maximum, we require the KPS utility (3) to be strictly quasiconcave in $(c_1, c_{21}, ..., c_{2S})$.

The solution to (4) - (5) can be expressed as the period 1 consumption $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and asset demand function $\mathbf{z}(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$. It will be understood that when we write $\pi_1, ..., \pi_S$, one can always replace π_S by $1 - \sum_{s=1}^{S-1} \pi_s$. Consistent with the above simplex normalization of probabilities, corresponding to any change in π_s ($s \neq S$) it will be understood that π_S will have a compensating change. Given this convention, $\partial c_1/\partial \pi_s$ and $\partial \mathbf{z}/\partial \pi_s$ are defined for $s = 1, ..., S - 1$.

Throughout most of this paper, we assume that one is given the functions $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and $\mathbf{z}(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ on an open set of period one consumption price, no-arbitrage asset prices, probabilities and incomes. These sets are denoted respectively by $\mathcal{P} \subset \mathbb{R}_{++}$, $\mathcal{Q} \subset \mathbb{R}_{++}^J$, $\Pi \subset \Delta_{++}^{S-1}$ and $\mathcal{I} \subset \mathbb{R}_{++}$. We assume that the function is given on the product on these sets, $\mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$ which we assume to be topologically connected. This assumption is not needed for our tests, but clearly one cannot uniquely (up to monotone or positive affine transformation) recover utility functions which are defined on different regions of the consumption space.

The key question we focus on is whether a given vector of demands $(c_1, \mathbf{z})$ is generated as the result of the optimization (4) - (5) and hence can be said to be rationalized by KPS preferences. As mentioned in the introduction, it is crucial to assume that variations of probabilities are observable on an open set.

2.2 Two Stage Optimization

The optimization (4) - (5) can be decomposed into a two stage problem.[5] First conditional on a given value of c_1, one solves the single period problem

$$\max_{\mathbf{z}} \sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s}) = \sum_{s=1}^{S} \pi_s V_{c_1}(\boldsymbol{\xi}_s \cdot \mathbf{z}) \quad S.T. \sum_{j=1}^{J} q_j z_j = I - p_1 c_1 = I_2, \quad (6)$$

where I_2 denotes period 2 residual income. The solution to (6) is referred to as the conditional asset demand function and denoted by $\mathbf{z}(\mathbf{q}, \boldsymbol{\pi}, I_2 | c_1)$. Then the second stage optimization is

$$\max_{c_1} U(c_1, \widehat{c}_2(\mathbf{z}(\mathbf{q}, \boldsymbol{\pi}, I_2 | c_1))) \quad (7)$$

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[5] A necessary and sufficient condition for being able to perform this decomposition is given in Remark 8, Appendix A.2.

is solved. It should be noted that in this formulation $\widehat{c}_2$ is a function of z and c_1. The resulting optimal period 1 consumption demand $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ can be substituted into $\mathbf{z}(\mathbf{q}, \boldsymbol{\pi}, I_2|\, c_1)$ yielding the unconditional asset demand $\mathbf{z}(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$. It will prove convenient to base our demand test for and identification of conditional risk preferences, corresponding to $\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})$, on the solution to the conditional asset optimization problem (6). Then for the existence and identification of time preferences represented by $U(c_1, c_2)$, we utilize the solution to (7).

For the overall optimization, suppose that $\mathcal{U}$ in (3) is quasiconcave in $(c_1, c_{21}, ..., c_{2S})$, then a solution to (6) - (7) exists, but the U implicitly defined by $U(c_1, c_2) = \mathcal{U}(c_1, c_{21}, ..., c_{2S})$ with $c_{21} = ... = c_{2S}$, need not be strictly increasing and quasiconcave, as is required for a suitable representation of time preferences. (See Example B.1 in Supplemental Appendix B.1.) For the two stage optimization, it is standard to assume that U is quasiconcave. However, this is not sufficient to ensure that the solution to the first order condition also satisfies the second order condition. (See Example B.2 in Supplemental Appendix B.1.)

Since for our primary integrability results, we assume the two stage formulation, we need an assumption on the certainty equivalent to guarantee overall strict quasiconcavity.

Assumption 1 *The period two certainty equivalent*

$$\widehat{c}_2 = V_{c_1}^{-1} \sum_{s=1}^{S} \pi_s V_{c_1}\left(\boldsymbol{\xi}_s \cdot \mathbf{z}\left(\mathbf{q}, \boldsymbol{\pi}, I_2|\, c_1\right)\right)$$

is weakly concave in c_1.

Proposition 1 *Suppose $J \geq 2$ and $S \geq J$ and one is given twice continuously differentiable demand functions $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and $\mathbf{z}(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$. Further assume that the conditional asset demand function $\mathbf{z}(\mathbf{q}, \boldsymbol{\pi}, I_2|\, c_1)$ is rationalizable by an EU function with a twice continuously differentiable NM index V_{c_1} and inverse conditional demand exists. Moreover, there exists a unique twice continuously differentiable, strictly increasing, strictly quasiconcave representation of time preferences $U(c_1, c_2)$ rationalizing the certainty demand. Then $(c_1, \mathbf{z})$ can be rationalized by a KPS utility (3) if Assumption 1 holds.*

Remark 1 *As proved in Selden (1980, Corollary), if V_{c_1} is a member of the HARA (hyperbolic absolute risk aversion) class, then $\widehat{c}_2$ is a linear function of c_1 and Proposition 1 is automatically satisfied.*

3 Motivating Examples

The most natural and direct way to solve the integrability problem would be to prove the existence of a rationalizing utility defined over period 1 consumption and assets rather than period 1 consumption and contingent claims, for instance, using Hurwicz and Uzawa (1971). However, even if one could prove the existence of an increasing and quasiconcave utility over assets, there is no guarantee that this would imply the existence of an increasing and quasiconcave utility over consumption. This difficulty is illustrated by the following simple example based on the consumption-portfolio problem (4) - (5). The Slutsky symmetry and negative semi-definiteness conditions necessary and sufficient for the existence of a utility over assets are satisfied, but the induced preferences over contingent claims are not increasing and quasiconcave.

Example 1 *Assume three states with a risk free asset, a risky asset, asset payoffs*

$$\xi_{11} = 1, \ \xi_{12} = 1, \ \xi_{13} = 1, \ \xi_{21} = 2, \ \xi_{22} = 0, \ \xi_{23} = \frac{1}{2}$$

and probabilities (π_1, π_2, π_3). *Suppose demand takes the following form*

$$c_1\left(p_1, \mathbf{q}, I\right) = \frac{I}{3p_1}, \ z_1\left(p_1, \mathbf{q}, I\right) = \frac{2\left(\pi_1\xi_{11} + \pi_2\xi_{12} + \pi_3\xi_{13}\right)I}{3\left(\pi_1\left(\xi_{11} + \xi_{21}\right) + \pi_2\left(\xi_{12} + \xi_{22}\right) + \pi_3\left(\xi_{13} + \xi_{23}\right)\right)q_1}$$

and

$$z_2\left(p_1, \mathbf{q}, I\right) = \frac{2\left(\pi_1\xi_{21} + \pi_2\xi_{22} + \pi_3\xi_{23}\right)I}{3\left(\pi_1\left(\xi_{11} + \xi_{21}\right) + \pi_2\left(\xi_{12} + \xi_{22}\right) + \pi_3\left(\xi_{13} + \xi_{23}\right)\right)q_2}.$$

It can be verified that the conditional asset demands are given by

$$z_1\left(\mathbf{q}, I_2 \middle| \ c_1\right) = \frac{\left(\pi_1\xi_{11} + \pi_2\xi_{12} + \pi_3\xi_{13}\right)I_2}{\left(\pi_1\left(\xi_{11} + \xi_{21}\right) + \pi_2\left(\xi_{12} + \xi_{22}\right) + \pi_3\left(\xi_{13} + \xi_{23}\right)\right)q_1}$$

and

$$z_2\left(\mathbf{q}, I_2 \middle| \ c_1\right) = \frac{\left(\pi_1\xi_{21} + \pi_2\xi_{22} + \pi_3\xi_{23}\right)I_2}{\left(\pi_1\left(\xi_{11} + \xi_{21}\right) + \pi_2\left(\xi_{12} + \xi_{22}\right) + \pi_3\left(\xi_{13} + \xi_{23}\right)\right)q_1}.$$

These demands satisfy Slutsky symmetry and negative semidefiniteness conditions where the former holds automatically since there are only two assets. Applying the Hurwicz and Uzawa (1971) recovery process yields the familiar Cobb-Douglas form defined over assets,

$$\overline{\mathcal{V}}_{c_1}(z_1, z_2) = \left(\pi_1\xi_{11} + \pi_2\xi_{12} + \pi_3\xi_{13}\right)\ln z_1 + \left(\pi_1\xi_{21} + \pi_2\xi_{22} + \pi_3\xi_{23}\right)\ln z_2.$$

This in turn, corresponds to a utility function over the contingent claim domain $\left\{ (c_{21}, c_{22}, c_{23}) \in \mathbb{R}^3_{++} \middle| \ c_{23} = \frac{c_{21}}{4} + \frac{3c_{22}}{4} \right\}$ *which is given by*

$$\mathcal{V}_{c_1}(c_{21}, c_{22}, c_{23}) = \ln \frac{c_{21} - c_{22}}{2} + \ln c_{22}.$$

By the Tietze extension theorem (see e.g., Hazewinkel 2001), this can be extended to a continuous utility function over the entire contingent claim space. However, it is easy to see that the utility is not everywhere increasing in c_{22}.

This example clearly demonstrates that any asset demand test has to work in the contingent claim setting since even if one can recover a well defined utility over assets, it may have no economic meaning when defined over contingent claims, which are associated with the consumers' real consumption. However, it is well known that when markets are incomplete one cannot uniquely identify the preferences over contingent claims from demand of assets. The following example which focuses on the conditional asset demand optimization (6) illustrates this point.[6]

Example 2 *Consider the following non-EU function representing conditional risk preferences*

$$\mathcal{V}_{c_1}(c_{21}, c_{22}, c_{23}; \pi_1, \pi_2, \pi_3) = \sum_{s=1}^{3} \pi_s \ln c_{2s} + \sqrt{c_{21} + c_{22} - 2c_{23}}, \qquad (8)$$

where

$$\xi_{11} = 1, \ \xi_{12} = 0, \ \xi_{13} = \frac{1}{2}, \ \xi_{21} = 0, \ \xi_{22} = 1, \ \xi_{23} = \frac{1}{2}.$$

Since

$$c_{21} + c_{22} - 2c_{23} = z_1 + z_2 - (z_1 + z_2) = 0,$$

it is clear that the non-EU function (8) expressed as a function of assets

$$\overline{\mathcal{V}}_{c_1}(z_1, z_2; \pi_1, \pi_2, \pi_3) = \sum_{s=1}^{3} \pi_s \ln (\xi_{1s} z_1 + \xi_{2s} z_2)$$

takes the same form as the conditional EU representation defined over contingent claims

$$\mathcal{V}_{c_1}(c_{21}, c_{22}, c_{23}; \pi_1, \pi_2, \pi_3) = \sum_{s=1}^{3} \pi_s \ln c_{2s}.$$

Therefore in this case, maximizing the non-EU and EU representations results in same conditional asset demand functions.

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[6]A similar example is given in Polemarchakis and Selden (1981).

The examples suggest that when markets are incomplete it is very difficult to derive necessary and sufficient conditions for intertemporal demand to be rationalized by a well-behaved utility function. Clearly, the Slutsky condition on asset demand is a necessary but not a sufficient condition. The difficulty in deriving sufficient conditions lies in the fact that there are many utility functions that rationalize asset demand in incomplete markets and it is generally impossible to ensure that one of them is increasing and quasiconcave.[7]

To address this problem, we note that assuming that a consumer's preferences are EU goes a long way to solve this problem. Although for this example, it is impossible to tell whether the true underlying representation of conditional risk preferences is the EU function, one can still ask whether the EU function rationalizing the observed asset demand is (i) unique in the class of EU functions and (ii) can be recovered from the demands.

4 Integrability

In this section we solve the integrability problem for KPS preferences by providing conditions such that there exists a KPS utility (3) defined by $(U, \{V_{c_1}\})$ which rationalizes given demands $(c_1, \mathbf{z})$, is unique and can be recovered from the demands. Our approach follows the two stage consumption-portfolio optimization (6) - (7). Based on the first stage conditional portfolio problem, we give conditions for the existence and uniqueness of the $\{V_{c_1}\}$ and a means for recovering the NM indices. Then, utilizing the second stage consumption-saving problem, we derive conditions for the existence and uniqueness of U and a means for recovering time preference utility. Finally, we provide conditions such that the KPS utility defined by the $(U, \{V_{c_1}\})$ obtained does indeed rationalize the given demands. A comprehensive example is provided which illustrates the application of the tests derived in this section.

4.1 Verifying That Conditional Asset Demands Are Generated by EU Risk Preferences

The question of the existence of a rationalizing conditional EU representation can only be answered in terms of restrictions on conditional asset demands.[8] Since

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[7]For the revealed preference analysis, matters are quite different. It is straightforward to derive the Afriat inequalities for this case. We illustrate this in Supplemental Appendix B.2.

[8]If the unconditional demands can be rationalized by a twice continuously differentiable, strictly increasing and strictly quasiconcave utility function, then as argued in Remark 8 in

it is more reasonable to suppose that one is given unconditional demands for period 1 consumption and assets, it is necessary to ensure that a unique twice continuously differentiable conditional asset demand function can be derived from the unconditional demand function. Sufficient conditions for this derivation are given by Lemma 2 in Appendix A.2. This is a mild technical condition and for simplicity it will be assumed to be satisfied throughout this paper.

In order to derive our integrability result for conditional asset demand, it will prove useful to consider the inverse demand function which maps asset demand, probabilities and income into a supporting price vector. To simplify notation, normalize $I_2 = 1$ and denote the conditional inverse demand function by $q_j(\mathbf{z}, \boldsymbol{\pi})$ ($j = 1, 2, ..., J$). It should be noted that although $q_j(.)$ refers to both the unconditional and conditional inverse demand, they can easily be distinguished, respectively, by the inclusion of I for the case of unconditional demand. For the analysis below, we require taking the partial derivatives of $\mathbf{q}(\mathbf{z}, \boldsymbol{\pi})$ with respect to probabilities. In Appendix A.2, sufficient conditions are given in Lemma 3 for the existences of twice continuously differentiable inverse demand. Again this is a mild technical condition that will be assumed to hold throughout the paper.

As explained in Section 2 above, conditional demand is assumed to be given in an open and topologically connected set of asset prices and probabilities. It follows from Lemma 3 that for each $\boldsymbol{\pi} \in \Pi$, the range of the inverse demand function is an open and connected subset of $\mathbb{R}^J$ and we can define the domain of inverse demand as $\mathcal{D} \subset \mathbb{R}^J \times \Delta_+^{S-1}$, an open and connected set.

In order to derive necessary and sufficient conditions for EU-rationalizability based on inverse demand, consider the conditional portfolio optimization problem in (6) - (7). The first order conditions for the optimization problem are

$$\sum_{s=1}^{S} \pi_s \xi_{js} V'_{c_1}(c_{2s}) = \mu q_j \ (j = 1, .., J), \tag{9}$$

where μ is the Lagrange multiplier. It is easy to see that when markets are complete the system (9) has a unique solution in $\pi_s V'_{c_1}(c_{2s})/\mu$, $s = 1, \ldots, S$. These are the contingent claim prices, p_{2s}. Kubler, Selden and Wei (2014, Theorem 2) derive simple necessary and sufficient conditions for the existence of a rationalizing EU function in complete markets based on the following relation between contingent claim demands

$$c_{2s} = f(c_{21}, k_s), \tag{10}$$

where

Appendix A.2, one can always consider the two stage optimization and hence unique twice continuously differentiable conditional asset demand exists.

$$k_s = \frac{\pi_s p_{21}}{\pi_1 p_{2s}}, \tag{11}$$

and f is strictly increasing in k_s and $f(c_{2s}, 1) = c_{2s}$ for all c_{2s}. The demand restriction (10) is referred to as the k-test. Unfortunately when markets are incomplete, (9) has many solutions and it is not feasible to determine whether there exists a particular solution that satisfies (10).

However as we next show in Theorem 1 below, it is possible to extend the complete market EU asset demand test to incomplete markets if we assume that probabilities can be varied. Indeed the three conditions stated in the theorem below comprise a formal analogue to the complete market k-test. The method gives us a way to uniquely pin down $\pi_s V'_{c_1}(c_{2s})/\mu$ which we denote by ρ_{2s}: Assuming that for some j, $\xi_{jS} - c_{2S}q_j \neq 0$, this allows us to define for all $(\mathbf{z}, \boldsymbol{\pi}) \in \mathcal{D}$,

$$\rho_{2S}(\mathbf{z}, \boldsymbol{\pi}) = \frac{\sum_{s=1}^{S-1} \pi_s \frac{\partial q_j}{\partial \pi_s}}{-\frac{1}{\pi_S}\left(\xi_{jS} - \frac{c_{2S}q_j}{I_2}\right)} = -\frac{\pi_S \sum_{l=1}^{S-1} \pi_l \frac{\partial q_j}{\partial \pi_l}}{\xi_{jS} - c_{2S}q_j} \tag{12}$$

and (assuming $\xi_{js} - c_{2s}q_j \neq 0$) for $s = 1, \ldots, S-1$,

$$\rho_{2s}(\mathbf{z}, \boldsymbol{\pi}) = \frac{\pi_s\left(\frac{\partial q_j}{\partial \pi_s} - \sum_{l=1}^{S-1} \pi_l \frac{\partial q_j}{\partial \pi_l}\right)}{\xi_{js} - c_{2s}q_j}. \tag{13}$$

In the proof of Theorem 1, we show that these are the analogues of contingent claim prices for the incomplete market setting.

For two states s and s', define

$$\mathcal{M}_{s,s'}(\mathbf{z}, \boldsymbol{\pi}) = \frac{\pi_{s'}\rho_{2s}(\mathbf{z}, \boldsymbol{\pi})}{\pi_s\rho_{2s'}(\mathbf{z}, \boldsymbol{\pi})}. \tag{14}$$

We have the following theorem.

Theorem 1 *Assume $J \geq 2$ and $S \geq J$ and that for some $j \in \{1, ..., J\}$, $\xi_{js} - c_{2s}q_j \neq 0$ ($\forall s = 1, ..., S$). Then conditional asset demand $\mathbf{z}(\mathbf{q}, \boldsymbol{\pi}, I_2 | c_1)$ can be rationalized by a unique EU representation if and only if the following three conditions hold.*

(i) *For all $(\mathbf{z}, \boldsymbol{\pi}) \in \mathcal{D}$, and all $j = 1, \ldots, J$,*

$$q_j(\mathbf{z}, \boldsymbol{\pi}) = \sum_{s=1}^{S} \xi_{js}\rho_{2s}(\mathbf{z}, \boldsymbol{\pi}).$$

(ii) *For all $(\mathbf{z}, \boldsymbol{\pi}) \in \mathcal{D}$, and all arbitrary vectors $\boldsymbol{\zeta} \in \mathbb{R}^J$ with $\boldsymbol{\xi}_s \cdot \boldsymbol{\zeta} = \boldsymbol{\xi}_{s'} \cdot \boldsymbol{\zeta} = 0$, the derivative in the direction of these vectors satisfies*

$$\frac{d}{d\delta} \mathcal{M}_{s,s'}(\mathbf{z} + \delta\boldsymbol{\zeta}, \boldsymbol{\pi})|_{\delta=0} = 0,$$

where δ is a scalar.

Moreover, for all $(\mathbf{z}, \boldsymbol{\pi}) \in \mathcal{D}$,

$$D_{\boldsymbol{\pi}} \mathcal{M}_{s,s'}(\mathbf{z}, \boldsymbol{\pi}) = 0.$$

(iii) *For all $(\mathbf{z}, \boldsymbol{\pi}), (\widetilde{\mathbf{z}}, \widetilde{\boldsymbol{\pi}}) \in \mathcal{D}$ and all $s, \widetilde{s}, s', \widetilde{s}' \in \{1, \ldots, S\}$, if $c_{2s} = \mathbf{z} \cdot \boldsymbol{\xi}_s \geq \widetilde{c}_{2\widetilde{s}} = \widetilde{\mathbf{z}} \cdot \boldsymbol{\xi}_{\widetilde{s}}$ and $c_{2s'} = \mathbf{z} \cdot \boldsymbol{\xi}_{s'} = \widetilde{c}_{2\widetilde{s}'} = \widetilde{\mathbf{z}} \cdot \boldsymbol{\xi}_{\widetilde{s}'}$ then*

$$\mathcal{M}_{s,s'}(\mathbf{z}, \boldsymbol{\pi}) \leq \mathcal{M}_{\widetilde{s}, \widetilde{s}'}(\widetilde{\mathbf{z}}, \widetilde{\boldsymbol{\pi}}),$$

where the inequality holds strictly if $c_{2s} = \mathbf{z} \cdot \boldsymbol{\xi}_s > \widetilde{c}_{2\widetilde{s}} = \widetilde{\mathbf{z}} \cdot \boldsymbol{\xi}_{\widetilde{s}}$.

Furthermore $\{V_{c_1}\}$ can be uniquely recovered up to a positive affine transformation on the intervals of consumption values demanded in states $s = 1, \ldots, S$ by integrating $\mathcal{M}_{s,s'}(\mathbf{z} + \delta\boldsymbol{\zeta}, \boldsymbol{\pi})$ with respect to δ for $\boldsymbol{\zeta} \in \mathbb{R}^J$ where $\boldsymbol{\zeta} \cdot \boldsymbol{\xi}_{s'} = 0$ and $\boldsymbol{\zeta} \cdot \boldsymbol{\xi}_s > 0$. In this case,

$$V_{c_1}(c_{2s}) = V_{c_1}\left(\boldsymbol{\xi}_s \cdot (\mathbf{z} + \delta\boldsymbol{\zeta})\right) = \int_{\delta} \mathcal{M}_{s,s'}(\mathbf{z} + \delta\boldsymbol{\zeta}, \boldsymbol{\pi}) d\delta.$$

.

One cannot directly integrate with respect to c_{2s}, since $\mathcal{M}_{s,s'}(\mathbf{z}, \boldsymbol{\pi})$ is a function of probabilities and asset demands instead of $(c_{2s}, c_{2s'})$, Define the marginal rate of substitution between c_{2s} and $c_{2s'}$ as $MRS_{ss'}$. If conditional asset demands are EU representable, we have

$$\mathcal{M}_{s,s'}(\mathbf{z}, \boldsymbol{\pi}) = \frac{\pi_{s'}}{\pi_s} MRS_{ss'} = \frac{V'_{c_1}(c_{2s})}{V'_{c_1}(c_{2s'})},$$

which is independent of probabilities. Here we choose a vector satisfying $\boldsymbol{\zeta} \cdot \boldsymbol{\xi}_{s'} = 0$ and $\boldsymbol{\zeta} \cdot \boldsymbol{\xi}_s > 0$. Since $c_{2s} = \mathbf{z} \cdot \boldsymbol{\xi}_s$, when integrating $\mathcal{M}_{s,s'}(\mathbf{z} + \delta\boldsymbol{\zeta}, \boldsymbol{\pi})$ with respect to δ, we effectively integrate for c_{2s}.

As argued in the proof of the theorem, Condition (i) follows from the first order condition for optimality. Condition (ii) follows because utility is assumed to be separable across states and the NM index does not depend on probabilities. Condition (iii) guarantees state independence and concavity of utility.

Remark 2 *To see more clearly the connection between Theorem 1 Conditions (i) - (iii) and the k-test (10), set $s = 1$ in (14) yielding*

$$\mathcal{M}_{1s'}(\mathbf{z}, \boldsymbol{\pi}) = \frac{\pi_{s'}\rho_{21}(\mathbf{z}, \boldsymbol{\pi})}{\pi_1 \rho_{2s'}(\mathbf{z}, \boldsymbol{\pi})} = k_{s'}. \tag{15}$$

Condition (ii) in Theorem 1 guarantees that $\mathcal{M}_{1s'}(\mathbf{z}, \boldsymbol{\pi})$ is a function of only c_{21} and $c_{2s'}$ and Condition (iii) ensures that this function is strictly decreasing in c_{21} and strictly increasing in $c_{2s'}$. By the Implicit Function Theorem, there exists a function f such that

$$c_{2s'} = f(c_{21}, k_{s'}).$$

Based on Condition (iii), it is easy to verify that $f(c_{21}, k_{s'})$ is strictly increasing in $k_{s'}$ and $f(c_{21}, 1) = c_{21}$ for all c_{21}. Therefore, the result in Theorem 1 converges to the complete market EU test in Theorem 2 of Kubler, Selden and Wei (2014).

Remark 3 *If there is a risk free asset and demand can be observed globally, then the process to recover the candidate V_{c_1} can be significantly simplified. Following Dybvig and Polemarchakis (1981), without loss of generality assume asset 1 is risk free. Then one can recover the candidate $\{V_{c_1}\}$ if the following inverse demand is known*

$$\mathbf{q}(\mathbf{z} = (\overline{z}, 0, \ldots, 0), \boldsymbol{\pi}, I_2) \in \mathcal{Q} \text{ for some } \overline{z}.$$

This recovery process rests crucially on the assumptions that (i) an EU representation exists and (ii) one can observe demand and prices where it is optimal to only demand the risk free asset.[9]

Remark 4 *McLennan (1979) provides an example in which asset demands can be rationalized by two EU representations with NM indices that differ by more than an affine transformation (also see Dybvig and Polemarchakis 1981, p. 165). It should be emphasized that one of the NM indices depend on probabilities. If, as McLennan (1979) assumes, probabilities are fixed, both representations can be viewed as standard EU functions. However because we allow probabilities to vary, a representation which takes the EU form but with a probability dependent NM index cannot rationalize the given demand.[10] Thus if there is a probability dependent NM index and a probability independent NM index to rationalize the demand for fixed probabilities, we will only recover the probability independent one based on Theorem 1.*

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[9] It should be noted that Green, Lau and Polemarchakis (1979) propose another approach for recovering V_{c_1} when there is no risk free asset and the analysis is local. This is based on the strong assumption that V_{c_1} is analytic in the nonnegative domain.

[10] This is discussed at length in Kubler, Selden and Wei (2017).

Next we apply the incomplete demand test associated with Conditions (i) - (iii) in Theorem 1 for demand to have been generated by the maximization of an EU function. Given that the asset demand function passes the incomplete market EU test, a unique EU representation exists and integration is shown to produce the corresponding NM index V_{c_1} (up to a positive affine transformation).

Example 3 *Assume three states and two assets where the payoffs are given by*[11]

$$\xi_{11} = 1, \ \xi_{12} = 0, \ \xi_{13} = \frac{1}{2}, \ \xi_{21} = 0, \ \xi_{22} = 1, \ \xi_{23} = \frac{1}{2}. \tag{16}$$

The period 1 consumption and unconditional asset demands are respectively given by[12,13]

$$c_1 = \frac{I}{p_1} - \frac{1}{p_1} \left(\begin{array}{c} \frac{1}{p_1} \left(\frac{\pi_1(1-1/B)}{q_1-q_2} \right)^{\pi_1} \left(\frac{\pi_2(B-1)}{q_1-q_2} \right)^{\pi_2} \\ \times \left(\frac{\pi_3(1-1/B)}{2(q_2-q_1/B)} \right)^{\pi_3} \end{array} \right)^{-\frac{1}{2}}, \tag{17}$$

$$z_1 = \frac{\pi_1(1-1/B)}{q_1-q_2} \left(\begin{array}{c} \frac{1}{p_1} \left(\frac{\pi_1(1-1/B)}{q_1-q_2} \right)^{\pi_1} \left(\frac{\pi_2(B-1)}{q_1-q_2} \right)^{\pi_2} \\ \times \left(\frac{\pi_3(1-1/B)}{2(q_2-q_1/B)} \right)^{\pi_3} \end{array} \right)^{-\frac{1}{2}} \tag{18}$$

and

$$z_2 = \frac{\pi_2(B-1)}{q_1-q_2} \left(\begin{array}{c} \frac{1}{p_1} \frac{\left(\frac{\pi_1(1-1/B)}{q_1-q_2} \right)^{\pi_1} \left(\frac{\pi_2(B-1)}{q_1-q_2} \right)^{\pi_2}}{ } \\ \times \left(\frac{\pi_3(1-1/B)}{2(q_2-q_1/B)} \right)^{\pi_3} \end{array} \right)^{-\frac{1}{2}}, \tag{19}$$

where

$$A = \sqrt{\left((1-\pi_1)q_1 - (1-\pi_2)q_2 \right)^2 + 4\pi_1\pi_2 q_1 q_2}$$

and

$$B = \frac{\pi_1 q_1 \left((1-\pi_1)q_1 - (1+\pi_2)q_2 + A \right)}{\pi_2 q_2 \left((1+\pi_1)q_1 - (1-\pi_2)q_2 - A \right)}.$$

Given that the unconditional demands satisfy the conditions in Lemma 2, and the conditions in Lemma 3 are also satisfied, one can first derive unique conditional

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[11] For the assumed payoffs, there exists an effective risk free asset, which corresponds to a portfolio of assets $\mathbf{z}$ satisfying the following condition of having the same payoff in each state

$$\sum_{j=1}^{J} z_j \xi_{js} = 1 \quad \text{for all } s.$$

However the effective risk free asset is not used in this example.

[12] It may strike the reader as surprising that asset demand is independent of income I. This will be clarified in Example 4 below. Nevertheless, in Supplemental Appendix B.3 the associated conditional asset demand is shown to depend on period 2 income I_2 before normalization.

[13] It should be noted that the complete market unconditional demands generated by the same rationalizing utility derived at the end of this example, are considerably simpler than (17) - (19). The considerable increase in complexity of the demand functions is a common consequence of incomplete markets.

asset demand functions and then obtain the following inverse conditional demands (with the normalization $I_2 = 1$)

$$q_1 = \frac{(1 - \pi_2)\, z_1 + \pi_1 z_2}{(z_1 + z_2)\, z_1} \quad and \quad q_2 = \frac{(1 - \pi_1)\, z_2 + \pi_2 z_1}{(z_1 + z_2)\, z_2}. \tag{20}$$

Applying the approach outlined in Theorem 1 and its proof, it first can be verified that

$$\frac{\partial q_1}{\partial \pi_1} = \frac{z_2}{(z_1 + z_2)\, z_1} \quad and \quad \frac{\partial q_1}{\partial \pi_2} = -\frac{1}{z_1 + z_2}$$

Noticing that

$$c_{21} = z_1, \quad c_{22} = z_2 \quad and \quad c_{23} = \frac{1}{2}(z_1 + z_2),$$

we have

$$\rho_{21}(\mathbf{z}, \boldsymbol{\pi}) = \frac{\pi_1 \left(\frac{\partial q_1}{\partial \pi_1} - \sum_{l=1}^{2} \pi_l \frac{\partial q_1}{\partial \pi_l} \right)}{\xi_{11} - c_{21} q_1} = \frac{\pi_1 \left((1 - \pi_1)\, z_2 + \pi_2 z_1 \right)}{(1 - q_1 z_1)\,(z_1 + z_2)\, z_1}.$$

and

$$\rho_{22}(\mathbf{z}, \boldsymbol{\pi}) = \frac{\pi_2 \left(\frac{\partial q_1}{\partial \pi_2} - \sum_{l=1}^{2} \pi_l \frac{\partial q_1}{\partial \pi_l} \right)}{\xi_{12} - c_{22} q_1} = \frac{\pi_2 \left(\pi_1 z_2 + (1 - \pi_2)\, z_1 \right)}{q_1 z_2 \,(z_1 + z_2)\, z_1}.$$

It can be verified that

$$q_j(\mathbf{z}, \boldsymbol{\pi}) = \sum_{s=1}^{S} \xi_{js} \pi_s \rho_{2s}(\mathbf{z}, \boldsymbol{\pi})$$

and hence Condition (i) in Theorem 1 is satisfied. Next we verify Condition (ii). Inserting the inverse demands (20) into the following yields

$$\mathcal{M}_{12}(\mathbf{z}, \boldsymbol{\pi}) = \frac{\pi_2}{\pi_1} \frac{\rho_{21}(\mathbf{z}, \boldsymbol{\pi})}{\rho_{22}(\mathbf{z}, \boldsymbol{\pi})} = \frac{((1 - \pi_1)\, z_2 + \pi_2 z_1)\, q_1 z_2}{(\pi_1 z_2 + (1 - \pi_2)\, z_1)\,(1 - q_1 z_1)} = \frac{z_2}{z_1}.$$

Next assume $\boldsymbol{\zeta} \in \mathbb{R}^2$ satisfying

$$\zeta_1 \xi_{11} + \zeta_2 \xi_{21} = 0 \quad and \quad \zeta_1 \xi_{12} + \zeta_2 \xi_{22} = 0.$$

It follows that $\boldsymbol{\zeta} = (0, 0)$ implies that

$$\frac{d}{d\delta} \mathcal{M}_{12}(\mathbf{z} + \delta \boldsymbol{\zeta}, \boldsymbol{\pi})|_{\delta=0} = 0.$$

Clearly

$$D_{\boldsymbol{\pi}} \mathcal{M}_{12}(\mathbf{z}, \boldsymbol{\pi}) = \mathbf{0}.$$

Verifying that the same conclusion holds for $\mathcal{M}_{s,s'}(\mathbf{z}, \boldsymbol{\pi})$ with other s and s', Condition (ii) is satisfied. Finally, it can be verified that

$$\mathcal{M}_{12}(\mathbf{z}, \boldsymbol{\pi}) = \frac{c_{22}}{c_{21}} \quad and \quad \mathcal{M}_{23}(\mathbf{z}, \boldsymbol{\pi}) = \frac{c_{23}}{c_{22}}.$$

Therefore, if $c_{21} \geq \widetilde{c}_{22}$ and $c_{22} = \widetilde{c}_{23}$, then for any $\widetilde{\boldsymbol{\pi}} \in \Pi$,

$$\mathcal{M}_{12}(\widetilde{\mathbf{z}}, \boldsymbol{\pi}) \leq \mathcal{M}_{23}(\widetilde{\mathbf{z}}, \widetilde{\boldsymbol{\pi}}),$$

with the strict inequality if and only if $c_{21} > \widetilde{c}_{22}$. Verifying that the same conclusion holds for $\mathcal{M}_{s,s'}(\mathbf{z}, \boldsymbol{\pi})$ with other s and s', Condition (iii) is satisfied.

To see the connection to the complete market k-test (10), notice that

$$k_s = \frac{\pi_{2s}\rho_{21}}{\pi_{21}\rho_{2s}} = \mathcal{M}_{1s}(\mathbf{z}, \boldsymbol{\pi}) = \frac{c_{2s}}{c_{21}},$$

implying that $c_{2s} = f(c_{21}, k_s) = c_{21}k_s$.

Given that a V_{c_1} exists, one can in fact recover the form that rationalizes the given conditional asset demands. Following the logic in the discussion after Theorem 1, define the dummy variable $c_{2s} = \boldsymbol{\xi}_s \cdot (\mathbf{z} + \delta\boldsymbol{\zeta})$. Then to identify V_{c_1}, taking $\boldsymbol{\zeta} = (\zeta_1, 0)$, we have

$$V_{c_1}(c_{2s}) = \int_\delta \mathcal{M}_{s,s'}(\mathbf{z} + \delta\boldsymbol{\zeta}, \boldsymbol{\pi}) d\delta = \int_\delta \frac{z_2}{z_1 + \zeta_1\delta} d\delta = \frac{z_2}{\zeta_1} \ln(\xi_{11}(z_1 + \zeta_1\delta)),$$

which is defined up to a positive affine transformation. (See Supplemental Appendix B.3 for supporting calculations.)

4.2 Verifying Certainty Consumption Demands Generated by Ordinal Representation of Time Preferences

In the prior subsection, Theorem 1 provides necessary and sufficient conditions such that for each c_1-value asset demands $\mathbf{z}$ are generated by the maximization of an EU representation where each V_{c_1} is increasing and strictly concave. In order to go further and show that the given $(c_1, \mathbf{z})$ were generated by the maximization of KPS utility (3), we have to show that (i) there exists a well behaved representation U of time preferences and (ii) the collection $(U, \{V_{c_1}\})$ define a KPS utility that rationalizes the given unconditional demands. Several obstacles need to be overcome in order to demonstrate that (i) and (ii) are satisfied.

The main obstacle lies in the fact that the function $U(.,.)$ is defined over first period consumption and the certainty equivalent of risky second period consumption. Unfortunately, this cannot be observed locally – given the demand function at a price and a small open neighborhood around that price does not pin down the certainty equivalent. Theorem 1 allows us to uniquely recover the risk preference $\{V_{c_1}\}$ locally for the consumption values demanded - this generally does not suffice to determine the certainty equivalent since the certainty equivalent may lie outside the domain of the locally defined $\{V_{c_1}\}$.

It remains an open question to derive local conditions on demand. However it is clear that if we want to recover time preference utility U, we need to assume that risky asset demand is given on a large enough set of prices so that the certainty equivalent, $\widehat{c}_2$ can be uniquely recovered from conditional demand. In the following we make this assumption.

As in Subsection 4.1 above it turns out to be useful to work with inverse demand. We assume in the following that the conditions in Lemma 3 hold and we are given inverse demand functions for prices as a function of choices

$$p_1(c_1, \mathbf{z}, I, \boldsymbol{\pi}), \quad \mathbf{q}(c_1, \mathbf{z}, I, \boldsymbol{\pi})$$

on some open an connected set $\widehat{\mathcal{D}} \subset \mathbb{R}_+ \times \mathbb{R}^J \times \mathbb{R}_+ \times \Pi$. We also assume that we are given

$$\widehat{c}_2(c_1, z_1, \ldots, z_J, \boldsymbol{\pi}) = V_{c_1}^{-1}\left(\sum_{s=1}^{S} \pi_s V_{c_1}\left(\boldsymbol{\xi}_s \cdot \mathbf{z}\right)\right), \qquad (21)$$

for all $c_1, \mathbf{z}, \boldsymbol{\pi}$ with $(c_1, \mathbf{z}, I, \boldsymbol{\pi}) \in \widehat{\mathcal{D}}$ for some I.

The main technical problem for a demand test is then that we cannot define a natural "price" for the certainty equivalent $\widehat{c}_2$ – if we could, the natural test would require demand for period one consumption and the certainty equivalent to satisfy the Slutsky equation.

Instead, similar to above we derive the marginal rate of substitution $MRS = \frac{\partial U/\partial c_1}{\partial U/\partial \widehat{c}_2}$ from inverse demand and the certainty equivalent.

For asset $j = 1$ we can define[14]

$$f(c_1, \mathbf{z}, \boldsymbol{\pi}) = \left(\frac{p_1(c_1, \mathbf{z}, I, \boldsymbol{\pi})}{q_j(c_1, \mathbf{z}, I, \boldsymbol{\pi})} - \frac{\partial \widehat{c}_2(c_1, \mathbf{z}, \boldsymbol{\pi})/\partial c_1}{\partial \widehat{c}_2(c_1, \mathbf{z}, \boldsymbol{\pi})/\partial z_j}\right)\frac{\partial \widehat{c}_2(c_1, \mathbf{z}, \boldsymbol{\pi})}{\partial z_j}. \qquad (22)$$

We show below that if demand is generated by the KPS utility (3), $U(c_1, \widehat{c}_2)$, then the function f only depends on $\mathbf{z}$ through $\widehat{c}_2$, does not depend on $\boldsymbol{\pi}$ and one can express it as a continuously differentiable function of $(c_1, \widehat{c}_2)$ denoted by $\widetilde{f}(c_1, \widehat{c}_2)$. We then have that

$$\widetilde{f}(c_1, \widehat{c}_2) = MRS = \frac{\partial U/\partial c_1}{\partial U/\partial \widehat{c}_2}. \qquad (23)$$

and utility can be recovered through integration.

The following theorem states this formally and provides necessary and sufficient conditions.

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[14]Since p_1 and $\mathbf{q}$ are homogenous of degree 1 in I, the function f does not depend on I. Furthermore, f could have been defined using any other asset $j > 1$, as this turns out to be irrelevant.

Theorem 2 *Suppose $J \geq 2$ and $S \geq J$ and one is given twice continuously differentiable inverse demand and the certainty equivalent defined on some open and connected set $\widehat{\mathcal{D}}$. Further assume that Assumption 1 is satisfied.*[15] *Then there exists a unique twice continuously differentiable, strictly increasing, strictly quasiconcave, probability independent representation of time preferences $U(c_1, c_2) : C_1 \times C_2 \to \mathbb{R}$ rationalizing the certainty demand $(c_1, \widehat{c}_2)$ if and only if the function f defined in (22) satisfies*

(i) *For all $i, j = 1, \ldots, J$ and all $(c_1, \mathbf{z}, I, \boldsymbol{\pi}) \in \widehat{\mathcal{D}}$*

$$\frac{\partial f(c_1, \mathbf{z}, \boldsymbol{\pi})/\partial z_i}{\partial f(c_1, \mathbf{z}, \boldsymbol{\pi})/\partial z_j} = \frac{\partial \widehat{c}_2(c_1, \mathbf{z}, \boldsymbol{\pi})/\partial z_i}{\partial \widehat{c}_2(c_1, \mathbf{z}, \boldsymbol{\pi})/\partial z_j};$$

(ii) *For all $(c_1, \mathbf{z}, I, \boldsymbol{\pi}) \in \widehat{\mathcal{D}}$, $f(c_1, \mathbf{z}, \boldsymbol{\pi}) > 0$ and for all $j = 1, \ldots, J$,*

$$D_{\boldsymbol{\pi}} \left(\frac{\frac{\partial f(c_1, \mathbf{z}, \boldsymbol{\pi})}{\partial c_1}}{\frac{\partial f(c_1, \mathbf{z}, \boldsymbol{\pi})/\partial z_j}{\partial \widehat{c}_2(c_1, \mathbf{z}, \boldsymbol{\pi})/\partial z_j}} - \frac{\partial \widehat{c}_2(c_1, \mathbf{z}, \boldsymbol{\pi})}{\partial c_1} \right) = \mathbf{0};$$

(iii) *For all $(c_1, \mathbf{z}, I, \boldsymbol{\pi}) \in \widehat{\mathcal{D}}$ and for all $j = 1, \ldots, J$,*

$$\frac{\partial f(c_1, \mathbf{z}, \boldsymbol{\pi})/\partial z_j}{\partial \widehat{c}_2(c_1, \mathbf{z}, \boldsymbol{\pi})/\partial z_j} > \frac{\frac{\partial f(c_1, \mathbf{z}, \boldsymbol{\pi})}{\partial c_1} - \frac{\partial f(c_1, \mathbf{z}, \boldsymbol{\pi})/\partial z_j}{\partial \widehat{c}_2(c_1, \mathbf{z}, \boldsymbol{\pi})/\partial z_j} \frac{\partial \widehat{c}_2(c_1, \mathbf{z}, \boldsymbol{\pi})}{\partial c_1}}{f(c_1, \mathbf{z}, \boldsymbol{\pi})}. \tag{24}$$

Furthermore U can be uniquely recovered up to an increasing transformation.

To implement Theorem 2, first note that on the right hand side of eqn. (22), $p_1(c_1, \mathbf{z})/q_1(c_1, \mathbf{z})$ can be always calculated when inverse demands exist. Given $\{V_{c_1}\}$, $\partial \widehat{c}_2/\partial c_1$ and $\partial \widehat{c}_2/\partial z_1$ can be also computed. Therefore, one can always calculate the right hand side of eqn. (22) as a function of $(c_1, \mathbf{z})$. In the above theorem, Condition (i) guarantees that $f(c_1, z_1, ..., z_J)$ can be expressed as a function of $(c_1, \widehat{c}_2)$. Condition (ii) ensures that the corresponding utility function is strictly increasing and independent of probabilities and Condition (iii) ensures the strict quasiconcavity of the utility. Moreover, if the function $\widetilde{f}(c_1, \widehat{c}_2)$ exists, then as proved in Theorem 2, the following partial differential equation

$$\frac{\partial U}{\partial c_1} - \widetilde{f}(c_1, \widehat{c}_2) \frac{\partial U}{\partial \widehat{c}_2} = 0 \tag{25}$$

always has a unique solution $U(c_1, \widehat{c}_2)$, which is defined up to a increasing transformation. Note that eqn. (23) defines an indifference curve. As already noted by

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[15]It should be noted that the constraint $\widehat{c}_2(c_1)$ referred to in Assumption 1 is based on optimal conditional asset demand $\mathbf{z}(\mathbf{q}, \boldsymbol{\pi}, I_2 | c_1)$. However, the certainty equivalent function $\widehat{c}_2(c_1, \mathbf{z}, \boldsymbol{\pi})$ used in Conditions (i) - (iii) below is based on the definition (21) without optimization.

Samuelson (1950) in the two dimensional case, existence and uniqueness of a solution to (25) follows from the fact that $\widetilde{f}$ is Lipschitz and that therefore the ordinary differential equation $d\widehat{c}_2/dc_1 = \widetilde{f}$ together with the boundary condition $(c_1^\circ, \widehat{c}_2^\circ)$ has a unique solution describing the indifference curve $U(c_1, \widehat{c}_2) = U(c_1^\circ, \widehat{c}_2^\circ)$.

Remark 5 *Theorem 2 differs from the classic Hurwicz and Uzawa (1971) integrability result in several key ways. A different approach is required due in part to an inability to identify a price for $\widehat{c}_2$ since the constraint $\widehat{c}_2(c_1)$ referred to in Assumption 1 may not be linear in c_1. However since we only have two goods $(c_1, \widehat{c}_2)$, the existence of U can be always be derived from the MRS analogous to the conclusion that in the standard linear budget constraint setting Slutsky symmetry is always satisfied for two goods. Condition (iii) in Theorem 2 plays the role of negative semidefiniteness of the Slutsky matrix in Hurwicz and Uzawa (1971) in guaranteeing quasiconcavity of U.*

Remark 6 *If there is a risk free asset (without loss of generality, asset 1) and demand is defined globally, then following Polemarchakis and Selden (1984) we can simplify the test in Theorem 2 to the following condition. In this case, the certainty equivalent $\widehat{c}_2$ will have an implicit price p_2. A well behaved U exists if and only if the Slutsky matrix associated with the demand function (c_1, c_2) with respect to (p_1, p_2) derived by solving*

$$\frac{p_2}{p_1} = \left.\frac{q_1(c_1, z_1, ..., z_J, I)}{p_1(c_1, z_1, ..., z_J, I)}\right|_{z_1 = c_2, z_2 = ... = z_J = 0} \tag{26}$$

is negative semidefinite and symmetric, and if p_2/p_1 is probability independent.[16]

If Assumption 1 does not hold, then in order to guarantee that the KPS representation is well defined, we have to recover both $\{V_{c_1}\}$ and U and then directly verify that the KPS utility is strictly quasiconcave. Thus we have the following theorem.

Theorem 3 *Suppose $J \geq 2$ and $S \geq J$ and one is given twice continuously differentiable inverse demand and the certainty equivalent defined on some open and connected set $\widehat{\mathcal{D}}$. Further assume that Conditions (i), (ii) and (iii) in Theorem 2 are satisfied. Then there exists a unique twice continuously differentiable, strictly increasing, strictly quasiconcave, probability independent representation of time preferences $U(c_1, c_2) : C_1 \times C_2 \to \mathbb{R}$ rationalizing the certainty demand if and only*

---

[16]Since continuous differentiability implies local Lipschitz continuity, we obtain uniqueness of the representation U without having to assume Lipschitz continuity as in Mas-Colell (1977).

if the KPS utility (3) constructed from $\{V_{c_1}\}$ and the solution U to the following partial differential equation

$$\frac{\partial U}{\partial c_1} - \widetilde{f}(c_1, \widehat{c}_2) \frac{\partial U}{\partial \widehat{c}_2} = 0$$

is strictly quasiconcave.

Based on Theorems 1 and 2, we are guaranteed that a well behaved $(U, \{V_{c_1}\})$ set exists. Moreover, Assumption 1 ensures that the resulting KPS utility (3) satisfies strict quasiconcavity and the given demands maximize KPS utility.

Remark 7 *Polemarchakis and Selden (1984) assume the existence of a strictly quasiconcave $U(c_1, \widehat{c}_2)$ and discuss how to identify it. Although the identification process of $U(c_1, \widehat{c}_2)$ provided by Theorem 2 is similar to that in Polemarchakis and Selden (1984), Theorem 2 also gives an analytical test for the existence of a certainty $U(c_1, \widehat{c}_2)$. The recovery process for U can be quite complicated or sometimes not solvable analytically. (See Example B.3 in Supplemental Appendix B.4). However Theorem 2 provides tests for the existence of U, which can be readily verified without going through the recovery process.*

Although based on Theorems 1 and 2 a unique U exists, in general it is not possible to determine its analytic form. However in some cases utilizing Theorem 3, it is possible to recover U. To illustrate this, we consider the following extension of Example 3, where we demonstrated the existence of a rationalizing EU representation of risk preferences and recovered the specific NM index $\{V_{c_1}\}$. In the following extension, we first calculate the right hand side of eqn. (22). Then $f(c_1, z_1, z_2)$ is shown to satisfy Conditions (i), (ii) and (iii) in Theorem 2 implying the existence of a U. Finally $\widetilde{f}(c_1, \widehat{c}_2)$ is derived and used following Theorem 3 to solve for the representation of certainty time preferences. Thus, we establish the existence of a KPS utility and also identify the defining representations $\{V_{c_1}\}$ and U.

Example 4 *Deriving the inverse demand functions from the given demands (17) - (19) in Example 3, it can be verified that*

$$\frac{p_1(c_1, z_1, z_2, I)}{q_1(c_1, z_1, z_2, I)} = \frac{\exp\left(\pi_1 \ln z_1 + \pi_2 \ln z_2 + \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)\right)}{\frac{\pi_1}{z_1} + \frac{\pi_3}{z_1 + z_1}}.$$

Since $V_{c_1}(c_2) = \ln c_2$ is independent of c_1,[17]

$$\widehat{c}_2 = \exp\left(\pi_1 \ln z_1 + \pi_2 \ln z_2 + \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)\right), \tag{27}$$

---

[17]It should be noted that the period two certainty equivalent in eqn. (27) corresponds to the function (21) and is not based on the first stage portfolio optimization.

implying that

$$\frac{\partial \widehat{c}_2}{\partial c_1} = 0 \quad and$$

$$\frac{\partial \widehat{c}_2}{\partial z_1} = \exp\left(\pi_1 \ln z_1 + \pi_2 \ln z_2 + \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)\right)\left(\frac{\pi_1}{z_1} + \frac{\pi_3}{z_1 + z_1}\right).$$

Then it follows from eqn. (22) that

$$f(c_1, z_1, z_2) = \exp\left(2\left(\pi_1 \ln z_1 + \pi_2 \ln z_2 + \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)\right)\right).$$

Next we verify Conditions (i), (ii) and (iii) in Theorem 2. First,

$$\frac{\partial f/\partial z_1}{\partial f/\partial z_2} = \frac{\exp\left(2\left(\begin{array}{c}\pi_1 \ln z_1 + \pi_2 \ln z_2 + \\ \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)\end{array}\right)\right)\left(\frac{2\pi_1}{z_1} + \frac{\pi_3}{\frac{1}{2}z_1 + \frac{1}{2}z_2}\right)}{\exp\left(2\left(\begin{array}{c}\pi_1 \ln z_1 + \pi_2 \ln z_2 + \\ \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)\end{array}\right)\right)\left(\frac{2\pi_2}{z_2} + \frac{\pi_3}{\frac{1}{2}z_1 + \frac{1}{2}z_2}\right)}$$

$$= \frac{2\pi_1/z_1 + \pi_3/\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)}{2\pi_2/z_2 + \pi_3/\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)} = \frac{\partial \widehat{c}_2/\partial z_1}{\partial \widehat{c}_2/\partial z_2}$$

and hence Condition (i) is satisfied. Second, if $j = 1$ or 2,

$$D_\pi\left(\frac{\partial f}{\partial c_1}\bigg/\frac{\partial f/\partial z_j}{\partial \widehat{c}_2/\partial z_j} - \frac{\partial \widehat{c}_2}{\partial c_1}\right) = \mathbf{0}$$

and hence Condition (ii) holds. Third, if $j = 1$ or 2,

$$\frac{\partial f/\partial z_j}{\partial \widehat{c}_2/\partial z_j} = 2\exp\left(\pi_1 \ln z_1 + \pi_2 \ln z_2 + \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)\right)$$

$$> \frac{1}{f}\left(\frac{\partial f}{\partial c_1} - \frac{\partial f/\partial z_j}{\partial \widehat{c}_2/\partial z_j}\frac{\partial \widehat{c}_2}{\partial c_1}\right) = 0$$

and hence Condition (iii) is satisfied. Thus one can conclude that there exists a unique twice continuously differentiable, strictly increasing, strictly quasiconcave representation of time preferences $U(c_1, \widehat{c}_2)$ rationalizing the certainty demand. Next we solve for U directly to verify our conclusion. Actually, it is easy to see that $\widetilde{f}(c_1, \widehat{c}_2) = \widehat{c}_2^2$ and hence the partial differential equation (25) becomes

$$\frac{\partial U}{\partial c_1} - \widehat{c}_2^2 \frac{\partial U}{\partial \widehat{c}_2} = 0.$$

Solving this partial differential equation yields

$$U(c_1, \widehat{c}_2) = c_1 - \frac{1}{\widehat{c}_2}, \tag{28}$$

which is twice continuously differentiable, strictly increasing, strictly quasiconcave and probability independent. Thus the reason why the unconditional asset demands (18) - (19) that we started with in Example 3 are independent of income is the quasilinearity of the certainty utility, (28). Finally, since $V_{c_1}(c_2) = \ln c_2$, which is a member of HARA class, (U, V_{c_1}) represents a KPS representation.

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5 Revealed Preference Tests

Theorems 1 - 3 are fully consistent with the integrability analysis of Hurwicz and Uzawa (1971) and Mas-Colell (1978) and give a theoretical answer to the question of observable restrictions imposed by the maximization of KPS utility. But what if in an applied setting one only has finite data rather than full demand functions, can revealed preference analysis be applied to address the weaker question of whether a finite number of observations on prices, demands, and possibly probabilities, are consistent with the maximization of KPS preferences? Along the lines of Mas-Colell (1978), one can extend the analysis by considering the case where the observations become dense (in an appropriate sense as defined in Mas-Colell (1978)) and can recover the KPS utility function. However, the purpose of section is to consider the case where the data is finite.

For our non-parametric analysis we need to derive Afriat inequalities (Afriat 1967). These non-linear inequalities completely characterize choices which are consistent with utility maximization. Varian (1983) showed that in revealed preference analyses, the Afriat inequalities for asset demand can be used to test whether demand and price observations are consistent with the maximization of an EU representation in an incomplete market setting. Kubler (2004) derives the Afriat inequalities for asset demand under Kreps and Porteus (1978) utility for the case of risk preference independence. Unfortunately from a practical point of view his results are useless since the inequalities are nonlinear and he does not provide an efficient algorithm to solve them.

The Afriat inequalities can be efficiently solved if one can find equivalent, quantifier free conditions as in Kubler, Selden and Wei (2014), or if they can be reduced to a system of linear inequalities. In the latter case, interior-point methods or the simplex methods can be used to solve systems even with a very large number of observations.

In Lemma 1 below, we propose a traditional revealed preference test based on a system of nonlinear inequalities. To make the computational process efficient, one solution would be to design a lab experiment such that one can directly observe period two certainty equivalent consumption based on the subjects' responses. Then the utility maximizing choices would be characterized by the computationally simpler linear system of inequalities if one assumes that the overall utility function is quasiconcave.

We start our analysis with a general characterization of rationalizable observations. It is helpful to use the budget constraint to transform asset demands directly into contingent claim demands as the latter are the objects that impose

restrictions on our data set. We consider the general RPD case and assume that the NM index $V_{c_1}(c_{2s})$ is jointly concave in c_1 and c_{2s}.[18] We assume throughout that consumption choices are interior. Moreover, the certainty equivalent is not observable in the lab experiment and needs to fulfill certain conditions. Consider the case where there are N observations of prices and demands with $i \in \{1, ..., N\}$. For the following lemma and theorem below, we simplify notation by defining V_0^i, V_s^i and c_{20}^i, respectively to be the period two EU-value based on the ith observation of contingent claim demands, the utility-value $V_{c_1^i}(c_{2s}^i)$, the certainty equivalent associated with the vector $(c_{21}^i, \ldots, c_{2S}^i)$ and probabilities, $\boldsymbol{\pi}^i$. U^i denotes the two period certainty utility-value associated with the ith observation of period one consumption c_1^i and period two certainty equivalent c_{20}^i. Also, we use the following definitions

$$u_1^i = U_1(c_1^i, \widehat{c}_2^i), \;\; u_2^i = U_2(c_1^i, \widehat{c}_2^i), \;\; v_{1s}^i = \frac{\partial V_{c_1^i}(c_{2s}^i)}{\partial c_1^i}, \;\; v_{2s}^i = \frac{\partial V_{c_1^i}(c_{2s}^i)}{\partial c_{2s}^i}, \;\; V_s^i = V_{c_1^i}(c_{2s}^i).$$

We have the following lemma.

Lemma 1 *The data set $(c_1^i, \mathbf{c}_2^i, p_1^i, \mathbf{q}^i, \boldsymbol{\pi}^i)_{i=1}^N$ is consistent with maximization of the two period KPS utility (3) defined by a concave time preference utility U and the NM index $\{V_{c_1}\}$, that is jointly concave in first and second period consumption, if and only if for each $i = 1, \ldots, N$ there exist $V_s^i, v_{1s}^i, v_{2s}^i > 0$, $s = 0, \ldots, S$, $U^i, u_1^i > 0, u_2^i > 0, c_{20}^i > 0$, $i = 1, \ldots, N$, such that*

(i) *For all $i = 1, \ldots, N$,*

$$\frac{\mathbf{q}^i}{p_1^i} \left(u_1^i + u_2^i \left(\frac{1}{v_{20}^i} \sum_{s=1}^S \pi_s^i v_{1s}^i + \frac{v_{10}^i}{v_{20}^i} \right) \right) = u_2^i \frac{1}{v_{20}^i} \sum_{s=1}^S \pi_s^i \boldsymbol{\xi}_s v_{2s}^i;$$

(ii) *For all $i, j = 1, \ldots, N$, and all $s, t = 0, \ldots, S$,*

$$V_s^i - V_t^j \le \begin{pmatrix} v_{1t}^j \\ v_{2t}^j \end{pmatrix} \cdot \left(\begin{pmatrix} c_1^i \\ c_{2s}^i \end{pmatrix} - \begin{pmatrix} c_1^j \\ c_{2t}^j \end{pmatrix} \right);$$

---

[18]In principle, when making the two stage identification, one does not require $V_{c_1}(c_{2s})$ to be concave in c_1. But for the revealed preference tests, it is standard for the Afriat concavity inequality to impose concavity restrictions for both arguments. One interesting example of KPS preferences that satisfies the joint concavity assumption is where the NM index $V_{c_1}(c_{2s})$ takes the following internal habit formation form analogous to the EU representation of Constantinides (1990)

$$V_{c_1}(c_{2s}) = V(c_{2s} - \beta c_1),$$

where $V'' < 0$ and $\beta > 0$. In this case, if the Arrow-Pratt measure of absolute risk aversion $-V''(c_{2s} - \beta c_1)/V'(c_{2s} - \beta c_1)$ is decreasing in c_{2s} then it will be increasing in c_1. In other words, the more consumed in period one, the more risk averse the consumer becomes.

For all $i = 1, \ldots, N$

$$V_0^i = \sum_{s=1}^{S} \pi_s^i V_s^i;$$

(iii) *For all $i, j = 1, \ldots, N$*

$$U^i - U^j \leq \begin{pmatrix} u_1^j \\ u_2^j \end{pmatrix} \cdot \left(\begin{pmatrix} c_1^i \\ c_{20}^i \end{pmatrix} - \begin{pmatrix} c_1^j \\ c_{20}^j \end{pmatrix} \right);$$

(iv) *For all $i, j = 1, \ldots, N$*

$$U^i - U^j \leq \begin{pmatrix} u_1^j + u_2^j \left(\frac{1}{v_{20}^j} \sum_{s=1}^{S} \pi_s^j v_{1s}^j + \frac{v_{10}^j}{v_{20}^j} \right) \\ u_2^j \frac{1}{v_{20}^j} v_{21}^j \\ \vdots \\ u_2^j \frac{1}{v_{20}^j} v_{2S}^j \end{pmatrix} \cdot \left(\begin{pmatrix} c_1^i \\ c_{21}^i \\ \vdots \\ c_{2S}^i \end{pmatrix} - \begin{pmatrix} c_1^j \\ c_{21}^j \\ \vdots \\ c_{2S}^j \end{pmatrix} \right).$$

The revealed preference Conditions (i)–(iv) in the lemma roughly parallel the conditions in the infinitesimal Theorems 1, 2 and 3. Since we do not assume that conditional asset demand is observable (this would correspond to the special case where the c_1^i are identical across observations i), the conditions ensure both the existence of a risk averse conditional second period conditional NM index (in Condition (ii)) and the existence of a concave time preference utility (Condition (iii)). In addition since there is no simple analogue to Assumption 1 in this setting, we need to impose separately that overall utility is concave in first and second period consumption – Condition (iv) is therefore similar to the requirement in Theorem 3.

In the lemma we assume observations on demand, prices and probabilities. However unlike in our analysis in the infinitesimal case, it is irrelevant whether probabilities vary or not. If we consider the case where observations become dense, we clearly need to assume that probabilities vary in order to be able to recover preferences uniquely.

Clearly the inequalities in Lemma 1 are nonlinear and it seems unlikely that there is a tractable algorithm to solve them. One obstacle to the solution of the full system in Lemma 1 lies in the fact that the certainty equivalents $(c_{20}^i)_{i=1}^N$ are unknown. It would seem possible, however, that in a lab experiment one might be able to solicit the certainty equivalents directly from the subjects. One way of doing this would be to assume that only a risk free asset is available for trade. Then each subject could be asked to specify the risk free asset price such that the subject is indifferent between purchasing the risky asset portfolio

and the risk free asset.[19] Then, observations would consist of consumption, asset demands, prices, certainty equivalents c_{20}^i and supporting risk free asset prices $\widehat{q}_1^i$ for all $i = 1, \ldots, N$. Building on the two stage identification process employed in Theorems 1-3, we derive Theorem 4 (and Lemma 1 above). Condition (ii) of the theorem is associated with the first stage optimization with respect to $V_{c_1}(c_{2s})$. Condition (iii) is associated with the second stage optimization with respect to $U(c_1, c_2)$, where $\widehat{q}_1/p_1$ can be viewed as a pseudo price ratio corresponding to the first order condition in eqn. (A.9) in Appendix A.4. Conditions (i) and (iv) are associated with overall utility properties.

Theorem 4 *The data set* $(c_1^i, \mathbf{c}_2^i, c_{20}^i, p_1^i, \mathbf{q}^i, \widehat{q}_1^i, \boldsymbol{\pi}^i)_{i=1}^N$ *is consistent with maximization of the two period KPS utility (3) defined by a concave time preference utility* U *and the NM index* $\{V_{c_1}\}$, *that is jointly concave in first and second period consumption, if and only if for each* $i = 1, \ldots, N$ *there exist* $V_s^i, v_{1s}^i, v_{2s}^i > 0$, $s = 0, \ldots, S$, $U^i, u_1^i > 0$, $i = 1, \ldots, N$, *such that*

(i) *For all* $i = 1, \ldots, N$,

$$\frac{\mathbf{q}^i}{p_1^i} \left(\frac{p_1^i}{\widehat{q}_1^i} v_{20}^i + \sum_{s=1}^S \pi_s^i v_{1s}^i + v_{10}^i \right) = \sum_{s=1}^S \pi_s^i \boldsymbol{\xi}_s v_{2s}^i;$$

(ii) *For all* $i, j = 1, \ldots, N$, *and all* $s, t = 0, \ldots, S$,

$$V_s^i - V_t^j \leq \begin{pmatrix} v_{1t}^j \\ v_{2t}^j \end{pmatrix} \cdot \left(\begin{pmatrix} c_1^i \\ c_{2s}^i \end{pmatrix} - \begin{pmatrix} c_1^j \\ c_{2t}^j \end{pmatrix} \right);$$

For all $i = 1, \ldots, N$

$$V_0^i = \sum_{s=1}^S \pi_s^i V_s^i;$$

(iii) *For all* $i, j = 1, \ldots, N$,

$$U^i - U^j \leq \begin{pmatrix} u_1^j \\ u_1^j \frac{\widehat{q}_1^j}{p_1^j} \end{pmatrix} \cdot \left(\begin{pmatrix} c_1^i \\ c_{20}^i \end{pmatrix} - \begin{pmatrix} c_1^j \\ c_{20}^j \end{pmatrix} \right);$$

---

[19]Suppose one is given the observation $(c_1^i, \mathbf{c}_2^i, p_1^i, \mathbf{q}^i, \boldsymbol{\pi}^i)$ for a subject, then the individual could be asked to give the risk free asset price $\widehat{q}_1^i$ such that she is indifferent between purchasing the period 1 consumption and risky asset pair $(c_1^i, \mathbf{c}_2^i, p_1^i, \mathbf{q}^i, \boldsymbol{\pi}^i)$ and the period 1 consumption and risk free asset pair $(c_1^i, c_{20}^i, p_1^i, \widehat{q}_1^i)$, where

$$c_{20}^i = \frac{\mathbf{c}_2^i \cdot \mathbf{q}^i}{\widehat{q}_1^i}.$$

(iv) *For all $i, j = 1, \ldots, N$,*

$$
U^i - U^j \leq
\begin{pmatrix}
u_1^j + u_1^j \frac{\widehat{q}_1^j}{p_1^j} \left(\frac{1}{v_{20}^j} \sum_{s=1}^{S} \pi_s^j v_{1s}^j + \frac{v_{10}^j}{v_{20}^j} \right) \\
u_1^j \frac{\widehat{q}_1^j}{p_1^j} \frac{1}{v_{20}^j} v_{21}^j \\
\vdots \\
u_1^j \frac{\widehat{q}_1^j}{p_1^j} \frac{1}{v_{20}^j} v_{2S}^j
\end{pmatrix}
\cdot
\left(
\begin{pmatrix}
c_1^i \\
c_{21}^i \\
\vdots \\
c_{2S}^i
\end{pmatrix}
-
\begin{pmatrix}
c_1^j \\
c_{21}^j \\
\vdots \\
c_{2S}^j
\end{pmatrix}
\right).
$$

The proof of Theorem 4 follows from the observation that for the certainty equivalents to be supported, we must have for all $i = 1, \ldots, N$, $\widehat{q}_1^i U_1^i / p_1^i = U_2^i$.

Theorem 4 Conditions (i), (ii) and (iii) can be written as a linear system of inequalities that can be solved efficiently using methods from numerical linear algebra. The additional system in Condition (iv) that ensures overall concavity unfortunately is nonlinear and it remains an open question as to how to verify it efficiently. Clearly Conditions (i)-(iii) are necessary conditions and they are sufficient if one assumes that overall utility is quasiconcave.

6 Conclusion

In this paper, we give the necessary and sufficient integrability conditions such that asset demand functions can be rationalized by a KPS utility function in an incomplete market setting without requiring the existence of a risk free asset but assuming probabilities can be varied. Moreover, a means for recovering the corresponding KPS utility function is proposed if the above conditions hold. In order to implement tests of whether in a lab setting the demands of individual subjects are consistent with KPS preferences when markets are incomplete, the results cannot be applied directly. One can either resort to a revealed preference analysis (as is suggested in Section 5), or one can use our theoretical results to obtain asset demand systems. Unlike in the case of demand for commodities under certainty, no convenient functional forms are known for asset demand. Theorems 1-3 can be used in principle to develop such demand systems which then can be estimated from experimental data. This is a subject for further research.

A good deal of the existing lab results questioning EU maximization, at least of which we are aware, is based on lotteries. However in the evolving experimental research based on contingent claim (asset demand) non-parametric tests (e.g., Polisson, Quah, and Renou 2019), the case against EU seems less clear than the tests based on lotteries. Implementing our non-parametric tests based on asset demands rather than lotteries in an experimental setting such as Choi, et al. (2007) would seem to provide a useful addition to the existing literature. Finally, it would

also seem quite interesting and potentially feasible to test whether consumers are less likely to exhibit conditional asset demand behavior which is consistent with EU maximizing behavior in incomplete versus complete markets perhaps due to the extra complexity of more states than assets.

Appendix

A Proofs

A.1 Proof of Proposition 1

The optimal demand satisfies $dU\left(c_1, \widehat{c}_2\left(c_1\right)\right)/dc_1 = 0$, which implies that

$$\frac{\frac{\partial U(c_1, \widehat{c}_2)}{\partial c_1}}{\frac{\partial U(c_1, \widehat{c}_2)}{\partial \widehat{c}_2}} = -\frac{d\widehat{c}_2}{dc_1}.$$

Moreover, we have

$$\frac{d^2 U\left(c_1, \widehat{c}_2\left(c_1\right)\right)}{dc_1^2} = \frac{d}{dc_1}\left(\frac{\partial U\left(c_1, \widehat{c}_2\left(c_1\right)\right)}{\partial c_1} + \frac{\partial U\left(c_1, \widehat{c}_2\left(c_1\right)\right)}{\partial \widehat{c}_2}\frac{d\widehat{c}_2}{dc_1}\right)$$

$$= \frac{\partial^2 U\left(c_1, \widehat{c}_2\right)}{\partial c_1^2} + 2\frac{\partial^2 U\left(c_1, \widehat{c}_2\right)}{\partial c_1 \partial \widehat{c}_2}\frac{d\widehat{c}_2}{dc_1} +$$

$$\frac{\partial^2 U\left(c_1, \widehat{c}_2\right)}{\partial \widehat{c}_2^2}\left(\frac{d\widehat{c}_2}{dc_1}\right)^2 + \frac{\partial U\left(c_1, \widehat{c}_2\right)}{\partial \widehat{c}_2}\frac{d^2\widehat{c}_2}{dc_1^2}$$

$$= \frac{\partial^2 U\left(c_1, \widehat{c}_2\right)}{\partial c_1^2} - 2\frac{\partial^2 U\left(c_1, \widehat{c}_2\right)}{\partial c_1 \partial \widehat{c}_2}\frac{\frac{\partial U(c_1, \widehat{c}_2)}{\partial c_1}}{\frac{\partial U(c_1, \widehat{c}_2)}{\partial \widehat{c}_2}} +$$

$$\frac{\partial^2 U\left(c_1, \widehat{c}_2\right)}{\partial \widehat{c}_2^2}\left(\frac{\frac{\partial U(c_1, \widehat{c}_2)}{\partial c_1}}{\frac{\partial U(c_1, \widehat{c}_2)}{\partial \widehat{c}_2}}\right)^2 + \frac{\partial U\left(c_1, \widehat{c}_2\right)}{\partial \widehat{c}_2}\frac{d^2\widehat{c}_2}{dc_1^2}.$$

Since U is strictly increasing and strictly quasiconcave, we have

$$2\frac{\partial^2 U}{\partial c_1 \partial \widehat{c}_2}\frac{\partial U}{\partial c_1}\frac{\partial U}{\partial \widehat{c}_2} - \left(\frac{\partial U}{\partial c_1}\right)^2\frac{\partial^2 U}{\partial \widehat{c}_2^2} - \left(\frac{\partial U}{\partial \widehat{c}_2}\right)^2\frac{\partial^2 U}{\partial c_1^2} > 0,$$

implying that

$$\frac{\partial^2 U}{\partial c_1^2} - 2\frac{\partial^2 U}{\partial c_1 \partial \widehat{c}_2}\frac{\frac{\partial U}{\partial c_1}}{\frac{\partial U}{\partial \widehat{c}_2}} + \frac{\partial^2 U}{\partial \widehat{c}_2^2}\left(\frac{\frac{\partial U}{\partial c_1}}{\frac{\partial U}{\partial \widehat{c}_2}}\right)^2 < 0.$$

Since $d^2\widehat{c}_2/dc_1^2 < 0$, we have

$$\frac{d^2 U\left(c_1, \widehat{c}_2\left(c_1\right)\right)}{dc_1^2} < 0.$$

Thus the second order condition is satisfied and the optimal demand maximizes the utility function U. Next we want to argue that if this solution always exists, then the KPS representation must be strictly quasiconcave. For the KPS representation $U(c_1, \widehat{c}_2)$, the local maximum/minimum can be always derived from the two stage optimization (6) - (7), where the first order conditions are satisfied. If $U(c_1, \widehat{c}_2)$ is not strictly quasiconcave, then there exists at least one local extremum which is not a local maximum and hence will violate the second order condition. This contradicts our argument above that the second order condition is always satisfied and hence $U(c_1, \widehat{c}_2)$ must be strictly quasiconcave.

A.2 Existence of Conditional Demands and Inverse Demands

The following lemma provides a sufficient condition for the existence of unique twice continuously differentiable conditional asset demand functions. It will prove useful to denote the Jacobian matrix of derivatives of the function (c_1, I_2) with respect to (p_1, I) as

$$J_c = \frac{\partial(c_1, I_2)}{\partial(p_1, I)}. \tag{A.1}$$

Since (c_1, I_2) can be viewed as a function of $(p_1, I, \mathbf{q}, \boldsymbol{\pi})$, the nonsingularity of the Jacobian matrix (A.1) ensures that the inverse function exists, i.e., (p_1, I) can be uniquely expressed as functions of $(c_1, I_2, \mathbf{q}, \boldsymbol{\pi})$.[20] Substituting these functions into the unconditional demand $(z_1, ..., z_J)$, one obtains the conditional demand.

Lemma 2 *For given twice continuously differentiable demands $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and $\mathbf{z}(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$, if (i) $\forall(\mathbf{q}, \boldsymbol{\pi}) \in \mathcal{Q} \times \Pi$, $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and $I_2(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ are proper maps with respect to (p_1, I)[21] and (ii) $\forall(p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$, $\det J_c \neq 0$, then $\forall(p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$, there exists unique twice continuously differentiable conditional asset demand*

$$z_i(\mathbf{q}, \boldsymbol{\pi}, I_2 | c_1) = z_i(p_1, \mathbf{q}, \boldsymbol{\pi}, I) \quad (i = 1, ..., J). \tag{A.2}$$

Proof. Consider the following equations

$$c_1 = c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I) \quad \text{and} \quad I_2 = I - p_1 c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I). \tag{A.3}$$

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[20] The reason for including $\mathbf{q}$ and $\boldsymbol{\pi}$ as arguments in the inverse demand functions is to ensure that $\mathbf{q}$ and $\boldsymbol{\pi}$ will enter into the unconditional demand for assets $(z_1, ..., z_J)$ as parameters.

[21] A map between topological spaces is called proper if inverse images of compact subsets are compact. A special case which is more economically intuitive is desirability, i.e., when some price goes to zero, the corresponding demand goes to infinity.

If $\forall (p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$,

$$\det \frac{\partial (c_1, I_2)}{\partial (p_1, I)} \neq 0,$$

and the map $(c_1 (p_1, I), I_2 (p_1, I))$ is proper, then following Gordon (1972, Theorem B) and Wagstaff (1975, p. 524), (p_1, I) can be solved for as a unique twice continuously differentiable function of $(c_1, \mathbf{q}, \boldsymbol{\pi}, I_2)$ from the set of equations (A.3). Substituting

$$p_1 (c_1, \mathbf{q}, \boldsymbol{\pi}, I_2) \quad \text{and} \quad I (c_1, \mathbf{q}, \boldsymbol{\pi}, I_2)$$

into the unconditional asset demand $z_i (p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ $(i = 1, ..., J)$, $\forall (p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$, one obtains the unique continuously differentiable conditional demand $z_i (\mathbf{q}, \boldsymbol{\pi}, I_2 | c_1)$ $(i = 1, ..., J)$. ∎

Condition (ii) ensures the local existence of conditional demand and Condition (i) guarantees that conditional demand exists globally.[22]

Remark 8 *Under the assumptions made in Section 2, namely that the U and $\{V_{c_1}\}$ defining KPS utility being, respectively, strictly increasing and strictly quasiconcave and strictly increasing and strictly concave, and overall KPS utility being quasi-concave in $c_1, c_{21}, \ldots, c_{2S}$ then it is always possible to express the consumption-portfolio optimization in two stages and there will always be a unique conditional asset demand. Also, see the last paragraph in Appendix A.1.*

Next we consider the existence of the inverse demand function which maps asset demand, probabilities and income into a supporting price vector. Denote the Jacobian matrix of derivatives of the vector function $(c_1, \mathbf{z})$ with respect to $(p_1, \mathbf{q})$ as

$$J_u = \frac{\partial (c_1, z_1, ..., z_J)}{\partial (p_1, q_1, ..., q_J)}.$$

Then the following ensures the global existence of unique inverse demand.

Lemma 3 *Assume $c_1 (p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and $z (p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ are twice continuously differentiable over prices, probabilities and income. If (i) $\forall (\boldsymbol{\pi}, I) \in \Pi \times \mathcal{I}$, $c_1 (p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and $z (p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ are proper maps with respect to $(p_1, \mathbf{q})$ and (ii) $\forall (p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$, $\det J_u \neq 0$, then $\forall (p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$, there exists unique twice continuously differentiable inverse demands $p_1 (c_1, \mathbf{z}, \boldsymbol{\pi}, I)$ and $q_i (c_1, \mathbf{z}, \boldsymbol{\pi}, I)$ $(i = 1, ..., J)$.*

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[22]Without condition (i), we cannot ensure that the conditional demands exist in the full domain of all (no arbitrage) prices and probabilities. Then our result becomes local and we cannot guarantee the uniqueness of conditional demand functions. A similar argument applies to the discussion for Lemma 3.

Proof. Consider the following set of equations

$$c_1 = c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I) \quad \text{and} \quad \mathbf{z} = \mathbf{z}(p_1, \mathbf{q}, \boldsymbol{\pi}, I). \tag{A.4}$$

If $\forall (p_1, \mathbf{q}, \boldsymbol{\pi}, I) \in \mathcal{P} \times \mathcal{Q} \times \Pi \times \mathcal{I}$,

$$\det \frac{\partial(c_1, z_1, ..., z_J)}{\partial(p_1, q_1, ..., q_J)} \neq 0,$$

and $\forall(\boldsymbol{\pi}, I) \in \Pi \times \mathcal{I}$, $c_1(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ and $z(p_1, \mathbf{q}, \boldsymbol{\pi}, I)$ are proper maps with respect to $(p_1, \mathbf{q})$, then following Gordon (1972, Theorem B) and Wagstaff (1975, p. 524), $(p_1, \mathbf{q})$ can be solved for as a unique twice continuously differentiable function of $(c_1, \mathbf{z}, \boldsymbol{\pi}, I)$ from the set of equations (A.4). $\blacksquare$

Although Lemma 3 is stated in terms of the unconditional demands, one can prove that conditional asset demand, if it exists, inherits the properties (i) and (ii) as well as being twice continuous differentiability.[23] Therefore, if the conditions in Lemma 3 are satisfied, the conditional demand is also globally invertible. If the preferences are represented by a twice continuously differentiable KPS utility function, then Lemmas 2 and 3 are automatically satisfied. First, the maps are clearly proper. Second, the conditional asset demand exists and is twice continuously differentiable. This implies that $\det J_c \neq 0$ as in Lemma 2. Finally, it follows from the first order condition that inverse conditional asset demand also exists. Therefore, $\det J_u \neq 0$ as in Lemma 3.

A.3 Proof of Theorem 1

To prove necessity observe that, using $\pi_S = 1 - \sum_{s=1}^{S-1} \pi_s$ and given the inverse demand function $\mathbf{q}(\mathbf{z}, \boldsymbol{\pi}, I_2)$, differentiating

$$\sum_{s=1}^{S} \pi_s \xi_{js} V'_{c_1}(c_{2s}) = \mu q_j,$$

with respect to π_s $(s \in \{1, ..., S-1\})$, one obtains

$$\xi_{js} V'_{c_1}(c_{2s}) - \xi_{jS} V'_{c_1}(c_{2S}) = \frac{\partial \mu}{\partial \pi_s} q_j + \frac{\partial q_j}{\partial \pi_s} \mu. \tag{A.5}$$

Differentiating the budget constraint with respect to π_s $(s \in \{1, ..., S-1\})$, it follows that

$$\sum_{j=1}^{J} \frac{\partial q_j}{\partial \pi_s} z_j = 0. \tag{A.6}$$

---

[23] The inheritance of twice continuous differentiability is obvious. For a formal proof for the inheritance of properties (i) and (ii), refer to Kannai, Selden and Wei (2017, Claims 2 and 3 in the proof of Theorem 3).

Combining eqn. (A.5) with (A.6) yields

$$V'_{c_1}(c_{2s})\,c_{2s} - V'_{c_1}(c_{2S})\,c_{2S} = \frac{\partial \mu}{\partial \pi_s} \sum_{j=1}^{J} q_j z_j = \frac{\partial \mu}{\partial \pi_s}.$$

Substituting the above equation into (A.5) one obtains

$$\xi_{js} V'_{c_1}(c_{2s}) - \xi_{jS} V'_{c_1}(c_{2S}) = (V'_{c_1}(c_{2s})\,c_{2s} - V'_{c_1}(c_{2S})\,c_{2S})\frac{1}{q_j} + \frac{\partial q_j}{\partial \pi_s}\mu.$$

Defining

$$\rho_{2s} = \frac{\pi_s V'_{c_1}(c_{2s})}{\mu},$$

we obtain

$$\frac{\partial q_j}{\partial \pi_s} = \left(\xi_{js} - c_{2s}q_j\right)\frac{\rho_{2s}}{\pi_s} - \left(\xi_{jS} - c_{2S}q_j\right)\frac{\rho_{2S}}{\pi_S} \quad (s = 1,...,S-1). \qquad (A.7)$$

Using $\sum_s \rho_{2s} c_{2s} = 1$ and $\sum_s \xi_{js}\rho_{2s} = q_j$ and summing over all $s = 1,\ldots,S-1$, it follows that

$$\sum_{s=1}^{S-1} \pi_s \frac{\partial q_j}{\partial \pi_s} = q_j - \xi_{jS}\rho_{2S} - q_j(1 - \rho_{2S}c_{2S}) - (1 - \pi_S)\left(\xi_{jS} - c_{2S}q_j\right)\frac{\rho_{2S}}{\pi_S}$$

or ρ_{2S} as defined in equation (12). Substituting the above equation into (A.7) yields

$$\frac{\partial q_j}{\partial \pi_s} = \left(\xi_{js} - c_{2s}q_j\right)\frac{\rho_{2s}}{\pi_s} + \sum_{l=1}^{S-1} \pi_l \frac{\partial q_j}{\partial \pi_l},$$

implying that

$$\rho_{2s} = \frac{\pi_s \left(\frac{\partial q_j}{\partial \pi_s} - \sum_{l=1}^{S-1} \pi_l \frac{\partial q_j}{\partial \pi_l}\right)}{\xi_{js} - c_{2s}q_j} \quad (s = 1,...,S-1),$$

as defined in equation (13).

Hence $\mathcal{M}_{s,s'}$ denotes the marginal rate of substitution between consumption in s and consumption in s' and necessity of the three conditions in the theorem now follows directly. Condition (i) follows from the first order condition for optimality. Condition (ii) follows because utility is assumed to be separable across states and the NM index does not depend on probabilities. Condition (iii) follows from state independence and concavity of utility.

To prove sufficiency we prove that each ρ_{2s} as defined in equation (12) and in equation (13) can be written as the fraction of a continuous, positive valued and decreasing function that only depends on c_{2s} (call that function $\pi_s V_{c_1}$) and a continuous function that is the same for all $s = 1,\ldots S$. This proves the result

since Condition (i) in the theorem ensures that the first order conditions hold and Condition (iii) ensures that the utility is state independent. This also proves that the second part of the theorem, namely that utility can be uniquely recovered.

Since we assume that $\mathcal{D}$ is topologically connected, it follows that for each $s = 2, \ldots, S$ the set of consumptions

$$\mathcal{C}_s = \{(c_{21}, c_{2s}) \in \mathbb{R}_+^2 : \exists (\mathbf{z}, \boldsymbol{\pi}) \in \mathcal{D} \text{ with } c_{21} = \boldsymbol{\xi}_1 \cdot \mathbf{z}, c_{2s} = \boldsymbol{\xi}_s \cdot \mathbf{z}\}$$

is an open and connected set in $\mathbb{R}_+^2$. Therefore it suffices to normalize $V'_{c_1}(\widehat{c}_{21}) = 1$ for some value of $\widehat{c}_{21} = \boldsymbol{\xi}_1 \cdot \widehat{\mathbf{z}}$ that is in the projection of the set onto c_{21}. From this, with the assumption of openness we can recover $V_{c_1}(c_{2s})$ locally by integrating $\mathcal{M}_{s1}(\widehat{\mathbf{z}} + \boldsymbol{\zeta}\delta)$ with respect to $\delta \in \mathbb{R}$ for any $\boldsymbol{\zeta} \in \mathbb{R}^J$ that satisfies $\boldsymbol{\zeta} \cdot \boldsymbol{\xi}_1 = 0, \boldsymbol{\zeta} \cdot \boldsymbol{\xi}_s > 0$. Since $\mathcal{C}_s$ is connected one can find a path of these integrals to obtain $V_{c_1}(c_{2s})$ for all c_{2s} that are observed.

Condition (ii) in the theorem ensures that this function only depends on c_{2s} and Condition (iii) ensures that it is concave and continuous.

A.4   Proof of Theorem 2

First prove necessity. If U exists, then it follows from the first order condition that

$$\frac{p_1(c_1, z_1, ..., z_J)}{q_1(c_1, z_1, ..., z_J)} = \frac{\frac{\partial U}{\partial c_1} + \frac{\partial U}{\partial \widehat{c}_2}\frac{\partial \widehat{c}_2}{\partial c_1}}{\frac{\partial U}{\partial \widehat{c}_2}\frac{\partial \widehat{c}_2}{\partial z_1}} = \frac{1}{\frac{\partial \widehat{c}_2}{\partial z_1}\frac{\partial U}{\partial \widehat{c}_2}}\frac{\partial U}{\partial c_1} + \frac{\frac{\partial \widehat{c}_2}{\partial c_1}}{\frac{\partial \widehat{c}_2}{\partial z_1}}, \tag{A.8}$$

implying that

$$\frac{\frac{\partial U}{\partial c_1}}{\frac{\partial U}{\partial \widehat{c}_2}} = \frac{\partial \widehat{c}_2}{\partial z_1}\left(\frac{p_1(c_1, z_1, ..., z_J)}{q_1(c_1, z_1, ..., z_J)} - \frac{\frac{\partial \widehat{c}_2}{\partial c_1}}{\frac{\partial \widehat{c}_2}{\partial z_1}}\right), \tag{A.9}$$

which is a continuously differentiable positive function of $(c_1, \widehat{c}_2)$. Define this function as $\widetilde{f}(c_1, \widehat{c}_2)$. First,

$$\frac{\partial f/\partial z_i}{\partial f/\partial z_j} = \frac{\partial \widetilde{f}/\partial \widehat{c}_2 \cdot \partial \widehat{c}_2/\partial z_i}{\partial \widetilde{f}/\partial \widehat{c}_2 \cdot \partial \widehat{c}_2/\partial z_j} = \frac{\partial \widehat{c}_2/\partial z_i}{\partial \widehat{c}_2/\partial z_j},$$

which is Condition (i). Second, $\widetilde{f}(c_1, \widehat{c}_2)$ is independent of probabilities, which is equivalent to

$$D_{\boldsymbol{\pi}}\left(\frac{\partial \widetilde{f}/\partial c_1}{\partial \widetilde{f}/\partial \widehat{c}_2}\right) = \mathbf{0}. \tag{A.10}$$

To convert the above condition to a condition based on f, notice that

$$\frac{\partial f}{\partial z_j} = \frac{\partial \widetilde{f}}{\partial \widehat{c}_2}\frac{\partial \widehat{c}_2}{\partial z_j} \quad \text{and} \quad \frac{\partial f}{\partial c_1} = \frac{\partial \widetilde{f}}{\partial c_1} + \frac{\partial \widetilde{f}}{\partial \widehat{c}_2}\frac{\partial \widehat{c}_2}{\partial c_1}. \tag{A.11}$$

Therefore, eqn. (A.10) can be rewritten as

$$D_{\boldsymbol{\pi}} \left(\frac{\partial f}{\partial c_1} \bigg/ \frac{\partial f/\partial z_j}{\partial \widehat{c}_2/\partial z_j} - \frac{\partial \widehat{c}_2}{\partial c_1} \right) = \mathbf{0},$$

which is Condition (ii). Finally, it follows from Arrow and Enthoven (1961, p. 797 Theorem 5) that the strict quasiconcavity of U is equivalent to $\det B_H > 0$, where B_H is the bordered Hessian matrix

$$B_H = \begin{pmatrix} 0 & U_1 & U_2 \\ U_1 & U_{11} & U_{12} \\ U_2 & U_{12} & U_{22} \end{pmatrix}.$$

It can be verified that

$$\det B_H = 2U_1 U_2 U_{12} - U_2^2 U_{11} - U_1^2 U_{22}.$$

Moreover,

$$\frac{\partial \widetilde{f}(c_1, c_2)}{\partial c_2} = \frac{U_{12} U_2 - U_1 U_{22}}{U_2^2}$$

and

$$\frac{\partial \ln \widetilde{f}(c_1, c_2)}{\partial c_1} = \frac{\partial (\ln U_1 - \ln U_2)}{\partial c_1} = \frac{U_{11}}{U_1} - \frac{U_{12}}{U_2}.$$

Since $U_1, U_2 > 0$,

$$\frac{\partial \widetilde{f}(c_1, c_2)}{\partial c_2} > \frac{\partial \ln \widetilde{f}(c_1, c_2)}{\partial c_1} \tag{A.12}$$

is equivalent to

$$\det B_H = 2U_1 U_2 U_{12} - U_2^2 U_{11} - U_1^2 U_{22} > 0.$$

Using eqn. (A.11), condition (A.12) can be transformed into

$$\frac{\partial f/\partial z_j}{\partial \widehat{c}_2/\partial z_j} > \frac{1}{f} \left(\frac{\partial f}{\partial c_1} - \frac{\partial f/\partial z_j}{\partial \widehat{c}_2/\partial z_j} \frac{\partial \widehat{c}_2}{\partial c_1} \right),$$

which is Condition (iii). Next prove sufficiency. Since $\widehat{c}_2$ is a twice continuously differentiable function of $(z_1, ..., z_J)$ and

$$\partial \widehat{c}_2/\partial z_j \neq 0 \quad (j = 1, ..., J),$$

it follows from Mas-Colell (1977) that $\widehat{c}_2$ is a Lipschitzian and regular function. Consider the following set of partial differential equations

$$\frac{\partial \widehat{c}_2/\partial z_1}{\partial \widehat{c}_2/\partial z_j} = g_j (z_1, ..., z_J) \quad (j = 2, ..., J), \tag{A.13}$$

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which define the shape of $\widehat{c}_2 = const$ curves on the full $(z_1, ..., z_J)$ space. Combining Theorems 1 and 2 in Mas-Colell (1977), one can conclude that the solution to the set of partial differential equations (A.13) is $T \circ \widehat{c}_2$, where T is a monotone transformation.[24] Since $f(c_1, z_1, ..., z_J)$ also satisfies (A.13), we have

$$f(c_1, z_1, ..., z_J) = \widetilde{f}_{c_1}(\widehat{c}_2) = \widetilde{f}(c_1, \widehat{c}_2).$$

Since $\widetilde{f}(c_1, \widehat{c}_2)$ exists and is continuously differentiable, implying that the Lipschitz condition is satisfied, it follows from Schaeffer and Cain (2016, Theorems 3.2.2 and 3.3.4) that the ordinary differential equation

$$\frac{d\widehat{c}_2}{dc_1} = -\widetilde{f}(c_1, \widehat{c}_2) \tag{A.14}$$

has a unique solution, which can be denoted by $U(c_1, \widehat{c}_2) = 0$. Since eqn. (A.14) can be viewed as the characteristic equation of the following first order homogeneous linear partial differential equation

$$\frac{\partial U}{\partial c_1} - \widetilde{f}(c_1, \widehat{c}_2)\frac{\partial U}{\partial \widehat{c}_2} = 0, \tag{A.15}$$

and the ordinary differential equation (A.14) has a unique solution, it follows from Polyanin and Nazaikinskii (2016, pp. 1123-1124) that the partial differential equation (A.15) has a unique solution $T \circ U(c_1, \widehat{c}_2)$, where T is an increasing transformation. Since $\widetilde{f}(c_1, \widehat{c}_2)$ is a continuously differentiable positive function satisfying condition (A.12), $U(c_1, \widehat{c}_2)$ is twice continuously differentiable, strictly increasing and strictly quasiconcave. Since condition (A.10) holds, $\widetilde{f}(c_1, \widehat{c}_2)$ is independent of probabilities and hence U is also independent of probabilities. Moreover, it can be seen that the first order condition (A.8) is satisfied for this $U(c_1, \widehat{c}_2)$. Thus there exists a unique twice continuously differentiable, strictly increasing and strictly quasiconcave time preference representation $U(c_1, c_2) : C_1 \times C_2 \to \mathbb{R}$ rationalizing the certainty demand.

A.5 Proof of Lemma 1

For necessity, suppose demand is rationalized by a KPS utility function. The necessary and sufficient first order conditions can be written as follows (for simplicity the superscript i is not included)

$$\frac{\mathbf{q}}{p_1}\left(U_1(c_1, \widehat{c}_2) + U_2(c_1, \widehat{c}_2)\frac{\partial \widehat{c}_2}{\partial c_1}\right) = U_2(c_1, \widehat{c}_2)\left(V_{c_1}^{-1}\right)'\left(\sum_{s=1}^{S}\pi_s V_{c_1}(c_{2s})\right)\sum_{s=1}^{S}\pi_s \boldsymbol{\xi}_s V_{c_1}'(c_{2s}),$$

---

[24]Mas-Colell does not argue directly that the partial differential equation (A.13) has a unique solution. Instead, he proves that the preference relation is unique when the preference is Lipschitzian. We apply his conclusion by viewing (A.13) as the marginal rate of substitution of the time preference utility. Then combining his Theorems 1 and 2 gives us the desired result.

where

$$\frac{\partial \widehat{c}_2}{\partial c_1} = \frac{\partial V_{c_1}^{-1}\left(\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})\right)}{\partial c_1} + \left(V_{c_1}^{-1}\right)'\left(\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})\right)\sum_{s=1}^{S}\pi_s\frac{\partial V_{c_1}(c_{2s})}{\partial c_1}.$$

Since

$$\left(V_{c_1}^{-1}\right)'\left(\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})\right) = \frac{1}{V_{c_1}'(\widehat{c}_2)}$$

and

$$\frac{\partial V_{c_1}^{-1}\left(\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})\right)}{\partial c_1} = \frac{\partial V_{c_1}^{-1}\left(V_{c_1}(\widehat{c}_2)\right)}{\partial c_1} = -\left(V_{c_1}^{-1}\right)'\left(\sum_{s=1}^{S} \pi_s V_{c_1}(c_{2s})\right)\frac{\partial V_{c_1}(\widehat{c}_2)}{\partial c_1},$$

the equations in Lemma 1 Condition (i) follow from the definitions of u_1^i, u_2^i, v_{1s}^i and v_{2s}^i, where $v_{10}^i = -\partial V_{c_1^i}(\widehat{c}_2^i)/\partial c_1^i$. Conditions (ii) and (iii) follow from concavity of $\{V_{c_1}\}$ and the concavity of U. Condition (iv) follows from the overall concavity of the KPS utility function.

For sufficiency, as in Afriat (1967), Conditions (ii) and (iii) allow us to construct piecewise linear and concave function $\widetilde{U}(c_1, \widehat{c}_2)$ and $\widetilde{V}_{c_1}(c_2)$. This implies a piecewise linear $\widetilde{V}_{c_1}^{-1}$, so overall utility is piecewise linear. It is concave if for all c_1 and $\mathbf{c}_2$, the gradient inequalities in Condition (iv) are satisfied. However, since it is piecewise linear they must be satisfied everywhere if they are satisfied at all $c_1^i, \mathbf{c}_2^i$, $i = 1, \ldots, N$.

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B For Online Publication: Supplemental Appendix

B.1 Properties of KPS Utility and $(U, \{V_{c_1}\})$

Example B.1 *Assume that*

$$U(c_1, c_2) = c_2 - 0.001c_1 \quad and \quad V_{c_1}(c_2) = \sqrt{c_2 + 0.5c_1}.$$

Clearly $U(c_1, c_2)$ always decreases with c_1 and hence it is not well behaved in the full consumption space. Consider the simple case of two states with equal probabilities. Then we have

$$\widehat{c}_2 = \left(0.5\sqrt{c_{21} + 0.5c_1} + 0.5\sqrt{c_{22} + 0.5c_1}\right)^2 - 0.5c_1.$$

Therefore, the KPS representation is

$$U(c_1, \widehat{c}_2) = \widehat{c}_2 - 0.001c_1 = \left(0.5\sqrt{c_{21} + 0.5c_1} + 0.5\sqrt{c_{22} + 0.5c_1}\right)^2 - 0.501c_1.$$

It can be verified that

$$\frac{\partial U(c_1, \widehat{c}_2)}{\partial c_1} = \frac{0.125\sqrt{c_{21} + 0.5c_1}}{\sqrt{c_{22} + 0.5c_1}} + \frac{0.125\sqrt{c_{22} + 0.5c_1}}{\sqrt{c_{21} + 0.5c_1}} - 0.251.$$

This value is positive if

$$\frac{c_{21} + 0.5c_1}{c_{22} + 0.5c_1} > 1.1.$$

It can be also verified that

$$\frac{\partial^2 U(c_1, \widehat{c}_2)}{\partial c_1^2} = -\frac{(c_{21} - c_{22})^2}{4(c_1 + c_{21})^{\frac{3}{2}}(c_1 + c_{22})^{\frac{3}{2}}} < 0,$$

$$\frac{\partial^2 U(c_1, \widehat{c}_2)}{\partial c_{21}^2} = -\frac{\sqrt{2c_{22} + c_1}}{4(c_1 + c_{21})^{\frac{3}{2}}} < 0,$$

$$\frac{\partial^2 U(c_1, \widehat{c}_2)}{\partial c_{22}^2} = -\frac{\sqrt{2c_{21} + c_1}}{4(c_1 + c_{22})^{\frac{3}{2}}} < 0,$$

$$Det\begin{bmatrix} \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_1^2} & \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_1 \partial c_{21}} \\ \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_1 \partial c_{21}} & \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_{21}^2} \end{bmatrix} = 0,$$

$$Det\begin{bmatrix} \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_1^2} & \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_1 \partial c_{22}} \\ \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_1 \partial c_{22}} & \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_{22}^2} \end{bmatrix} = 0,$$

$$Det\begin{bmatrix} \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_{21}^2} & \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_{21} \partial c_{22}} \\ \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_{21} \partial c_{22}} & \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_{22}^2} \end{bmatrix} = 0$$

and

$$Det \begin{bmatrix} \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_1^2} & \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_1 \partial c_{21}} & \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_1 \partial c_{22}} \\ \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_1 \partial c_{21}} & \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_{21}^2} & \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_{21} \partial c_{22}} \\ \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_1 \partial c_{22}} & \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_{21} \partial c_{22}} & \frac{\partial^2 U(c_1,\widehat{c}_2)}{\partial c_{22}^2} \end{bmatrix} = 0.$$

Therefore, $U(c_1,\widehat{c}_2)$ is concave and is strictly increasing if

$$\frac{c_{21} + 0.5c_1}{c_{22} + 0.5c_1} > 1.1. \tag{B.1}$$

In other words, although $U(c_1, c_2)$ is not well defined over the full consumption space, $U(c_1,\widehat{c}_2)$ is well defined in the region where the inequality (B.1) holds.

Example B.2 *Assume that*

$$V(c_2) = -\exp(-c_2) - \exp(-2c_2).$$

It can be easily verified that $V' > 0$ and $V'' < 0$. Assume two states of nature, period two certainty equivalent is given by

$$\widehat{c}_2 = \ln \frac{\sqrt{1 + 4(\pi_1 \exp(-c_{21}) + \pi_1 \exp(-2c_{21}) + \pi_2 \exp(-c_{22}) + \pi_2 \exp(-2c_{22}))} + 1}{2(\pi_1 \exp(-c_{21}) + \pi_1 \exp(-2c_{21}) + \pi_2 \exp(-c_{22}) + \pi_2 \exp(-2c_{22}))}. \tag{B.2}$$

Assuming

$$U(c_1, c_2) = c_1^{0.99} + c_2,$$

then the KPS utility is given by

$$U(c_1, \widehat{c}_2) = c_1^{0.99} + \widehat{c}_2,$$

where $\widehat{c}_2$ is defined by eqn. (B.2), Although U is strictly increasing and strictly quasiconcave in $\mathbb{R}_+^2$ and V is strictly increasing and strictly concave in $\mathbb{R}_+$, numerically it can be verified that the KPS utility is not strictly quasiconcave in $(c_1, c_{21}, c_{22}) \in \{5\} \times \{1\} \times [0, 0.4]$.

B.2 A Revealed Preference Test for the General Case

Following the methods from Afriat (1967) and Lemma 1, it is easy to derive Afriat inequalities for the general utility over $c_1, c_{21}, \ldots, c2S$ that is increasing and concave. In particular a data set $(c_1^i, \mathbf{c}_2^i, p_1^i, \mathbf{q}^i, \boldsymbol{\pi}^i)_{i=1}^N$ is consistent with maximization of the two period increasing and concave utility if and only if for each $i = 1, \ldots, N$ there exist $U^i, u_1^i > 0, u_{2s}^i > 0, s = 1, \ldots, S, i = 1, \ldots, N$, such that

(i) For all $i = 1, \ldots, N$,

$$\frac{\mathbf{q}^i}{p_1^i} u_1^i = \sum_{s=1}^S \boldsymbol{\xi}_s u_{2s}^i;$$

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(ii) For all $i, j = 1, \ldots, N$

$$
U^i - U^j \leq \begin{pmatrix} u_1^j \\ u_{21}^j \\ \vdots \\ u_{2S}^j \end{pmatrix} \cdot \left(\begin{pmatrix} c_1^i \\ c_{21}^i \\ \vdots \\ c_{2S}^i \end{pmatrix} - \begin{pmatrix} c_1^j \\ c_{21}^j \\ \vdots \\ c_{2S}^j \end{pmatrix} \right).
$$

B.3 Supporting Calculations for Example 3

Based on the unconditional demand functions, it is possible to next derive the corresponding unique conditional asset demand functions. First, solve for the period 2 income

$$
I_2 = I - p_1 c_1 = \left(\frac{1}{p_1} \frac{\left(\frac{\pi_1(1-1/B)}{q_1-q_2} \right)^{\pi_1} \left(\frac{\pi_2(B-1)}{q_1-q_2} \right)^{\pi_2}}{\times \left(\frac{\pi_3(1-1/B)}{2(q_2-q_1/B)} \right)^{\pi_3}} \right)^{-\frac{1}{2}},
$$

from which we obtain

$$
p_1 = I_2^2 \left(\frac{\pi_1\left(1-1/B\right)}{q_1-q_2} \right)^{\pi_1} \left(\frac{\pi_2\left(B-1\right)}{q_1-q_2} \right)^{\pi_2} \left(\frac{\left(1-\pi_1-\pi_2\right)\left(1-1/B\right)}{2\left(q_2-q_1/B\right)} \right)^{\pi_3}.
$$

Substituting the above equation into the unconditional asset demand yields

$$
z_1 = \frac{\pi_1\left(1-1/B\right)I_2}{q_1-q_2} \quad \text{and} \quad z_2 = \frac{\pi_2\left(B-1\right)I_2}{q_1-q_2}. \tag{B.3}
$$

Deriving the inverse conditional asset demand functions from eqn. (B.3) yields

$$
q_1 = \frac{\left(1-\pi_2\right)z_1 + \pi_1 z_2}{\left(z_1+z_2\right)z_1}I_2 \quad \text{and} \quad q_2 = \frac{\left(1-\pi_1\right)z_2 + \pi_2 z_1}{\left(z_1+z_2\right)z_2}I_2.
$$

B.4 Example: Difficulties to Recover U

Example B.3 *Suppose one is given $(c_1, \mathbf{z})$ and recovers based on Theorem 1 that $V_{c_1}(c_2) = \ln c_2$, implying that*

$$
\widehat{c}_2 = \exp\left(\pi_1 \ln z_1 + \pi_2 \ln z_2 + \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2 \right) \right).
$$

Then suppose that the following is derived based on (22) prior to Theorem 2

$$
f\left(c_1, z_1, z_2\right) = \ln\left(3 + c_1 + \exp\left(\pi_1 \ln z_1 + \pi_2 \ln z_2 + \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2 \right) \right) \right).
$$

We verify Conditions (i), (ii) and (iii) in Theorem 2. First,

$$\frac{\partial f/\partial z_1}{\partial f/\partial z_2} = \frac{\left(3 + c_1 + \exp\left(\begin{array}{c}\pi_1 \ln z_1 + \pi_2 \ln z_2 + \\ \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)\end{array}\right)\right)\left(\frac{2\pi_1}{z_1} + \frac{\pi_3}{\frac{1}{2}z_1 + \frac{1}{2}z_2}\right)}{\left(3 + c_1 + \exp\left(\begin{array}{c}\pi_1 \ln z_1 + \pi_2 \ln z_2 + \\ \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)\end{array}\right)\right)\left(\frac{2\pi_2}{z_2} + \frac{\pi_3}{\frac{1}{2}z_1 + \frac{1}{2}z_2}\right)}$$

$$= \frac{2\pi_1/z_1 + \pi_3/\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)}{2\pi_2/z_2 + \pi_3/\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)} = \frac{\partial \widehat{c}_2/\partial z_1}{\partial \widehat{c}_2/\partial z_2}$$

and hence Condition (i) is satisfied. Second, if $j = 1$ or 2,

$$D_{\boldsymbol{\pi}}\left(\frac{\partial f}{\partial c_1} \bigg/ \frac{\partial f/\partial z_j}{\partial \widehat{c}_2/\partial z_j} - \frac{\partial \widehat{c}_2}{\partial c_1}\right) = D_{\boldsymbol{\pi}} 1 = \mathbf{0}$$

and hence Condition (ii) holds. Third, if $j = 1$ or 2,

$$\frac{\partial f/\partial z_j}{\partial \widehat{c}_2/\partial z_j} = \frac{1}{3 + c_1 + \exp\left(\pi_1 \ln z_1 + \pi_2 \ln z_2 + \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)\right)}$$

and

$$\frac{1}{f}\left(\frac{\partial f}{\partial c_1} - \frac{\partial f/\partial z_j}{\partial \widehat{c}_2/\partial z_j}\frac{\partial \widehat{c}_2}{\partial c_1}\right) = \frac{\left(\ln\left(3 + c_1 + \exp\left(\begin{array}{c}\pi_1 \ln z_1 + \pi_2 \ln z_2 + \\ \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)\end{array}\right)\right)\right)^{-1}}{3 + c_1 + \exp\left(\begin{array}{c}\pi_1 \ln z_1 + \pi_2 \ln z_2 + \\ \pi_3 \ln\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right)\end{array}\right)},$$

implying that

$$\frac{\partial f/\partial z_j}{\partial \widehat{c}_2/\partial z_j} > \frac{1}{f}\left(\frac{\partial f}{\partial c_1} - \frac{\partial f/\partial z_j}{\partial \widehat{c}_2/\partial z_j}\frac{\partial \widehat{c}_2}{\partial c_1}\right)$$

and hence Condition (iii) is satisfied. Thus there exists a unique twice continuously differentiable, strictly increasing, strictly quasiconcave representation of time preferences $U(c_1, \widehat{c}_2)$ rationalizing the certainty demand. Next we argue that U cannot be derived analytically. Actually, it is easy to see that

$$\widetilde{f}(c_1, \widehat{c}_2) = \ln(3 + c_1 + \widehat{c}_2).$$

But no analytical solution is known for the following partial differential equation[25]

$$\frac{\partial U}{\partial c_1} - \widetilde{f}(c_1, \widehat{c}_2)\frac{\partial U}{\partial \widehat{c}_2} = 0.$$

---

[25] To see the problem, note first that one only needs to solve the following ordinary differential equation

$$\frac{d\widehat{c}_2}{dc_1} = -\ln(3 + c_1 + \widehat{c}_2).$$

Defining $x = c_1 + \widehat{c}_2$, the above equation becomes $dx/dc_1 = -\ln(3 + x)$. However, $-\int \frac{dx}{\ln(3+x)}$ is not integratable analytically.