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Product Line Positioning without Market Information

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# Product Line Positioning without Market Information

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## Abstract

Traditional product line positioning and pricing models assume that firms have full information about the market demand and consumer preferences. In this paper we consider a setting where the firm has limited market information and tries to select its product positioning and pricing strategy optimally in light of this missing market information. To do this, we use competitive ratio and maximum regret criteria, which measure (respectively) the percentage and absolute loss relative to the benchmark case where the firm has full knowledge of customer preferences. We provide closed form solutions for the optimal product line positioning policies under limited market information for both the case of horizontal and vertically differentiated product lines. Our analysis, while stylized, provides insights into practices observed in many real world markets. In the case of horizontal differentiation, we show that the optimal decision for both the competitive ratio and maximum regret criteria is to position products at equal intervals in the attribute space and to price them identically; that is, to evenly span the product space with uniformly priced versions of the product. For the vertically differentiated case, we show that the optimal policy consists of offering some number of highest quality versions, which we call *nested quality offerings*. We show in this case that the more ambiguity there is over customers' taste for quality, the more product versions the firm should offer.

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# 1 Introduction

Positioning and pricing a product line is a central problem in marketing. (See Dobson and Yano [9], Green and Kieger [13], Krishnan and Ulrich [18], Ramdas [26].) In product line positioning, a firm must decide which different versions of a product to offer. Versions may differ in ways that appeal to the heterogeneous tastes of consumers, such as offering different colors, sizes or flavors - called *horizontal differentiation* - or they may differ in their level of quality and performance - called *vertical differentiation*. Product line positioning decisions are intimately related to the prices charged, too; because product versions closer to customers' preferences for variety or quality provide them more utility and can therefore support higher prices. At the same time, differentiated versions of the same product are almost always substitutes, so that lower prices for one version cannibalize demand from the other versions in the product line. Hence, positioning and pricing a product line should be coordinated decisions.

The difficulty in practice is that often little is known about customer preferences for various product attributes and the testing necessary to determine their preferences (e.g. conjoint analysis, (see Green and Srinivasan [14], Green, Carroll and Goldberg [12])) may be either too costly, too time consuming or otherwise unreliable, as in the case of an innovative product that customers have never experienced using. In such cases, firms must make product line positioning and pricing decisions with essentially no (or very minimal) information about customer preferences. This is the problem we address in this paper; it is a real and vexing problem and one that has received no attention to date in the research literature.

How does one analyze a decision problem with “no information”? And how does one do so in a way that is both practically meaningful and conceptually sound?

One approach is to use a max-min criterion. That is, one treats the unknown information as being controlled by an adversary (“nature”) who seeks to construct instances that minimize the firm's profits for any given decision it makes. The firm then seeks policies (product positions and prices in our case) that produce the maximum profit against this adversary. The difficulty with this approach is that it can lead to decisions that are driven by excessively extreme and pessimistic scenarios about the unknown information.

*Robust optimization* is another approach to lack of information; it attempts to reduce the inherent conservatism of the max-min criterion by imposing constraints on the adversary. The approach assumes that an adversary controls certain parameters of the problem (e.g. customer valuations), but that these parameters are constrained to an “uncertainty set” in which the total deviation of the actual parameter values from their nominal values is upper bounded by a constant, interpreted as an “uncertainty budget”. Depending on

the norm used to measure deviations, the constraints are either ellipsoidal (see BenTal and Nemirovski [3], and ElGhaoui and Lebret [10]), or polyhedral (see Bertsimas and Sim [5], as well as Bertsimas and Thiele [6], Perakis and Sood [25]). The idea is that it is not usually realistic that all parameters will simultaneously be at their “worst possible” values and the uncertainty budget in effect defines how many values may be extreme ones. The approach is appealing computationally, but it suffers from the defect that it becomes something of an ad hoc choice to select the magnitude of the uncertainty budget - or said differently, selecting the uncertainty budget amounts to a statement (and not an easy-to-interpret one at that) about the degree of uncertainty in the information. More to the point, robust optimization is primarily a computational formalism and doesn’t naturally lend itself to finding analytic solutions that provide structural insights. In a similar vein, Lim and Shanthikumar [21] recently studied a robust dynamic pricing problem that imposed a relative entropy constraint to bound the distance of the probabilistic model uncertainty from a nominal distribution.

A second approach to overcome the excessive conservatism of max-min formulations is to measure performance relative to a fully-informed decision maker. That is, this approach assumes what we truly care about is reducing the gap between the performance we achieve with *no information* relative to what we could have achieved with *full information*. It has the advantage of filtering out instances that produce bad outcomes no matter what the policy (e.g. consumers who have zero utility for any version we create); clearly we wouldn’t want to “blame” the decision maker for a bad outcome in such cases and using relative performance measures reflects this sensibility.

Two such measures are the *competitive ratio* and *maximum regret* criteria. Specifically in our case, suppose the firm first selects a positioning and pricing policy  $\pi$ , and the adversary selects a worst-case distribution for the unknown consumer preferences,  $F(\cdot)$ . (These constructs will be made precise below.) Let  $R(\pi, F)$  be the *actual revenue earned* for the pair of actions  $\pi$  and  $F(\cdot)$ , and  $R(\pi^*(F), F)$  be the maximum revenue the firm *could have extracted* if it knew the selected distribution  $F(\cdot)$ ; i.e.,  $\pi^*(F)$  denotes the optimal policy if  $F$  was known. Then, the competitive ratio criterion formulation is

$$c = \max_{\pi} \min_F \frac{R(\pi, F)}{R(\pi^*(F), F)},$$

while the maximum regret criterion is

$$r = \min_{\pi} \max_F [R(\pi^*(F), F) - R(\pi, F)].$$

That is, the firm strives to either minimize the relative or the absolute difference from the theoretical

maximum revenues it could have extracted in the full-information case. To contrast, the max-min criterion takes the form

$$\max_{\pi} \min_F R(\pi, F) .$$

The use of these two criteria implicitly constrains the actions of the adversary and typically results in more intuitive policy recommendations. These criteria have been used extensively in the computer science literature and have recently been applied in pricing and operations management problems. Specifically, Ball and Queyranne [2] used a competitive ratio criterion for a single-resource capacity allocation problem, while Bergemann and Schlag [4] and Perakis and Roels [24] adopted the regret criterion to study the monopolist pricing and the newsvendor problems, respectively. Relative performance measures retain the feature of being analyzable and yielding good structural insight while constituting a more realistic notion of “worst-case performance”. Still, relative worst-case performance is largely a stylized formalism and the model and results of our work should be interpreted in this light.

We use these competitive ratio and maximum regret criteria to study product line positioning and pricing decisions in a setting where the firm does not have information about the distribution that characterizes the consumers’ preferences over product variety (in the case of horizontal differentiation) or quality (in the case of vertical differentiation).

Our approach to modeling horizontal product variety is based on classical locational models, in which each version of the product is mapped into a location in a product attribute space and each consumer’s location represents her most preferred values of the product attributes (her “ideal point”). The distance between a customer’s ideal point and the locations of different product versions provides a measure of their disutility from consuming a less-than-ideal product. Given a distribution of customer locations, one can then analyze issues of optimal product line design, positioning and pricing. Of course in our case, this distributional information is what is unknown.

The seminal paper in this area belongs to Hotelling [15], who considers a linear one-dimensional attribute space. Vickrey [29] is credited for formulating the circular model, which gets rid of the boundary effects of the linear model. In 1966, Lancaster [19] offered an alternative approach in which the utility of a choice is determined by a parametric function of the consumer characteristics and product attributes. This theory significantly increased interest in locational product differentiation models. Salop [27] provides the most complete treatment of this setup in his seminal paper for monopolistic competition, where he shows a symmetric equilibrium along the product space (i.e. the circle) exists with equal prices, and all the consumers in the market (i.e. around the circle) are served. Several extensions of this basic game theoretical model has

been made covering different scenarios of the market structure. Further interested reader is referred to the excellent survey paper by Lancaster [20].

Our work is also closely related to the product line positioning literature. The product line positioning problem itself is a complex multi-facet problem touching on marketing concerns (consumer utility and product attributes modeling, pricing and positioning to maximize market share or revenues, etc.), operations concerns (costs of production, configuration of supply chains for variety, etc.), and engineering design concerns (product architecture, shared versus unique components, etc.). There are several survey papers structuring the literature around these three areas. (Again, see Dobson and Yano [9], Green and Kieger [13], Krishnan and Ulrich [18], Ramdas [26].) Prior work in the literature can be roughly characterized as follows: most directly uses or is inspired by the utility framework of spatial differentiation models. Generally, models consider a finite number of possible product offerings and the aim is to choose which of these to offer with the objective of maximizing market share, revenues, or profits. However, there are a number of papers in operations literature that solely consider cost minimization or supply chain configuration. It is assumed in all this literature that customer preferences/valuations for each possible product offering and the potential market size at each such point are known or can be estimated using conjoint analysis. Moreover, most of the models do not treat price as a decision variable, though several treat it as part of the attribute space. The corresponding problems are usually mixed integer programs and exact analytical solutions are generally not available. Algorithmic solutions usually consist of greedy or simulation based heuristics (e.g. simulated annealing).

Our work is aligned more closely with the marketing view of the problem explained above. We base our model on the classical locational attributes and linear utility framework as above. However, unlike the product line positioning literature above, we try to avoid additional assumptions on top of this basic set up. Specifically, we allow for uncertainty/ambiguity in customer valuations, and are interested in analyzing the effects of this uncertainty on optimal policies. Our objective is to maximize revenues; we do not consider production costs (though variable costs can be added without much change). Further, we only consider a single dimension of variety for simplicity. Unlike most of the literature, we treat both variety and price as continuous decision variables; hence, an infinite number of possible product offerings/locations is allowed. While these assumptions are highly stylized, we are able to derive analytical closed-form solutions for optimal positioning and pricing policies which provide structural insights and permit sensitivity analysis with respect to problem primitives.

As mentioned, we also analyze the case of vertical positioning, in which all consumers have a common ordering of their preferences for different versions, i.e. every consumer agrees that a certain version  $i$  is better

(in terms of what literature denotes as “quality”) than another version  $j$ . Our model of vertical positioning is also classical, and is based on the early influential work of Mussa and Rosen [23], who introduce a framework that makes use of a *linear utility function of quality*. This utility framework has been widely adopted by both following researchers and practitioners. In their seminal paper, Shaked and Sutton [28] use this model to show that firms can sustain positive profits even under price competition, contrary to the classical “zero-profit” result of Bertrand price competition, when they are allowed to choose the quality levels of their products under monopolistic competition. Assuming convex costs as a function of product quality, Mussa and Rosen, and later Maskin and Riley [22] and Kim [17], show that a firm can increase profits by offering vertically differentiated products to customers with heterogeneous tastes for quality.

These models were widely adopted as they successfully explained product differentiation seen in the traditional manufactured goods. However, the theory fell short of explaining product differentiation, or “versioning”, for information goods. Information goods are particularly relevant in our case since most production costs are sunk development costs and thus profits maximization coincides with revenue maximization, which is the case we analyze.

Several early works (see Bhargava and Choudhary [8], Jones and Mendelson [16], Acharyya [1]) use the linear utility function for quality and conclude that there are no gains to product differentiation for information goods and the product should be produced and supplied only at the highest quality levels to prevent cannibalization. Yet empirical evidence [11] shows that information goods have high levels of quality differentiation. Only the highest quality is produced, as predicted by the above works, but the product is then degraded afterwards and offered at varying levels of quality in the market. In order to explain this practice, several models with different and more complex utility functions have been proposed, but none has been as widely accepted as the linear utility function of quality. Finally, Bhargava and Choudhary [7] demonstrated that previous insights about suboptimality of vertical differentiation are not robust, and showed that quality differentiation does indeed occur under a general class of utility functions and sunk costs assumptions. They conclude that the higher the heterogeneity in consumers’ taste for quality, the more likely vertical differentiation is and that the highest quality version is always offered in the product bundle. Other popular marketing texts (see for example, Lilien et al. [?]) also conclude that versioning is attractive when consumers are sufficiently heterogeneous.

The remainder of the paper is organized as follows: in Section 2, we consider the horizontal positioning of a monopolist’s product line using Salop’s classical circular model of spatial differentiation. For both competitive ratio and maximum regret criteria, we first derive the optimal pricing policy for a given product line with fixed attributes. We show that the optimal price vector depends on the maximum attribute difference

among neighboring products. We also show that the worst-case performance decreases (i.e. the competitive ratio decreases or the regret increases) as the consumers’ sensitivity for differentiating attribute increases, consumers’ nominal valuations for their ideal product decreases, or the maximum attribute difference among neighboring products increases. Then, we show that for both criteria, the optimal positioning and pricing policy is to position products at equal intervals in the attribute space and to price them identically; that is, to evenly span the product space with uniformly priced versions of the product. This type of positioning of a product line is frequently observed in practice and our results provide one explanation for why it occurs.

In Section 3, we study the vertical positioning of a product line using the linear utility of quality framework of Mussa and Rosen. Again, we first derive the optimal pricing policy for a fixed number of versions with given quality levels. Furthermore, consistent with Bhargava and Choudhary’s results, we show that the optimal policy consists of offering some number of highest quality versions, which we call *nested quality offerings*; and that the number of versions offered increase as the heterogeneity and ambiguity in consumers’ taste for quality increases. This result is consistent with empirical evidence and conclusions about versioning in information goods in literature. (See again [11] and [?].)

For both the case of horizontal and vertical positioning, we solve for the optimal policy in closed form, which provides a detailed description of the optimal pricing and positioning strategies for the firm and the resulting worst-case revenue loss. Our analysis, while stylized, provides insights into practices observed in many real world markets. In order to provide a continuous exposition, proofs of all results are relegated to the Appendix.

## 2 Horizontal Product Line Positioning

We first look at the case of horizontal positioning. We begin by defining the model and the formulation of the basic competitive ratio and maximum regret problems. We then analyze each of these problems in turn.

### 2.1 Model

We use Salop’s classical model of spatial differentiation, also known as the *circular Hotelling model*. In it, versions of the product are differentiated along a single attribute, represented by a location on the unit circle (*attribute space*). Each version  $j$  is represented by a location  $l_j$  in this attribute space.

We assume the firm can offer at most  $K$  versions of a product. Such a limitation could, for example, reflect



minimum efficient economies of scale needed for production. We do not incorporate fixed or variable costs of producing up to this upper limit of  $K$  versions; rather, our analysis concentrates on revenue maximization. However, it is easy to incorporate linear variable costs in our analysis simply by rescaling the nominal valuations for products. If one wanted to analyze the optimal level of variety given such costs, it is possible to solve our model for different values of  $K$  and use the solutions to determine if changes in revenue off-set the costs of production and warrant increasing or decreasing the level of variety. But again in what follows below, we shall assume  $K$  is fixed.

Consumer preferences are also represented by locations on the unit circle. A randomly selected customer location is denoted  $l$  with distribution  $F(\cdot)$ . Our key assumption is that this preference distribution  $F(\cdot)$  is unknown to the firm and it must therefore make its positioning and pricing decisions accounting for this missing information.

Consumers have a common deterministic nominal valuation,  $v$ , for the product, which represents the maximum amount they would be willing to pay if the product were at their preferred location. The market size (“number of customers”), denoted  $\Lambda$ , is assumed fixed and continuous. Hence, the distribution of valuations is the only missing information to the firm in our model. We do this primarily to focus on understanding preference uncertainty and secondarily to simplify the analysis. Still, in many retail contexts the assumptions are not unreasonable. Retailers often do have a sense of the overall market size from past experience even though the preferences for individual version is highly uncertain (e.g. different styles of ski jackets). One can see evidence of this in a typical retail aggregate product planning process, where budgets are allocated to categories of products based on past sales and the primary uncertainty is how to allocate that budget to different products. Assuming the valuation  $v$  is known is arguably less realistic and certainly affects the pricing decision. But again this valuation could reflect a historical “price point” for the given category of products that is relatively well understood from historical experience even though the preference for individual version is unknown.

Consumers incur a disutility that depends on the distance the product location to their preferred location, given by  $\theta|l-l_j|$  where  $\theta$  is a known disutility coefficient, which can be interpreted as *the consumers’ sensitivity for variety* and  $|l-l_j|$  denotes the distance between  $l$  and  $l_j$  on the unit circle. Therefore the net utility for a consumer of type  $l$  buying a product located at  $l_j$  offered at price  $p_j$  is given by  $u(l, j) \stackrel{def}{=} v - \theta|l-l_j| - p_j$ . Note that the higher the nominal valuation, the lower the consumers’ sensitivity for variety, and/or the closer the product is to the consumer’s ideal point, the more they are willing to pay for it. For the distance between two versions to be equal to the directional distance, either clockwise or counter-clockwise, we assume that  $K \geq 3$  and that the directional distance between any two versions is not greater than 0.5. This is really

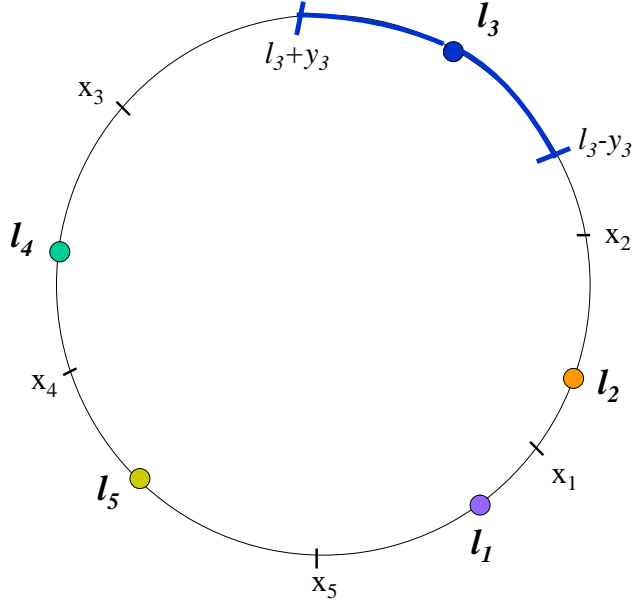


Figure 1: Horizontal Differentiation on a Unit Circle

a “cosmetic” assumption: all the results in this section can be extended to the general case, but it would require additional notation and case-wise definitions of all functions.

A consumer of type  $l$  between  $l_j$  and  $l_{j+1}$  prefers  $l_j$  if  $v - \theta(l - l_j) - p_j \geq v - \theta(l_{j+1} - l) - p_{j+1}$ ; that is if the  $j$ -th *indifference point*  $x_j \stackrel{def}{=} \frac{p_{j+1} - p_j}{2\theta} + \frac{l_{j+1} + l_j}{2} \geq l$ . Similarly a consumer with type  $l$  between  $l_j$  and  $l_{j-1}$  prefers  $l_j$  if  $v - \theta(l - l_{j-1}) - p_{j-1} \leq v - \theta(l_j - l) - p_j$ ; that is if  $x_{j-1} = \frac{p_j - p_{j-1}}{2\theta} + \frac{l_j + l_{j-1}}{2} \leq l$ . As a result, a consumer of type  $l \in [x_{j-1}, x_j]$  prefers product  $j$  to its neighboring products. Without loss of generality we denote the location of first version to be  $l_1 = 0$  (thus, also  $l_1 = 1$  depending on the context). Consequently, in the remainder of the section whenever  $j = K$ ,  $j + 1$  denotes 1; and  $x_1 := \frac{p_2 - p_1}{2\theta} + \frac{l_2}{2}$  and  $x_K := \frac{p_1 - p_K}{2\theta} + \frac{1 + l_K}{2}$  above. Observe that the coefficient  $\theta$  effects the aggregate demand in the sense that higher values of  $\theta$  make all consumers less willing to pay for each version irrespective of its location and price, reducing the over all demand faced by the firm.

Each consumer has a most preferred product, which yields a utility of  $u(l) \stackrel{def}{=} \max_j \{u(l, j)\}$  for a given price vector  $p$ , and which is purchased iff  $u(l)$  is nonnegative. Note that if a version with a relatively low price is placed close enough to a version with a high price, the low price version may be preferred over the high price version by every consumer location on the unit circle. In this case, we say that the low price version *dominates* the high price version.

## 2.2 Product line positioning and pricing decision

Having introduced the model, let us describe the firm’s decision making process. First, the firm chooses how many of the  $K$  possible versions to offer, the positions of these versions, i.e. vector  $l = \{l_j\}_{j=1..K}$ , on the unit circle, and their prices, i.e. the vector  $p$ . Then we evaluate this decision against the “worst-case” outcome of customer preferences. To do so, we imagine an adversary (nature) that selects a worst-case preference distribution  $F(\cdot)$  in order to minimize the competitive ratio or maximum regret. Thus, the competitive ratio problem is

$$\max_{p,l} \min_F \frac{R(l,p;F)}{R(l^*(F),p^*(F);F)},$$

where  $(l^*(F),p^*(F))$  is the revenue maximizing strategy of the firm given the distribution  $F(\cdot)$ . Here we seek a policy which maximizes our revenue as a percentage of the full-information revenue under worst-case outcomes. For example, a competitive ratio of  $1/2$  would say we are guaranteed to get at least 50% of the revenue we could have gotten had we known the exact customer preference distribution. Similarly, the maximum regret problem is

$$\min_p \max_F [R(l^*(F),p^*(F);F) - R(l,p;F)].$$

Here we seek a policy which minimizes the difference between our revenues and the full-information revenue under worst-case outcomes. For example, a maximum regret value of \$1,000 would say that the revenue we achieve with our policy is guaranteed to be no more than \$1,000 below the revenue we would have obtained with exact information on the customer preference distribution.

Our goal is to specify the optimal pricing and product line positioning strategy of the firm for these two relative performance objectives. Specifically, how many versions should the firm offer? Where should these versions be positioned on the unit circle? What prices should we charge for the different versions? We also would like to explain how the answers depend on the nominal valuation, the disutility coefficient, the maximum number of products, and their relative positions. We begin with the competitive ratio case.

## 2.3 Competitive Ratio

We begin by assuming a fixed number of versions,  $K$ , is offered with fixed attributes (locations on the unit circle), denoted by vector  $l \in [0,1]^K$ , and consider the pricing decision of the firm. We say a product  $i$  at

price  $p_i$  *dominates* product  $j$  at price  $p_j$  if every customer (regardless of location) prefers  $i$  to  $j$ . Firstly, we have the following result:

**Proposition 2.1** *There exists an optimal pricing policy where no version dominates any other version.*

Thus, for the rest of the analysis, we restrict our attention only to such non-dominating pricing policies, in which case all the consumers of type  $l \in [x_{j-1}, x_j]$  prefer product version  $j$ , and  $x_j \in [l_j, l_{j+1}]$  holds for all  $j$ . Also, defining the lengths  $y_j \stackrel{\text{def}}{=} \frac{v-p_j}{\theta}$ , we see that everyone within  $y_j$  distance of the location  $l_j$  finds that buying version  $j$  yields a nonnegative net utility. Therefore, everyone with  $l \in [k_j, z_j]$  prefers and buys product  $j$ , where  $k_j$  and  $z_j$  are the points defined as  $z_j := \min(x_j, l_j + y_j)$ ,  $k_j := \max(x_{j-1}, l_j - y_j)$  for  $j = 2 \dots K$ , and  $z_1 := \min(x_1, y_1)$  and  $k_1 := \max(x_K, 1 - y_1)$ . Consequently, the revenue of the firm can be written as

$$R(p, F) = \Lambda \sum_{j=1}^K p_j (F(z_j) - F(k_j)) . \quad (1)$$

For the competitive ratio problem, we can further characterize the revenue functions using the following result:

**Lemma 2.1** *At the optimal solution, the firm sets its prices such that the whole market is served, i.e.  $x_j \leq l_j + y_j$  and  $x_{j-1} \geq l_j - y_j$  holds for all  $j$ , and consequently, we have  $[z_j, k_j] = [x_j, x_{j+1}]$  for all  $j$ .*

As a result of this lemma, we can restrict our attention to policies where the firm collects  $p_j$  from everybody within the interval  $[x_j, x_{j+1}]$  for all  $j$ . For a distribution function  $F(\cdot)$ , define the probability mass allocated to each such interval as  $f_j := F(x_{j+1}) - F(x_j)$  for  $j = 1 \dots K - 1$  and  $f_K := F(x_1) + 1 - F(x_K)$ . Using the above result, we can characterize the worst-case distribution as follows:

**Lemma 2.2** *The worst-case competitive ratio distribution function  $F(\cdot)$  allocates the mass  $f_j$  to location  $l_j$  for all  $j$ .*

**Corollary 2.1** *The revenue function can be written as  $R(p, F) = \Lambda \sum_{j=1}^K p_j f_j$ , and the maximum revenue the firm could extract under full information is  $R(p^*(F), F) = \Lambda v$ .*

As a consequence of all these results, we can write down the competitive ratio problem as the following maxmin formulation

$$\max_p \min_f \left\{ \frac{\Lambda \sum_j p_j f_j}{\Lambda v} : p_j + p_{j-1} + \theta |l_j - l_{j-1}| \leq 2v \quad \forall j, \quad p \geq 0, \quad \sum_j f_j = 1, \quad f \geq 0 \right\} ,$$

where constraints  $p_j + p_{j-1} + \theta|l_j - l_{j-1}| \leq 2v$  impose the condition the whole market must be served and where  $|l_1 - l_K|$  equals to  $1 - l_K$ . This formulation can be reduced to a linear program, and can be solved analytically as shown in the proof of the following proposition. The intuition behind the analysis is that the worst-case distributions are extreme point distributions, and the firm's optimal response is to choose prices so that the adversary is indifferent among these extreme point distributions.

**Proposition 2.2** *For fixed number of versions  $K$ , and their attributes  $l$ , consider the following indexing of the price vector that satisfies  $p_{[1]} \leq p_{[2]} \leq p_{[3]} \leq \dots \leq p_{[K]}$ , and the vector  $\bar{p}$  solving the equations  $\bar{p}_{[j]} = 2v - \bar{p}_{[j-1]} - \theta |l_{[j]} - l_{[j-1]}| \quad \forall j$ , then the optimal price vector  $p$  satisfies*

$$p_{[1]} = p_{[2]} = \bar{p}_{[1]} = \bar{p}_{[2]} = v - \frac{\theta \max_j |l_j - l_{j-1}|}{2} \quad \text{and} \quad p_{[j]} \in [p_{[1]}, \bar{p}_{[j]}] \quad \text{for} \quad j = 3..K,$$

and the resulting optimal competitive ratio is  $c = 1 - \frac{\theta \max_j |l_j - l_{j-1}|}{2v}$ .

Thus, we see that the minimum price is charged at two neighboring locations which are the furthest apart on the unit circle, and the competitive ratio depends on these minimum prices. The competitive ratio decreases as the consumers' sensitivity for differentiating attribute increases, consumers' nominal valuations for their ideal product decreases, or the maximum attribute difference among neighboring products increases.

Inspecting the above formula for the optimal competitive ratio, we see that it is maximized when the term  $\max_j |l_j - l_{j-1}|$  is minimized, which would occur when  $|l_j - l_{j-1}| = 1/K$  for all  $j$ . In other words, we have the following result:

**Proposition 2.3** *For fixed  $K$ , the optimal product line positioning and pricing decision is to locate products at equally spaced intervals along the attribute space and price them equally, in which case the optimal competitive ratio is  $c = 1 - \frac{\theta}{2vK}$ .*

The proof of this result is straight forward and omitted.

## 2.4 Maximum Regret

For the maximum regret problem, we consider again the case with fixed number of versions  $K$  and fixed attributes for different versions,  $l \in [0, 1]^K$ , and focus on the pricing strategy first. It is easy to see that the worst-case distribution can again be reduced to deciding how much probability mass to allocate to the intervals  $[z_j, k_j]$ ,  $j = 1, \dots, K$ . However, unlike the competitive ratio case, it is not necessarily the case

that  $[z_j, k_j] = [x_j, x_{j+1}]$  for all  $j$ . In other words, it might be the case that not everybody chooses to buy at the optimal prices. We will denote such a pricing strategy where some consumers choose not to buy a *non-spanning* pricing policy. Formally, defining the set of indices  $S(p) = \{j \mid x_j > l_j + y_j, \text{ or } x_{j-1} < l_j - y_j\}$  for a given price vector  $p$ , if the set  $S(p)$  is empty then  $p$  is a spanning price policy. The structure of the worst-case distribution changes according to whether the pricing policy used is spanning or non-spanning in the above sense. The following result formalizes this claim:

**Lemma 2.3** *For a given price vector  $p$ , the maximum regret is*

$$r(p) \stackrel{\text{def}}{=} \begin{cases} \Lambda (v - \min_j p_j) & \text{if } S(p) \text{ is empty} \\ \Lambda \max\{(v - \min_j p_j), \max_{j \in S(p)} p_j\} & \text{otherwise} \end{cases}$$

Furthermore, using this lemma one can easily prove the following:

**Lemma 2.4** *There exists an optimal price vector with equal components.*

Thus, finding the optimal prices reduces to a single dimensional decision of finding the best uniform price to offer for all versions. Using Lemmas 2.3 and 2.4, we can now prove our main result for the maximum regret problem:

**Proposition 2.4** *For fixed number of versions,  $K$ , with fixed attributes,  $l$ , the optimal regret value is*

$$r = \Lambda \min \left\{ \frac{\theta \max_j |l_j - l_{j-1}|}{2}, \frac{v}{2} \right\} .$$

*which is achieved by charging equal prices for different versions*

$$p_j = \begin{cases} v - \frac{\theta \max_j |l_j - l_{j-1}|}{2}, & \forall j, \text{ when } v \geq \theta \max_j |l_j - l_{j-1}| \text{ (spanning optimal policy)} \\ v/2, & \forall j, \text{ when } v < \theta \max_j |l_j - l_{j-1}| \text{ (non-spanning optimal policy)} . \end{cases}$$

The intuition lying behind this result is as follows: when  $\theta$  is high or  $v$  is low, the firm faces a linear demand curve with downward slope of  $\theta$  for each version, because cannibalization is not an issue as it is too costly for a customer close to a version to consider some other version. As a result, the firm charges the revenue maximizing price for a monopolist, i.e.  $v/2$ , which corresponds to a non-spanning pricing policy. On the other hand, when  $\theta$  is low or  $v$  is high, it is less costly for consumers close to a version to consider other versions; and therefore, cannibalization is an issue. Charging the revenue maximizing price  $v/2$  for

a version cannibalizes neighboring versions. Therefore, the firm charges a higher price for the version, i.e.  $v - \frac{\theta \max_j |l_j - l_{j-1}|}{2} > v/2$ .

Note the regret above is minimized irrespective of  $v$  when  $\max_j |l_j - l_{j-1}|$  is minimized, which again occurs when  $|l_j - l_{j-1}| = 1/K$  for all  $j$ . Thus, we have the same result as in the competitive ratio case:

**Proposition 2.5** *For fixed  $K$ , the optimal product line positioning decision is to locate versions at equally spaced intervals along the attribute space and to price them equally according to Proposition 2.4, in which case the optimal regret is  $r = \Lambda \min \left\{ \frac{\theta}{2K}, \frac{v}{2} \right\}$ .*

We therefore have shown that for both the competitive ratio and maximum regret criteria, the optimal pricing policy is a function of attribute differences between neighboring versions. This policy reflects the firm’s sensitivity for potential cannibalization among versions (e.g. non-dominating pricing policies). The worst-case performance decreases (i.e. the competitive ratio decreases or the regret increases) as the consumers’ sensitivity for differentiating attribute increases, consumers’ nominal valuations for their ideal product decreases, or the maximum attribute difference among neighboring products increases. The optimal positioning decision is to span the attribute space with equally spaced product versions and to sell them at a uniform price. In other words, to cover the product space uniformly with a collection of products offered at a single price. Interestingly, this “equally-space and equally-price” strategy is a celebrated Nash Equilibrium for the classical horizontal differentiation game where each version is controlled by a different firm, which is well studied by Salop and others. (See Salop [27].) In the competitive case, this arises as an equilibrium response among competing firms. In our case, it arises as an optimal hedge against another sort of adversary: “nature” - the unknown preferences of customers. While stylized, this sort of positioning and pricing structure is seen commonly in real-world markets. For example, iTunes offers music downloads spanning all genres of music at a uniform price of \$0.99; apparel retailers normally charge to same price for a garment sold in different colors and sizes; ice cream vendors carry a range of flavors sold at the same price-per-cone, etc.

Notice also that the optimal prices charged, and thus, revenues earned increases with the number of versions  $K$ , increasing the competitive ratio and decreasing the regret. This is the typical profit maximization response when there are no economies of scale to production of variety. However, when there are economies of scale, one needs to balance the revenue gain of more variety against the cost of producing many versions.

### 3 Vertical Product Line Positioning

We next consider the case of vertical product positioning, in which versions correspond to different quality levels over which consumers share a common ranking (though with heterogeneous willingness-to-pay for quality). Specifically, there are  $K$  distinct quality versions of a product that the firm considers potentially offering. The quality level of version  $j$  is denoted by  $l_j$ . The indexing satisfies  $l_1 < l_2 < \dots < l_K$ , so that  $l_1$  is the lowest quality level, whereas  $l_K$  is the highest. In order to reduce the heavy notation induced by the analysis of this setting, we assume there is a continuum of consumers of unit mass. All the results can be scaled to any market size without changing the optimal policy.

Each consumer has a distinct taste for quality, denoted by  $v$ , which is an independent and identically distributed random variable with unknown distribution  $F(\cdot)$  on a known interval  $[\underline{v}, \bar{v}]$ . This distribution is the key missing information to the firm in our model.

If a consumer has a quality taste level of  $v$  and purchases product version  $j$ , her net utility is given by  $u(v, j) \stackrel{def}{=} v l_j - p_j$ , where  $p_j$  is the price of product  $j$ . Consequently, the higher the taste for quality or the higher the quality level, the more a consumer is willing to pay for a particular version. A consumer prefers version  $j$  over version  $i$  if  $u(v, j) = v l_j - p_j > v l_i - p_i = u(v, i)$  holds. Therefore, unlike the horizontal differentiation model in the previous section, each consumer enjoys the higher quality versions more. As a tie-breaking assumption, we assume that a consumer prefers the higher quality version when faced with two versions giving the same utility. For a given vector of prices  $p$ , a consumer buys the version of the product which yields the highest net utility provided that is nonnegative, i.e.  $u(v) \stackrel{def}{=} \max_j \{v l_j - p_j\} \geq 0$ , or will not buy any product otherwise.

The sequence of events is as follows: first, the firm decides on how many versions to offer at what quality levels, and chooses their prices without knowing the distribution of the random taste for quality. Then, we evaluate the decision against the worst-case distribution  $F(\cdot)$  of the taste for quality. Our goal is again to find the optimal product line positioning (this time in terms of quality levels) and pricing policy of the firm. Specifically, we want to answer the following questions: How many versions should the firm offer? What quality levels should be offered and how should they be priced? And how do these decisions change with respect to the uncertainty level of customer taste for quality? We begin with the competitive ratio case and then analyze the case of maximum regret.



### 3.1 Competitive Ratio

First, let us assume for the moment that the firm offers all  $K$  quality levels of the product. If we define  $v_j$  as the quality level where a consumer switches from version  $j$  to version  $(j + 1)$ , then  $v_j$  must satisfy  $v_j l_j - p_j = v_j l_{j+1} - p_{j+1}$ , yielding  $v_j = \frac{p_{j+1} - p_j}{l_{j+1} - l_j}$  for  $j = 1 \dots K - 1$ . Using the definition of  $u(v, j)$ , it is easy to see that version  $j$  is preferred over version  $(j + 1)$  by those consumers with  $v < v_j$ , and similarly, version  $(j + 1)$  is preferred over version  $j$ , when  $v \geq v_j$ . We say version  $j$  is *strictly dominated* by versions  $(j - 1)$  and  $(j + 1)$ , if version  $(j - 1)$  or version  $(j + 1)$  yields strictly more utility for every consumers type  $v \in [v, \bar{v}]$  than version  $j$ . The necessary and sufficient condition for this is  $v_{j-1} > v_j$ , which can be easily seen graphically in Figure 2.

Note the product line positioning and pricing decisions of the firm can be aggregated into the single decision of offering all  $K$  quality levels of the product and choosing only the prices offered. To see this, assume that some version  $j$  is not offered under some policy. Analytically, this is equivalent to offering that version at some price  $p_j$  that satisfies  $v_{j-1} = v_j$ , because at that price level no consumer strictly prefers version  $j$ , but a consumer of type  $v = v_{j-1} = v_j$  has the same value of utility for consuming versions  $(j - 1)$ ,  $j$  and  $(j + 1)$ . However, she prefers version  $(j + 1)$  over others due to the tie-breaking assumption. Therefore, at price  $p_j$  version  $j$  is effectively a “dummy version” which is not chosen by any consumer, and this case is therefore analytically equivalent to not offering version  $j$ .

For the competitive ratio problem, we first prove that there exists an optimal price vector where no product strictly dominates any other product at the optimal solution and further that the consumer with the lowest taste for quality earns zero utility at the optimal solution. These properties are formalized in the following proposition:

**Proposition 3.1** *There exists an optimal price vector where no product strictly dominates any other product and which satisfies  $v \leq v_1 \leq v_2 \leq \dots \leq v_{K-1} \leq \bar{v}$ .*

The next result shows that there exists an optimal policy where only some number of highest quality versions of the product are offered:

**Proposition 3.2 (Nested Quality Offers)** *There exists an optimal solution where only the  $(K - j + 1)$  highest quality products are offered for some  $j = 1 \dots K$ .*

Using Proposition 3.1 and our previous observations, we conclude that offering only the  $(K - j + 1)$  highest quality products is analytically equivalent to offering all  $K$  quality levels of the product, but fixing

the prices of the first  $j$  quality versions to be  $p_1 = \underline{v}l_1$ ,  $p_2 = \underline{v}l_2, \dots$ ,  $p_j = \underline{v}l_j$ . This is illustrated in Figure 2. The remainder of the competitive ratio section tries to identify the best competitive ratio that can be

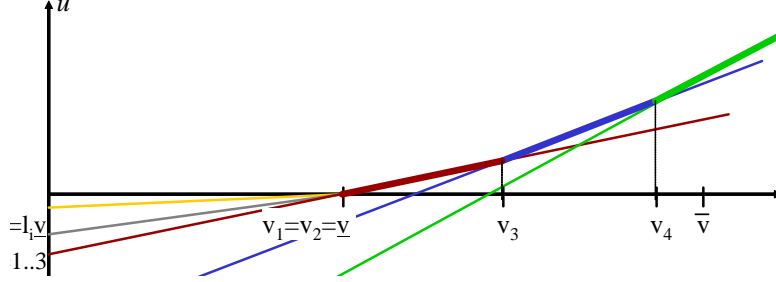


Figure 2: An example plotting the utilities by different versions of a feasible policy for  $c_3$  when  $K = 5$ .

achieved by the firm using a policy which offers only the  $(K - j + 1)$  highest quality versions, denoted by  $c_j$ . Then, the optimal value of the competitive ratio problem satisfies  $c = \max_j \{c_j\}$ .

For the problem of finding  $c_j$ , we show that for a given pricing policy  $p$  satisfying above structure, the worst-case distribution allocates the whole unit probability mass arbitrarily close to  $\bar{v}$  or to some  $v_j$ , where a consumer is indifferent between two neighboring quality offers. The resulting value of the competitive ratio for this response is characterized as follows:

**Lemma 3.1** *When a pricing policy  $p$  offering only the  $(K - j + 1)$  highest quality products is used, the worst-case distribution yields a competitive ratio arbitrarily close to*

$$c_j(p) \stackrel{def}{=} \min \left\{ \min_{i=j \dots K-1} \{p_i / l_K v_i\}, p_K / l_K \bar{v} \right\} .$$

This result helps us write-down the competitive ratio problem when  $(K - j + 1)$  highest quality versions are offered as

$$\begin{aligned} c_j &= \max_p \min \left\{ \min_{i=j \dots K-1} \{p_i / l_K v_i\}, p_K / l_K \bar{v} \right\} \\ &s.t. \quad p_1 = \underline{v}l_1, p_2 = \underline{v}l_2, \dots, p_j = \underline{v}l_j \\ &\quad \underline{v} < v_j \leq v_{j+1} \leq \dots \leq v_{K-1} \leq \bar{v} \end{aligned}$$

In the proof of the following proposition in the appendix, we first relax constraint  $\underline{v} < v_j$  above, and define  $c_j^r$  as the best ratio that can be achieved by this relaxed formulation, which yields an upper bound

on  $c_j$ . Then, we convert the formulation of  $c_j^r$  to an equivalent linear program, and use weak duality to upper bound  $c_j^r$ , which in turn yields another upper bound on  $c_j$ . Finally, using strong duality on this dual problem, we show that  $c_j$  achieves these upper bounds for all  $j$ , letting us characterize the optimal policy and the corresponding competitive ratio as follows:

**Proposition 3.3** *The optimal competitive ratio for the vertical product positioning problem is given by  $c = \max_j \{c_j\}$ , where  $c_j$  satisfies*

$$\frac{\underline{v}}{\bar{v}} l_j \frac{\prod_{i=j+1}^K (l_i - l_{i-1} + c_j l_K)}{(c_j l_K)^{K+1-j}} = 1 \text{ for } j = 1 \dots K-1 \quad \text{and} \quad c_K = \frac{\underline{v}}{\bar{v}}.$$

If  $c = c_j$ , only the  $(K - j + 1)$  highest quality version are offered with the following optimal prices

$$p_j = \underline{v} l_j, \quad \text{and} \quad p_n = \underline{v} l_j \frac{\prod_{i=j+1}^n (l_i - l_{i-1} + c_j l_K)}{(c_j l_K)^{n-j}} \text{ for } n = j+1 \dots K.$$

We see that the optimal policy charges a price premium for each higher quality level as a function of the quality difference offered at each step,  $l_i - l_{i-1}$ .

Define  $\gamma = \underline{v}/\bar{v}$  and  $\gamma_j = \frac{l_j^{K-j}}{\prod_{i=j+1}^K (l_i - l_{i-1} + l_j)}$  for  $j = 1 \dots K-1$ . Observe that  $\gamma$  is a measure of potential market heterogeneity for quality: as  $\gamma$  decreases, the relative difference between the lowest and the highest taste for quality,  $\underline{v}$  and  $\bar{v}$ , increases, and vice versa. We further prove the following result tying the heterogeneity level to the optimal portfolio of offered quality levels:

**Proposition 3.4** *As the potential market heterogeneity for quality increases (which can also be interpreted as increasing uncertainty in taste for quality for this model), i.e. as  $\gamma$  decreases, it is optimal to offer more quality versions of the product. Specifically, for  $\gamma \in [0, \gamma_1)$  the optimal solution offers all the  $K$  versions of the product; for  $\gamma \in [\gamma_j, \gamma_{j+1})$  the optimal solution offers  $(K-j)$ -highest quality versions of the product for  $j = 2 \dots K-2$ ; and for  $\gamma \in [\gamma_{K-1}, 1]$  the optimal solution offers only the highest quality version of the product.*

Just as suggested by Bhargava and Choudhary [7] and Lilien et al. [?], this result shows that versioning becomes more attractive when consumers are sufficiently heterogeneous, which is reflected by the range of valuations for quality,  $\underline{v}$  and  $\bar{v}$ , in this model. When the potential market heterogeneity for quality increases, i.e.  $\gamma$  decreases, and the exact valuations of customers for quality are unknown, it is optimal to offer more and more quality levels.

## 3.2 Maximum Regret

For the maximum regret problem, it is no longer the case that  $u(\underline{v}) = 0$  and  $u(v) > 0$  for  $v \in (\underline{v}, \bar{v}]$  at the optimal solution. The firm can choose a pricing policy where consumers of type  $v \in [\underline{v}, v_0)$  choose not to buy any version for some  $v_0 \in [\underline{v}, \bar{v}]$ , i.e.  $u(v) < 0$  for  $v \in [\underline{v}, v_0)$ . Every feasible price vector  $p$  implies a corresponding point  $v_0$ . Thus, we can divide the decision of the firm into two parts: first selecting the optimal value of  $v_0$ , i.e. selecting which segment of the market to serve, then selecting optimal price vector which serves that particular segment. That is defining

$$r(v_0) := \min_{p \in P(v_0)} \max_F [R(p^*(F), F) - R(p, F)] \quad (2)$$

where  $P(v_0) = \{p \mid u(v_0) = 0\}$ , we can rewrite maximum regret problem as

$$r^* = \min_p \max_F [R(p^*(F), F) - R(p, F)] = \min_{v_0 \in [\underline{v}, \bar{v}]} r(v_0) .$$

For fixed  $v_0$ , there again exists a non-dominating optimal pricing policy, denoted by the vector  $p(v_0) \in P(v_0)$ , offering only the  $(K - j + 1)$  highest quality versions for the problem of identifying  $r(v_0)$ , as stated by the following results, proofs of which follow identical reasoning with the competitive ratio case and are thus omitted.

**Proposition 3.5** *There exists an optimal price vector  $p(v_0)$  for the problem in (2) above, where no product strictly dominates any other product, and which satisfies  $v_0 \leq v_1 \leq v_2 \leq \dots \leq v_{K-1} \leq \bar{v}$ .*

**Proposition 3.6 (Nested Quality Offers)** *There exists an optimal solution  $p(v_0)$  for the problem in (2) where only the  $(K - j + 1)$ -highest quality versions are offered for some  $j = 1 \dots K$ .*

For fixed  $v_0$ , let us again confine our attention to pricing policies offering only the  $(K - j + 1)$  highest quality products for  $j = 1 \dots K$ , and define the minimum regret that can be achieved using such a pricing policy as  $r_j(v_0)$ , then by definition the equality  $r(v_0) = \min_j \{ r_j(v_0) \}$  holds. The optimal response by the adversary under the problem of identifying  $r_j(v_0)$  is to choose a deterministic valuation arbitrarily close to  $v_0$ ,  $v_K$  or  $v_j$  for some  $j = 1 \dots K - 1$ , which is formalized by the following result. The proof again uses identical logic and steps with the corresponding lemma for the competitive ratio case, and therefore is omitted.

**Lemma 3.2** *For fixed  $v_0$ , when a pricing policy  $p \in P(v_0)$  offering only the  $(K - j + 1)$  highest quality*

products is used, the regret, denoted by  $r_j(v_0, p)$ , is given by

$$r_j(v_0, p) = \max_F [R(p^*(F), F) - R(p, F)] = \begin{cases} \max \{v_0 l_K, \max_{i=j \dots K-1} \{l_K v_i - p_i\}, l_K \bar{v} - p_K\} & \text{if } v_0 > \underline{v} \\ \max \{\max_{i=j \dots K-1} \{l_K v_i - p_i\}, l_K \bar{v} - p_K\} & \text{if } v_0 = \underline{v} . \end{cases}$$

Then, we have  $r_j(v_0) = \min_{p \in P_j(v_0)} r_j(v_0, p)$  by definition. However, the exact specification of  $r_j(v_0)$  depends on whether  $v_0 > \underline{v}$  or  $v_0 = \underline{v}$ , as shown above.

Using the above results, the optimal regret value the firm can achieve satisfies

$$\begin{aligned} r^* &= \min_{v_0 \in [\underline{v}, \bar{v}]} r(v_0) = \min \{ r(\underline{v}), \inf_{v_0 \in (\underline{v}, \bar{v}]} r(v_0) \} = \min \{ \min_j \{r_j(\underline{v})\}, \inf_{v_0 \in (\underline{v}, \bar{v}]} \min_j \{r_j(v_0)\} \} \\ &= \min \{ \min_j \{r_j(\underline{v})\}, \min_j \{ \inf_{v_0 \in (\underline{v}, \bar{v}]} r_j(v_0) \} \} = \min_j \{ \min \{ r_j(\underline{v}), \inf_{v_0 \in (\underline{v}, \bar{v}]} r_j(v_0) \} \} . \end{aligned}$$

However, using Lemma 3.2, we can characterize the terms at the right hand side of the last equality as

$$\begin{aligned} \inf_{v_0 \in (\underline{v}, \bar{v}]} r_j(v_0) &= \inf_{v_0 \in (\underline{v}, \bar{v}]} \min_{p \in P_j(v_0)} r_j(v_0, p) = \inf_{v_0 \in (\underline{v}, \bar{v}]} \min_{p \in P_j(v_0)} \max \left\{ v_0 l_K, \max_{i=j \dots K-1} \{l_K v_i - p_i\}, l_K \bar{v} - p_K \right\}, \\ \text{and } r_j(\underline{v}) &= \min_{p \in P_j(\underline{v})} r_j(\underline{v}, p) = \min_{p \in P_j(\underline{v})} \max \left\{ \max_{i=j \dots K-1} \{l_K v_i - p_i\}, l_K \bar{v} - p_K \right\} . \end{aligned}$$

These characterizations can be written as equivalent linear programs. For example, for  $\inf_{v_0 \in (\underline{v}, \bar{v}]} r_j(v_0)$ , we have the following formulation:

$$\inf_{v_0 \in (\underline{v}, \bar{v}]} r_j(v_0) \stackrel{\text{def}}{=} r_0^j = \min_{p, v_0, r} r \tag{3}$$

$$\text{s.t. } r \geq v_0 l_K \tag{4}$$

$$r \geq v_n l_K - p_n \quad n = j \dots K-1 \tag{5}$$

$$r \geq \bar{v} l_K - p_K \tag{6}$$

$$v_{n-1} = \frac{p_n - p_{n-1}}{x_n} \leq \frac{p_{n+1} - p_n}{x_{n+1}} = v_n \quad n = j+1 \dots K-1 \tag{7}$$

$$v_{K-1} = \frac{p_K - p_{K-1}}{x_K} \leq \bar{v} \tag{8}$$

$$v_j = \frac{p_{j+1} - p_j}{x_{j+1}} \geq v_0 \tag{9}$$

$$p_j = v_0 l_j \tag{10}$$

$$v_0 > \underline{v} \tag{11}$$

where the constraints (4),(5) and (6) reflect the possible choices of the adversary that appear in the characterization of  $\inf_{v_0 \in (\underline{v}, \bar{v})} r_j(v_0)$  above, and the constraints (7),(8), (9) and (11) define the feasible region,  $v_1 = v_2 = \dots = v_{j-1} = v_0 < v_j \leq v_{j+1} \leq \dots \leq v_{K-1} \leq \bar{v}$ , for the prices.

Similarly,  $r_j(\underline{v})$  can be written as an equivalent linear program that is almost identical to the above LP but without the constraints (4) and (11) and with the value of the variable  $v_0$  fixed at  $\underline{v}$ . Thus, the feasible region for the prices is given by inequalities  $v_1 = v_2 = \dots = v_{j-1} = \underline{v} < v_j \leq v_{j+1} \leq \dots \leq v_{K-1} \leq \bar{v}$ , in this case, and we denote the optimal value to this LP as  $\underline{r}^j$ . This formulation is given in equation (31) in the Appendix.

In the proof of Proposition 3.7 below, we first relax the strict inequalities ( $v_0 > \underline{v}$  in (3) and  $v_j > \underline{v}$  in (31)), and solve for the relaxed versions of the LPs in (3) and (31), establishing lower bounds on  $r_0^j$  and  $\underline{r}^j$ . Then, we show that the optimal values, denoted respectively by  $r_0'^j$  and  $\underline{r}'^j$ , satisfy

$$r_0'^j := \frac{\bar{v} l_K^{K+2-j}}{(l_j + l_K) \prod_{i=j+1}^K (h_i + l_K)} \quad \text{and} \quad \underline{r}'^j := \frac{\bar{v} l_K^{K+1-j}}{\prod_{i=j+1}^K (h_i + l_K)} - \underline{v} l_j \quad \text{for } j = 1 \dots K-1, \text{ with,}$$

$$r_0'^K := \bar{v} l_K / 2, \quad \text{and} \quad \underline{r}'^K := (\bar{v} - \underline{v}) l_K .$$

where  $h_j = l_j - l_{j-1}$  is the quality difference between two neighboring versions.

With these values, we are ready to state our main result of this section:

**Proposition 3.7** *The optimal regret for the vertical product positioning problem satisfies*

$$r^* = \min_j \{ \min\{r_j(\underline{v}), \inf_{v_0 \in (\underline{v}, \bar{v})} r_j(v_0)\} \} = \min_j \{ \min\{\underline{r}^j, r_0^j\} \} = \min_j \{ \min\{\underline{r}'^j, r_0'^j\} \} ,$$

with the optimal price vector having one of the forms

$$p_j = r_0'^j l_j / l_K, \quad \text{and} \quad p_{n+1} = \frac{r_0'^j h_{n+1} + p_n(l_K + h_{n+1})}{l_K} \quad \text{for } n = j \dots K-1 \quad \text{if } r^* = r_0'^j \text{ for some } j, \text{ or}$$

$$p_j = \underline{v} l_j, \quad \text{and} \quad p_{n+1} = \frac{\underline{r}'^j h_{n+1} + p_n(l_K + h_{n+1})}{l_K} \quad \text{for } n = j \dots K-1 \quad \text{if } r^* = \underline{r}'^j \text{ for some } j .$$

For example, if  $r^* = \underline{r}^j = \underline{r}'^j$ , the optimal policy is to offer  $(K - j + 1)$  versions such that all consumers choose to buy one version, i.e. the whole market is served. The lowest consumer type  $\underline{v}$  chooses version  $j$  which is priced at  $p_j = \underline{v} l_j$ . The market is segmented with respect to quality types in a monotone manner, and segments with higher quality taste choose higher quality versions and pay higher prices. On the other hand, if  $r^* = r_0^j = r_0'^j$ , the whole market is not served. Consumers of type  $v \in [\underline{v}, v_0)$  choose not to buy any

version, where  $v_0 = r_0^{l_j} / l_K = \frac{\bar{v} l_K^{K+1-j}}{(l_j + l_K) \prod_{i=j+1}^K (h_i + l_K)}$ , and consumer type  $v_0$  chooses version  $j$  which is priced at  $p_j = v_0 l_j$ .

Below we show that the firm can choose not to serve the whole market only when she offers all  $K$  versions, which further reduces the optimal regret formula. This means that some lowest quality types are not served only when the potential market heterogeneity for quality is so large that even offering all  $K$  versions is not enough to span the market in an optimal manner.

**Proposition 3.8** *The following inequalities hold  $\bar{v} l_K / 2 > r_0^{K-1} > \dots > r_0^1$ , and thus, the optimal regret can be reduced to the following form*

$$r^* = \min \left\{ \frac{\bar{v} l_K^{K+1}}{(l_1 + l_K) \prod_{i=2}^K (h_i + l_K)}, \left\{ \frac{\bar{v} l_K^{K+1-j}}{\prod_{i=j+1}^K (h_i + l_K)} - \underline{v} l_j \right\} \quad j = 1 \dots K-1, (\bar{v} - \underline{v}) l_K \right\}$$

Define again  $\gamma = \underline{v} / \bar{v}$  and  $\gamma_j = \frac{l_K^{K-j+1}}{\prod_{i=j}^K (h_i + l_K)} = \frac{l_K^{K-j+1}}{\prod_{i=j}^K (l_i - l_{i-1} + l_K)}$  for  $j = 1 \dots K$ . Remember that  $\gamma$  is a measure of the potential market heterogeneity for quality. Similar to the competitive ratio case, we prove the following result tying the degree of heterogeneity to the number of versions offered and their quality levels:

**Proposition 3.9** *As the potential market heterogeneity for quality increases, i.e. as  $\gamma$  decreases, it is optimal to offer more quality versions of the product. Specifically, for  $\gamma \in [0, \gamma_1)$  the optimal solution offers all the  $K$  versions of the product; and consumers of type  $v \in [\underline{v}, v_0)$  choose not to buy any version, where  $v_0 = \frac{\bar{v} l_K^{K+1-j}}{(l_1 + l_K) \prod_{i=1}^K (h_i + l_K)}$ , i.e. the whole market is not served. For  $\gamma \geq \gamma_1$  the whole market is served. Specifically, for  $\gamma \in [\gamma_j, \gamma_{j+1})$  there exists an optimal solution offering  $(K+1-j)$ -highest quality versions of the product for  $j = 1 \dots K-1$ ; and for  $\gamma \in [\gamma_K, 1]$  there exists an optimal solution offering only the highest quality version of the product.*

For both the competitive ratio and maximum regret criteria, we have shown that the optimal pricing policy charges a price-premium for each higher quality level as a function of the quality difference between neighboring versions. The optimal pricing and versioning policies are coordinated and take possible cannibalization effects among versions into account (e.g. non-dominating pricing policies again). The optimal positioning policy first decides on how much of the potential market to serve, and segments the market by offering only some number of highest quality versions, i.e. nested quality offers, so that consumers with higher taste for quality choose higher quality versions and pay higher prices. The number of versions offered depends on the market heterogeneity for quality. For both criteria, the number of quality levels offered increases as the uncertainty in the consumers' preferences increases. This again leads to similar conclusions

as the one in the previous section: offering different quality versions can be a response to uncertainty and/or lack of information with respect to consumer preferences as well as serving as a price discrimination mechanism. These results are also consistent with the vertical differentiation strategies for many products we observe in daily life. For products with large and heterogeneous customer bases, such as micro processors with different speeds and high or low cache memories, professional, standard and student versions of software, and consumer electronics product lines with models having increasing functionality with higher prices, we see many different quality versions offered at different prices.

## 4 Conclusion and Future Research Directions

Relative performance criteria seem a useful device for analyzing problems with limited market information; they have good practical motivation yet are simple enough to yield meaningful structural insights. Having these closed-form solutions readily available makes it easy to perform sensitivity analysis with respect to changes in these problem primitives. Thus, questions like how the optimal policy changes with respect to ambiguity level, or how the product variety effects the pricing policy can easily be answered. In addition, the analysis of these criteria sheds light on the risks firms face due to limited information and the related trade-offs. Lastly, structurally the results – like the equally-spaced, equally-priced strategy in the horizontal differentiation case, or more versioning for larger market heterogeneity for quality in the vertical differentiation case– correspond well with what we see in real world markets. Our results in this sense help show that these everyday strategies may be good responses to limited information from a worst-case perspective.

However, such stylized analysis is not suited to computing recommendations for real-life problems. In many real-life situations, firms has some limited (albeit often unstructured) information rather than “no information” at all. Incorporating such limited additional information into an analysis using relative performance criteria is not easy, as most of the analysis depends on extreme point optimality, and the number of extreme points can grow very quickly in the presence of additional information. It would be interesting to see if these criteria and the settings considered in this paper can be extended to include further information. As another future research direction, one might look into the reasons why the sorts of robust strategies we derive are actually observed in daily practice. Studies on the incentive structures for the decision makers, and empirical research to see to what extent firms are risk-neutral vs. risk-averse in decision making could be helpful in explaining this.



## References

- [1] R. Acharyya. Monopoly and product quality: Separating or pooling menu? *Economics Letters*, 61:187–194, 1998.
- [2] M. Ball and M. Queyranne. Competitive analysis of on-line booking: Dynamic policies. *Working Paper, University of Maryland, College Park, MD*, 2004.
- [3] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Math. Oper. Res.*, 23(4):769–805, 1998.
- [4] D. Bergemann and K. Schlag. Robust monopoly pricing: The case of regret. *Working Paper, Yale University, New Haven, Connecticut 06520-8281*, 2005.
- [5] D. Bertsimas and M. Sim. The price of robustness. *Oper. Res.*, 52:35–53, 2003.
- [6] D. Bertsimas and A. Thiele. A robust optimization approach to supply chain management. *Proceedings of the 14th IPCO*, pages 86–100, 2004.
- [7] H. Bhargava and V. Choudhary. Research note: One size fits all? optimality conditions for versioning information goods. *University of California Irvine, Irvine, CA 92697*.
- [8] H. Bhargava and V. Choudhary. Information goods and vertical differentiation. *Journal of Management Information Systems*, 18(2):85–102, 2001.
- [9] G. Dobson and C. A. Yano. Profit optimizing product line design, selection and pricing with manufacturing cost considerations: A survey. in *Product Variety Management: Research Advances (T.-H. Ho and C.S. Tang, eds.)*, Kluwer Academic Publisher, pages 145–175, 1998.
- [10] L. El Ghaoui and H. Lebret. Robust solutions to least-squares problems with uncertain data. *SIAM J. on Matrix Analysis and Applications*, 18(4):1035–1064, 1997.
- [11] A. Ghose and A. Sundararajan. Software versioning and quality degradation? an exploratory study of the evidence. *Working Paper, New York University, New York, NY 10012*, 2005.
- [12] J. Green, P. E. Carroll and S. Goldberg. A general approach to product design optimization via conjoint analysis. *Journal of Marketing*, 43:17–35, 1981.
- [13] P. E. Green and A. M. Krieger. Recent contributions to optimal product positioning and buyer segmentation. *European Journal Operational Research*, 41(2):127–141, 1989.
- [14] P. E. Green and V. Srinivasan. Conjoint analysis in consumer research: Issues and outlook. *Journal of Consumer Research*, 5:103–123, 1978.

- [15] H. Hotelling. Stability in competition. *Economic Journal*, 39:41–57, 1929.
- [16] R. Jones and H. Mendelson. Product and price competition for information goods. *Discussion paper, Stanford University*.
- [17] J. S. Kim. Optimal price-quality schedules and sustainability. *Journal of Industrial Economics*, 36(2):231–44, 1987.
- [18] V. Krishnan and K. T. Ulrich. Product development decisions: A review of the literature. *Management Science*, 47(1):1–21, 1990.
- [19] K. Lancaster. A new approach to consumer theory. *Journal of Political Economy*, 74:132–157, 1966.
- [20] K. Lancaster. The economics of product variety: A survey. *Marketing Science*, 9(3):189–206, 1990.
- [21] A. Lim and J. Shanthikumar. Relative entropy, exponential utility, and robust dynamic pricing. *Working Paper, University of California Berkeley, CA*, 2004.
- [22] E. Maskin and J. Riley. Monopoly with incomplete information. *RAND Journal of Economics*, 15(2):171–196, 1984.
- [23] M. Mussa and S. Rosen. Monopoly and product quality. *Journal of Economic Theory*, 18:301–317, 1978.
- [24] G. Perakis and G. Roels. Regret in the newsvendor model with partial information. *Working Paper, MIT, Cambridge, MA*, 2004.
- [25] G. Perakis and A. Sood. Competitive multi-period pricing with fixed inventories. *Working Paper, MIT, Cambridge, MA*, 2003.
- [26] K. Ramdas. Managing product variety: An integrative review and research directions. *Production and Operations Management*, 12(1):79101, 2003.
- [27] S. Salop. Monopolistic competition with outside goods. *Bell Journal of Economics*, 10:141–156, 1979.
- [28] A. Shaked and J. Sutton. Relaxing price competition through product differentiation. *Review of Economic Studies*, 49:3–14, 1982.
- [29] W. S. Vickrey. [1964] microstatics. harcourt, brace and world, new york. selection "spatial competition and monopolistic competition" reprinted (with foreword) in. *International Journal of Industrial Organization*, 17:953–963, 1999.

## 5 APPENDIX

**Proof of Proposition 2.1:** By contradiction assume at all optimal solutions, there exist a price,  $p_j$  which dominates its neighboring price  $p_{j+1}$ , i.e.  $x_j > l_{j+1}$  or equivalently  $p_{j+1} > p_j + \theta(l_{j+1} - l_j)$ . Then, for any distribution the adversary chooses, the firm earns  $p_j$  from those who choose to buy in the interval,  $[x_{j-1}, x_j]$ . However, decreasing the price  $p_{j+1}$  to  $p_j + \theta(l_{j+1} - l_j)$ , the firm can earn  $p_j + \theta(l_{j+1} - l_j)$  from those in the interval  $[x_j, l_{j+1}]$ , plus  $p_j$  from those in the interval  $[x_{j-1}, l_{j+1}]$  without changing any other prices or affecting revenues of other regions, which yields a revenue as at least high as the optimal revenue. Repeating this augmentation for every dominating price we achieve a price vector where there is no price strictly dominating any other.  $\square$

**Proof of Lemma 2.1:** Assume the optimal prices are such that for some  $l_j$ , we have  $x_j > l_j + y_j$  (or  $x_{j-1} < l_j - y_j$ ). Then the adversary could select a distribution that puts all the probability mass to the interval  $[l_j + y_j, x_j]$  ( or  $[x_{j-1}, l_j - y_j]$ ), causing zero sales, and thus, achieving a competitive ratio of zero. On the other hand, any price policy that satisfies inequalities  $x_j \leq l_j + y_j$  and  $x_{j-1} \geq l_j - y_j$  for all  $j$  guarantees a positive ratio.  $\square$

**Proof of Lemma 2.2:** Under a price policy that satisfies  $x_j \leq l_j + y_j$  and  $x_{j-1} \geq l_j - y_j$  for all  $j$ , the revenue in equation-(1) depends only on how much probability mass exists within each interval  $[x_j, x_{j+1}]$ , call  $f_j := F(z_j) - F(k_j) = F(x_{j+1}) - F(x_j)$ . However, the specifics of the distribution function matters for the adversary, as the maximum revenue that can be achieved, which appears in the denominator of the competitive ratio to be minimized, depends on this distribution. On the other hand, the value of the maximum revenue is always less than  $\Lambda v$ . Also note that given an aggregate decision to how much probability mass to put on each interval, the adversary can achieve the maximum revenue  $\Lambda v$  in the denominator of the competitive ratio by putting each mass  $f_j$  on the point  $l_j$  for all  $j$ . Therefore, the adversary does not sacrifice in terms of minimization by restricting her decision to how much probability mass to put on each interval.  $\square$

**Proof of Proposition 2.2:**

$$\begin{aligned}
 c &= \max_p \min_f \left\{ \frac{\Lambda \sum_j p_j f_j}{\Lambda v} : p_j + p_{j-1} + \theta|l_j - l_{j-1}| \leq 2v \quad \forall j, \quad p \geq 0, \quad \sum_j f_j = 1, \quad f \geq 0 \right\} \\
 &= \frac{1}{v} \max_p \left\{ \min_j \{p_j\} : p_j + p_{j-1} + \theta|l_j - l_{j-1}| \leq 2v \quad \forall j, \quad p \geq 0 \right\} \\
 &= \frac{1}{v} \max_{p,s} \left\{ s : s \leq p_j \quad \forall j, \quad p_j + p_{j-1} + \theta|l_j - l_{j-1}| \leq 2v \quad \forall j, \quad p \geq 0 \right\}
 \end{aligned}$$

As the objective  $\min_j \{p_j\}$  is homogeneous in the components of price vector  $p$ , it is obvious that  $p_{[1]} + p_{[2]} + \max_j |l_j - l_{j-1}| = 2v$  at any optimal solution, and the upper bounds for other prices are given by setting  $\bar{p}_{[j]} = 2v - \bar{p}_{[j-1]} - \theta|l_{[j]} - l_{[j-1]}| \quad \forall j$ , which yields the result.  $\square$

**Proof of Lemma 2.3:** The part of the result for when  $S(p)$  is empty is obvious using the arguments of the previous section as explained above: when  $S(p)$  is empty, we have  $[z_j, k_j] = [x_j, x_{j+1}]$  for all  $j$ , and the firm collects  $p_j$  from everybody within the interval  $[x_j, x_{j+1}]$  for all  $j$ . Also, the maximum revenue that can be achieved is always less than  $\Lambda v$ . Thus the adversary restricts her decision to how much probability mass  $f_j$  to put on each interval  $[x_j, x_{j+1}]$ , and allocates each mass  $f_j$  on the point  $l_j$  for all  $j$ , achieving the maximum regret of  $\Lambda (v - \min_j p_j)$ .

For the other part, fix some price  $p$  for which  $S(p)$  is not empty (i.e.  $p$  is a non-spanning price policy), and define the regret as a function of distribution  $F$  as  $r(p, F) := R(p^*(F), F) - R(p, F)$ . For a given  $F$ , define  $o_j = F(z_j) - F(k_j)$  and  $r_j = F(k_{j+1}) - F(z_j)$ , i.e.  $o_j$  is the fraction of customers that are served by price  $p_j$ , and  $r_j$  is the fraction of customers that are not served either by price  $p_j$  or  $p_{j+1}$ , so that  $\sum o_j + \sum r_j = 1$ .

Now, we can see the adversary's problem of finding the best distribution to maximize the regret function as a two-step optimization of first choosing  $o$  and  $r$  vectors, then choosing a distribution function satisfying the requirements of these vectors. That is,

$$\max_F r(p, F) = \max_{o, r} \max_F \{ r(p, F) \quad : \quad o_j = F(z_j) - F(k_j), \quad r_j = F(k_{j+1}) - F(z_j) \quad \forall j \} \quad (12)$$

If we concentrate on the inner maximization for given vectors of  $o$  and  $r$ , we see that  $R(p, F) = \Lambda \sum o_j p_j$ . As for the function  $R(p^*(F), F)$ , the revenue generated by the optimal price vector  $p^*(F)$  at each region  $[k_j, z_j]$  is obviously bounded by  $\Lambda o_j v$ . In addition, for the vector  $p^*(F)$  to be able to sell to customers with preferences in the region  $[z_j, k_{j+1}]$  either  $p_j^*(F) < p_j$  or  $p_{j+1}^*(F) < p_{j+1}$  must hold. Therefore, the revenue generated by the optimal price vector  $p^*(F)$  at each region  $[z_j, k_{j+1}]$  is bounded by  $\Lambda r_j \max(p_j, p_{j+1})$ . Also observe that whenever  $r_j > 0$ , the prices  $p_j$  and  $p_{j+1}$  has to satisfy  $p_j + p_{j+1} + \theta|l_{j+1} - l_j| > 2v$ , which implies  $x_j > l_j + y_j$ , and  $x_j < l_{j+1} - y_{j+1}$ , thus both  $j, j+1 \in S(p)$ . Combining these observations, we see

that the regret function for any given  $F$  satisfies:

$$\begin{aligned}
r(p, F) &\leq \sum \Lambda o_j v + \sum \Lambda r_j \max(p_j, p_{j+1}) - \sum \Lambda o_j p_j \\
&\leq \sum \Lambda o_j v + \sum \Lambda r_j \max_{j \in S(p)} \{p_j\} - \sum \Lambda o_j \min\{p_j\} \\
&= \Lambda \sum o_j (v - \min_j \{p_j\}) + \Lambda \sum r_j \max_{j \in S(p)} \{p_j\} \\
&\leq \Lambda \max\{(v - \min_j \{p_j\}), \max_{j \in S(p)} \{p_j\}\}.
\end{aligned}$$

Therefore, the regret function  $r(p, F)$  is upper-bounded by  $\Lambda \max\{(v - \min_j \{p_j\}), \max_{j \in S(p)} \{p_j\}\}$  independent of  $F$ . However, as explained above, the adversary can achieve this bound by putting the whole unit mass to the location of the product with the minimum price achieving a maximum regret of  $\Lambda(v - \min_j p_j)$  when  $S(p)$  is empty, or she can select the maximum price whose index- $i$  is in the set  $S(p)$ , and put the whole mass arbitrarily close to point  $l_j + y_j$  (if  $x_j > l_j + y_j$ ) or  $l_j - y_j$  (if  $x_{j-1} < l_j - y_j$ ), which ever gives a greater regret value, achieving (or coming arbitrarily close to) a maximum regret of  $\Lambda \max\{(v - \min_j p_j), \max_{j \in S(p)} p_j\}$ .  $\square$

**Proof of Lemma 2.4:** Consider an optimal solution  $p^*$ . If the set  $S(p^*)$  is empty, the result is obvious in that for the optimal vector  $p^*$ , the vector  $p$  with equal components of  $p_j = \min_j p_j^* \forall j$  has the same regret by definition in equation (2) and by the fact that  $S(p)$  is necessarily empty as  $p \leq p^*$ . Therefore, assume the set  $S(p^*)$  is not empty for the optimal price vector  $p^*$  whose components are not equal. Then we have  $r(p^*) = \Lambda \max\{\max_{j \in S(p^*)} p_j^*, (v - \min_j p_j^*)\}$ . However, the vector  $p$  with equal components of  $p_j = \min_j p_j^* \forall j$  has at most the same regret as shown below, and therefore, is also optimal

$$\begin{aligned}
r(p^*) &= \Lambda \max\{\max_{j \in S(p^*)} p_j^*, (v - \min_j p_j^*)\} \\
&\geq \Lambda \max\{\min_j p_j^*, (v - \min_j p_j^*)\} && \text{as } \min_j p_j^* \leq \max_{j \in S(p^*)} p_j^* \\
&\geq r(p) && \text{by definition of } r(p) \text{ for } p_j = \min_j p_j^* \forall j. \square
\end{aligned}$$

**Proof of Proposition 2.4:** As a consequence of Lemma 2.4, we can restrict our attention to the pricing policies with equal components. This reduces the decision vector to one dimension. Thus, given a decision  $p \in R_+$ , the maximum regret as a function of is:

$$r(p) = \begin{cases} \Lambda (v - p) & \text{if } S(\{p\}) \text{ is empty} \\ \Lambda \max\{(v - p), p\} & \text{otherwise} \end{cases} \quad (13)$$

We prove the resulting in two parts. First, we show that for  $v \geq \theta \max_j |l_j - l_{j-1}|$ , the optimal price vector has  $p_j = v - \frac{\theta \max_j |l_j - l_{j-1}|}{2} \quad \forall j$ : If we can show that at the optimal price  $\{p\}$  with equal components of  $p$ , the set  $S(\{p\})$  is empty, then the result follows directly from the analog steps of competitive ratio analysis which shows that minimum value of maximum regret is achieved for  $p = (v - \frac{\theta \max_j |l_j - l_{j-1}|}{2})$  which yields minimum value of  $\Lambda[v - (v - \frac{\theta \max_j |l_j - l_{j-1}|}{2})] = \Lambda \frac{\theta \max_j |l_j - l_{j-1}|}{2}$ . Therefore assume by contradiction that the set  $S(\{p\})$  is not empty. Then, we have

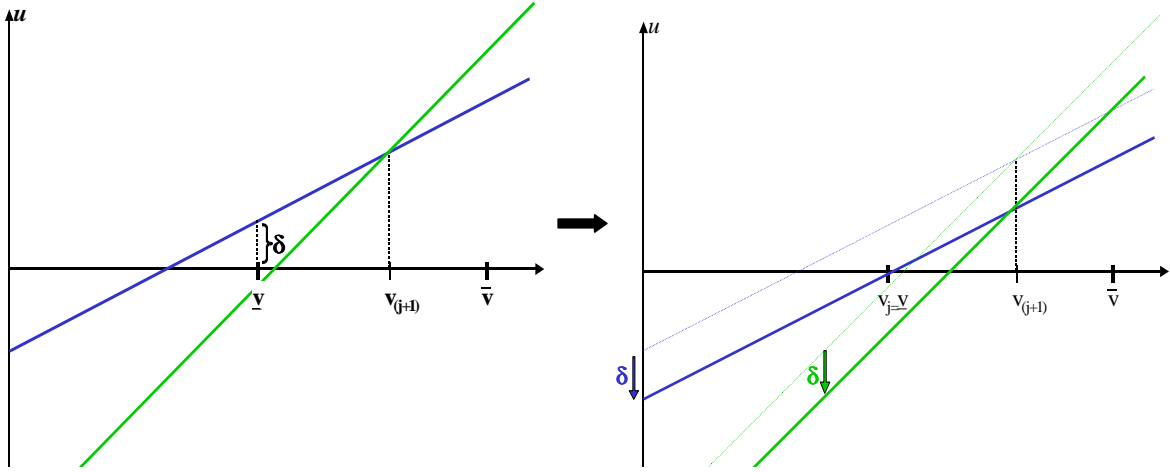
$$r(p) \geq \Lambda p_j > \Lambda (v - \frac{\theta \max_j |l_j - l_{j-1}|}{2}) \geq \Lambda \frac{\theta \max_j |l_j - l_{j-1}|}{2}$$

where the first inequality follows from definition of  $r(p)$  in equation (13), the strict inequality follows from the fact that  $\{p\}$  must violate the inequality  $p_j + p_{j+1} + \theta \max_j |l_j - l_{j-1}| = 2p + \theta \max_j |l_j - l_{j-1}| \leq 2v$  as the set  $S(\{p\})$  is not empty, and last inequality follows by the specification of  $v$  in this region. Therefore, the non-spanning policy  $\{p\}$  gives strictly more regret, and cannot be optimal in this case.

Second, for  $v < \theta \max_j |l_j - l_{j-1}|$ , we show that the optimal price vector has  $p_j = \frac{v}{2} \quad \forall j$  below. Using the previous case, we know that the firm can guarantee a regret value of  $\Lambda \frac{\theta \max_j |l_j - l_{j-1}|}{2}$  using a price policy low enough to span the whole unit circle. Now, we show that a strictly less regret is possible by charging  $p_j = \frac{v}{2} \quad \forall j$ , which is a non-spanning policy for values of  $v$  in this region, therefore  $S(\{\frac{v}{2}\})$  is not empty. If the firm uses a non-spanning pricing policy the maximum regret is given by  $\Lambda \max\{p, (v - p)\}$  by equation (13). However, this function is minimized when  $p = \frac{v}{2}$ , as it is the maximum of two linear functions one increasing the other decreasing in  $p$ . As this yields a regret value of  $\Lambda \frac{v}{2} < \Lambda \frac{\theta \max_j |l_j - l_{j-1}|}{2}$ , the optimal price of  $p_j = \frac{v}{2} \quad \forall j$  for this region. Combining the two cases yields the result.  $\square$

**Proof of Proposition 3.1:** We prove the result in several steps. As the first step, let us show that for every optimal price vector  $p$  where a version is strictly dominated by others, there exists another optimal price vector  $\hat{p}$  where no version strictly dominates (i.e., yields strictly more utility for all quality types) any other version. It is sufficient to show the result for any neighboring product pairs. Assume at the optimal solution, some version  $j$  is strictly dominated by version  $j - 1$  and version  $j + 1$ . The necessary and sufficient condition for this is easily seen to be  $v_{j-1} > v_j$  using the definition of  $v_j$ . Consider such an optimal solution with  $v_{j-1} > v_j$ . Decreasing only the price for version  $j$ ,  $p_j$ , to a level where  $\hat{v}_{j-1} = \hat{v}_j$ , while keeping the other prices fixed, the revenues earned by the firm does not change regardless of the quality taste distribution selected by the adversary. Decreasing any dominated price in such a fashion yields another optimal price vector,  $\hat{p}$  where  $\hat{v}_1 \leq \hat{v}_2 \leq \dots \leq \hat{v}_{K-1}$  holds, i.e. an optimal price vector where no version strictly dominates any other.

Next step is to show that the customer with lowest taste for quality (type) earns zero utility at any optimal price, i.e.  $u(\underline{v}) = 0$ , and that there exists an optimal price vector where  $u(\underline{v}) = \underline{v} l_1 - p_1 = 0$ . Notice that  $u(\underline{v}) = \max_j \{\underline{v} l_j - p_j\} \geq 0$  must hold at the optimal price  $p$ , as otherwise the adversary would guarantee a competitive ratio of zero by selecting the distribution with  $\mathbf{P}(v = \underline{v}) = 1$ . Assume  $\delta \stackrel{def}{=} \max_j \{\underline{v} l_j - p_j\} > 0$ , which would imply that  $u(v) > 0$  for  $\forall v \in [\underline{v}, \bar{v}]$ , i.e. all consumers have strictly positive utility or surplus, and consider the price vector  $\hat{p}$  with  $\hat{p}_j = p_j + \delta \quad \forall j$ . Notice that the resulting indifferent customer types satisfy  $\hat{v}_j = v_j \quad \forall j$ . Thus,  $\hat{p}$  strictly increases the revenues earned by the firm whatever the distribution selected by the adversary is. Consequently,  $\hat{p}$  would yield a strictly better competitive ratio leading to a



contradiction of the optimality of  $p$ .

As a result, we have  $u(\underline{v}) = 0$ . Now, using the first step's conclusion, we have an optimal solution where  $\underline{v} l_1 - p_1 \leq 0 = \max_j \{\underline{v} l_j - p_j\}$ , as otherwise  $p_1$  would be strictly dominated. Assume  $\underline{v} l_1 - p_1 < 0$ , decreasing only the first price to  $\underline{v} l_1$  does not change the revenues earned by the firm under any distribution selected by the adversary, yielding the same optimal competitive ratio, and thus, we always have an optimal price vector with  $\underline{v} l_1 - p_1 = 0$ . Combining all these results so far, we conclude that there exists an optimal solution where  $\underline{v} \leq v_1 \leq v_2 \leq \dots \leq v_{K-1}$  holds. Observe that all customer types choose to buy under such an optimal solution.

The final step is to show that there exists an optimal solution with  $v_{K-1} \leq \bar{v}$ . Consider an optimal solution with  $v_{K-1} > \bar{v}$ , that is  $p_K > \bar{v} (l_K - l_{K-1}) + p_{K-1}$ . The highest quality product is not sold under this optima, and the highest type customer buys version  $K - 1$ . Reducing only the price of the highest quality version to a level  $\hat{p}_K = \bar{v} (l_K - l_{K-1}) + p_{K-1}$ , so that  $\hat{v}_{K-1} = \bar{v}$ , the firm guarantees to earn at least the same revenue regardless of the distribution selected by the adversary. Thus, we achieve another optimal solution satisfying  $v_{K-1} \leq \bar{v}$ . Combining all of the arguments above yields the result of the proposition.

**Proof of Proposition 3.2:** Using the previous proposition, the result is trivially true for  $j = 1$  if  $\underline{v} < v_1$  in the optimal solution. If  $\underline{v} = v_1 = \dots = v_{j-1} < v_j$  for some  $j = 2 \dots K - 1$ , we can solve for the optimal price components of the first  $j$  versions as  $p_1 = \underline{v}l_1$ ,  $p_2 = \underline{v}l_2, \dots$ ,  $p_j = \underline{v}l_j$  using the definitions of  $v_i$  for  $i = 1 \dots j$ . Thus, the customer with lowest quality taste chooses version  $j$  by the tie-breaking assumption as  $u(\underline{v}, 1) = u(\underline{v}, 2) = \dots = u(\underline{v}, j)$ . Furthermore,  $u(\underline{v}, 1) < u(\underline{v}, 2) < \dots < u(\underline{v}, j)$  for  $\forall v > \underline{v}$  as  $l_1 < l_2 < \dots < l_j$ . Consequently, the versions 1 to  $j - 1$  are not bought by any type of customers. Therefore, offering only the  $(K - j + 1)$  highest quality version with their optimal prices achieves the same product selection for each type of customer for  $\forall v \in [\underline{v}, \bar{v}]$  with same prices, resulting in the same revenue collected by the firm and the same response chosen by the adversary, therefore achieving the optimal competitive ratio.  $\square$

**Proof of Lemma 3.1:** For the problem where only the  $(K - j + 1)$  highest quality versions are offered, observe that the best competitive ratio satisfies  $c_K = \frac{p_K}{\bar{v}l_K} = \frac{\underline{v}l_K}{\bar{v}l_K} = \frac{\underline{v}}{\bar{v}}$ . Let us now consider the problem of finding  $c_j$  for  $j = 1 \dots K - 1$ . Defining  $f_j = \mathbf{P}(v < v_j)$ , and  $f_i = \mathbf{P}(v \geq v_{i-1}) - \mathbf{P}(v \geq v_i)$  for  $i = j + 1 \dots K - 1$ , and  $f_K = \mathbf{P}(v \geq v_{K-1})$  the competitive ratio achieved by a pricing policy satisfying  $\underline{v} = \dots = v_{j-1} < v_j \leq v_{j+1} \leq \dots \leq v_{K-1} \leq \bar{v}$  is given by:

$$c_j = \max_p \min_F \frac{\sum_{i=j}^K f_i p_i}{\int_{\underline{v}}^{v_j} p_F(v) dF(v) + \sum_{i=j}^{K-2} \int_{v_i}^{v_{i+1}} p_F(v) dF(v) + \int_{v_{K-1}}^{\bar{v}} p_F(v) dF(v)} \quad (14)$$

where  $p_F(v)$  is the price paid by a customer of type  $v$  for her best choice under the revenue maximizing pricing scheme  $p_F(\cdot)$  for a given quality taste distribution  $F(\cdot)$ .

Now fix some  $p$  offering only the  $(K - j + 1)$  highest quality versions, then the ratio for this fixed price, denoted by  $c_j(p)$ , is given by

$$\begin{aligned} c_j(p) &= \min_F \frac{\sum_{i=j}^K f_i p_i}{\int_{\underline{v}}^{v_j} p_F(v) dF(v) + \sum_{i=j}^{K-2} \int_{v_i}^{v_{i+1}} p_F(v) dF(v) + \int_{v_{K-1}}^{\bar{v}} p_F(v) dF(v)} \\ &\geq \min_F \frac{\sum_{i=j}^{K-1} f_i p_i + f_K p_K}{\sum_{i=j}^{K-1} f_i l_K v_i + f_K l_K \bar{v}} \\ &\geq \min \left\{ \min_{i=j \dots K-1} \{p_i / l_K v_i\}, p_K / l_K \bar{v} \right\} \end{aligned}$$

where the first inequality follows from the fact that every fixed  $F(\cdot)$ ,  $p_F(v) \leq l_K v_j$  for  $v \in [\underline{v}, v_j]$ ,  $p_F(v) \leq l_K v_{i+1}$  for  $v \in [v_i, v_{i+1}]$ ,  $i = j \dots K - 2$ , and  $p_F(v) \leq l_K \bar{v}$  for  $v \in [v_{K-1}, \bar{v}]$ . However, selecting a distribution satisfying  $\Pr(v = v_{i^*} - \epsilon) = 1$ , where  $v_{i^*}$  is the point where the lower bound above achieves its minima, for some arbitrarily small  $\epsilon \in \mathbb{R}_+$ , the adversary can achieve a competitive ratio that is arbitrarily



close to the lower bound above.  $\square$

**Proof of Proposition 3.3:** Using the previous lemma, the problem of finding  $c_j$  can be written as:

$$c_j = \max_p \min \left\{ \min_{i=j \dots K-1} \{p_i/l_K v_i\}, p_K/l_K \bar{v} \right\} \quad (15)$$

$$s.t. \quad v_{n-1} = \frac{p_n - p_{n-1}}{l_n - l_{n-1}} \leq \frac{p_{n+1} - p_n}{l_{n+1} - l_n} = v_n \quad n = j + 1 \dots K - 1 \quad (16)$$

$$v_{K-1} = \frac{p_K - p_{K-1}}{l_K - l_{K-1}} \leq \bar{v} \quad (17)$$

$$v_j = \frac{p_{j+1} - p_j}{l_{j+1} - l_j} > \underline{v} \quad (18)$$

$$p_j = \underline{v} l_j$$

where the constraints (16), (17) and (18) imposes the structure  $\underline{v} < v_j \leq v_{j+1} \leq \dots \leq v_{K-1} \leq \bar{v}$  for the pricing policy.

We will first relax constraint (18) above. Define  $c_j^r$  as the best ratio that can be achieved by relaxing the constraint (18), which yields an upper bound on  $c_j$ . Then, we will convert the formulation of  $c_j^r$  to an equivalent linear program, and use weak duality to upper bound  $c_j^r$ , which in turn yields another upper bound on  $c_j$ . Finally, using strong duality on this dual problem, we will show that  $c_j$  achieves these upper bounds.

Let us first relax constraint (18), and consider

$$c_j^r = \max_p \min \left\{ \min_{i=j \dots K-1} \{p_i/l_K v_i\}, p_K/l_K \bar{v} \right\} \quad (19)$$

$$s.t. \quad v_{n-1} = \frac{p_n - p_{n-1}}{l_n - l_{n-1}} \leq \frac{p_{n+1} - p_n}{l_{n+1} - l_n} = v_n \quad n = j + 1 \dots K - 1 \quad (20)$$

$$v_{K-1} = \frac{p_K - p_{K-1}}{l_K - l_{K-1}} \leq \bar{v} \quad (21)$$

$$p_j = \underline{v} l_j$$

or equivalently,

$$\begin{aligned}
c_j^r &= \max_{p,c} && c \\
&s.t. && c \geq \frac{p_n}{l_K v_n} && n = j+1 \dots K-1 \\
&&& c \geq \frac{p_K}{l_K \bar{v}} \\
&&& v_{n-1} = \frac{p_n - p_{n-1}}{l_n - l_{n-1}} \leq \frac{p_{n+1} - p_n}{l_{n+1} - l_n} = v_n && n = j+1 \dots K-1 \\
&&& v_{K-1} = \frac{p_K - p_{K-1}}{l_K - l_{K-1}} \leq \bar{v} \\
&&& p_j = \underline{v} l_j .
\end{aligned}$$

Renaming the following variables as  $h_j \stackrel{def}{=} l_j$ ,  $y_j \stackrel{def}{=} p_j - \underline{v} l_j$  and  $h_n \stackrel{def}{=} l_n - l_{n-1}$ ,  $y_n = p_n - p_{n-1}$   $n = j+1 \dots K$ , we can reexpress the problem as the following linear program

$$c_j^r = \max_c \max_y c \tag{22}$$

$$\begin{aligned}
&s.t. && - \sum_{i=j+1}^K y_i \leq \underline{v} l_j - c l_K \bar{v} \\
&&& c l_K y_{n+1} - h_{n+1} \sum_{i=j+1}^n y_i \leq \underline{v} l_j h_{n+1} && n = j+1 \dots K-1
\end{aligned}$$

$$\begin{aligned}
&&& c l_K y_{j+1} \leq \underline{v} l_j h_{j+1} \\
&&& v_{n-1} = \frac{y_n}{h_n} \leq \frac{y_{n+1}}{h_{n+1}} = v_n && n = j+1 \dots K-1 \tag{23}
\end{aligned}$$

$$v_{K-1} = \frac{y_K}{h_K} \leq \bar{v} \tag{24}$$

$$y, c \geq 0 .$$

For any fixed  $c$ , the inner maximization problem above is an LP and is feasible for some  $y$  iff  $c \leq c_j^r$ . For

fixed  $c$ , consider the dual of this LP:

$$\begin{aligned}
\min_{w,z,t,q} \quad & (\underline{v} l_j - c l_K \bar{v}) w + \underline{v} l_j \sum_{i=j}^{K-1} h_{i+1} z_i + \bar{v} q \\
\text{s.t.} \quad & -w + c l_K z_j - \sum_{i=j+1}^{K-1} h_{i+1} z_i + \frac{t_{j+1}}{h_{j+1}} \geq 0 \\
& -w + c l_K z_{n-1} - \sum_{i=n}^{K-1} h_{i+1} z_i + \frac{t_n}{h_n} - \frac{t_{n-1}}{h_n} \geq 0 \quad n = j+2 \dots K-1 \\
& -w + c l_K z_{K-1} + \frac{q}{h_K} - \frac{t_{K-1}}{h_K} \geq 0 \\
& w, z, t, q \geq 0
\end{aligned} \tag{25}$$

Observe that for any fixed  $c$ , the primal problem is always bounded and the dual problem is always feasible. Therefore, for any given  $c$ , the primal problem is infeasible, i.e.  $c > c_j^r$ , if and only if the dual is unbounded .

Consider the dual feasible vectors of the form

$$z_{K-1} = \frac{w}{c l_K} \quad \text{and} \quad z_n = \frac{w \prod_{i=n+2}^K (h_i + c l_K)}{(c l_K)^{K-n}} \quad n = j \dots K-2$$

that are parameterized over variable  $w$  yielding dual objective values

$$w \left[ \underline{v} l_j \frac{\prod_{i=j+1}^K (h_i + c l_K)}{(c l_K)^{K-j}} - c l_K \bar{v} \right] \tag{26}$$

Then, let  $\gamma = \underline{v}/\bar{v}$  and define  $c_j^d$  such that

$$\gamma l_j \frac{\prod_{i=j+1}^K (h_i + c_j^d l_K)}{(c_j^d l_K)^{K+1-j}} = 1 \tag{27}$$

The objective function in (26) is decreasing in  $c$  and is equal to 0 whenever  $c = c_j^d$ . Thus, above dual solution makes the problem unbounded, whenever  $c > c_j^d$  by selecting and infinitely large  $w$ . Consequently,  $c_j^r$  cannot be larger than  $c_j^d$ . Thus, using the primal-dual feasibility arguments above, we have

$$c_j \leq c_j^r \leq c_j^d, \quad j = 1 \dots K-1 \tag{28}$$

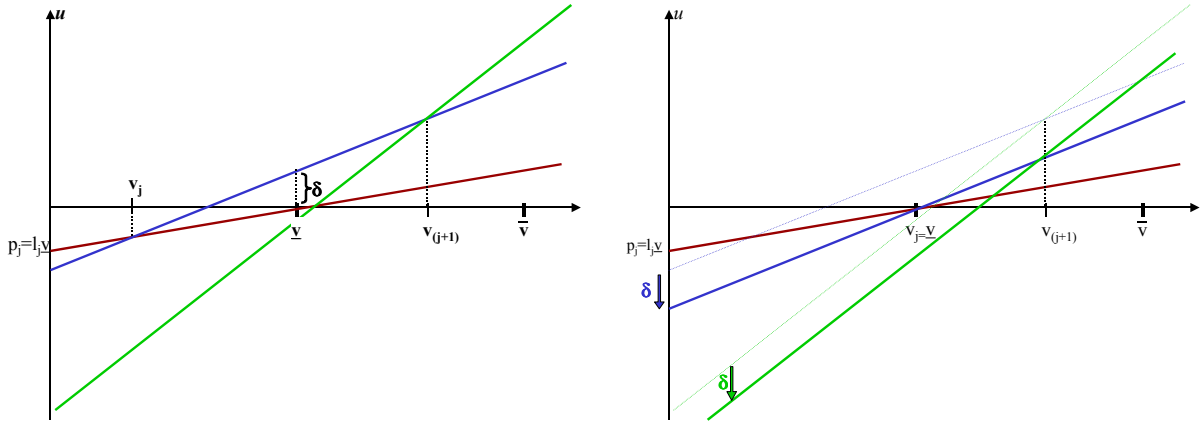
Now consider the following price vector for the problem in (19)

$$\begin{aligned}
 p_n^d &= v l_n, & n &= 1 \dots j \\
 p_n^d &= v l_j \frac{\prod_{i=j+1}^n (h_i + c_j^d l_K)}{(c_j^d l_K)^{n-j}} = p_{n-1}^d \frac{(h_n + c_j^d l_K)}{(c_j^d l_K)} & n &= j+1 \dots K-1
 \end{aligned}$$

which yields the objective value of  $c_j^d$ . Thus  $p^d$  is optimal for the problem of finding  $c_j^d$  by inequality (28) if we can show that it is feasible for formulation (19). Therefore, we proceed to show to feasibility of  $p^d$ .  $p^d$  satisfies the constraints (20) as  $v_{n-1} = \frac{p_n^d - p_{n-1}^d}{h_n} = \frac{p_{n-1}^d}{c_j^d l_K} \leq \frac{p_n^d}{c_j^d l_K} = \frac{p_{n+1}^d - p_n^d}{l_{n+1} - l_n} = v_n$   $n = j+1 \dots K-1$ , and also the constraints (21), as  $v_{K-1} = \frac{p_K^d - p_{K-1}^d}{h_K} = \frac{p_{K-1}^d}{c_j^d l_K} \leq \bar{v}$ , where the inequality follows from the fact that  $\frac{p_{K-1}^d}{\bar{v} l_K} \leq \frac{p_K^d}{\bar{v} l_K} \leq c_j^d$ . As a result,  $p^d$  is feasible and optimal for formulation (19) with the optimal objective function value of  $c_j^d$ , i.e.  $c_j^r = c_j^d$ . Observe that above arguments are valid for every  $j$ , and  $c_j \leq c_j^d = c_j^r$   $j = 1 \dots K-1$ .

Finally, defining  $c_K^d \stackrel{def}{=} v/\bar{v} = c_K \stackrel{def}{=} c_K^r$ , and  $c^r = \max_j \{c_j^r\}$ ,  $c^d = \max_j \{c_j^d\}$  we proceed to show  $c = c^r = c^d$ , which will complete the proof.

So far, we have  $c \leq c^r = c^d$ . Assume  $c_j \leq c < c^d = c^r = c_j^r$  for some  $j$ , which is achieved only if the relaxed constraint is violated, i.e.  $v_j \leq v$  or equivalently  $p_{j+1} \leq p_j + v h_{j+1}$ . Define  $\delta = p_j + v h_{j+1} - p_{j+1}$ , and consider  $\hat{p}$  constructed as  $\hat{p}_i = p_i + \delta$  for  $i = j+1 \dots K$ , and  $\hat{p}_i = p_i$  for  $i \leq j$ . Then,  $\hat{v}_1 = \dots = \hat{v}_j = v$  and  $\hat{v}_i = v_i$  for  $i = j+1 \dots K-1$ , yielding  $p_i/v_i \leq \hat{p}_i/\hat{v}_i$  for  $i = j+1 \dots K-1$ , and  $p_K/\bar{v} \leq \hat{p}_K/\bar{v}$ . Consequently, the competitive ratio achieved by  $\hat{p}$  ( call  $\hat{c}_j$  ) is at least as large as  $c_j^d$ . On the other hand, observe that the price vector  $\hat{p}$  is feasible for the primal problem where only the highest  $K-j$  quality products are offered as  $v = \hat{v}_j < \hat{v}_{j+1} \leq \hat{v}_n$   $n = j+2 \dots K-1$ , resulting in  $c_j^r \leq \hat{c}_j \leq c_{j+1} \leq c$  which yields a contradiction.  $\square$



**Proof of Proposition 3.4:** For  $\gamma_j = l_j^{K-j} / \prod_{i=j+1}^K (h_i + l_j)$ , observe that  $c_j^d = c_{j+1}^d = l_j / l_K$  by equation (27). Also by differentiating equation (27), we have

$$\frac{dc_j^d}{d\gamma} = \frac{c_j^d}{\gamma \left( K + 1 - j - c_j^d l_K \sum_{i=j+1}^K \frac{1}{h_i + c_j^d l_K} \right)}$$

Combining these,

$$\frac{dc_{j+1}^d}{d\gamma} \Big|_{\gamma=\gamma_j} - \frac{dc_j^d}{d\gamma} \Big|_{\gamma=\gamma_j} = \frac{l_j \left( 1 - \frac{l_j}{l_{j+1}} \right)}{\gamma_j l_K \left( K + 1 - j - l_j \sum_{i=j+1}^K \frac{1}{h_i + l_j} \right) \left( K - j - l_j \sum_{i=j+2}^K \frac{1}{h_i + l_j} \right)} > 0$$

which shows that for some  $\epsilon_j^+, \epsilon_j^- \in R_+$ ,  $c_{j+1}^d > c_j^d \forall \gamma \in (\gamma_j, \gamma_j + \epsilon_j^+)$ , and  $c_{j+1}^d < c_j^d \forall \gamma \in (\gamma_j - \epsilon_j^-, \gamma_j)$ . However, whenever  $c_{j+1}^d > c_j^d$  for some  $\gamma$ , we have

$$\begin{aligned} & \frac{dc_{j+1}^d}{d\gamma} \Big|_{\gamma} - \frac{dc_j^d}{d\gamma} \Big|_{\gamma} \\ &= \frac{c_{j+1}^d l_K \left( K + 1 - j - c_j^d l_K \sum_{i=j+1}^K \frac{1}{h_i + c_j^d l_K} \right) - c_j^d l_K \left( K - j - c_{j+1}^d l_K \sum_{i=j+2}^K \frac{1}{h_i + c_{j+1}^d l_K} \right)}{tl_K \left( K + 1 - j - c_j^d l_K \sum_{i=j+1}^K \frac{1}{h_i + c_j^d l_K} \right) \left( K - j - c_{j+1}^d l_K \sum_{i=j+2}^K \frac{1}{h_i + c_{j+1}^d l_K} \right)} \\ &\geq \frac{c_{j+1}^d l_K \left[ \left( K + 1 - j - c_j^d l_K \sum_{i=j+1}^K \frac{1}{h_i + c_j^d l_K} \right) - \left( K - j - c_{j+1}^d l_K \sum_{i=j+2}^K \frac{1}{h_i + c_{j+1}^d l_K} \right) \right]}{tl_K \left( K + 1 - j - c_j^d l_K \sum_{i=j+1}^K \frac{1}{h_i + c_j^d l_K} \right) \left( K - j - c_{j+1}^d l_K \sum_{i=j+2}^K \frac{1}{h_i + c_{j+1}^d l_K} \right)} \\ &\geq \frac{c_{j+1}^d l_K \left( 1 - \frac{c_j^d l_K}{h_{j+1} + c_j^d l_K} \right)}{tl_K \left( K + 1 - j - c_j^d l_K \sum_{i=j+1}^K \frac{1}{h_i + c_j^d l_K} \right) \left( K - j - c_{j+1}^d l_K \sum_{i=j+2}^K \frac{1}{h_i + c_{j+1}^d l_K} \right)} > 0 \end{aligned}$$

As a result, we conclude that  $c_{j+1}^d > c_j^d$  for  $\forall \gamma > \gamma_j$  using the first theorem of calculus. Also similarly we can show that  $c_{j+1}^d < c_j^d$  for  $\forall \gamma < \gamma_j$ . Then, the result follows as the above arguments are valid for all  $j = 1 \dots K - 1$ .  $\square$

**Proof of Proposition 3.7:** The result will be established in several steps. First we will analyze the relaxed versions of the problems  $\inf_{v_0 \in (v, \bar{v})} r(v_0)$  and  $r(v)$ , then we will show that these relaxed versions are sufficient to characterize the optima.

In this respect, let us first consider the relaxed version of the LP in (3) that ignores (11). To find the optimal solution to this relaxed version, we first ignore the constraints (7),(8) and (9); and find  $r$  and  $p$  that solves for constraints (4), (5), (6) and (10) as equalities. This yields the optimal value denoted by  $r_0^j$  below

$$r_0^j = \frac{\bar{v} l_K^{K+2-j}}{(l_j + l_K) \prod_{i=j+1}^K (h_i + l_K)} \quad (29)$$

which is attained by the following price vector

$$p_j = v_0 l_j = r_0'^j l_j / l_K \quad \text{and} \quad p_{n+1} = \frac{r_0'^j h_{n+1} + p_n(l_K + h_{n+1})}{l_K} \quad n = j \dots K-1 \quad (30)$$

Below we show this solution satisfies constraints (7), (8) and (9), justifying its optimality for the relaxed LP that ignores constraint (11). For inequality (7), we use the definition of  $p$  in (30):

$$v_{n-1} = \frac{p_n - p_{n-1}}{h_n} = \frac{r_0'^j + p_{n-1}}{l_K} \leq \frac{r_0'^j + p_n}{l_K} = \frac{p_{n+1} - p_n}{h_{n+1}} = v_n \quad n = j+1 \dots K-1$$

For inequality (8), we use constraint (6)

$$v_{K-1} = \frac{p_K - p_{K-1}}{h_K} = \frac{r_0'^j + p_{K-1}}{l_K} \leq \frac{r_0'^j + p_K}{l_K} = \frac{\bar{v} l_K}{l_K} = \bar{v}$$

Finally, for inequality (9), we first use the definition of  $p$  in (30), then constraint (4) to get:

$$v_j = \frac{p_{j+1} - p_j}{h_K} = \frac{r_0'^j + p_j}{l_K} = \frac{r_0'^j + v_0 l_j}{l_K} = \frac{v_0 l_K + v_0 l_j}{l_K} \geq v_0$$

Similarly, let us now consider finding  $r_j(v)$ . We have

$$r_j(v) = \min_{p \in P_j(v)} r_j(v, p) = \min_{p \in P_j(v)} \max \left\{ \max_{i=j \dots K-1} \{l_K v_i - p_i\}, l_K \bar{v} - p_K \right\}$$

which also can be reexpress with the following LP formulation

$$\underline{r}^j \stackrel{\text{def}}{=} r_j(v) = \min_{p, r} \quad r \quad (31)$$

$$\text{s.t.} \quad r \geq v_n l_K - p_n \quad n = j \dots K-1 \quad (32)$$

$$r \geq \bar{v} l_K - p_K \quad (33)$$

$$v_{n-1} = \frac{p_n - p_{n-1}}{h_n} \leq \frac{p_{n+1} - p_n}{h_{n+1}} = v_n \quad n = j+1 \dots K-1 \quad (34)$$

$$v_{K-1} = \frac{p_K - p_{K-1}}{h_K} \leq \bar{v} \quad (35)$$

$$v_j = \frac{p_{j+1} - p_j}{h_{j+1}} \geq v \quad (36)$$

$$p_j = v l_j \quad (37)$$

Consider now the relaxed version of this LP which ignores constraint (36). To find the optimal solution to

this relaxed version, we first ignore the constraints (34) and (35), and find  $r$  and  $p$  that solves for constraints (32), (33) and (37) as equalities. This yields,

$$\underline{r}^j \stackrel{def}{=} \frac{\bar{v} l_K^{K+1-j}}{\prod_{i=j+1}^K (h_i + l_K)} - \underline{v} l_j \quad (38)$$

with the following optimal price vector

$$p_j = \underline{v} l_j \quad \text{and} \quad p_{n+1} = \frac{\underline{r}^j h_{n+1} + p_n (l_K + h_{n+1})}{l_K} \quad n = j \dots K - 1 \quad (39)$$

Below we show this solution satisfies constraints (34) and (35), justifying its optimality for the relaxed LP that ignores constraint (36). For inequality (34), we use the definition of  $p$  in (39):

$$v_{n-1} = \frac{p_n - p_{n-1}}{h_n} = \frac{\underline{r}^j + p_{n-1}}{l_K} \leq \frac{\underline{r}^j + p_n}{l_K} = \frac{p_{n+1} - p_n}{h_{n+1}} = v_n \quad n = j + 1 \dots K - 1$$

and for inequality (35), we use constraint (33)

$$v_{K-1} = \frac{p_K - p_{K-1}}{h_K} = \frac{\underline{r}^j + p_{K-1}}{l_K} \leq \frac{\underline{r}^j + p_K}{l_K} = \frac{\bar{v} l_K}{l_K} = \bar{v}$$

Finally, we establish that considering these relaxed versions are sufficient to characterize the optima, by showing that

$$r^* = \min_j \{ \min \{ r_j(\underline{v}), \inf_{v_0 \in (\underline{v}, \bar{v})} r_j(v_0) \} \} = \min_j \{ \min \{ \underline{r}^j, r_0^j \} \} = \min_j \{ \min \{ \underline{r}^j, r_0^j \} \} .$$

We have  $\min_j \{ \min \{ \underline{r}^j, r_0^j \} \} \leq \min_j \{ \min \{ \underline{r}^j, r_0^j \} \} = \min_j \{ \min \{ r_j(\underline{v}), \inf_{v_0 \in (\underline{v}, \bar{v})} r_j(v_0) \} \}$ . Assume  $\min \{ \underline{r}^j, r_0^j \} = \min_j \{ \min \{ \underline{r}^j, r_0^j \} \} < \min_j \{ \min \{ r_j(\underline{v}), \inf_{v_0 \in (\underline{v}, \bar{v})} r_j(v_0) \} \}$  for some  $j$ , which implies  $\underline{r}^j < r_j(\underline{v})$  or  $r_0^j < \inf_{v_0 \in (\underline{v}, \bar{v})} r_j(v_0)$  or both.

If  $\underline{r}^j < r_j(\underline{v})$ , we must have  $v_j < \underline{v}$  at optimal price  $p$  for the relaxed problem of finding  $\underline{r}^j$ , or equivalently  $p_{j+1} < p_j + \underline{v} h_{j+1}$ . Also, optimality conditions impose that  $\underline{r}^j = v_i l_K - p_i$  for  $i = j \dots K - 1$ . Define  $\delta = p_j + \underline{v} h_{j+1} - p_{j+1}$ , and consider some  $\hat{p}$  constructed as  $\hat{p}_i = p_i + \delta$  for  $i = j + 1 \dots K$ , and  $\hat{p}_i = p_i$  for  $i \leq j$ . Then,  $\hat{v}_1 = \dots = \hat{v}_j = \underline{v}$  and  $\hat{v}_i = v_i$  for  $i = j + 1 \dots K - 1$ , yielding  $\hat{v}_i l_K - \hat{p}_i < v_i l_K - p_i$  for  $i = j + 1 \dots K - 1$ . Consequently,  $r_{j+1}(\underline{v}) \leq \max \{ \max_{i=j+1 \dots K-1} \{ \hat{v}_i l_K - \hat{p}_i \}, \bar{v} l_K - \hat{p}_K \} < \underline{r}^j$  as  $\hat{p}$  is feasible for the problem of finding  $r_{j+1}(\underline{v})$ .

On the other hand, if  $r_0^j < \inf_{v_0 \in (\underline{v}, \bar{v})} r_j(v_0)$ , we must have  $v_0 \leq \underline{v}$  at optimal price  $p$  for the relaxed

problem of finding  $\underline{r}^j$ ; which implies  $u(\underline{v}) = \underline{v} l_{j+1} - p_{j+1} \geq 0$ . Also, optimality conditions impose that  $r_0^{j,j} = v_i l_K - p_i$  for  $i = j \dots K-1$ . Define  $\delta = u(\underline{v})$ , and consider, some  $\hat{p}$  constructed as  $\hat{p}_i = p_i + \delta$  for  $i = j+1 \dots K$ , and  $\hat{p}_i = p_i$  for  $i \leq j$ . Then,  $\hat{v}_1 = \dots = \hat{v}_j = \underline{v}$  and  $\hat{v}_i = v_i$  for  $i = j+1 \dots K-1$ , yielding  $\hat{v}_i l_K - \hat{p}_i \leq v_i l_K - p_i$  for  $i = j+1 \dots K-1$ . Consequently,  $r_{j+1}(\underline{v}) \leq \max \{ \max_{i=j+1 \dots K-1} \{ \hat{v}_i l_K - \hat{p}_i \}, \bar{v} l_K - \hat{p}_K \} \leq r_0^{j,j}$  as  $\hat{p}$  is feasible for the problem of finding  $r_{j+1}(\underline{v})$ . Combining these two cases we have  $\min \{ \underline{r}^j, r_0^{j,j} \} \geq r_{j+1}(\underline{v}) \geq r^*$  which yields a contradiction.  $\square$

**Proof of Proposition 3.9:** The first inequality is obtained as follows

$$\begin{aligned} \frac{\bar{v} l_K}{2} &= \frac{\bar{v} l_K^3}{2l_K^2} > \frac{\bar{v} l_K^3}{2l_K^2 + h_K l_{K-1}} = \frac{\bar{v} l_K^3}{l_K^2 + l_K(l_K + h_K) + h_K l_{K-1}} \\ &= \frac{\bar{v} l_K^3}{(l_{K-1} + l_K)(l_K + h_K)} = r_0^{K-1} \end{aligned}$$

Similarly, for the other inequalities, we note that

$$\frac{1}{l_{j+1} + l_K} > \frac{l_K}{l_K^2 + l_K l_{j+1} + h_{j+1} l_j} = \frac{l_K}{l_K^2 + l_K(l_j + h_{j+1}) + h_{j+1} l_j} = \frac{l_K}{(l_j + l_K)(l_K + h_{j+1})},$$

multiplying both sides by  $\frac{\bar{v} l_K^{K+1-j}}{\prod_{i=j+2}^K (h_i + l_K)}$  yields  $r_0^{j+1} > r_0^{j,j}$ , which is valid for  $j = 1 \dots K-1$ .  $\square$

**Proof of Proposition 3.4:** For  $\gamma < \gamma_1$ , we have

$$\begin{aligned} \gamma = \underline{v}/\bar{v} &< \frac{l_K^K}{\prod_{i=1}^K (h_i + l_K)} \\ \Rightarrow \underline{v} &< \bar{v} \frac{l_K^K}{\prod_{i=1}^K (h_i + l_K)} \\ \Rightarrow \underline{v} l_1 &< \bar{v} l_1 \frac{l_K^K}{\prod_{i=1}^K (h_i + l_K)} \\ \Rightarrow 0 &< \bar{v} \frac{l_1}{h_1 + l_K} \frac{l_K^K}{\prod_{i=2}^K (h_i + l_K)} - \underline{v} l_1 \\ \Rightarrow 0 &< \bar{v} \frac{l_1}{l_1 + l_K} \frac{l_K^K}{\prod_{i=2}^K (h_i + l_K)} - \underline{v} l_1 && \text{as } h_1 = l_1 \text{ by definition} \\ \Rightarrow \bar{v} \frac{l_K}{l_1 + l_K} \frac{l_K^K}{\prod_{i=2}^K (h_i + l_K)} &< \bar{v} \frac{l_K^K}{\prod_{i=2}^K (h_i + l_K)} - \underline{v} l_1 \\ \Rightarrow r_0^{j+1} &< \underline{r}^1. \end{aligned}$$

Similarly, using the reverse arguments, we can show that for  $\gamma \geq \gamma_1$ , we have  $r_0^{j+1} \geq \underline{r}^1$ .



Now, let us show that for  $\gamma < \gamma_{j+1}$ , we have  $\underline{r}^j < \underline{r}^{j+1}$ ; and  $\gamma \geq \gamma_{j+1}$ , we have  $\underline{r}^j \geq \underline{r}^{j+1}$ . For  $\gamma < \gamma_{j+1}$ ,

$$\begin{aligned}
\gamma = \underline{v}/\bar{v} &< \frac{l_K^{K-j}}{\prod_{i=j+1}^K (h_i + l_K)} \\
\Rightarrow \underline{v} h_{j+1} &< \bar{v} h_{j+1} \frac{l_K^{K-j}}{\prod_{i=j+1}^K (h_i + l_K)} \\
\Rightarrow \underline{v} h_{j+1} &< \bar{v} \frac{h_{j+1}}{h_{j+1} + l_K} \frac{l_K^{K-j}}{\prod_{i=j+2}^K (h_i + l_K)} \\
\Rightarrow \underline{v} h_{j+1} &< \bar{v} \left(1 - \frac{l_K}{h_{j+1} + l_K}\right) \frac{l_K^{K-j}}{\prod_{i=j+2}^K (h_i + l_K)} \\
\Rightarrow \underline{v} (l_{j+1} - l_j) &< \bar{v} \left(1 - \frac{l_K}{h_{j+1} + l_K}\right) \frac{l_K^{K-j}}{\prod_{i=j+2}^K (h_i + l_K)} \\
\Rightarrow \bar{v} \frac{l_K^{K-j+1}}{\prod_{i=j+1}^K (h_i + l_K)} - \underline{v} l_j &< \bar{v} \frac{l_K^{K-j}}{\prod_{i=j+2}^K (h_i + l_K)} - \underline{v} l_{j+1} \\
\Rightarrow \underline{r}^j &< \underline{r}^{j+1} .
\end{aligned}$$

Similarly, using the reverse arguments, we can show that for  $\gamma \geq \gamma_{j+1}$ , we have  $\underline{r}^j \geq \underline{r}^{j+1}$ . Then, the result follows as the above arguments are valid for all  $j = 1 \dots K$ .  $\square$