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ABSTRACT

An optimal tax and government borrowing plan in a setting with tax distortions (Barro, 1979) locally pin down the marginal cost of servicing government debt, called marginal p. An option to default determines the government's debt capacity and its optimal state-contingent risk management policies make its debt risk-free. Optimal debt-GDP ratio dynamics are driven not only by three widely discussed forces, 1.) a primary deficit, 2.) interest payments, and 3.) GDP growth, but also by 4.) hedging costs. Hedging fundamentally alters debt transition dynamics and equilibrium debt-capacity, which are at the center of the recent 'r-g' and debt sustainability discussions. We calibrate our model and make comparative dynamic quantitative statements about the debt-GDP ratio transition dynamics, equilibrium debt capacity, and how long it will take the US to attain debt capacity.

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1 Introduction

In Q4 2021 US federal debt is 123% of US GDP. In some developed European and Asian countries it is even higher. What is a maximum sustainable government debt-to-GDP ratio? Under a good policy, how long will it take for the US debt to GDP ratio to attain that maximum? How costly is it for a government to service its debt and how does that depend on its current debt-GDP ratio? Should a government plan to borrow more when, as in the US today, interest rates on government debt are lower than prospective GDP growth rates? Under an optimal policy, how much will US tax rates have to rise over time in order to finance the $29.6 trillion dollar debt outstanding as of Q4 2021?[1]

To answer such questions, we construct a tractable stochastic continuous-time model of taxes and government debt that adds three features to a deterministic model of Barro (1979). We retain a key assumption of Barro (1979) that deadweight losses from distortionary taxes are convex in tax revenue and homogeneous of degree one in output and tax revenue. The debt-GDP ratio b emerges as a state variable. A government optimally smooths the household's tax burdens over time by equating the marginal cost of taxing the household with the marginal benefit of using tax proceeds to service government debt. While in Barro (1979) a government solves a discounted deadweight loss minimization problem, the structure of our model impels us instead to ask a government to maximize a risk-adjusted present value of total cashflow payoffs to the household.[2]

In addition to being set within an explicitly stochastic environment, the three features appearing in our model but not in Barro (1979) include options for the government to default on its debt as in Eaton and Gersovitz (1981), complete financial spanning and risk premia, and a government that is impatient relative to the representative household that is paying taxes to the government as in Aguiar and Amador (2021) and DeMarzo, He, and Tourre (2021).[3] We show how our no-commitment-to-repay assumption shapes a government's equilibrium debt capacity.[4] Upon default, the government's debt balance drops to zero,

[1]The numbers quoted here are from Fred at https://fred.stlouisfed.org.

[2]In Barro (1979), the household's value maximization problem is equivalent to the tax distortion cost minimization problem because the government full commits to repay its debt and output is exogenous. Therefore, the solution in Barro (1979) is indeed welfare maximizing. However, in our model, we have to work with the value maximization problem as the government's limited commitment to repay its debt causes output to be endogenous. We cannot simply follow Barro (1979) to solve the distortion cost minimization problem.

[3]While there is no default in equilibrium in our model, the default option induces a limited-commitment constraint. Outcomes in our model differ from Eaton and Gersovitz (1981). Aguiar and Amador (2021) present sovereign debt models with limited commitment.

[4]Our model shares an emphasis on the effects of financial constraints on sovereign finance with Bolton (2016), Bolton and Huang (2018), and Rebelo, Wang, and Yang (2021).

output decreases, and the government permanently loses access to the debt market, with the consequence that thereafter it must set the primary government surplus to zero each period and also face a more adverse tax distortion function.[5] As in Thomas and Worrall (1988), Worrall (1990), and Kehoe and Levine (1993), adverse continuation values consequent upon default deter a borrower from reneging on its debt while bounding from above its sustainable debt.[6]

Continuous time facilitates a sharp characterization of debt limits and debt dynamics.[7] Two conditions allow us to characterize the maximally sustainable risk-free debt-to-GDP ratio $\overline{b}$: 1.) the government's indifference condition between defaulting and servicing its debt induced by its limited commitment and 2.) a zero-drift condition for the debt-GDP ratio b at debt capacity $\overline{b}$, which boils down to an equivalent perpetual (Gordon) growth valuation formula at a steady state $\overline{b}$.[8] We find that the quantitative effect of the limited-commitment constraint is substantial. Only by incorporating this limited-commitment constraint, can we generate a debt-GDP capacity $\overline{b}$ in a plausible range of 150-300%. If we withdraw our limited-commitment debt-market participation constraint, our model becomes a stochastic version of Barro's that shares his commitment-to-repay assumption. That version of the model predicts debt capacity that we think is implausibly high, in the range of 10-15 times GDP.

Our second amendment relative to Barro (1979) is that we assume that government debt bears a risk premium that reflects the correlation between a country's GDP growth rate and an aggregate stock market return. In the spirit of arrangements proposed by Shiller (1994), we assume that the government trades assets that allow it to insure itself against risk in GDP growth rates. We take as exogenous a stochastic discount factor (SDF) process implied in Black and Scholes (1973) and Merton (1973), a process that we assume is not affected by

[5]Our main qualitative results are robust to the detailed specification of punishments for default. The key is that default is costly and hence the government faces a consequence from default. The costly default supports a debt capacity. Otherwise, optimal debt capacity would be zero as shown by Bulow and Rogoff (1989).

[6]Our model shares some of the structure of the simple villager-money-lender model that Ljungqvist and Sargent (2023, ch. 22) use to introduce some of the ideas in the closed economy model of Kocherlakota (1996) that builds on and reinterprets Thomas and Worrall (1988).

[7]DeMarzo, He, and Tourre (2021) construct a continuous-time sovereign-debt model that generates equilibrium debt ratcheting. Rebelo, Wang, and Yang (2021) construct a continuous-time sovereign-debt model in which a country's degree of financial development, defined as how easily it can issue debt denominated in domestic currency in international capital markets, generates "debt intolerance" in the sense of Rogoff, Reinhart, and Savastano (2003).

[8]The zero-drift condition at $\overline{b}$ is an equilibrium argument based on local changes. The Gordon growth model at the steady state is a forward-looking present value calculation argument for the determination of $\overline{b}$. They are equivalent. A non-zero drift of b at $\overline{b}$ would be inconsistent with the notion of debt capacity.

the government's tax and borrowing policy.[9] Our use of a SDF brings insights about how government debt is evaluated in complete markets settings in ways also studied empirically by Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019).

Two important implications of our complete-market SDF formulation are that 1.) the optimal debt-GDP ratio b process evolves deterministically and 2.) the risk premium has a first-order effect on the dynamics of the debt-GDP ratio b. Both implications follow from the government's incentive to reduce the household's tax burdens. Thus, to smooth taxes over time and across states, it is optimal to make the b process deterministic – this follows from applying Jensen's inequality to the first-order condition for the tax rate. With complete financial spanning, it is feasible to make contributions to the volatility of b from both the systematic and idiosyncratic risks be zero; optimal risk management policies do indeed set them both to zero. Finally, while it is costless to hedge idiosyncratic risk, the government has to pay a risk premium to hedge the systematic risk component of its GDP shock by trading in markets for GDP growth rate instruments like those described by Shiller (1994).[10]

Dynamics of the debt-GDP ratio b under optimal policies is deterministic and driven by four forces. In addition to 1.) the primary deficit, 2.) interest payments, and 3.) GDP growth, our model also features a fourth: hedging costs. We summarize the equilibrium debt-GDP process b under optimality as follows:

$$\text{change of } b = \text{primary deficit} + \text{interest rate } (r) \times b - \text{growth } (g) \times b + \text{hedging cost}. \quad (1)$$

The first term on the right side of (1) is the scaled primary deficit, the difference between government spending and tax revenues, divided by contemporaneous GDP. The second term is the (scaled) interest payment, which equals b multiplied by the risk-free rate r. The third term describes debt reduction due to growth, which equals b multiplied by the expected GDP growth rate. These widely acknowledged three terms are discussed, for example, by Blanchard (2019) and Mehrotra and Sergeyev (2021). In addition to those three terms, our model contains a fourth terms because it is optimal for the government to hedge its GDP process in a way that makes b evolve deterministically. The associated hedging cost, the fourth term in (1), equals b multiplied by the risk premium of a risky asset whose cash flow process is the same as the GDP process. Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2020) put related risk-pricing formulas to work to value government debt. See Barro

[9]The SDF process that we use is the endogenous SDF that emerges from the equilibrium asset-pricing model of Lucas (1978).

[10]States in which the stock market return is high are also ones in which investors' marginal utility (equivalently the SDF) is low. That is why the SDF and the market return are negatively correlated.

(2020), Van Wijnbergen, Olijslagers, and de Vette (2020), Aguiar, Amador, and Arellano (2021), Mian, Straub, and Sufi (2021), and Reis (2021) for more about '$r-g$' and debt sustainability.

A third amendment relative to Barro (1979) is that our government is impatient as in Aguiar and Amador (2021). By impatience, we mean that a government's discount rate exceeds the interest rate. With impatience, our model generates a backloaded tax schedule so that the optimal tax rate on output increases over time. This makes fiscal deficits scaled by GDP decrease over time and eventually turn into fiscal surpluses. The debt-GDP ratio moves towards a steady state in which it attains its maximally sustainable level $\overline{b}$. When it is sufficiently impatient, a government with a sufficiently low level of debt immediately increases its b to an optimal target level $\underline{b} > 0$ in which the government's marginal cost of servicing debt equals one.[11] Thus, optimal debt-GDP dynamics reside in three disjoint regions:[12] 1.) a lumpy debt issuance and payout region in which $b < \underline{b}$; 2.) a default region in which debt is unsustainable ($b > \overline{b}$); and 3.) the interior region in which $b \in [\underline{b}, \overline{b}]$.

An optimum is described by 1.) a first-order nonlinear ordinary differential equation (ODE) for the government's (scaled) value $p(b)$; 2.) a first-order condition for the optimal smooth tax rate $\tau(b)$; 3.) a zero-drift condition and the indifference condition between defaulting and not that characterize the steady state where debt is at the maximally sustainable level $\overline{b}$; 4.) value-matching and smooth-pasting conditions that characterize the lumpy debt issuance and payout boundary $\underline{b}$. The upper debt-capacity boundary $\overline{b}$ is an absorbing state and the lower lumpy debt issuance boundary $\underline{b}$ is a reflecting barrier. These two are very different types of boundaries that reflect different economic mechanisms for the government's maximally sustainable debt and its optimal policy for lumpy payouts to the household.[13]

The government's marginal cost of servicing debt $-p'(b) \geqslant 1$ measures how much the government's value decreases in the absolute value when the government debt-GDP ratio increases by one unit. This marginal cost of debt servicing appears in both the first-order

[11] To construct an optimal fiscal plan, our government uses both singular control (lumpy debt issuance and payout to the household) and convex control (tax smoothing). The US government's 2020 and 2021 covid stimulus checks and related transfers might be interpreted as examples of such payouts financed by lumpy debt issuances.

[12] The lumpy debt issuance and payout region and the default region are only possible at time 0. If starting in the lumpy debt issuance and payout region where $b < \underline{b}$, the government increases its debt so that its b instantly equals $\underline{b}$ after time 0 and then the b process is dictated by the law of motion in the interior region. If starting in the default region where $b > \overline{b}$, the government immediately defaults and sets taxes to its expenditure so that its primary deficit is zero at all time.

[13] Our baseline model is amenable to extensions that will allow additional sources of randomness not included in the baseline model – e.g., a Markov process for the government expenditures/GDP ratio rather than the fixed ratio in the baseline model.

condition for the optimal tax rate and an equation restricting the government's optimal value function.

We use a calibrated version of our model to approximate how long it will take for the US to exhaust its debt capacity. Such calculations help sort through current debates about debt sustainability. We show that the time to reach debt capacity critically depends both on a government's impatience and on the prevailing interest rate.[14] Holding a government's impatience fixed, the lower is the interest rate, the higher is a government's debt capacity. So in an economy in which the interest rate on government debt is low, a government taxes less and borrows more now, making the debt-GDP ratio increases at a faster rate. In this situation, a direct debt-capacity effect dominates an indirect (debt-GDP ratio) drift effect so that it takes longer time to reach its debt capacity. Such logic underlies an argument that a government should borrow more when debt is cheap, e.g., Blanchard (2019).

However, if we hold a government's discount rate fixed, a lower interest rate also makes a government more impatient – impatience introduces a wedge between the discount rate and the interest rate. In a quantitative exercise in Section 7, we show that a government reaches its debt capacity faster in a lower interest rate environment. This is because when a government is sufficiently impatient the indirect drift effect dominates the direct debt capacity effect. Cheaper debt (a lower interest rate) causes an impatient government to accelerate its borrowing and consequently exhaust its debt capacity sooner. These comparative dynamic analyses with respect to impatience and interest rate highlight roles that key structural parameters play in shaping policy responses.

By deploying a continuous-time contracting framework like that used by DeMarzo and Sannikov (2006) and Sannikov (2008), we show how to formulate the government's optimal debt management problem as a dual problem for a planner facing a government that has default opportunities.[15] In the dual problem, the key state variable is the government's promised value and the well-diversified planner maximizes the present value of the cash flows subject to optimally managing the government's promised value.

Related Literature. By taking a stochastic discount factor process as exogenous, our model contrasts with the Lucas and Stokey (1983) model in which a government's tax and borrowing strategy affects the stochastic discount factor process, motivating the government to manipulate equilibrium prices of its debts. Like Lucas and Stokey (1983), we assume

[14] Bohn (1998) described measures that the US took in response to the accumulation of debt during the 1970s and 1980s that are broadly consistent with dynamics prescribed by our model.

[15] Ai and Li (2015) and Bolton, Wang, and Yang (2019) construct recursive contracts to cope with limited-commitment problems in corporate finance.

complete financial markets that allow the government to make its debt fully state contingent. By staying within the Barro tradition of an exogenous SDF process,[16] we remove the dynamic inconsistencies that arise from the price-manipulation motives central to models in the Lucas-Stokey tradition. Thus, we focus on implications of limited commitment for debt capacity and debt dynamics. Our model blends key building blocks from Lucas and Stokey (1983) (complete state-contingent debt) and Barro (1979) (tax distortion costs) in a tractable continuous-time framework with an exogenously specified SDF along lines of Black and Scholes (1973), Merton (1973), and Harrison and Kreps (1979).

Bohn (1990) studies the role of hedging with financial instruments in shaping optimal fiscal policy of a risk-neutral government in a stochastic reformulation of Barro (1979). A difference between our paper and Bohn (1990) is that hedging costs play a key role in debt-GDP dynamics in our model. Bohn (1995) was among the first researchers to value government debt with an SDF like that of Lucas (1978). We extend Bohn's insights by incorporating effects of default opportunities on debt dynamics and sustainability. Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) analyze how the covariance between an intertemporal marginal rate of substitution and a primary government surplus ought to affect the value of government debt. Brunnermeier, Merkel, and Sannikov (2022) develop a model of safe assets with a negative beta in an incomplete-markets setting and analyze implications for debt sustainability. Reis (2021) studies debt capacity in a related model with a bubble on government debt. D'Erasmo, Mendoza, and Zhang (2016) review the literature on government debt sustainability. Abel, Mankiw, Summers, and Zeckhauser (1989) and Abel and Panageas (2022) analyze maximum budget-feasible government debt in overlapping generations models with perpetually zero primary budget surpluses.

We call it a p theory of taxes and government debt because a key outcome in our model is a marginal cost of servicing debt $-p'(b)$ and also because we can invoke an analogy with a q theory of investment. The convex tax distortion cost inherited from Barro (1979) in our model serves as a counterpart to the convex capital adjustment cost in the q theory of investment, e.g., Hayashi (1982). In q theory, marginal q (marginal value of capital) equals the marginal cost of investing. In our p theory, the marginal cost of taxing equals marginal cost of servicing government debt, $-p'(b)$. In a q theory, a firm's asset is productive capital that generates a cash flow. In our p theory, government debt is both "backward" and "forward looking": it cumulates past primary government deficits and has to be serviced from prospective primary surpluses. Marginal q exceeds one in q theory because it is costly

[16] Because the Barro (1979) model is deterministic, his SDF is an exponential function that decays at the risk-free rate per unit of time.

to adjust productive capital, while the marginal cost of servicing debt, $-p'(b)$, exceeds one in our p theory because the prospective taxes that service government are distortionary. Thus, it is useful to watch our model enlist features and unleash forces that resemble ones appearing in the q-theories of costly capital stock adjustment of Lucas and Prescott (1971), Hayashi (1982), and Abel and Eberly (1994). Tax distortions in our model affect asset valuations and act in ways similar to the costs of capital adjustment in the q theories.

2 The Setting

Time $t \in [0, +\infty)$ is continuous. A government maximizes the net present value of the household's payoffs while financing a stream of exogenous stochastic government spending by levying taxes, issuing and servicing risk-free debt, and hedging risk in prospective GDP growth rates. We generalize Barro (1979) along the following three aspects. First, we introduce both idiosyncratic and systematic shocks that allow us to analyze how risks affect taxation and debt management. Second, in the spirit of Thomas and Worrall (1988), Worrall (1990), Kehoe and Levine (1993), Kocherlakota (1996), Alvarez and Jermann (2000), and Chien and Lustig (2010), our decision maker – the government – cannot commit and is free at each instant to default; that constrains its ability to borrow and induces an endogenous debt capacity. Third, we assume that the government is impatient.

We describe two interrelated "regimes". In a "normal" regime, the government services its debt obligations and chooses how much to tax. At every instant, the government can default on its debt, with the consequence that it enters a "default" regime from which it can never leave.

2.1 Output, Government Spending, and Taxation

After describing GDP, government spending, and taxation in the normal regime, we'll describe them in the default regime.

2.1.1 Output, Government Spending, and Taxation in the Normal Regime

Output Process. GDP $\{Y_t; t \geqslant 0\}$ is exogenous and follows a geometric Brownian motion (GBM) process

$$\frac{dY_t}{Y_t} = g dt + \sigma_Y d\mathcal{Z}_t^Y \,, \tag{2}$$

where $\mathcal{Z}_t^Y$ is a standard Brownian motion under the physical measure $\mathbb{P}$, g is the expected GDP growth rate, $\sigma_Y > 0$ is the growth volatility, and $Y_0 > 0$ is the known initial value of Y_t.

GDP Y_t is subject to both idiosyncratic shocks that bear no risk premium, and systematic shocks that bear a risk premium. Let the standard Brownian motion $\mathcal{Z}_t^h$ represent the idiosyncratic shock and the standard Brownian motion $\mathcal{Z}_t^m$ represent the systemic shock under a physical measure $\mathbb{P}$, respectively. We also refer to the systematic shock $d\mathcal{Z}_t^m$ as the market shock.[17] Without loss of generality, we can decompose the output shock $d\mathcal{Z}_t^Y$ over dt under the physical measure $\mathbb{P}$ as

$$d\mathcal{Z}_t^Y = \sqrt{1-\rho^2}\, d\mathcal{Z}_t^h + \rho\, d\mathcal{Z}_t^m\,, \tag{3}$$

where ρ is the constant correlation coefficient between the output shock $d\mathcal{Z}_t^Y$ and the aggregate (market) shock $d\mathcal{Z}_t^m$. For convenience, we also equivalently write the output process $\{Y_t; t \geqslant 0\}$ given in (2) as

$$\frac{dY_t}{Y_t} = gdt + \left(\psi_h d\mathcal{Z}_t^h + \psi_m d\mathcal{Z}_t^m\right)\,, \tag{4}$$

where ψ_m and ψ_h are the systematic and idiosyncratic volatility parameters given by

$$\psi_m = \rho\sigma_Y \qquad \text{and} \qquad \psi_h = \sqrt{1-\rho^2}\,\sigma_Y\,, \tag{5}$$

respectively. Expressions (4)-(5) for $\{Y_t; t \geqslant 0\}$ are convenient for analyzing distinct roles of systematic and idiosyncratic shocks.

Government Spending and Debt. Let $\{\Gamma_t; t \geqslant 0\}$ denote the government spending process that is exogenous and does not bring utils to the household. For tractability, we assume that Γ_t depends on contemporaneous output Y_t in the normal regime as

$$\Gamma_t = \gamma_t Y_t\,, \tag{6}$$

where γ_t is exogenous. For expositional simplicity, we set $\gamma_t = \gamma \in [0,1]$ so that government spending is proportional to GDP in the normal regime. The government finances its spending Γ_t with taxes and debts.

[17] For mnemonic purposes, we use superscript m to refer to the *market* shock and the superscript h to refer to the *hedgeable* idiosyncratic shock.

Debt and Taxes. Let $\{B_t; t \geqslant 0\}$ denote the government's debt balance and $\{\mathcal{T}_t; t \geqslant 0\}$ denote the tax revenue process, respectively. As in Barro (1979), we assume that taxes are distortionary. Let $C_t = C(\mathcal{T}_t, Y_t)$ denote the deadweight loss in units of consumption goods when the government collects tax revenue $\mathcal{T}_t$ and GDP is Y_t in the normal regime. Following Barro (1979), we assume that the deadweight loss function, $C(\mathcal{T}_t, Y_t)$, is homogeneous in output Y_t and tax revenue $\mathcal{T}_t$ of degree one:

$$C_t = C(\mathcal{T}_t, Y_t) = c(\tau_t) Y_t \,, \tag{7}$$

where $\tau_t = \mathcal{T}_t / Y_t$ is the (average) tax rate on output. As in Barro (1979), we assume that the scaled deadweight loss, $c(\tau)$, is increasing, convex, and smooth.

As the tax revenue at any time t cannot exceed the contemporaneous net output, $Y_t - \Gamma_t$, we require $\mathcal{T}_t \leqslant \overline{\tau}\, Y_t$, which is equivalent to the following constraint on the tax rate τ_t:

$$\tau_t \leqslant \overline{\tau} \,, \tag{8}$$

where $\overline{\tau}$ is a maximal politically feasible tax rate on GDP Y_t in the normal regime. Keynes (1923, pp.56–62) inferred limits on a country's debt-GDP ratio partly from an upper bound like $\overline{\tau}$ based on political considerations.

2.1.2 Output, Government Spending, and Taxation in the Default Regime

Defaulting causes an output loss that proxies for associated disruptions in economic activity. Let $\widehat{Y}_t$ denote GDP in the default regime and let $T^{\mathcal{D}}$ denote an endogenous time when the government defaults. Following Aguiar and Gopinath (2006) and Rebelo, Wang, and Yang (2021), we assume that upon defaulting, the government completely reneges on its debt, GDP immediately drops from $Y_{T^{\mathcal{D}}-} = \lim_{s \uparrow T^{\mathcal{D}}-} Y_s$, the pre-default GDP level, to $\widehat{Y}_{T^{\mathcal{D}}} = \alpha Y_{T^{\mathcal{D}}-}$, and the economy permanently enters the default regime.[18]

In this regime ($t \geqslant T^{\mathcal{D}}$), the government cannot issue debt at all ($B_t = 0$) and output $\widehat{Y}_t$ follows the same GBM process (4) as in the normal regime. Therefore,

$$\widehat{Y}_t = \alpha Y_t \,, \quad t \geqslant T^{\mathcal{D}} \,, \tag{9}$$

where $\alpha \in (0,1)$ is a constant.[19] So output in the default regime equals an α fraction of Y_t

[18] To ease exposition we assume that the economy upon default never exits from the default regime. We can relax this assumption and allow the economy to return to the normal regime.

[19] Hébert and Schreger (2017) provide supporting empirical evidence.

given in (2), where $\{Y_t; t \geqslant 0\}$ would have been GDP had the economy permanently stayed in the normal regime.

Let $\widehat{\mathcal{T}}_t$ denote tax revenue in the default regime. Since the government has no debt in the default regime, it has to finance its spending period by period according to

$$\widehat{\mathcal{T}}_t = \Gamma_t = \gamma_t Y_t \,, \quad t \geqslant T^{\mathcal{D}} \,. \tag{10}$$

Note that government spending $\{\Gamma_t; t \geqslant 0\}$ is exogenous and solely depends on the exogenous $\{Y_t; t \geqslant 0\}$ process given in (2) regardless of the government's default decision.

As in the normal regime, taxation is distortionary in the default regime. Let $\widehat{C}_t = \widehat{C}(\widehat{\mathcal{T}}_t, \widehat{Y}_t)$ denote the deadweight loss in the default regime, when the government collects tax revenue $\widehat{\mathcal{T}}_t$ and output is $\widehat{Y}_t$. We assume that $\widehat{C}(\widehat{\mathcal{T}}_t, \widehat{Y}_t)$ is also homogeneous of degree one in tax revenue $\widehat{\mathcal{T}}_t$ and output $\widehat{Y}_t$:

$$\widehat{C}_t = \widehat{C}(\widehat{\mathcal{T}}_t, \widehat{Y}_t) = \widehat{c}(\widehat{\tau}_t)\widehat{Y}_t \,, \tag{11}$$

where $\widehat{\tau}_t = \widehat{\mathcal{T}}_t / \widehat{Y}_t$ is the tax rate in the default regime. We assume that $\widehat{c}(\widehat{\tau})$ is increasing, convex, and smooth.

Deadweight loss functions in the two regimes are connected as

$$\widehat{c}(\,\cdot\,) = \kappa\, c(\,\cdot\,) \,. \tag{12}$$

The parameter $\kappa > 1$ measures how much more costly taxation is in the default regime than in the normal regime.

As in the normal regime, we require $\widehat{\mathcal{T}}_t \leqslant \overline{\tau}\, \widehat{Y}_t$, which is equivalent to the following constraint on the tax rate $\widehat{\tau}_t$ in the default regime:

$$\widehat{\tau}_t \leqslant \overline{\tau} \,, \quad t \geqslant T^{\mathcal{D}} \,, \tag{13}$$

where $\overline{\tau}$ is the same maximum politically feasible tax rate described above.

In sum, while the government enjoys debt relief via default, it has to bear three costs if defaulting on its debt: 1.) a loss of output (as $\widehat{Y}_t = \alpha Y_t < Y_t$); 2.) a worse deadweight loss function than it faced in the normal regime ($\kappa > 1$); and 3.) the loss of the option to use tax smoothing over time because it must balance its budget period by period.

Next, we introduce financial markets. In answering the question "what is the government's maximally sustainable debt," we grant the government access to a complete set of financial securities subject only to the participation constraint associated with the default option. Equivalently, we assume that the government can dynamically trade a complete set of Arrow securities subject to limited commitment in the normal regime (before defaulting).[20] Thus, in terms of financial markets, we follow Lucas and Stokey (1983).

2.2 Financial Markets

In the normal regime, the government has the following investment and financing opportunities: (1) it can insure its idiosyncratic risk through actuarially fairly priced hedging contracts; (2) it can invest in the market portfolio; and (3) it can issue debt in the international market. We assume that debt that matures instantaneously and is continuously rolled over. However, because markets are dynamically complete, outcomes would not changes if we were to include longer term government debt too.

Idiosyncratic Risk Hedging Asset. We assume that there is a perfectly competitive market in a financial asset that is perfectly correlated with the idiosyncratic diffusive shock $\mathcal{Z}_t^h$. An investor who holds one unit of this asset at time t receives no up-front payment, since there is no risk premium for bearing idiosyncratic risk, and receives a gain or loss equal to $d\mathcal{Z}_t^h = \left(\mathcal{Z}_{t+dt}^h - \mathcal{Z}_t^h\right)$ at time $t+dt$. We normalize the volatility parameter of this hedging contract to be one. We denote the government's holdings of this idiosyncratic risk hedging asset at time t by $-\Pi_t^h$, which implies that the government's idiosyncratic risk exposure in levels is $-\Pi_t^h d\mathcal{Z}_t^h$ over dt.

Stock Market Portfolio and Equivalent Stock Market Futures Contract. Investors, the household, and the government can manage their exposures to the aggregate shock by investing in financial assets whose returns are solely driven by the aggregate shock. A natural example is the stock market portfolio. As in Merton (1971) and Black and Scholes (1973), we assume that the stock market return, which we denote by dR_t over dt, is independently and identically distributed (i.i.d) with the drift parameter μ_m and the volatility parameter σ_m under the physical measure $\mathbb{P}$:[21]

$$dR_t = \mu_m dt + \sigma_m d\mathcal{Z}_t^m \,, \tag{14}$$

[20] Our argument builds on the dynamic replicating portfolio argument used in Black and Scholes (1973) and Harrison and Kreps (1979) under complete markets with full commitment.

[21] This widely used geometric Brownian motion process for stock price is fully consistent with the asset pricing model of Lucas (1978). We can generalize our model to allow for disasters/jumps as in Barro (2006). All our insights will remain valid.

where $\mathcal{Z}_t^m$ is a standard Brownian motion under the physical measure $\mathbb{P}$.

We can rewrite the return process (14) as $dR_t = rdt + \sigma_m d\widetilde{\mathcal{Z}}_t^m$, where η is the Sharpe ratio of the market portfolio

$$\eta = \frac{\mu_m - r}{\sigma_m} \tag{15}$$

and $\widetilde{\mathcal{Z}}_t^m$ represents the risk-adjusted aggregate shock[22]

$$d\widetilde{\mathcal{Z}}_t^m = \eta dt + d\mathcal{Z}_t^m \,. \tag{16}$$

We interpret $d\widetilde{\mathcal{Z}}_t^m = \eta dt + d\mathcal{Z}_t^m$ as the payoff on a unit of the futures contract on the stock market (an example of a one-step-ahead Arrow security.) The value of this futures contract on the stock market with payoff (16) is zero (Cox, Ingersoll, and Ross, 1981). Thus, a risk-averse investor requires a payment of ηdt to bear a unit of the aggregate shock $d\mathcal{Z}_t^m$. Once we add the drift payoff ηdt with the aggregate shock exposure $d\mathcal{Z}_t^m$, the investor is indifferent between investing in this futures contract and not participating, implying that the value of the futures contract is zero.

As for the idiosyncratic risk hedging position, we denote the government's holdings of this stock market futures contract at time t by $-\Pi_t^m$, which implies that the government's systematic risk exposure in levels is $-\Pi_t^m(\eta dt + d\mathcal{Z}_t^m)$ over dt. A government or citizen could just as well have used the stock market portfolio rather than stock futures to manage aggregate shocks because financial market risk spanning is complete. We choose tje futures contract in order to preserve the expositional symmetry in our treatment of idiosyncratic risk and systematic risk management.

Stochastic Discount Factor. To ease exposition, we have set up our model with only one source of aggregate shock, $\mathcal{Z}_t^m$, which drives the stock market portfolio.[23] Using the standard no-arbitrage argument for complete-markets economies, we obtain the following unique stochastic discount factor (SDF), which we denote by $\mathbb{M}_t$:

$$\frac{d\mathbb{M}_t}{\mathbb{M}_t} = -rdt - \eta d\mathcal{Z}_t^m \,, \quad \mathbb{M}_0 = 1 \,. \tag{17}$$

No arbitrage requires that the drift of $d\mathbb{M}_t/\mathbb{M}_t$ equals $-r$. Additionally, in our one-factor model, the volatility of $d\mathbb{M}_t/\mathbb{M}_t$ equals $-\eta$, where $\eta = (\mu_m - r)/\sigma_m$ is the market price of

[22]In Appendix B, we show that $\widetilde{\mathcal{Z}}_t^m$ is a standard Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. The drift of the price of the stock futures contract is zero under $\widetilde{\mathbb{P}}$ (Duffie, 2001).

[23]We can generalize our model to allow for a richer model for aggregate risk.

risk, which is also the Sharpe ratio for the market portfolio (Duffie, 2001).

Next, we describe the government's budget constraint and optimization problem.

2.3 Government Budget and Objective

Budget Constraints. At $t = 0$, given the initial debt level (B_0), the government has the following budget constraint:

$$B_0 + \mathbb{E}_0 \int_0^{T^{\mathcal{D}}} \mathbb{M}_t dU_t \leqslant \mathbb{E}_0 \int_0^{T^{\mathcal{D}}} \mathbb{M}_t \left(\mathcal{T}_t - \Gamma_t\right) dt\,, \tag{18}$$

where $\{U_t; t \geqslant 0\}$ is the undiscounted (cumulative) debt issuances and dU_t is the net debt issuance over dt.[24] The right side of (18) is the present value of the government's primary surplus ($\mathcal{T}_t - \Gamma_t$). The left side of (18) is the sum of the initial debt level B_0 and the present value of all future state-contingent debt issuances dU_t. When calculating present values, we use the SDF $\mathbb{M}_t$ to discount payoffs for risk and remoteness in time. Inequality (18) states that the present value of all debt issued until the default time $T^{\mathcal{D}}$ cannot exceed the present value of the primary government surplus.

Flow Payoffs to the Household Let $\mathbf{1}_t^{\mathcal{D}}$ be an indicator function that equals one in the default regime when $t \geqslant T^{\mathcal{D}}$ and zero in the normal regime when $t < T^{\mathcal{D}}$. In the default regime ($\mathbf{1}_t^{\mathcal{D}} = 1$), the government has no debt and the household continuously receives payments at the rate of $(\widehat{Y}_t - (\widehat{\mathcal{T}}_t + \widehat{C}_t))$ where $\widehat{\mathcal{T}}_t = \Gamma_t$. In the normal regime ($\mathbf{1}_t^{\mathcal{D}} = 0$), the household continuously receives payments at the rate $(Y_t - (\mathcal{T}_t + C_t))$, which equals the difference between GDP Y_t and the total taxation cost $(\mathcal{T}_t + C_t)$. The household may also occasionally receive a lumpy payment dU_t if the government issues debt dU_t and distributes the proceeds. As we show later, this lumpy payment can occur under an optimal government plan when the household is impatient.

In sum, the household receives flow payments from three sources: 1.) lumpy payments to the household financed by debt issuance dU_t in the normal regime; 2.) recurrent payments in the normal regime $(Y_t - (\mathcal{T}_t + C_t))$; and 3.) recurrent payments in the default regime $(\widehat{Y}_t - (\widehat{\mathcal{T}}_t + \widehat{C}_t))$.

Intertemporal Discounting and Risk Premium Specifications. Let $(\zeta + r)$ denote the rate at which the household discounts future payoffs. We assume that the household

[24] Technically, $\{U_t; t \geqslant 0\}$ is a singular control process. As we will show, at an optimum U_t is non-decreasing.

values risk in the same way as investors and hence use the same market price η for the aggregate risk.[25] As a result, when $\zeta = 0$, the government and the market are equally patient. In this case, the household and investors use the same SDF $\mathbb{M}_t$ to value their respective payoffs. However, when the household is impatient ($\zeta > 0$), a common assumption in the sovereign debt literature (e.g., Aguiar and Gopinath, 2006), the government front loads consumption and tilts debt repayments towards the future generations.

In sum, for intertemporal discounting and risk specifications, we use $e^{-\zeta t}\mathbb{M}_t$ as the effective SDF for the household to value their risky payoffs, which differs from the SDF $\mathbb{M}_t$ price investors use to price payoffs. In Appendix B, we provide additional technical details.

Government Objective. Combining our assumptions about flow payoffs and the effective SDF for the household, we obtain the expression for the government's objective:

$$\mathbb{E}_0 \int_0^\infty e^{-\zeta t}\mathbb{M}_t \left(dU_t + (Y_t - (\mathcal{T}_t + C_t)) \left(1 - \mathbf{1}_t^{\mathcal{D}}\right) dt + [\widehat{Y}_t - (\widehat{\mathcal{T}}_t + \widehat{C}_t)]\mathbf{1}_t^{\mathcal{D}} dt \right) , \tag{19}$$

where $\zeta \geqslant 0$ measures the household's impatience. The government chooses debt issuance (dU_t), tax rates (τ_t and $\widehat{\tau}_t$), and idiosyncratic and systematic risk hedging demands (Π_t^h and Π_t^m) to maximize (19) subject to budget constraint (18), constraint (8) on the tax rate τ in the normal regime, and constraint (13) on $\widehat{\tau}$ in the default regime. Availability of full financial spanning and inefficiency of default means that government debt is risk free and the government chooses never to default. Optimal risk-free debt capacity $\overline{B}_t$ is part of an optimal plan.

Let $P_t = P(B_t, Y_t)$ denote the government's time-t value function.[26] Let $V_t = V(B_t, Y_t)$ denote the sum of debt value B_t and the government's value $P(B_t, Y_t)$:

$$V_t = V(B_t, Y_t) = P(B_t, Y_t) + B_t \, . \tag{20}$$

To obtain an optimal policy in the normal regime, we need the government's value function in the default regime, since the government's value function after a default affects the government's value and optimal decisions before it ever defaults. Since government debt is always zero in the default regime, the government's value function in that regime only depends on contemporaneous GDP $\widehat{Y}_t = \alpha Y_t$; we denote this value function $\widehat{P}(\widehat{Y}_t)$. Because default is

[25] Technically speaking, the household and the investors use the same Radon-Nikodym derivative that links the physical measure $\mathbb{P}$ to the risk-neutral measure $\widetilde{\mathbb{P}}$ (Duffie, 2001). Because of complete markets, this Radon-Nikodym derivative is unique.

[26] The value function $P_t = P(B_t, Y_t)$ is analogous to the levered "equity" value for a firm.

costly, the government wants to manage its state-contingent debt dynamics to avoid default. That gives rise to the following participation constraint:

$$P(B_t, Y_t) \geqslant \widehat{P}(\widehat{Y}_t)\,. \tag{21}$$

Before deducing an optimal government plan, we temporarily shut down all three of our frictions in order to recover a manifold of tax-debt profiles that support the same optimal plan, in the spirit of the Ricardian equivalence logic of Barro (1974).

3 Ricardian Equivalence

To uncover Ricardian equivalence, we turn off three frictions by 1.) setting $\zeta = 0$; 2.) endowing the government with the ability to commit always to repay its debt by setting $T^{\mathcal{D}} = \infty$ and equivalently $\mathbf{1}_t^{\mathcal{D}} = 0$ at all t; and 3.) removing deadweight losses by setting $C_t = 0$ for all t. Simplifying (19), we write the household's value as

$$P_0 = \mathbb{E}_0 \int_0^\infty \mathbb{M}_t \left[dU_t + (Y_t - \mathcal{T}_t)\, dt\right]\,, \tag{22}$$

subject to the following simplified budget constraint:

$$B_0 + \mathbb{E}_0 \int_0^\infty \mathbb{M}_t dU_t \leqslant \mathbb{E}_0 \int_0^\infty \mathbb{M}_t \left(\mathcal{T}_t - \Gamma_t\right) dt\,. \tag{23}$$

Combining (22) and (23), the latter of which binds due to local non-satiation, yields

$$P_0 + B_0 = \mathbb{E}_0 \int_0^\infty \mathbb{M}_t \left(Y_t - \Gamma_t\right) dt\,. \tag{24}$$

Expression (24) states that the total value $V_0^{FB} = P_0 + B_0$ is independent of policies $\{U_t, \mathcal{T}_t; t \geqslant 0\}$, an assertion of Ricardian equivalence. We use superscript FB for the total value V to denote the value attained when none of our three frictions is active.

In the spirit of Shiller (1994), consider a financial asset whose cash flow is almost surely equals net output $\{Y_t - \Gamma_t; t \geqslant 0\}$ process. The value of this financial asset equals the right side of (24). Let r_V and ξ denote this asset's expected return and risk premium, respectively. The unique SDF (17) implies that the CAPM holds for this asset:

$$r_V = r + \xi = \beta \times (\mu_m - r)\,, \tag{25}$$

where $\beta = \rho\sigma_Y/\sigma_m$ is the coefficient of regressing this asset's return on the market portfolio return. Equivalently, we can write this asset's risk premium ξ as follows:

$$\xi = \psi_m \eta = \rho\sigma_Y \eta \,. \tag{26}$$

Since tax and debt policies are irrelevant here, the total value $V_t^{FB} = P_t + B_t$ under Ricardian equivalence equals the value of this financial asset:

$$V_0^{FB} = \mathbb{E}_0 \int_0^\infty \mathbb{M}_t \left(Y_t - \Gamma_t\right) dt = \frac{1-\gamma}{r_V - g} Y_0 \,. \tag{27}$$

For the integral above to converge, we require the expected return r_V to be larger than the GDP growth rate g:

$$r_V > g \,. \tag{28}$$

We can rewrite budget constraint (18) in terms of r_V as

$$B_0 \leqslant \mathbb{E}_0 \left[\int_0^\infty \mathbb{M}_t \left(\mathcal{T}_t - \Gamma_t\right) dt - \int_0^\infty \mathbb{M}_t dU_t \right] = \mathbb{E}_0 \left[\int_0^\infty e^{-r_V t} \left(\left(\mathcal{T}_t - \Gamma_t\right) dt - dU_t \right) \right] . \tag{29}$$

Although debt is risk free, it is backed by a stochastic stream of primary surpluses. That explains the presence of risk premium ξ and the use of r_V to discount the primary surplus in (29).

In the next section, we provide a stochastic formulation of Barro (1979).

4 Stochastic Version of Barro (1979)

Our stochastic Barro model only has one friction: tax distortions/deadweight losses as in Barro (1979). We turn off the other two frictions in our baseline model of Section 2 by 1.) setting $\zeta = 0$ and 2.) imposing full commitment by setting $T^{\mathcal{D}} = \infty$ and equivalently setting $\mathbf{1}_t^{\mathcal{D}} = 0$ at all t. The government chooses a policy to maximize

$$\mathbb{E}_0 \int_0^\infty \mathbb{M}_t \left[dU_t + \left(Y_t - \left(\mathcal{T}_t + C_t\right)\right) dt \right] \,, \tag{30}$$

subject to the same budget constraint as (23) from our section 3 Ricardian equivalence setting and the constraint for the tax rate (8).

Substituting budget constraint (23), which binds under optimality, into the objective function (30), we obtain the following expression for the value of the government:

$$\mathbb{E}_0 \int_0^\infty \mathbb{M}_t \left(Y_t - \Gamma_t - C_t\right) dt - B_0\,. \tag{31}$$

Maximizing (31) is equivalent to minimizing the present value of deadweight losses, $\mathbb{E}_0 \int_0^\infty \mathbb{M}_t C_t dt$, by choosing $\{\mathcal{T}_t; t \geqslant 0\}$ subject to the constraint of honoring its outstanding debt B_0, which satisfies (23) with equality. That equivalence was Barro's justification for recasting the government's value maximization problem as a deadweight loss minimization problem. However, an analogous equivalence does not prevail in our model with its limited commitment; a government's option to default contributes endogenous distortion costs. For this reason, unlike Barro we have to work with a value-maximization problem rather than a cost-minimization problem.

It is useful to scale variables by contemporaneous GDP. Let b_t denote a debt-GDP ratio:

$$b_t = \frac{B_t}{Y_t}. \tag{32}$$

Similarly, let

$$p(b_t) = \frac{P(B_t, Y_t)}{Y_t} \qquad \text{and} \qquad v(b_t) = \frac{V(B_t, Y_t)}{Y_t} = p(b_t) + b_t\,. \tag{33}$$

Next, we summarize an optimal plan for our stochastic formulation of Barro (1979).

Proposition 4.1. Stochastic Barro (1979) Model. *Assuming $\zeta = 0$ and government commitment to service its debt, the optimal debt-GDP ratio b_t is constant over time, i.e., $b_t = b_0$ for all t; the optimal tax rate τ_t is constant over time and depends only on b_0:*

$$\tau(b_t) = \tau(b_0) = (r_V - g)b_0 + \gamma\,. \tag{34}$$

The government's scaled value function, $f(b_t)$, is also constant over time and given by

$$p(b_t) = p(b_0) = \frac{1 - \tau(b_0) - c(\tau(b_0))}{r_V - g}\,. \tag{35}$$

Any initial debt level b_0 satisfying $b_0 \leqslant \overline{b}^*$ is sustainable, where $\overline{b}^*$ is the maximally sustainable debt-output ratio given by:

$$\overline{b}^* = \frac{\overline{\tau} - \gamma}{r_V - g}\,. \tag{36}$$

We relegate a proof of Proposition 4.1 to Appendix A. In our stochastic Barro economy, the initial condition is the steady state, since $b_t = b_0$ and $p(b_t) = p(b_0)$. Therefore, the present value of the (scaled) primary surplus $\tau(b_t) - \gamma$ equals the (scaled) debt b_t at all t:

$$\frac{\tau(b_t) - \gamma}{r_V - g} = b_t = b_0 \,. \tag{37}$$

Notice that the discount rate that appears in present value equation (37) is r_V and not the risk-free rate r. The optimal tax rate $\tau(b_t)$ also satisfies the following first-order condition:

$$1 + c'(\tau(b_t)) = -p'(b_t) \,. \tag{38}$$

The government optimally equates the marginal cost $1 + c'(\tau(b_t))$ of taxing the household with the marginal benefit $-p'(b_t) > 0$ of reducing debt. This is a version of Barro's tax smoothing recommendation.

Were it to be given an option to choose its initial debt level, a government would optimally set $b_0 = 0$ because doing so maximizes $v(b_0) = p(b_0) + b_0$. Using (38), we obtain $v'(b_0) = p'(b_0) + 1 = -c'(\tau(b_0)) \leqslant 0$. Therefore, $b_0 = 0$, which follows from the assumption that $c(\,\cdot\,)$ is increasing and convex. The intuition is that issuing lumpy debt yields no benefit but induces distortionary debt servicing costs.

Next we show that when the government has the option to default as it does in our Section 2 model, equivalence between the government's value maximization and cost minimization problem no longer holds .

5 Optimal Government Plan

We formulate the optimum problem of our section 2 government as a dynamic program.

5.1 Normal Regime

First, we introduce the government's dynamic debt and risk management problem. Then we characterize the government's decisions in interior and lumpy payout regions.

5.1.1 Dynamic State-Contingent Debt Management

When managing its debt dynamics, the government also actively engages in idiosyncratic and systematic risk management by choosing Π_t^h and Π_t^m. The value of government debt B_t

evolves as follows:

$$dB_t = (rB_t + (\Gamma_t - \mathcal{T}_t))\,dt + dU_t - \Pi_t^h d\mathcal{Z}_t^h - \Pi_t^m\left(\eta dt + d\mathcal{Z}_t^m\right) . \tag{39}$$

The first term on the right side of (39) is government savings where $\Gamma_t - \mathcal{T}_t$ is the primary deficit and rB_t is the interest payment. The second term dU_t is the government's lumpy debt issuance. The third and fourth terms are gains or losses from government holdings of the idiosyncratic risk-hedging asset and stock market futures, respectively.

By trading an idiosyncratic risk hedging asset and stock market futures, the government makes its debt state-contingent. Its optimal use of these risk management tools shapes the government's debt capacity and also ensures that government debt ends up being risk-free at all time and across all states. While government debt is risk free, equation (39) shows that the quantity of debt is stochastic.

Let $\overline{B}_t$ denote the government's endogenous debt capacity (the maximally sustainable debt level), to be determined in Section 5.2. We show later that the government's optimal (lumpy) debt issuance policy $\{dU_t\}$ is characterized by an endogenous debt threshold level, $\underline{B}_t$, below which it issues and makes a payout to the household ($dU_t > 0$).

Next, we characterize the optimal policies and value function for the interior region ($B_t \in [\underline{B}_t, \overline{B}_t]$).

5.1.2 Interior Region ($\underline{B}_t \leqslant B \leqslant \overline{B}_t$)

In this region, the government relies exclusively on risk hedging strategies and taxation to manage its state-contingent debt dynamics. It abstains from making lumpy payouts to the household financed from a lumpy debt issuances, so $dU_t = 0$.

Dynamic Programming. The government chooses tax revenue $\mathcal{T}$, idiosyncratic-risk hedging demand Π^h, and the systematic risk hedging demand Π^m to maximize the value function $P(B,Y)$ by solving the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}(\zeta + r)P(B,Y) \quad = \quad & \max_{\mathcal{T}\leqslant\overline{\tau}Y,\,\Pi^h,\,\Pi^m} \; (Y - \mathcal{T} - C(\mathcal{T},Y)) + [rB + \Gamma - \mathcal{T}\ ]\,P_B(B,Y) \\ & + \frac{(\Pi^h)^2 + (\Pi^m)^2}{2}P_{BB}(B,Y) + (g - \rho\eta\sigma_Y)YP_Y(B,Y) \\ & + \frac{\sigma_Y^2Y^2}{2}P_{YY}(B,Y) - \left(\psi_h\Pi^h + \psi_m\Pi^m\right)YP_{BY}(B,Y)\,. \end{aligned} \tag{40}$$

The first term on the right side of (40), $(Y - \mathcal{T} - C(\mathcal{T},Y))$, is the net flow payment to the household. The second and third terms are the drift and diffusion volatility effects of increasing debt B on $P(B,Y)$. The fourth and fifth terms reflect the drift and volatility effects of GDP, Y, on $P(B,Y)$. The sixth term captures the effect of the intertemporal idiosyncratic and systematic risk hedging demands on $P(B,Y)$.

First-Order Conditions. Tax revenue $\mathcal{T}$ satisfies the FOC:

$$1 + C_{\mathcal{T}}(\mathcal{T}, Y) = -P_B(B,Y)\,. \tag{41}$$

It equates the marginal cost of taxing the household, $1 + C_{\mathcal{T}}(\mathcal{T}, Y)$, with the marginal benefit of using taxes to reduce debt, $-P_B(B,Y) > 0$.

As in Merton (1971), systematic risk intertemporal hedging demand Π^m satisfies:

$$\Pi^m = \psi_m \frac{Y P_{BY}(B,Y)}{P_{BB}(B,Y)}. \tag{42}$$

Similarly, the FOC for the intertemporal diffusion risk hedging demand is

$$\Pi^h = \psi_h \frac{Y P_{BY}(B,Y)}{P_{BB}(B,Y)}\,. \tag{43}$$

The cross partial derivative P_{BY} shapes the government's idiosyncratic and systematic risk intertemporal hedging demands in equations (42) and (43). Note the symmetry between (42) and (43).

We can use the FOCs (41), (42), and (43) to represent the HJB equation (40) as

$$\begin{aligned}(\zeta + r)P(B,Y) = \max_{\mathcal{T} \leqslant \overline{\tau} Y} \; & Y - \mathcal{T} - C(\mathcal{T}, Y) + [rB + \Gamma - \mathcal{T}\,]\, P_B(B,Y) + \widetilde{g} Y P_Y(B,Y)\,, \\ & + \frac{\sigma_Y^2 Y^2}{2} P_{YY}(B,Y) - \frac{\sigma^2 Y^2}{2} \frac{P_{BY}^2(B,Y)}{P_{BB}(B,Y)}\,,\end{aligned} \tag{44}$$

where $\widetilde{g} = g - \rho\eta\sigma_Y$ is a risk-adjusted growth rate. [27] We can verify that the government's value function $P(B,Y)$ is homogeneous of degree one in B and Y. Consequently the following expression holds:[28]

$$P_{YY}(B,Y) = \frac{P_{BY}^2(B,Y)}{P_{BB}(B,Y)}\,. \tag{45}$$

[27] Technically, it is the growth rate under the risk-neutral measure $\widetilde{\mathbb{P}}$.

[28] Using the homogeneity property $P(B,Y) = p(b)Y$, we obtain $P_B = f'(b)$, $P_{BB} = p''(b)/Y$, $P_Y = p(b) - p'(b)b$, $P_{YY} = p''(b)bB/Y^2 = p''(b)b^2/Y$, and $P_{BY} = -p''(b)b/Y$. Therefore, we can verify $P_{BB}P_{YY} = (p''(b)b/Y)^2 = P_{BY}^2$.

Using (45) to simplify (44), we obtain the following first-order partial differential equation:

$$(\zeta + r)P(B,Y) = \max_{\mathcal{T} \leqslant \overline{\tau} Y} \; (Y - \mathcal{T} - C(\mathcal{T},Y)) + (rB + \Gamma - \mathcal{T})\, P_B + (g - \rho\eta\sigma_Y) Y P_Y \,. \quad (46)$$

The first term on the right side of (46) is the flow payoff to the household. The second term captures the effect of fiscal deficit $(rB + \Gamma - \mathcal{T})$ on its value function $P(B,Y)$ and the last term describes the risk-adjusted growth effect of Y on the government's value. Optimality implies that the sum of these three terms equals $(\zeta + r)P(B,Y)$, the government's value $P(B,Y)$ multiplied by its discount rate $(\zeta + r)$. Full financial spanning allows the the government optimally to hedge so that its debt is risk free, so there are no diffusion terms (no P_{BB}, no P_{YY}, and no P_{BY}) in (46). Systematic volatility of output growth ψ_m appears in the last term because it influences the government's value via the standard discount rate channel as in CAPM.

Next, we turn to the region $(0 \leqslant B_t < \underline{B}_t)$, where the government issues a lumpy amount of debt and pay out to the household .

5.1.3 Lumpy Debt Issuance and Payout Region $(0 \leqslant B_t < \underline{B}_t)$

In this region the debt-output ratio $b_t = B_t/Y_t$ is so low that it is optimal for the government immediately to issue debt and pay out the proceeds to the household . The optimal lumpy debt-issue and payout policy for a given B_t is

$$dU_t = \max\left\{\underline{B}_t - B_t, 0\right\}. \quad (47)$$

Equation (47) implies the following value-matching condition when $B_t < \underline{B}_t$:

$$P(B_t, Y_t) = P(\underline{B}_t, Y_t) + (\underline{B}_t - B_t)\,. \quad (48)$$

Rewriting (48) and using the definitions $V(B_t, Y_t) = P(B_t, Y_t) + B_t$ and $V(\underline{B}_t, Y_t) = P(\underline{B}_t, Y_t) + \underline{B}_t$, we find that $V(B_t, Y_t) = V(\underline{B}_t, Y_t)$, so that sums of the government's value and debt value are equated before and after new debt issuances.

The government optimally chooses a new debt level $\underline{B}_t \geqslant 0$ (or equivalently the new debt issuance dU_t) to solve:

$$\max_{\underline{B} \geqslant 0} \quad V(\underline{B}_t, Y_t) = P(\underline{B}_t, Y_t) + \underline{B}_t\,. \quad (49)$$

If the optimal $\underline{B}_t$ is interior (i.e., if $\underline{B}_t > 0$), it satisfies the FOC:

$$P_B(\underline{B}_t, Y_t) = -1 \quad \text{equivalently} \;\; V_B(\underline{B}_t, Y_t) = 0\,. \tag{50}$$

Otherwise, the government issues no lumpy debt and $\underline{B}_t = 0$.

5.2 Debt Capacity $\overline{B}_t$ and Default Regime ($B_t > \overline{B}_t$)

Here we characterize the value function in the default regime where $B_t > \overline{B}_t$ and determine debt capacity $\overline{B}_t$.

5.2.1 Default Regime ($B_t > \overline{B}_t$)

When government debt B_t exceeds debt capacity $\overline{B}_t$, the government defaults and enters the (permanently) absorbing default regime.[29] The government's value function $P(B_t, Y_t)$ at $B_t > \overline{B}_t$ satisfies

$$P(B_t, Y_t) = \widehat{P}(\widehat{Y}_t)\,, \tag{51}$$

where $\widehat{Y}_t = \alpha Y_t$ and $\widehat{P}(\widehat{Y})$ is the government's value in the default regime given by

$$(\zeta + r)\widehat{P}(\widehat{Y}) = \left(\widehat{Y} - \Gamma - \widehat{C}(\Gamma, \widehat{Y})\right) + (g - \rho\eta\sigma_Y)\widehat{Y}\widehat{P}'(\widehat{Y}) + \frac{\sigma_Y^2 \widehat{Y}^2}{2}\widehat{P}''(\widehat{Y})\,. \tag{52}$$

The first term on the right side of (52) gives the net flow payment received by the household in the default regime. Since the government can neither borrow nor lend in this regime, its spending equals tax income, $\mathcal{T}_t = \Gamma_t$.) The second and third terms capture the impact of the risk-adjusted drift and volatility of output on the government's value function $\widehat{P}(\widehat{Y})$, respectively. The default regime is absorbing. Here for $t \geqslant T^{\mathcal{D}}$, output equals $\widehat{Y}_t = \alpha Y_t$, and there is no debt ($B_t = 0$). Let $\widehat{p}_t = \widehat{P}(\widehat{Y}_t)/\widehat{Y}_t$. Later we'll show that $\widehat{p}_t = \widehat{p}$, a constant. To ensure that the value function in the default regime is non-negative, we impose:[30]

$$1 - \gamma/\alpha - \kappa c(\gamma/\alpha) \geqslant 0\,. \tag{53}$$

[29] We can generalize our model to allow for the possibility where the government has a probability to exit the default regime and return to the normal regime.

[30] The value function in the default regime is non-negative if and only if the condition $\widehat{Y} - \Gamma - \widehat{C}(\Gamma, \widehat{Y}) \geqslant 0$ holds, which is equivalent to the condition given in (53) after we use the homogeneity property and $\widehat{Y}_t = \alpha Y_t$.

5.2.2 Debt Capacity $\overline{B}$

What is the maximal level of debt that the government can issue without provoking default? We refer to this debt level, denoted by $\overline{B}_t$, as the government's debt capacity. To characterize $\overline{B}_t$, we must respect two constraints: 1.) the government's incentive to renege on its debt, which gives rise to a limited-commitment constraint; and 2.) the "Keynesian" tax constraint $\tau \leqslant \overline{\tau}$, where $\overline{\tau}$ is the maximal rate at which the government can tax the household to support its debt repayment (again see Keynes (1923, pp.56–62).) Two outcomes are possible, depending on which one of these two constraints binds. If the government's default incentive is strong, the limited-commitment constraint binds at its debt capacity. If the government has limited power to tax output (i.e., when the maximally feasible tax rate on output, $\overline{\tau}$, is relatively low), the tax constraint $\tau \leqslant \overline{\tau}$ binds at debt capacity.

When Limited-commitment Constraint Binds at $\overline{B}_t$. When the government is indifferent between making its debt payments and defaulting, it has reached its debt capacity, $\overline{B}_t$, and the following value-matching condition prevails:

$$P(\overline{B}_t, Y_t) = \widehat{P}(\widehat{Y}_t)\,, \tag{54}$$

where $\widehat{Y}_t = \alpha Y_{t-}$ and $\widehat{P}(\widehat{Y}_t)$ satisfies (52). Here we obtain the government's risk-free debt limit (capacity) by adapting to our setting an off-an-optimal-path default consequence in the spirit of Worrall (1990) and Kehoe and Levine (1993).[31]

When Tax Constraint $\mathcal{T}(B, Y) \leqslant \overline{\tau} Y$ Binds at $\overline{B}_t$. When the government has limited power to tax output (i.e., when $\overline{\tau}$ is not high), the tax constraint $\tau_t \leqslant \overline{\tau}$ binds at debt capacity:

$$\mathcal{T}(\overline{B}_t, Y_t) = \overline{\tau} Y_t\,. \tag{55}$$

In sum, either (54) or (55) holds at debt capacity $\overline{B}_t$. Because $\overline{B}_t$ is a free boundary, we require one more condition to pins down its value. We supply this condition in the next subsection after describing some simplifications.

[31]Our approach is similar to but different from Bolton, Wang, and Yang (2019) and Rebelo, Wang, and Yang (2021) who incorporate the limited-commitment constraints into corporate finance and international finance in their continuous-time models.

5.3 Exploiting the Homogeneity Property

We take the debt-output ratio b as state variable. Let $du_t = dU_t/Y_t$ be the scaled lumpy debt issuance and $\overline{b}_t = \overline{B}_t/Y_t$ be the maximally feasible debt-GDP ratio.

Optimal Tax Rate $\tau(b)$. Substituting $P(B,Y) = p(b)Y$ into FOC (41) for tax revenue $\mathcal{T}$, we obtain the following simplified FOC for the tax rate $\tau(b)$:[32]

$$1 + c'(\tau(b)) = -p'(b)\,. \tag{56}$$

Since $c''(\,\cdot\,) > 0$, we can invert the marginal tax distortion cost function $c'(\,\cdot\,)$ to obtain the unique tax rate $\tau(b)$ for a given b.

Debt-GDP (b_t) Dynamics in the Interior Region: $b \in [\underline{b}, \overline{b}]$. When the debt-GDP ratio is not too low, i.e., $b \geqslant \underline{b}$, where $\underline{b}$ is endogenous, the government makes no lumpy payments to the household : $du_t = 0$, because the marginal benefit of financing an immediate payout to the household is smaller than the marginal cost (including deadweight losses) of financing debt. Using Ito's Lemma, we can show that in this interior region b_t evolves deterministically at the rate

$$\dot{b}_t \equiv \mu_t^b = \mu^b(b_t) = \underbrace{\gamma - \tau(b_t)}_{\text{primary deficit}} + \underbrace{r \times b_t}_{\text{interest payment}} - \underbrace{g \times b_t}_{\text{growth}} + \underbrace{\xi \times b_t}_{\text{hedging cost}}\,. \tag{57}$$

The first term on the right side of (57) is the scaled primary fiscal deficit $\gamma - \tau(b)$, also known as the scaled net-of-interest fiscal deficit. The second term is the interest cost of servicing debt. The sum of these two terms is the scaled fiscal deficit, gross of interest payments. The third term is a debt-GDP ratio reduction effect due to output growth. The last term captures the hedging cost due to the risk premium payment, a new term also included by Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) in a different setting. This new term arises because current debt B_t is backed by future stochastic primary surpluses that are discounted at r_V, the sum of the risk-free rate r and the risk premium ξ defined in (26).

Endogenous Debt-GDP Ratio Limit $\overline{b}$ and the Steady State. How do we pin down debt capacity $\overline{b}$? We set the drift for the debt-GDP ratio b_t to zero: $\dot{b}_t = 0$ at $\overline{b}$ according to the following logic. To be consistent with the notion that $\overline{b}$ is debt capacity b cannot exceed

[32] This condition holds regardless of whether the tax constraint (8) binds or not. The reason is that the tax constraint may only bind at $\overline{b}$. Tax smoothing implies that the FOC (56) holds also at the boundary $\overline{b}$.

$\bar{b}$, which implies $\mu^b(\bar{b}) \leqslant 0$. Additionally, with $\zeta \geqslant r$, the government weakly has incentives at the margin to postpone tax burdens, which suggests $\mu^b(\bar{b}) \geqslant 0$. These two inequalities jointly imply that the drift of b at debt capacity is zero: $\mu^b(\bar{b}) = 0$.

Substituting this $\mu^b(\bar{b}) = 0$ condition for b_t into (57) yields

$$\bar{b} = \frac{\tau(\bar{b}) - \gamma}{r_V - g} \,. \tag{58}$$

Equation (58) asserts that at the maximal sustainable debt-GDP ratio $\bar{b}$ equals the present value of the primary deficit $(\tau(\bar{b}) - \gamma)$ evaluated at discount rate $r_V = r + \xi$. This is the appropriate discount rate because the optimal primary deficit is risky and bears a risk premium of ξ. Condition (58) is condition that in Section 5.2 we promised to deliver to pin down the endogenous debt-GDP capacity $\bar{b}$.

Next, we characterize the government's lumpy debt issuance and payout decisions.

Scaled Lumpy Debt Issuance Boundary $\underline{b}$ and Payout Policy du_t. We can use the homogeneity property to simplify (49) and verify that the lumpy debt boundary $\underline{b}$ solves

$$\max_{\underline{b} \geqslant 0} \quad v(\underline{b}) = p(\underline{b}) + \underline{b} \,. \tag{59}$$

If the optimal $\underline{b}$ is interior (i.e., $\underline{b} > 0$), the cost of debt issuance must be zero at $\underline{b}$: $v'(\underline{b}) = 0$. Otherwise, the government issues no lumpy debt and $\underline{b} = 0$, as $v'(\underline{b}) < 0$. Thus, an optimal lumpy debt issuance policy satisfies

$$du_t = \max\{\underline{b} - b_t, 0\}. \tag{60}$$

Conditions characterizing optimal upper and lower boundaries embody distinct economic forces. The lower boundary $\underline{b}$ is about optimal lumpy debt issuance and payout, which is characterized by value-matching and smooth-pasting conditions. The upper boundary $\bar{b}$ is absorbing and can only be reached from the left where $b < \bar{b}$. That $\bar{b}$ is absorbing and can be reached only from the left certifies it as the maximally sustainable level of debt per unit of GDP.

If at $t = 0$ initial government debt is zero and if an optimal $\underline{b} > 0$, a government immediately issues debt and uses the proceeds to finance a lumpy immediate payment $dU_0 = \underline{b}Y_0$ to the household.[33] Afterwards, the government's b equals the optimal target level $\underline{b}$ and

[33] When the optimal b is strictly positive ($\underline{b} > 0$), there is no deadweight cost of debt and the marginal

remains inside $[\underline{b}, \overline{b}]$ until it reaches the maximally sustainable debt capacity $\overline{b}$.

5.4 Optimal Fiscal Plan

The following proposition describes an optimal tax and debt plan.

Theorem 5.1. *Under the $r_V > g$ restriction (28), $\kappa > 1$, $\alpha \leqslant 1$, and the condition $1-\gamma/\alpha-\kappa c(\gamma/\alpha) \geqslant 0$ given in (53), the scaled value function in the normal regime, $p(b)$, satisfies the first-order nonlinear differential equation:*

$$[\zeta + (r_V - g)]\, p(b) \quad = \quad 1 - \tau(b) - c(\tau(b)) + [(r_V - g)b + \gamma - \tau(b)]\, p'(b)\,, \qquad (61)$$

subject to the debt-sustainability condition (58) and one of the following two conditions for the scaled debt capacity $\overline{b}$:

$$p(\overline{b}) \quad = \quad \alpha\widehat{p}\,, \quad \textit{when the tax rate constraint (8) does not bind}; \qquad (62)$$

$$\tau(\overline{b}) \quad = \quad \overline{\tau}\,, \quad \textit{when the tax rate constraint (8) binds}. \qquad (63)$$

The scaled value in the default regime, $\widehat{p}$, is

$$\widehat{p} \quad = \quad \frac{1-\gamma/\alpha-\kappa c(\gamma/\alpha)}{\zeta + (r_V - g)}\,. \qquad (64)$$

The lumpy debt issuance boundary $\underline{b}$ is given by (59), and the optimal lumpy debt issuance policy, du_t, is given by (60). The optimal tax rate policy $\tau(b)$ is given by (56) and the debt-output ratio $\{b_t\}$ evolves deterministically at the rate of $\dot{b}_t$ given by (57).

Next, we report an optimal plan in closed-form for the special case with no impatience ($\zeta = 0$).

Lemma 5.2. *When $\zeta = 0$, $b_t = b_0$ and the optimal tax rate $\tau(b_t)$ is linear in b_t for all t: $\tau(b_t) = \tau(b_0) = (r_V - g)b_0 + \gamma$. The scaled value function in the normal regime, $p(b)$, is constant and given by*

$$p(b_t) = p(b_0) = \frac{1-\tau(b_0)-c(\tau(b_0))}{r_V - g}\,. \qquad (65)$$

The scaled value under autarky, $\widehat{p}$, is $\widehat{p} = \frac{1-\gamma/\alpha-\kappa c(\gamma/\alpha)}{r_V - g} > 0$. There is no lumpy debt-issuance and hence $\underline{b} = 0$. The scaled debt capacity is given by $\overline{b} = p^{-1}(\alpha\widehat{f})$, when the tax constraint

cost of servicing debt, $-p'(\underline{b})$, equals one. This outcome differs from the zero fiscal cost of debt asserted in Blanchard (2019) and Sims (2022). "Debt is cheap" statements like ones in those two papers apply when $b < \overline{b}$. Here the government has not borrowed enough and should increase its debt-GDP ratio to $\underline{b} > 0$.

(8) does not bind. Otherwise, $\overline{b} = \frac{\overline{\tau}-\gamma}{r_V-g}$. *Combining the two cases, we obtain the following expression for* $\overline{b}$*:*

$$\overline{b} = \min\left\{p^{-1}(\alpha\widehat{p}), \frac{\overline{\tau}-\gamma}{r_V-g}\right\}. \tag{66}$$

Thus, with $\zeta = 0$, an optimal plan entails complete tax smoothing result as in Barro (1979). However, unlike the full-commitment to repay assumption of Barro (1979), our model contains an endogenous debt capacity. This turns out to be important quantitatively. Later we offer calculations with calibrated parameter values that show that debt capacity is much smaller in our model than it would be in Barro (1979) . Further, because our model contains shocks to GDP growth rates, debt-GDP ratio dynamics and debt capacity depend on a risk premium. Moreover, the debt balance, B_t, is volatile and non-stationary since $B_t = b_0 Y_t$; B_t follows the same geometric Brownian motion process (2) as Y_t.

6 Dual Formulation

A government's dynamic debt management problem is equivalent to a planner's dynamic resource allocation problem.

6.1 Planner's Value and the Household's Promised Value

Consider a long-term resource allocation (contracting) problem between a planner and the household. The output process $\{Y_t; t \geqslant 0\}$ given in (2) is publicly observable and verifiable. The government spending process $\{\Gamma_t; t \geqslant 0\}$ is exogenous.

Optimal Contracting Problem. The contract specifies a tax revenue process $\{\mathcal{T}_t; t \geqslant 0\}$ that implies a smooth flow payment $(Y_t - \mathcal{T}_t - C_t)$ to the household, and a cumulative payment process to the household $\{J_t; t \geqslant 0\}$. Optimal policies, $\{\mathcal{T}_t; t \geqslant 0\}$ and $\{J_t; t \geqslant 0\}$, depend on an entire history of both idiosyncratic and systematic shocks $\{\mathcal{Z}_t^h, \mathcal{Z}_t^m; t \geqslant 0\}$. The maximum feasible tax rate that the planner can impose on the output process is $\overline{\tau} Y_t$ for all $t \geqslant 0$, i.e., $\mathcal{T}_t \leqslant \overline{\tau} Y_t$, the same as the constraint (8) that appeared in our primal dynamic debt management problem.

The planner maximizes the risk-adjusted present value of $(\mathcal{T}_s - \Gamma_s)\, ds - dJ_s$, the difference between the government's primary surplus $((\mathcal{T}_s - \Gamma_s)\, ds)$ and its distribution to the household

(dJ_s), at time 0. Let F_t denote the planner's value function at time t:

$$F_t = \max \quad \mathbb{E}_t \left[\int_t^{T^{\mathcal{D}}} \frac{\mathbb{M}_s}{\mathbb{M}_t} \left[(\mathcal{T}_s - \Gamma_s)\, ds - dJ_s \right] \right]. \tag{67}$$

We adopt the assumption of Green (1987), Phelan and Townsend (1991), and Atkeson (1991) that the planner is risk-neutral or has access to complete insurance markets. To accomplish this we use the same unique SDF $\mathbb{M}$ given in (17), as the one in our primal debt management problem. We assume that there is zero continuation value for the planner after $T^{\mathcal{D}}$. This assumption corresponds to our earlier assumption of no debt recovery upon default in the debt management problem.

Despite the rich history dependence of optimal policies, we can formulate the planner's optimization problem as a time consistent and recursive one using the household's promised value, denoted by $\{W_t; t \geqslant 0\}$, as the key state variable.[34]

Household's Promised Value $\{W_t; t \geqslant 0\}$. The household's promised value at time t, W_t, equals the present value of all future payments:

$$W_t = \mathbb{E}_t \int_t^{\infty} e^{-\zeta(s-t)} \frac{\mathbb{M}_s}{\mathbb{M}_t} \left(dJ_s + (Y_s - (\mathcal{T}_s + C_s)) \left(1 - \mathbf{1}_s^{\mathcal{D}}\right) ds + [\widehat{Y}_s - (\widehat{\mathcal{T}}_s + \widehat{C}_s)]\mathbf{1}_s^{\mathcal{D}} ds \right). \tag{68}$$

Using the Martingale Representation Theorem, without loss of generality, we can represent the dynamics of $\{W_t; t \geqslant 0\}$ as:

$$dW_t = [(\zeta + r)W_t - (Y_t - \mathcal{T}_t - C_t) - \eta \Phi_t^m]\, dt - dJ_t - \Phi_t^h d\mathcal{Z}_t^h - \Phi_t^m d\mathcal{Z}_t^m. \tag{69}$$

The planner chooses $\{\Phi_t^h; t \geqslant 0\}$ and $\{\Phi_t^m; t \geqslant 0\}$, exposures of the household's promised value $\{W_t; t \geqslant 0\}$ to idiosyncratic and systematic risks exposures, respectively.

As in our debt management problem, the planner and the household have diversified away risks, so we use the same risk adjustments, ones that are consistent with the SDF ($\mathbb{M}$ given in (17), to evaluate the risk premia for both of them. Note that the discount rate ζ for the household exceeds the risk-free rate r.

The planner's problem is subject to the limited-commitment constraint that the household faces at $t \geqslant 0$. Let $\underline{W}_t = \underline{W}(Y_t)$ denote the threshold for the household's promised value

[34] See DeMarzo and Sannikov (2006) and Sannikov (2008) for pioneering work on continuous-time recursive contracting formulations. See Ai and Li (2015) and Bolton, Wang, and Yang (2019) for continuous-time recursive formulations of contracting problems with limited commitment in Corporate Finance.

at which the household is indifferent between defaulting and not defaulting on government debt. This incentive induces the following limited-commitment constraint at all t:

$$W_t \geqslant \underline{W}(Y_t), \quad t \geqslant 0\,. \tag{70}$$

Next, we turn to the planner's choice of a lumpy payout to the household and the associated upper boundary for W. There is a cost of deferring payments to the household because it is impatient ($\zeta \geqslant 0$) and has a higher discount rate than the planner. Deferring payments to the household increases W_t, which relaxes the limited-commitment constraint and reduces the marginal cost of servicing debt. This trade-off suggests an endogenous threshold level, $\overline{W}_t = \overline{W}(Y_t)$, above which it is optimal for the planner to make a payment to the household and to defer payments otherwise. Therefore, we set

$$dJ_t = \max\{W_t - \overline{W}(Y_t), 0\}\,. \tag{71}$$

Let $F(W_t, Y_t)$ denote the planner's value function that solves the optimization problem (67). In the payout region where $W_t > \overline{W}(Y_t)$,

$$F(W_t, Y_t) = F(\overline{W}(Y_t), Y_t) - \left(W_t - \overline{W}(Y_t)\right)\,, \tag{72}$$

and the threshold level $\overline{W}$ solves

$$\max_{\overline{W}} \quad F(\overline{W}, Y) + \overline{W}\,. \tag{73}$$

In the interior region where $W \in [\underline{W}, \overline{W}]$, the planner optimally sets $dJ_t = 0$ and the value function $F(W, Y)$ satisfies the HJB equation:

$$\begin{aligned} rF(W,Y) = \max_{\mathcal{T},\Phi^h,\Phi^m} \quad & (\mathcal{T} - \Gamma) + ((\zeta + r)W - (Y - \mathcal{T} - C(\mathcal{T}, Y)))\,F_W + (g - \rho\eta\sigma_Y)YF_Y \\ & + \frac{\sigma_Y^2 Y^2 F_{YY}}{2} + \frac{((\Phi^h)^2 + (\Phi^m)^2)F_{WW}}{2} - \left(\psi_h \Phi^h + \psi_m \Phi^m\right) YF_{WY}. \end{aligned} \tag{74}$$

See Appendix C for details.

6.2 Planner's Optimal Value Function

Using the homogeneity property, we can simplify the planner's problem to a one-dimensional problem. Let $w_t = W_t/Y_t$ denote the scaled household's value and let

$$F(W_t, Y_t) = f(w_t) \cdot Y_t \,. \tag{75}$$

Let $\overline{w}_t = \overline{W}_t/Y_t$ denote the scaled upper boundary of w. We can show that $\overline{w}_t$ is constant so that we can drop the time subscript.

The scaled optimal lumpy payout to the household for w_t, $dj_t = dJ_t/Y_t$, at any t is

$$dj_t = \max\{w_t - \overline{w}_t, 0\}. \tag{76}$$

Interior Region: $w_t \in [\underline{w}, \overline{w}]$**.** Here there is no lumpy payout: $dj_t = 0$. Let $\theta_t = \theta(w_t) = \mathcal{T}_t/Y_t$ denote the optimal tax rate. Substituting (75) into (74) and simplifying yields the following implicit equation for $\theta(w)$:

$$1 + c'(\theta(w)) = -1/f'(w). \tag{77}$$

Using the optimal tax policy (77) and the optimal hedging strategies, (C.32) and (C.33), we obtain the following deterministic dynamics for the scaled promised value w_t:

$$\dot{w}_t \equiv \mu_t^w = \mu^w(w_t) = (\zeta + r_V - g)w_t - (1 - \theta_t - c(\theta_t)) \,. \tag{78}$$

Substituting $F(W_t, Y_t) = f(w_t) \cdot Y_t$ from (75) and the optimal policy functions (77), (C.32), and (C.33) for $\theta(w)$, $\phi^h(w)$, and $\phi^m(w)$, respectively, into the HJB equation (74), we obtain the following first-order nonlinear differential equation for the planner's scaled value $f(w)$:

$$(r_V - g)f(w) = \tau(w) - \gamma + \left[(\zeta + r_V - g)w - (1 - \theta(w) - c(\theta(w)))\right] f'(w) \,. \tag{79}$$

Lumpy-payout Region: $w > \overline{w}$**.** Here the planner's value function is $f(w) = f(\overline{w}) + \overline{w} - w$. The upper boundary $\overline{w}$ is constant and solves

$$\max_{\overline{w}} \quad f(\overline{w}) + \overline{w} \,. \tag{80}$$

Default Regime and Limited-commitment Constraint. The government has the option to renege on its debt at any t. Upon default, debt is zero, output drops to $\widehat{Y}_t = \alpha Y_t$, and the household has to pay for government spending period by period ($\mathcal{T}_t = \Gamma_t$). The government's (the household's) value in the default regime, $\widehat{W}(\widehat{Y}_t)$, is

$$\widehat{W}(\widehat{Y}_t) = \mathbb{E}\int_t^{\infty} e^{-\zeta(s-t)}\frac{\mathbb{M}_s}{\mathbb{M}_t}\left(\widehat{Y}_s - \Gamma_s - \widehat{C}(\Gamma_s, \widehat{Y}_s)\right) dt\,. \tag{81}$$

The limited-commitment constraint requires that the lower boundary of W_t in the interior region, $\underline{W}(Y_t)$, is greater than or equal to the value function in the default regime $\widehat{W}(\widehat{Y}_t)$:

$$W_t \geqslant \underline{W}(Y_t) \geqslant \widehat{W}(\widehat{Y}_t)\,. \tag{82}$$

The inequality $\underline{W}(Y_t) \geqslant \widehat{W}(\widehat{Y}_t)$ holds with equality when the tax constraint (8) is not binding. Otherwise, the tax constraint (8) pins down the lower boundary $\underline{W}(Y_t)$.

Let $\widehat{w}_t = \widehat{W}(\widehat{Y}_t)/\widehat{Y}_t$. Using the homogeneity property and solving (81), we obtain:

$$\widehat{w} = \frac{1-\gamma/\alpha - \kappa c(\gamma/\alpha)}{\zeta + r_V - g}\,. \tag{83}$$

Then the scaled promised outside value $\underline{w}$ is

$$\underline{w} = \alpha\widehat{w}\,, \quad \text{when the tax constraint (8) does not bind.} \tag{84}$$

Otherwise, (8) binds at the boundary and $\underline{w}$ is the root of the following equation:

$$\theta(\underline{w}) = \overline{\tau}\,. \tag{85}$$

To ensure that $w \geqslant \underline{w}$, using the same reasoning as in our primal formulation, we obtain the following zero-drift condition for w at $\underline{w}$:

$$\mu^w(\underline{w}) = (\zeta + r_V - g)\underline{w} - (1 - \theta(\underline{w}) - c(\theta(\underline{w}))) = 0. \tag{86}$$

The following theorem describes the optimal contract.

Theorem 6.1. *Under the $r_V > g$ condition given in (28), $\kappa > 1$, $\alpha \leqslant 1$, and the condition $1-\gamma/\alpha - \kappa c(\gamma/\alpha) \geqslant 0$ given in (53), the scaled value function in the normal regime, $f(w)$, satisfies the first-order nonlinear differential equation:*

$$(r_V - g)f(w) = \tau(w) - \gamma + \left[(\zeta + r_V - g)w - (1 - \theta(w) - c(\theta(w)))\right] f'(w)\,, \tag{87}$$

subject to the zero-drift condition (86) and one of the following two conditions for the scaled promised outside value $\underline{w}$*:*

$$\underline{w} = \alpha\widehat{w}\,, \quad \textit{when the tax constraint (8) does not bind}; \tag{88}$$

$$\theta(\underline{w}) = \overline{\tau}\,, \quad \textit{when the tax constraint (8) binds}. \tag{89}$$

The scaled value in the default regime, $\widehat{w}$*, is given by*

$$\widehat{w} = \frac{1-\gamma/\alpha-\kappa c(\gamma/\alpha)}{\zeta+(r_V-g)}\,. \tag{90}$$

The lumpy-payout boundary $\overline{w}$ *is given by (80), and the optimal lumpy-payout policy,* dj_t*, is given by (76). The optimal tax rate policy* $\theta(w)$ *is given by (77) and the scaled promised value* $\{w_t\}$ *evolves deterministically at the rate of* $\dot{w}_t$ *described by (78).*

6.3 Equivalence of Primal and Dual Taxes and Debts

The government's dynamic state-contingent debt management problem (the primal) and the planner's problem (the dual) yield identical outcomes with probability one. The state variable in the primal government debt management problem (scaled debt, b) equals the value function (scaled planner's value, $f(w)$) in the dual planner's problem. By symmetry, the state variable in the dual planner's problem (promised value for the household, w) equals the value function (investors' value, $p(b)$) in the primal government debt management problem. Thus, the following two equations hold:

$$b = f(w) \quad \text{and} \quad w = p(b). \tag{91}$$

Together these equations imply $f \circ p(b) = b$. The composition of $p(\,\cdot)$ from the primal debt management problem with $f(\,\cdot)$ from the dual planner's problem equals an identity function. Table 1 summarizes the one-to-one mapping for state variables, value functions, policy rules in the two problems.

Table 1: **Comparison of Primal and Dual Optimization Problems**

	Primal Debt Management	Dual Planner's Allocation
A. State variables	b	w
Drift	$\dot{b}_t$ given in (57)	$\dot{w}_t$ given in (78)
Admissible region	$b \in [\underline{b}, \bar{b}]$	$w \in [\underline{w}, \overline{w}]$
B. Value function	$p(b)$	$f(w)$
Interior region	ODE given in (61)	ODE given in (79)
C. Policy rules		
Lumpy payouts	du given in (60)	dj given in (76)
Payout boundaries	$\underline{b}$ given in (59)	$\overline{w}$ given in (80)
Tax rates	$\tau(b)$ given in (56)	$\theta(w)$ given in (77)
D. Limited commitment		
Boundary condition	$\mu^b(\bar{b}) = 0$	$\mu^w(\underline{w}) = 0$
Default value	$\widehat{p}$ given in (64)	$\widehat{w}$ given in (83)
Non-binding-tax-constraint case	$p(\bar{b}) = \alpha\widehat{p}$	$\underline{w} = \alpha\widehat{w}$
Binding-tax-constraint case	$\tau(\bar{b}) = \overline{\tau}$	$\theta(\underline{w}) = \overline{\tau}$

7 Quantitative Analysis

To prepare the way for some quantitative illustrations of some of our model's salient properties, we first describe how we set key parameters.

7.1 Parameters

We set the mean and volatility of output growth to $g = 3\%$ and $\sigma_Y = 5\%$ per annum in line with the estimates in Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2020). We set the annual risk-free rate r to 1%, the risk premium ξ to 4%, and the government spending/output ratio to $\gamma = 20\%$, in line with the estimates in Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019). We set the parameter that governs output loss after default to 5% by choosing $\alpha = 0.95$ (see Hébert and Schreger (2017) and Rebelo, Wang, and Yang (2021)). We set the upper bound for the maximum politically feasible tax rate $\overline{\tau}$ at 50%.[35] As benchmarks, Denmark has the highest average tax-output ratio: 46.3% and the average tax rate in OECD countries is 33.8%. We calibrated $\Xi = \{\zeta, \kappa, \varphi\}$ from the US debt data from 2000 to 2020 (see Appendix D). The impatience parameter is $\zeta = 0.1\%$ per annum.

[35] In the 1920's, Keynes had guessed .25 for this parameter.

Table 2: **Parameter Values.** This table summarizes the parameter values for our baseline quantitative analysis. Whenever applicable, parameter values are continuously compounded and annualized.

Parameter	Symbol	Value
A. Calibration inputs		
risk-free rate	r	1%
risk premium	ξ	4%
average output growth rate	g	3%
output growth volatility	σ_Y	5%
government spending to output ratio	γ	20%
output recovery in the default regime	α	0.95
B. Calibration outputs		
impatience	ζ	0.1%
tax deadweight loss	φ	2.9
default deadweight loss	κ	1.2

We follow Barro (1979) in using a quadratic deadweight loss function:

$$c(\tau) = \frac{\varphi}{2}\tau^2 , \tag{92}$$

where the parameter $\varphi > 0$ measures the deadweight cost caused by distortionary taxes. We calibrated tax distortion parameter at $\varphi = 2.9$. We assumed that the deadweight loss function (holding the tax rate on output fixed, $\tau = \widehat{\tau}$) increases from $c(\,\cdot\,)$ to $\widehat{c}(\,\cdot\,) = \kappa c(\,\cdot\,)$ by 20% if the government defaults: $\kappa = 1.2$. Table 2 summarizes parameter values deployed in our baseline quantitative analysis.

7.2 Government Value and Endogenous Debt Capacity

Figure 1 plots the government's value $p(b)$, the marginal cost of debt $-p'(b)$, the optimal tax policy $\tau(b)$, and the debt-GDP ratio drift $\mu^b(b)$. The higher debt b, the more constrained is the government, and thus $p(b)$ decreases (panel A). Debt capacity is $\overline{b} = 1.99$, which means that the government's maximal debt capacity $\overline{B}$ is 199% of output Y. The government's value $p(b)$ decreases with debt-GDP ratio b and equals $p(\overline{b}) = 32.2$ when the government reaches its debt capacity ($\overline{b} = 1.99$). Panel B shows that the marginal cost of debt $-p'(b)$ increases with b and reaches the maximum value of $-p'(1.99) = 1.69$ at $b = \overline{b} = 1.99$. Thus, the net social marginal cost of debt is 1.69 dollars at the debt limit. This number reflects tax distortion costs and the government's option to default on its debt. At the current

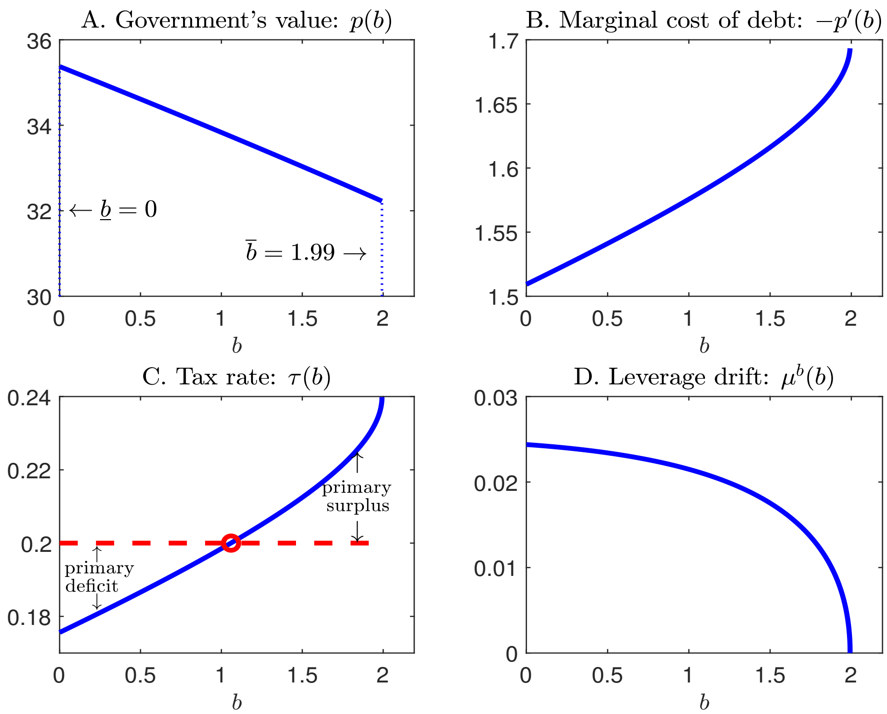

Figure 1: **Government's Value $p(b)$, Marginal Cost of (Servicing) Debt $-p'(b)$, Optimal Tax Rate $\tau(b)$, and Drift of Debt-GDP Ratio $\mu^b(b)$.** Debt capacity is $\overline{b} = 1.99$ and there is no lumpy debt issuance: $\underline{b} = 0$. Parameter value are reported in Table 2.

US debt-output ratio (1.08), the marginal cost of servicing debt is about $-p'(1.08) = 1.58$ dollars.

As $-p'(b)$ increases with b, the optimal tax rate $\tau(b)$ also increases and reaches its maximum value $\tau(\overline{b}) = 0.24$ at the debt limit $\overline{b} = 1.99$ (panel C). At the current debt-output ratio (1.08), the optimal tax rate on output is about $\tau(1.08) = 20\%$.

Note that $\dot{b}_t$, the rate at which the debt-GDP ratio increases, decreases with the level of b_t. As b increases, both the marginal cost of servicing debt $-p'(b)$ and the tax rate $\tau(b)$ increase. As a result, the debt-GDP ratio increases at a slower rate (i.e., $\dot{b}_t$ decreases) until it eventually reaches zero at debt capacity: $\mu^b(\overline{b}) = 0$ (panel D). This is because the government cannot exceed its debt limit. Finally, because the marginal cost of servicing debt $-p'(0) = 1.51$, the government does not want to issue lumpy debt.

Figure 2 shows that limited commitment significantly reduces the government's debt capacity (compare the solid blue with the dashed red lines). To isolate the effect of limited-commitment, we set $\zeta = 0$ in our baseline model. Notice that for all levels of b up to $\overline{b} = 1.99$,

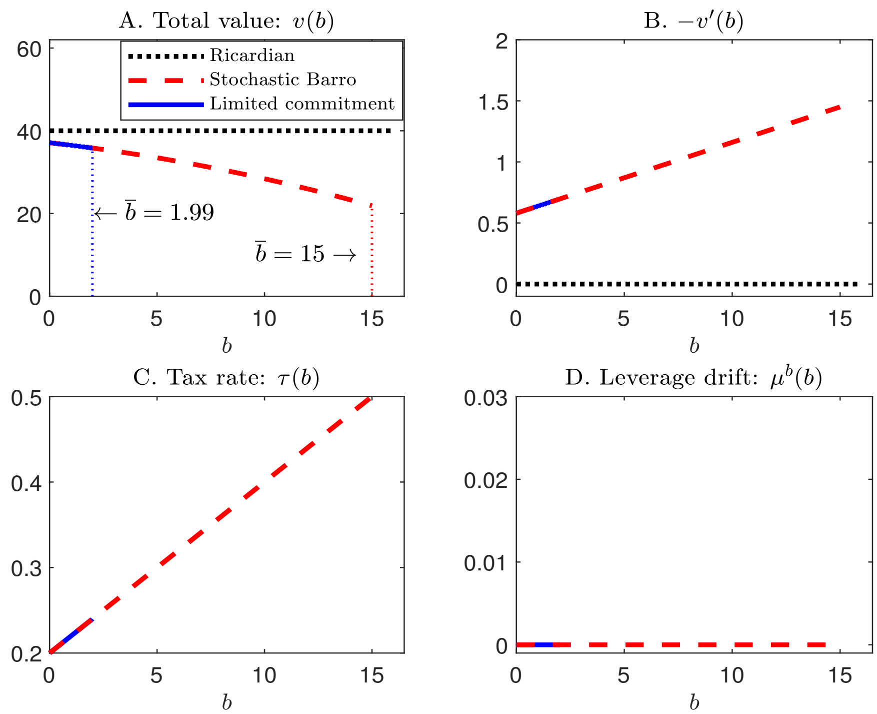

Figure 2: **Value of Commitment and Ricardian Equivalence.** For all three cases in this figure, there is no impatience ($\zeta = 0$). All other parameter value are reported in Table 2. In the stochastic Barro (full-commitment) model, debt capacity is $\bar{b} = 15$ with $\bar{\tau} = 0.5$. In our limited-commitment model, debt capacity is $\bar{b} = 1.99$. Under Ricardian equivalence, an outcome prevails at which $v(b) = v^{FB} = 40$ and $v'(b) = 0$.

the optimal government plan in our limited-commitment model coincides with that for our stochastic version of a Barro model, which assumes commitment and $\zeta = 0$); here $\bar{b} = 1.99$ is debt capacity in our limited-commitment model. A notable result from this figure is that the government's debt capacity is reduced by 87% from $\bar{b} = 15$ in the stochastic Barro model to 1.99 in our model.[36] This 87% reduction of debt capacity is attributable solely to the government having the option to default in our model.

Figure 2 also shows how taxes are distortionary. An undistorted outcome is attained under the special section 3 version of our model that we used to deliver a Ricardian equivalence outcome. In our model, the total scaled value in this case is $v^{FB} = (1-\gamma)/(r_V - g) = 40$. Under Ricardian equivalence, tax and debt policies are irrelevant and therefore the marginal deadweight cost of debt, $-v'(b) = 0$, is zero for all admissible levels of b (panel B). The

[36] The government's debt capacity for the stochastic Barro model equals $\bar{b}^* = \frac{\bar{\tau}-\gamma}{r_V-g} = \frac{0.5-0.2}{5\%-3\%} = 15$.

gap between the solid blue line (the $v(b)$ solution for the stochastic Barro model) and the horizontal Ricardian (dotted black) line $v(b) = v^{FB} = 40$ increases with b. In the special section 4 stochastic Barro (1979) version of our model, the marginal deadweight cost of debt increases with b and reaches $-v'(\overline{b}) = 1.45$ at its debt limit $\overline{b} = 15$. To sustain such a high level of debt, the government has to tax output at the very distortionary 50%: $\tau(\overline{b}) = 0.5$.

7.3 Comparative Dynamics

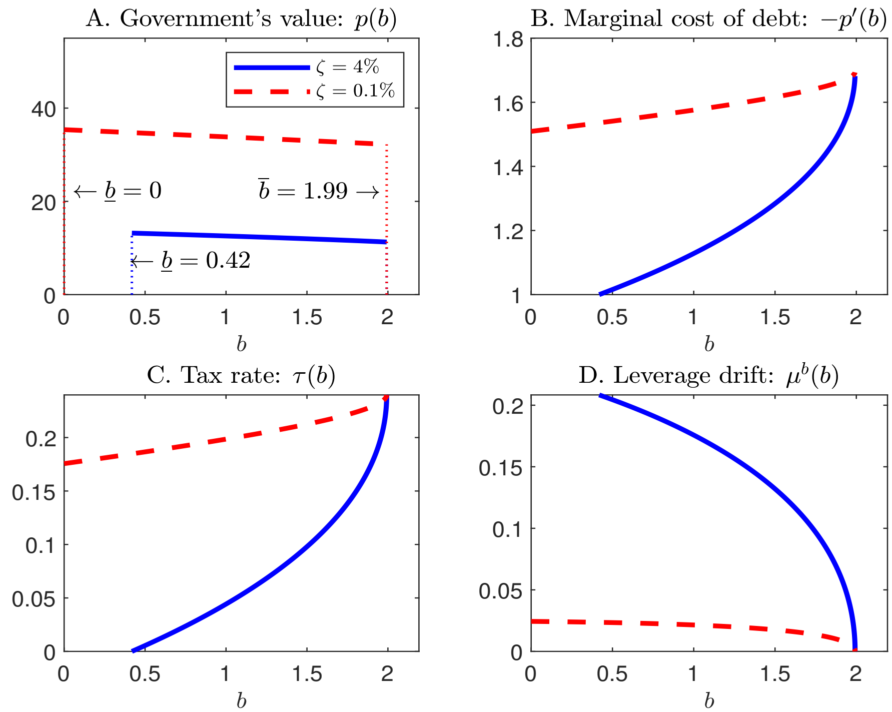

Figure 3: **Effects of Impatience ζ.** All parameter values other than ζ are reported in Table 2.

Effects of Impatience ζ. The parameter ζ measures the degree of the government's impatience. A higher ζ indicates more impatience. It introduces a wedge in first-order conditions that has quantitatively important effects on taxes and value functions. Figure 3 compares outcomes in our baseline ($\zeta = 0.1\%$) case with those from a $\zeta = 4\%$ case in which the government is much more impatient.

As ζ increases from 0.1% to 4%, the total value $p(b)$ decreases by about two thirds at all admissible levels of b (panel A.) This outcome emerges mostly from a typical discounting channel. More interesting to us is that both the marginal cost of debt ($-p'(b)$) and the optimal tax rate ($\tau(b)$) decrease substantially for most values of b (panels B and C). This happens because it is much less costly for the government to defer taxation. As a result, the marginal cost of debt ($-p'(b)$) at $b = 0.42$ is one when $\zeta = 4\%$ but equals 1.54 dollars in our baseline $\zeta = 0.1\%$ case. The optimal tax rate ($\tau(b)$) at $b = 0.42$ is zero when $\zeta = 4\%$ but equals 18% in our baseline $\zeta = 0.1\%$ case.

For both cases, as b increases, the tax rate $\tau(b)$ and the marginal cost of debt increase until debt has reached debt capacity $\bar{b} = 1.99$. While increasing ζ does not change the government's debt capacity, it does substantially increases the drift of the debt-GDP ratio $\mu^b(b)$, which in turn changes the time it takes for a government to reach its debt capacity, as we describe in Section 7.4.

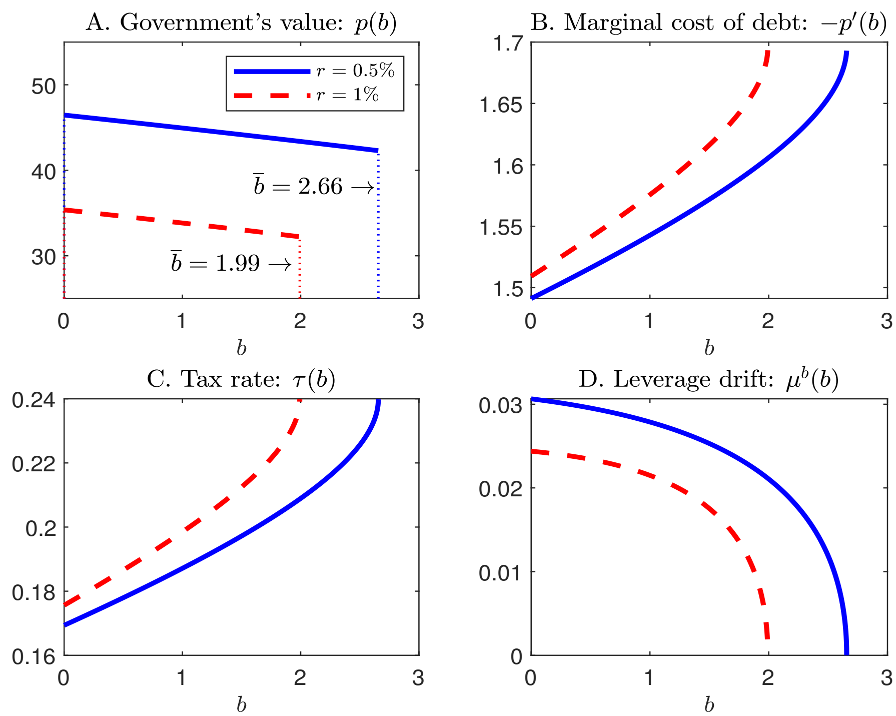

Figure 4: **Effects of Interest Rate r.** All parameter values other than r are reported in Table 2.

Effects of Risk-free Rate r. Figure 4 compares outcomes in our baseline ($r = 1\%$) case with those in an $r = 0.5\%$ case. When r decreases across economies from 1% to 0.5%, a government's debt capacity $\bar{b}$ increases substantially from 1.99 to 2.66. Importantly, both the marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ decrease substantially for the lower r economy. Because interest payments are smaller, debt burden is smaller and tax distortions are also smaller. As a result, a government is more willing to borrow causing the drift of the debt-GDP ratio $\mu^b(b)$ to increase as r falls for all levels of b (panel D).

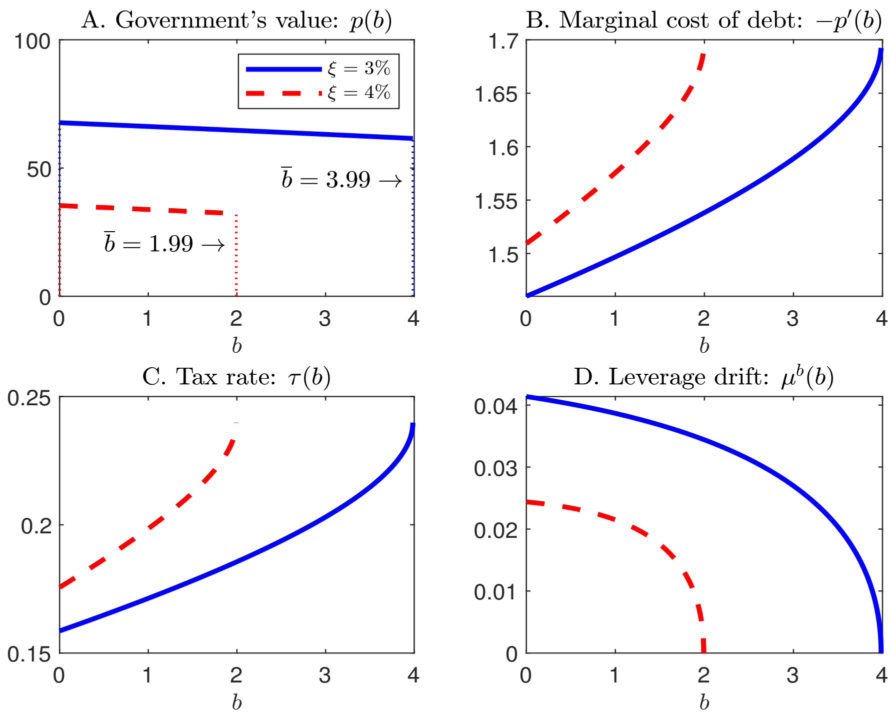

Figure 5: **Effects of Risk Premium ξ.** All parameter values other than ξ are reported in Table 2.

Effects of Risk Premium ξ. Figure 5 compares outcome under our baseline ($\xi = 4\%$) case with those of a $\xi = 3\%$ case. When across economies ξ decreases from 4% to 3%, a government's debt capacity $\bar{b}$ doubles from 1.99 to 3.99. Importantly, both the marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ decrease markedly as the risk premium ξ falls. Because systematic risk management costs are smaller, the debt burden and tax distortions are smaller. As a result, a government is more willing to borrow causing the drift of the debt-GDP ratio $\dot{b}_t = \mu^b(b)$ to increase as risk premium falls for all levels of b (panel D).

Effects of Output Growth Rate g. Figure 6 compares outcomes under our baseline ($g = 3\%$) case with those from a $g = 2\%$ economy. When the growth rate across economies

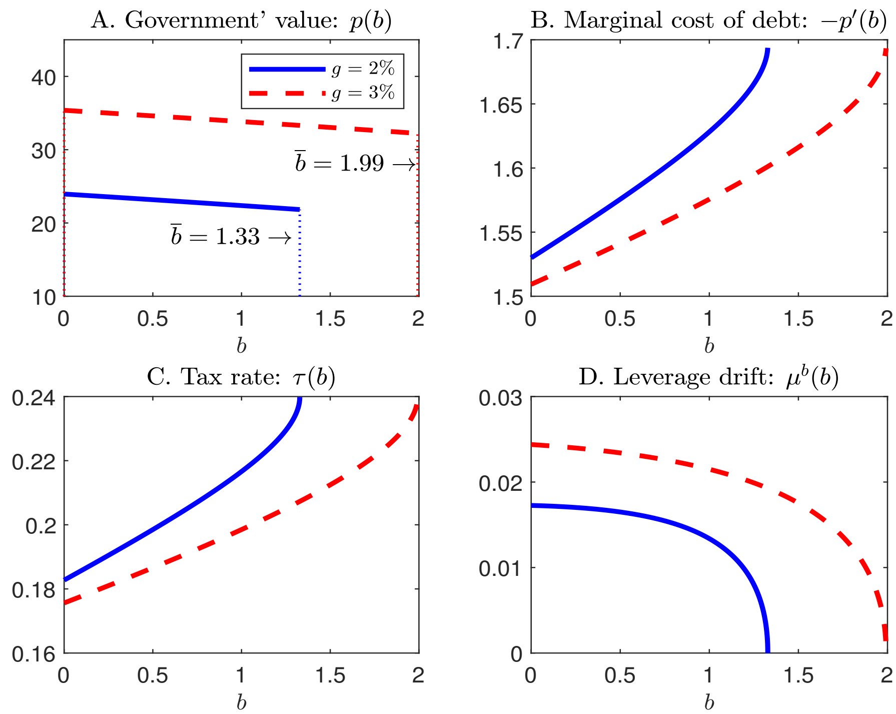

Figure 6: **Effect of Average Output Gtrowth Rate g.** All parameter values other than g are reported in Table 2.

decreases from 3% to 2%, a government's debt capacity $\bar{b}$ decreases by about one third from 1.99 to 1.33. The marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ both increase substantially as the growth rate falls from 3% to 2%. With slower growth, a government is less willing to borrow against the future, causing drift of the debt-GDP ratio $\dot{b}_t = \mu^b(b)$ to fall for all levels of b (panel D). That government response has important implications about the time it takes for a government to reach its debt limit.

Effects of Tax Distortion Cost φ. The parameter φ governs tax distortions in the deadweight loss function $c(\,\cdot\,)$. Figure 7 compares outcomes under our baseline ($\varphi = 2.9$) case with those from a $\varphi = 4$ case. When φ increases from 2.9 to 4, a government's debt capacity $\bar{b}$ decreases a little from 1.99 to 1.89 and a government's value function $p(b)$ decreases. The marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ both increase. When taxes are more distortionary, a government is less willing to borrow against the future, causing drift of the debt-GDP ratio $\dot{b}_t = \mu^b(b)$ to fall at all levels of b (panel D).

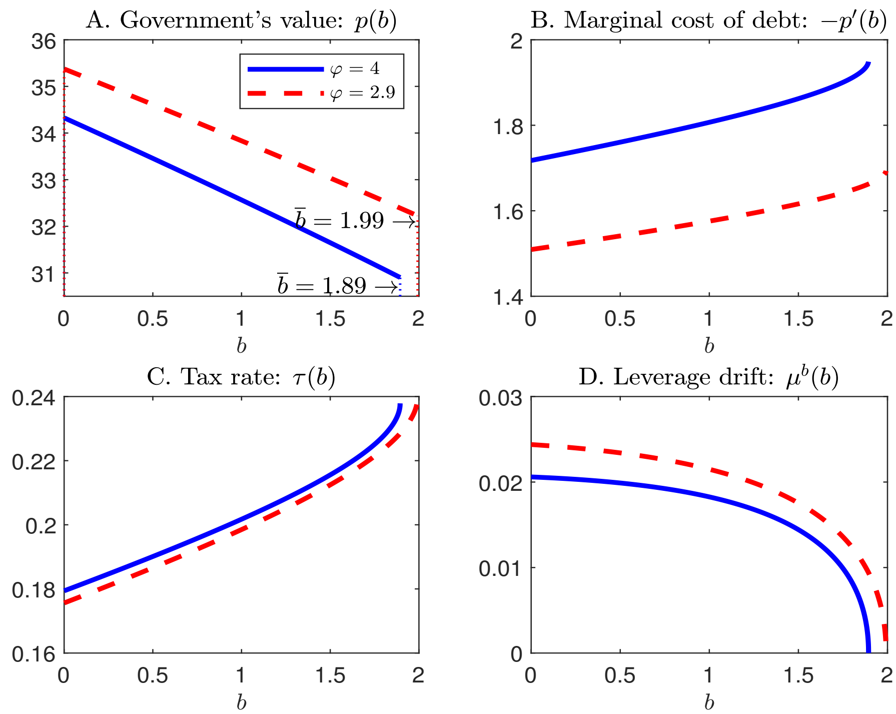

Figure 7: **Effect of Tax Distortion Cost φ.** All parameter values other than φ are reported in Table 2.

Effects of of Default Costs: (Increasing Tax Distortion Costs $\kappa > 1$). The parameter κ measures how much more distortionary taxes are in the default regime than in the service-debt regime. Figure 8 compares outcomes under our baseline ($\kappa = 1.2$) case with those under a $\kappa = 1.5$ case. When across economies κ increases from 1.2 to 1.5, a government's debt capacity $\bar{b}$ *increases* from 1.99 to 2.53 and a government's value function $p(b)$ increases slightly. The marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ both decrease. That is because when default is more costly, a government is more willing to repay debt, allowing it to borrow more. As κ increases across economies, the drift of the debt-GDP ratio $\dot{b}_t = \mu^b(b_t)$ is higher for all levels of b (panel D).

Effects of Default Costs: Output Loss $(1-\alpha)$. The parameter α measures the recovery of output in the default regime. Figure 9 compares outcomes under our baseline ($\alpha = 0.95$) case with those under an $\alpha = 0.9$ case. When across economies output loss $(1-\alpha)$ increases from 5% to 10%, a government's debt capacity $\bar{b}$ *increases* markedly from 1.99 to 3.55, but a government's value function $p(b)$ increases only slightly. The marginal cost of debt

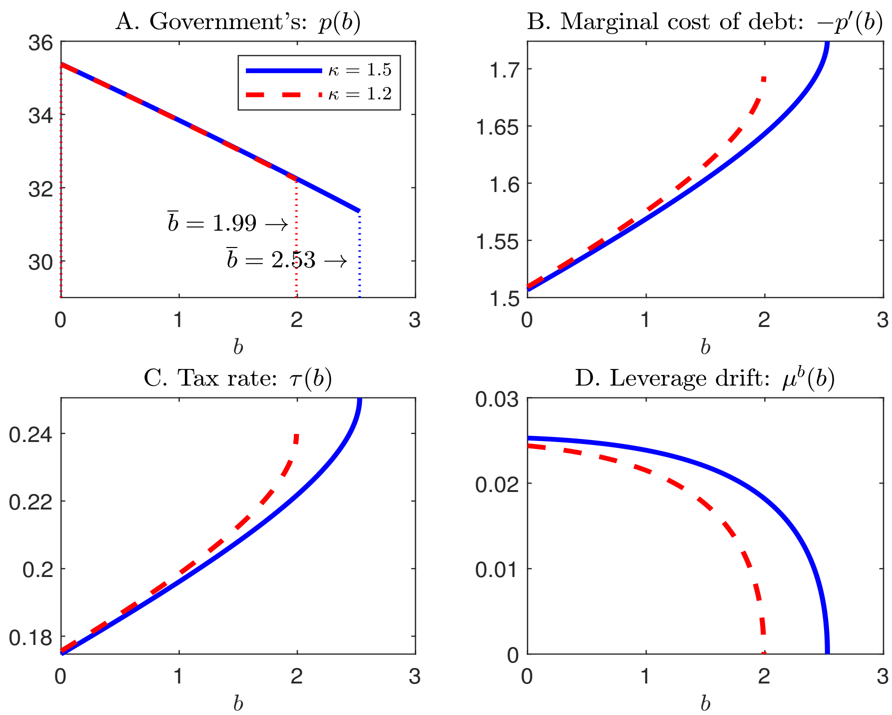

Figure 8: **Effects of Default Costs: (Increasing Tax Distortion Costs $\kappa > 1$).** All parameter values other than κ are reported in Table 2.

$-p'(b)$ and the tax rate $\tau(b)$ both decrease. This is because when default is more costly, the government is more willing to repay debt and hence is able to borrow more. Finally, the drift of the debt-GDP ratio $\dot{b}_t = \mu^b(b_t)$ is higher as we increase output loss $(1-\alpha)$ for all levels of b (panel D).

Our comparative static results with respect to $(1-\alpha)$ and κ are similar because increasing $(1-\alpha)$ directionally has the same effect as increasing κ. Both make default more costly, which in turn improves incentives to repay and therefore debt capacity.

Effects of Government Spending-GDP Ratio γ. The parameter γ measures a government spending as a fraction of output. Figure 10 compares outcomes under our baseline ($\gamma = 0.2$) case with those under a $\gamma = 0.25$ case. When across economies government spending γ increases from 0.2 to 0.25, a government's debt capacity $\bar{b}$ *increases* slightly from 1.99 to 2.07, but the government's value function $p(b)$ decreases markedly. The marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ both increase substantially. That is because when the government spending fraction is higher, a government's value in the default regime becomes

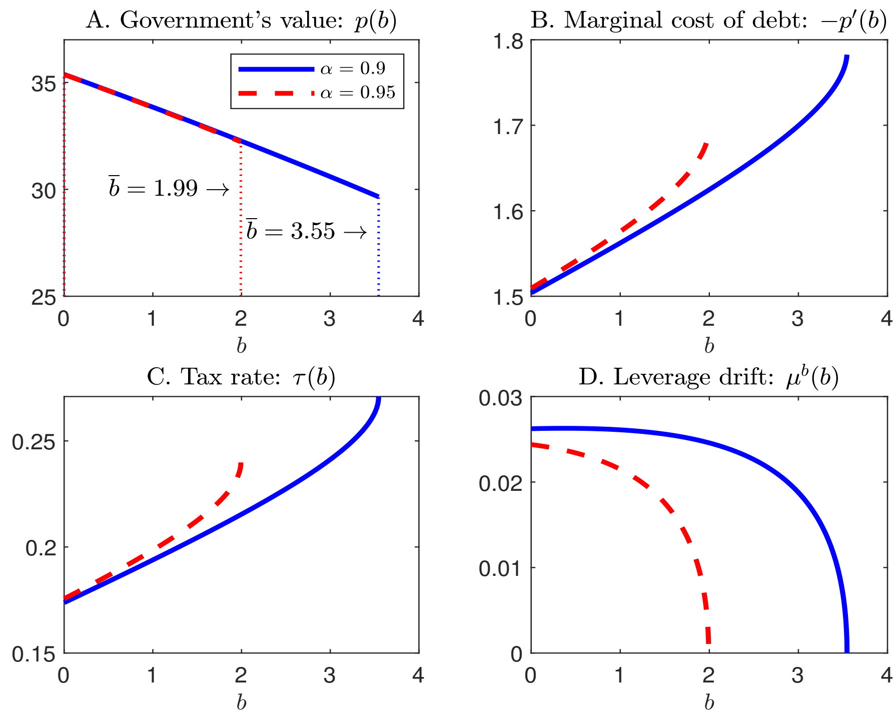

Figure 9: **Effect of Default Costs: Output Recovery** α**.** All parameter values other than α are reported in Table 2.

lower. Hence, a government is more willing to tax more in order to repay its debt. The enables it to borrow more. For all levels of b, the drift of the debt-GDP ratio $\dot{b}_t = \mu^b(b_t)$ is higher when the government spending fraction γ is higher (panel D).

7.4 Time to Reach Debt Capacity

Our model asserts that a government's debt-output ratio b_t evolves deterministically at rate $\dot{b}_t = \mu^b(b_t)$ described by (57). For a given initial b_0, the time it takes for the government to reach its debt capacity $\bar{b}$ is

$$\int_{b_0}^{\bar{b}} \frac{db_t}{\dot{b}_t} = \int_{b_0}^{\bar{b}} \frac{1}{(r_V - g)b_t + \gamma - \tau(b_t)} db_t. \tag{93}$$

Figure 11 shows that as governments become more impatient across economies (i.e., as ζ increases), the time it takes for the government to exhaust its debt capacity decreases. Even for seemingly small increase of impatience, effects of impatience are large In our calculation,

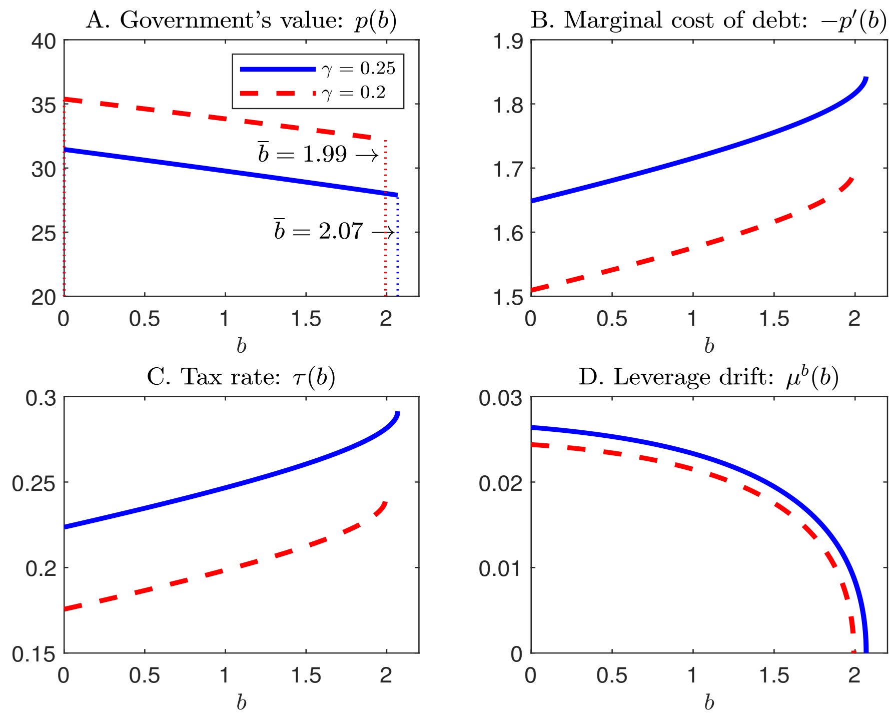

Figure 10: **Effect of Government Spending γ.** All parameter values other than γ are reported in Table 2.

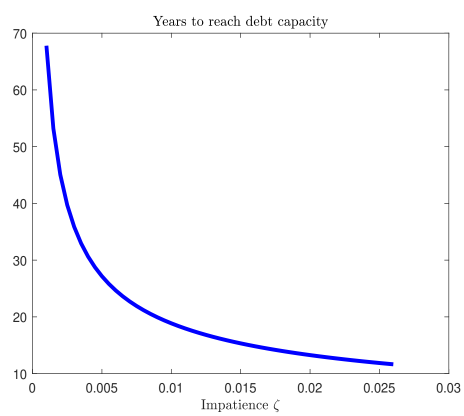

Figure 11: **Time to Reach Debt Capacity as a Function of Impatience ζ.** All other parameter values are reported in Table 2. The initial the debt-GDP ratio is $b_0 = 108.1\%$ and debt capacity is 199%.

starting from the current US debt level of $b = 108\%$, it will takes about 68 years to reach the debt limit in 2088 if $\zeta = 0.1\%$, but it would takes less than 20 years to reach the debt limit in 2039 if impatience were to increases to $\zeta = 1\%$. If we interpret populism as impatience, these comparative dynamics are consistent with a commonly held view that debt capacity is smaller for a populist government.

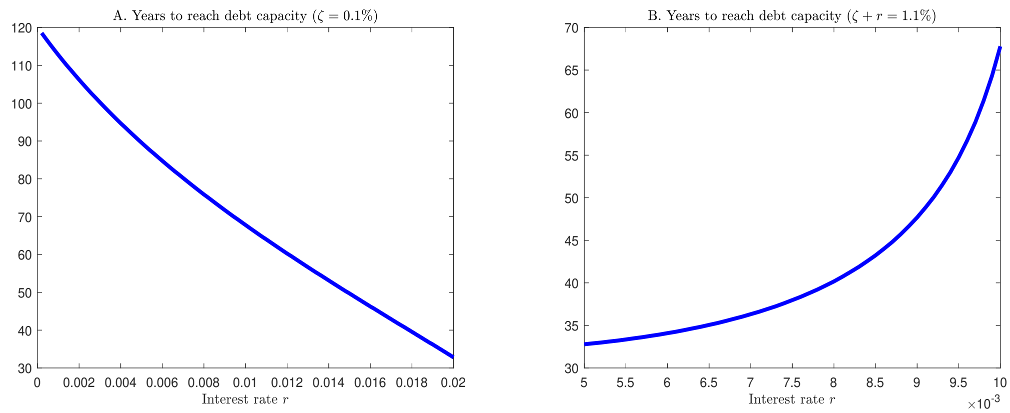

Figure 12: **Time to Reach Debt Capacity as a Function of Interest Rate** r. For both panels, the initial b is $b_0 = 108.1\%$. In panel A, the impatience parameter is fixed at $\zeta = 0.1\%$. In panel B, the discount rate is fixed at $\zeta + r = 1.1\%$. All other parameter values are reported in Table 2.

Figure 12 plots time it takes for the government to reach its debt capacity as a function of interest rate r. First recall that when facing a lower interest rate, a forward-looking government can finance its debt repayment with a lower tax rate $\tau(b)$, which is less distortionary (a lower marginal cost of debt, $-p'(b)$). As a result, debt is more sustainable, which means a larger debt capacity $\bar{b}$, but the debt-GDP ratio also drifts upward at a faster rate $\dot{b}_t$, *ceteris paribus*. Holding impatience ζ fixed, we see that it takes longer to reach the steady state and exhaust its debt capacity if interest rate is lower (panel A). This is because the debt capacity force is stronger than the drift effect. Across economies, the level of the interest rate has big consequences. With our parameter settings, starting from the current US debt level of $b = 108\%$, it takes about 90 years to reach the debt limit in 2110 if $r = 0.5\%$, but takes about 68 years to reach the debt limit in 2088 if $r = 1\%$. This pattern is in line with reasoning of Blanchard (2019) and Furman and Summers (2020).

We now perform a distinct calculation that holds a government's discount rate should be fixed even though we alter the interest rate. Under such an assumption, we hold a government's discount rate $(\zeta + r)$ fixed and plot time to reach debt capacity as a function of r in panel B of Figure 12. Evidently, it takes less time to reach steady-state debt capacity if interest rate is lower. This is because the drift effect (due to a corresponding increase in impatience ζ) becomes much stronger than the debt capacity effect. For a fixed value of $\zeta + r = 1.1\%$, starting from the current US debt level of $b = 108\%$, it would take about 33 years to reach the debt limit in 2053 if $r = 0.5\%$; but if $r = 1\%$, it would take about 68 years to reach the debt limit in 2088.

A key takeaway from the two panels of Figure 12 is that time to reach the steady-state debt capacity crucially depends on both how impatient the government is and the level of interest rate.

7.5 Quantitative Debt-GDP Ratio Dynamics

Next, we analyze the predicted debt-GDP ratio dynamics using our calibrated parameter values. Since we are interested in both the maximum sustainable debt $\overline{b}$ at yhr optimal steady state and transition dynamics towards $\overline{b}$, we assume that s government can completely hedge its exposures to risks, with the consequence that dynamics of the debt-GDP ratio are deterministic. We have designed our model parsimoniously in a way that can capture a long-run trend and the steady state of debt dynamics.

Panel A of Figure 13 plots the implied debt-GDP ratio dynamics from 2000 to 2020 using parameters from our baseline calibration.[37] Our model (the blue solid line) does a good job of approximating the trend of debt-GDP ratio dynamics $\{b_t\}$ over this 20-year period in the US (the black dashed line). Panels B, C, and D of Figure 13 plot the predicted debt-output ratio $\{b_t\}$ processes starting from 2021 until the government exhausts its debt capacity and reaches the steady state for various scenarios where we change interest rate r, growth rate g, and risk premium ξ.

Panel B shows that the government can be expected to reach its debt capacity ($\overline{b} = 1.99$) in 2088 if $r = 1\%$ as we noted earlier. The debt-GDP ratio gradually builds up until reaching the steady state where $\overline{b} = 1.99$ (the solid blue line.) But if the interest rate were unexpectedly and permanently decrease to $r = 0.5\%$, the debt-GDP ratio would increase at a much faster rate, so that a steady state $\overline{b} = 2.66$ (the dotted red line) would be reached in 2110.

Panel C shows that if a government's growth rate permanently drops to 2% from 3%, the government will reach its reduced debt capacity ($\overline{b} = 1.33$ from 1.99) in 2050. This result confirms the intuition that economic growth is a key source of servicing debt.

Panel D shows that if the risk premium ξ were unexpectedly and permanently to drop to 3% from 4%, the government's debt capacity would then increase to $\overline{b} = 3.98$ from 1.99; it would take almost 110 years to exhaust its debt limit around 2140. This result shows

[37]Recall that our calibration procedure did not target the debt-GDP ratio leverage dynamics that we plot, which only conditions on the initial condition. Our calibration procedure minimizes the sum of the squared of the difference between one-step-ahead model-predicted b_t and the realized b_t.

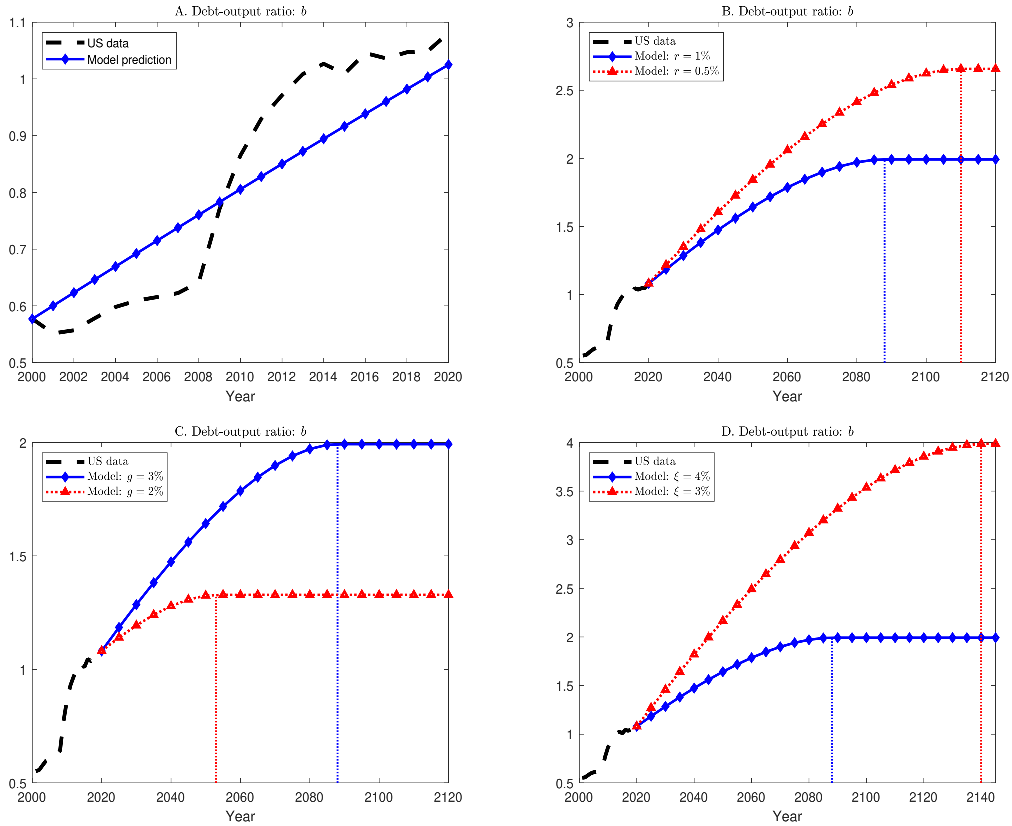

Figure 13: **Predicting Debt-GDP Ratio Dynamics for a Few Scenarios.** The US debt-output ratios in 2000 and 2020 are 57.5% and 108.1%, respectively. For all model-predicted b processes in panels B, C, and D, the left-end points of the horizontal lines are the corresponding levels of debt capacity $\overline{b}$.

that the risk premium ξ has a very large quantitative effect on both debt capacity and on transition dynamics to a steady state.

8 Concluding Remarks and Extensions

To construct streamlined formulas that allow us to isolate salient forces that determine optimal fiscal policy, debt capacity, and debt dynamics, we purposefully chose to work with a complete-markets limited-commitment model with only one aggregate shock (i.e., the stock market). We have neglected other sources of aggregate risks that governments face including stochastic interest rates, a stochastic government spending-GDP ratio γ, and market prices of risk (Jiang, Lustig, Van Nieuwerburgh, and Xiaolan, 2019). We can extend our model to include such risks by making γ, the risk-free rate r, or GDP growth g an n-state Markov process. The mathematical structure of such an extended model closely remains tractable. That extended model has richer dynamics of debt, debt capacity, and taxes and can be used to study various long-run risks that confront a government.

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Appendices

A Stochastic Barro Model (Section 4)

In this appendix, we compute an optimal fiscal policy for the Section 4 model, which is a stochastic formulation of Barro (1979). This model is a special case of our general model formulation with full commitment and no impatience ($\zeta = 0$). We characterize the government's value function and show that the government's tax policies are time consistent.

To solve the government's optimization problem given by (30) subject to the budget constraint (18), we introduce the following Lagrangian $\mathcal{L}$

$$\begin{aligned}\mathcal{L} \;=\; & \max_{\mathcal{T}_t, U_t; t \geqslant 0} \quad \mathbb{E}_0 \int_0^\infty \mathbb{M}_t \left[dU_t + \left(Y_t - (\mathcal{T}_t + C_t)\right) dt \right] \\ & +\lambda \left[\mathbb{E}_0 \int_0^\infty \mathbb{M}_t \left(\mathcal{T}_t - \Gamma_t\right) dt - \mathbb{E}_0 \int_0^\infty \mathbb{M}_t dU_t - B_{0-} \right] , \end{aligned} \tag{A.1}$$

where λ is the Lagrangian multiplier for the government's budget constraint (18).

The first order condition for the optimal tax rate at time t is given by

$$1 + C_{\mathcal{T}}(\mathcal{T}, Y) = \lambda \, . \tag{A.2}$$

Using the homogeneity property of the tax deadweight cost function (7) to simplify the FOC (A.2), we obtain $c'(\tau_t^*) = \lambda - 1$ for the optimal tax rate τ_t^* at any time t. Since λ is a constant, the optimal tax rate τ_t^* is constant at all t: $\tau_t^* = \tau^*$ for all t, where τ^* satisfies:

$$c'(\tau^*) = \lambda - 1 \, . \tag{A.3}$$

The (strict) convexity of the deadweight loss function $c'(\tau)$ implies that the Lagrangian multiplier for the government budget constraint is (strict) larger than one: $\lambda > 1$. Because tax is distortionary and there is no incentive for the government to front load consumption (as $\zeta = 0$), there is no lumpy debt issuance at any time t: $dU_t = 0$. (Moreover, the optimal debt target should be zero: $\underline{b} = 0$, if the government were given the option to chooses its initial debt b_0.) We obtain λ by using (A.3): $\lambda = 1 + c'(\tau^*)$. Next, we determine τ^*.

Because the government's budget constraint (18) holds with equality (as $\lambda > 1$), the present value of primary surplus $\{(\tau^* - \gamma)Y_t; t \geqslant 0\}$, discounted at the rate of r_V, the sum of the risk-free rate r and risk premium ξ, equals the outstanding debt balance, B_0. This calculation yields the following explicit equation:[38]

$$\tau^* = b_0(r_V - g) + \gamma\,. \tag{A.5}$$

Substituting (A.3) and $dU_t = 0$ into the Lagrangian (A.1) and using the homogeneity property, we obtain the following expression for the value function (also the Lagrangian) under optimal policies:

$$p(b_0) = \mathcal{L} = \frac{1 - \tau^* - c(\tau^*)}{r_V - g}\,, \tag{A.6}$$

where τ^* is given in (A.5). As the budget constraint (18) binds, we only need to calculate the first term in (A.1) under the optimal policies.

Using the tax policy given by (A.5), the government optimally adjusts its debt balance B_t in lock step with output Y_t so that the debt-GDP ratio is constant at all $t \geqslant 0$: $b_t = b_0$. The government in the future will follow the same strategy chosen by the time-0 government. Therefore, the government's optimization problem is time consistent Lucas and Stokey (1983).

Finally, we discuss the maximally sustainable debt under commitment. Suppose that the maximal tax burden that the household is willing to tolerate without triggering a revolution, denoted by $\overline{\mathcal{T}}_t^*$, is the level at which the household's value function is zero. Given the stationarity of our perpetual growth model, the household's net cash flow payoff in each period is zero:

$$Y_t - \overline{\mathcal{T}}_t^* - C(\overline{\mathcal{T}}_t^*, Y_t) = 0\,. \tag{A.7}$$

Let $\overline{B}^*$ denote the corresponding largest sustainable debt that the government can credibly honor. Then, $\overline{B}^*$ satisfies the following equation:

$$\overline{B}^* = \mathbb{E}_0 \int_0^\infty \mathbb{M}_t \left(\overline{\mathcal{T}}_t^* - \Gamma_t\right) dt. \tag{A.8}$$

The maximally sustainable debt-GDP ratio $\overline{b}^*$ is then given by $\overline{b}^* = (\overline{\tau}^* - \gamma)\,/\,(r_V - g)$, where $\overline{\tau}^* = \overline{\mathcal{T}}_t^*/Y_t$ solves the equation: $1 - \tau - c(\tau) = 0$.

[38]The present value formula is

$$\frac{\tau^* - \gamma}{r_V - g} = \frac{B_0}{Y_0} \equiv b_0 \tag{A.4}$$

under the condition that the tax policy τ^* is feasible.

B Optimal Fiscal Plan for Section 5 model

In this appendix, we describe the optimal plan that appeared in Section 5 for the primal dynamic debt management problem defined in Section 2.

HJB equation for $P(B,Y)$. Using Ito's formula, we obtain the following SDF-adjusted dynamics for the government's value function $P(B_t, Y_t)$:

$$d(\mathbb{M}_t P(B_t, Y_t)) = \mathbb{M}_t dP(B_t, Y_t) + P(B_t, Y_t) d\mathbb{M}_t + < d\mathbb{M}_t, dP(B_t, Y_t) >, \tag{B.9}$$

where the SDF $\{\mathbb{M}_t; t \geqslant 0\}$ is given in (17) and

$$\begin{aligned} dP(B_t, Y_t) &= P_B dB_t + \frac{P_{BB}}{2} < dB_t, dB_t > + P_Y dY_t + \frac{P_{YY}}{2} < dY_t, dY_t > + P_{BY} < dB_t, dY_t > \\ &= \left[\left((rB + (\Gamma - \mathcal{T})) - \Pi^m \eta \right) P_B + gY P_Y + \frac{\sigma_Y Y^2 P_{YY}}{2} \right] dt \\ &\quad + \left[\frac{\left((\Pi^h)^2 + (\Pi^m)^2 \right) P_{BB}}{2} - (\Pi^h \psi_h + \Pi^m \psi_m) Y P_{BY} \right] dt \\ &\quad - P_B (\Pi^h d\mathcal{Z}_t^h + \Pi^m d\mathcal{Z}_t^m) + Y P_Y (\psi_h d\mathcal{Z}_t^h + \psi_m d\mathcal{Z}_t^m) \, . \end{aligned} \tag{B.10}$$

Note that the process defined by

$$\int_0^t \left(e^{-\zeta s} \mathbb{M}_s \left(Y_s - \mathcal{T}_s - C(\mathcal{T}_s, Y_s) \right) ds \right) + e^{-\zeta s} \mathbb{M}_s dU_s + e^{-\zeta t} \mathbb{M}_t P(B_t, Y_t)$$

is a martingale under the physical measure $\mathbb{P}$. Therefore, its drift under $\mathbb{P}$ is zero:

$$\mathbb{E}_t \left[d \left(e^{-\zeta t} \mathbb{M}_t P(B_t, Y_t) \right) \right] + e^{-\zeta t} \mathbb{M}_t \left(Y_t - \mathcal{T}_t - C(\mathcal{T}_t, Y_t) \right) = 0. \tag{B.11}$$

Note that we have used the result that $dU_t = 0$ in the interior region. Simplifying (B.11) gives the HJB equation (40) for the government's value function $P(B_t, Y_t)$.

We do not repeat the first-order condition (FOC) for the tax rate and other derivations contained in the main body. Below we provide the details for risk management policies.

Stock market portfolio allocation π^m. Let $\pi_t^m = \Pi_t^m / Y_t$ denote the scaled stock market portfolio allocation. Using the homogeneity property, we show that π_t^m is a function of b_t, which we denote by $\pi^m(b_t)$. Simplifying the FOC given in (42) for Π^m, we obtain the following expression for $\pi^m(b)$:

$$\pi^m(b) = -\psi_m b\,. \tag{B.12}$$

Idiosyncratic hedging demand π^h. Let $\pi_t^h = \Pi_t^h/Y_t$ denote the scaled idiosyncratic risk hedging demand. Similarly, using the homogeneity property, we show that π_t^h is a function of b_t, which we denote by $\pi^h(b_t)$. Simplifying the FOC given in (43) for Π^h, we obtain the following expression for $\pi_t^h = \pi^h(b_t)$:

$$\pi^h(b) = -\psi_h b\,. \tag{B.13}$$

Debt-GDP ratio b_t dynamics. Applying Ito's lemma to $b_t = B_t/Y_t$, where B_t is given in (39) and Y_t is given in (2), we obtain

$$db_t \;=\; \mu_t^b\,dt + du_t + \sigma_t^{b,h}\,d\mathcal{Z}_t^h + \sigma_t^{b,m}\,d\mathcal{Z}_t^m\,, \tag{B.14}$$

where

$$\mu_t^b \;=\; (r-g)b_t + \gamma - \tau_t - \eta\pi_t^m + \left(\psi_h\pi_t^h + \psi_m\pi_t^m + b_t\sigma_Y^2\right) \tag{B.15}$$

$$\sigma_t^{b,h} \;=\; -\left(\pi_t^h + \psi_h b_t\right) \tag{B.16}$$

$$\sigma_t^{b,m} \;=\; -\left(\pi_t^m + \psi_m b_t\right)\,. \tag{B.17}$$

Substituting hedging policies (B.12) and (B.13) into (B.15), we show that the debt-output ratio, $\{b_t\}$, evolves deterministically at the rate given by:

$$\dot{b}_t = \mu_t^b = \mu^b(b_t) = (r_V - g)b_t + \gamma - \tau(b_t) \tag{B.18}$$

where $\tau(b_t)$ is given by (56).

Equivalent formulation of optimization problem under risk-neutral measure $\widetilde{\mathbb{P}}$. As is standard in macro research, we have formulated the government's optimization problem in Section 2 and provided the solution in Section 5 under the physical measure $\mathbb{P}$. We can equivalently formulate the problem and solve it under the risk-neutral measure $\widetilde{\mathbb{P}}$. Recall that under the physical measure $\mathbb{P}$, the Brownian motions for idiosyncratic shock and systemic shock are given by $\mathcal{Z}_t^h$ and $d\mathcal{Z}_t^m$, respectively. Because the shock to the market portfolio is systematic with a constant Sharpe ratio of η, using the standard Black-Merton-Scholes dynamic replication argument, we can show that the Brownian motion for systemic shock under the risk-neutral measure $\widetilde{\mathbb{P}}$, denoted by $\widetilde{\mathcal{Z}}_t^m$, is given by

$$d\widetilde{\mathcal{Z}}_t^m = d\mathcal{Z}_t^m + \eta dt\,. \tag{B.19}$$

This equation is also the reason why a well-diversified investor who holds a long position in the market futures contract demands a positive payment at the rate of ηdt to break even. This explains the last term in the law of motion (39) for B_t. The Brownian motion for the idiosyncratic shock under the risk-neutral measure $\widetilde{\mathbb{P}}$ is the same as that under the physical measure $\mathbb{P}$:

$$d\widetilde{\mathcal{Z}}_t^h = d\mathcal{Z}_t^h\,, \tag{B.20}$$

as there is no risk premium.

Using (B.19) amd (B.20) under the risk-neutral measure, we may express the output process (2) under the risk-neutral measure $\widetilde{\mathbb{P}}$ as follows:

$$\frac{dY_t}{Y_t} = \widetilde{g}dt + \sigma_Y\left(\sqrt{1-\rho^2}d\widetilde{\mathcal{Z}}_t^h + \rho d\widetilde{\mathcal{Z}}_t^m\right)\,, \tag{B.21}$$

where $\widetilde{g}$ is the average output growth rate under the risk-neutral measure $\widetilde{\mathbb{P}}$:

$$\widetilde{g} = g - \rho\sigma_Y\eta\,. \tag{B.22}$$

In the interior region where $dU_t = 0$, we may equivalently express the government's optimization problem under the risk-neutral measure $\widetilde{\mathbb{P}}$ as follows:

$$\max_{\mathcal{T}_t\leqslant\overline{\tau}Y_t,\Pi_t^h,\Pi_t^m} \widetilde{\mathbb{E}}_0\left[\int_0^\infty e^{-(\zeta+r)t}\left(\left(Y_t-\mathcal{T}_t-C(\mathcal{T}_t,Y_t)\right)\mathbf{1}_t^{\mathcal{D}} + \left(\widehat{Y}_t-\widehat{\mathcal{T}}_t-\widehat{C}(\widehat{\mathcal{T}}_t,\widehat{Y}_t)\right)\left(1-\mathbf{1}_t^{\mathcal{D}}\right)\right)dt\right]\,, \tag{B.23}$$

subject to the government's tax constraint $\mathcal{T}_t \leqslant \overline{\tau}Y_t$ and the budget constraint:

$$B_t = \widetilde{\mathbb{E}}_t\left[\int_t^{T^D} e^{-r(s-t)}\left(\mathcal{T}_s-\Gamma_s\right)ds\right]\,. \tag{B.24}$$

Note that the budget constraint (B.24) is under the risk-neutral measure $\widetilde{\mathbb{P}}$.

Equation (B.24) implies that $e^{-rt}B_t + \int_0^t e^{-rs}\left(\mathcal{T}_s-\Gamma_s\right)ds$ is a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$. Using the marginal representation theorem, we can equivalently express debt dynamics under the risk-neutral measure $\widetilde{\mathbb{P}}$ as:

$$dB_t = (rB_t + (\Gamma_t - \mathcal{T}_t))\, dt - \Pi_t^h \sigma \, d\widetilde{\mathcal{Z}}_t^h - \Pi_t^m \sigma_m d\widetilde{\mathcal{Z}}_t^m \,. \tag{B.25}$$

Using (B.23), (B.25), and (B.21) in the interior region, we use the following HJB equation to solve the government's value function$P(B,Y)$:

$$\begin{aligned}(\zeta + r)P(B,Y) = \max_{\mathcal{T}\leqslant\overline{\tau}Y,\,\Pi^h,\,\Pi^m} \quad & Y - \mathcal{T} - C(\mathcal{T},Y) + [rB + \Gamma - \mathcal{T}\,]\,P_B(B,Y) \\ & + (g - \rho\eta\sigma_Y)Y P_Y(B,Y) + \frac{(\Pi^h)^2 + (\Pi^m)^2}{2} P_{BB}(B,Y) \\ & + \frac{\sigma_Y^2 Y^2}{2} P_{YY}(B,Y) - \left(\psi_h \Pi^h + \psi_m \Pi^m\right) Y P_{BY}(B,Y)\,. \end{aligned} \tag{B.26}$$

Limited-commitment-induced boundary condition. We show that under the $\kappa > 1$ and $\alpha \leqslant 1$ conditions, there exists a strictly positive debt capacity: $\overline{b} > 0$ which satisfies $p(\overline{b}) \geqslant \alpha\widehat{p}$. When the tax constraint (8) does not bind, there exists a unique $\overline{b} > 0$ where $p(\overline{b}) = \alpha\widehat{p}$. The intuition for this result is as follows. The government is always better off not defaulting. This is because when taxes are more distortionary ($\kappa > 1$) or default causes output losses ($\alpha \leqslant 1$), then the government is always better off to avoid default by prudently managing risk exposures and debt dynamics. Below is a proof.

Equations (61) and (58) imply

$$p(\overline{b}) = \frac{1 - \tau(\overline{b}) - c(\tau(\overline{b}))}{\zeta + r_V - g}\,, \tag{B.27}$$

where $\tau(\overline{b}) = (r_V - g)\overline{b} + \gamma$. Therefore, to prove $p(\overline{b}) \geqslant \alpha\widehat{p}$, where $\widehat{p}$ is given in (64), is equivalent to show

$$1 - \tau(\overline{b}) - c(\tau(\overline{b})) \geqslant \alpha - \gamma - \alpha\kappa c(\gamma/\alpha)\,. \tag{B.28}$$

First, the left side of (B.28) is decreasing $\overline{b}$. Second, the left side of (B.28) when $\overline{b} = 0$ equals $1-\gamma-c(\gamma)$, which is strictly larger than the right side of (B.28). Third, the left side of (B.28) approaches negative infinity as $\overline{b} \to \infty$. Therefore, there exists a unique value of $\overline{b} > 0$ where (B.28) holds with equality. Of course, this solution is interesting when the tax constraint (8) does not bind. Otherwise, there exists a value of $\overline{b} > 0$ that satisfies (B.28) with inequality. In this case, the boundary condition at $\overline{b}$ is determined by the tax constraint (8).

C Duality

In this appendix, we derive the solution summarized in Section 6 for the dual planner's problem and then verify duality between the primal debt management problem and the dual planner's problem.

C.1 Planner's Problem from Section 6

HJB equation for the planner's value function $F(W, Y)$**.** Using Ito's formula, we obtain the following SDF-adjusted dynamics for the planner's value function $F(W_t, Y_t)$:

$$d(\mathbb{M}_t F(W_t, Y_t)) = \mathbb{M}_t dF(W_t, Y_t) + F(W_t, Y_t) d\mathbb{M}_t + < d\mathbb{M}_t, dF(W_t, Y_t) >, \tag{C.29}$$

where the SDF $\mathbb{M}_t$ is given in (17) and

$$\begin{aligned}
dF(W_t, Y_t) &= F_W dW_t + \frac{F_{WW}}{2} < dW_t, dW_t > + F_Y dY_t \\
&\quad + \frac{F_{YY}}{2} < dY_t, dY_t > + F_{WY} < dW_t, dY_t > \\
&= \left[\left(\zeta W_t - (Y_t - \mathcal{T} - C_t) - \Phi^m \eta \right) F_W + gY F_Y + \frac{\sigma_Y Y^2 F_{YY}}{2} \right] dt \\
&\quad + \left[\frac{\left((\Phi^h)^2 + (\Phi^m)^2 \right) F_{WW}}{2} - (\Phi^h \psi_h + \Phi^m \psi_m) Y F_{WY} \right] dt \\
&\quad - F_W (\Phi^h d\mathcal{Z}_t^h + \Phi^m d\mathcal{Z}_t^m) + Y F_Y (\psi_h d\mathcal{Z}_t^h + \psi_m d\mathcal{Z}_t^m) \,.
\end{aligned} \tag{C.30}$$

Note that the process defined by

$$\int_0^t \mathbb{M}_s \left(\mathcal{T}_s - \Gamma_t \right) ds + \mathbb{M}_s dJ_s + \mathbb{M}_t F(W_t, Y_t)$$

is a martingale under the physical measure $\mathbb{P}$. Therefore, its drift under $\mathbb{P}$ is zero:

$$\mathbb{E}_t \left[d \left(\mathbb{M}_t F(W_t, Y_t) \right) \right] + \mathbb{M}_t \left(\mathcal{T}_t - \Gamma_t \right) = 0. \tag{C.31}$$

Note that we have used the result that $dJ_t = 0$ in the interior region. Simplifying (C.31) gives the HJB equation (74) for the government's value function $F(W_t, Y_t)$.

We do not repeat FOC for the tax rate and other derivations contained in the main body. Below we provide the details for risk management policies.

Optimal hedging policies. The optimal idiosyncratic and systematic risk hedging demand functions, $\phi^h(w_t) = \Phi_t^h/Y_t$ and $\phi^m(w_t) = \Phi_t^m/Y_t$, are respectively given by

$$\begin{aligned}\phi^h(w) &= \frac{\psi_h Y F_{WY}(W,Y)}{F_{WW}(W,Y)} = -\psi_h w \text{ and} &\text{(C.32)}\\ \phi^m(w) &= \frac{\psi_m Y F_{WY}(W,Y)}{F_{WW}(W,Y)} = -\psi_m w\,. &\text{(C.33)}\end{aligned}$$

Household promised value w_t dynamics. Applying Ito's lemma to $w_t = W_t/Y_t$, where W_t is given in (69) and Y_t is given in (2), we obtain:

$$\begin{aligned}dw_t &= [(\zeta + r + \rho\eta\sigma_Y - g)w_t - (1-\theta_t - c(\theta_t))]dt + dj_t\\ &\quad + \left[\sigma_Y^2 w_t dt + \left(\sqrt{1-\rho^2}\sigma_Y\phi^h(w_t) + \rho\sigma_Y\phi^m(w_t)\right)dt\right]\\ &\quad -(\phi^h(w_t) + \sqrt{1-\rho^2}\sigma_Y w_t)d\mathcal{Z}_t^h - (\phi^m(w_t) + \rho\sigma_Y w_t)d\mathcal{Z}_t^m\,, &\text{(C.34)}\\ &= \mu^w(w_t)dt + dj_t + \sigma^{w,h}(w_t)d\mathcal{Z}_t^h + \sigma^{w,m}(w_t)d\mathcal{Z}_t^m\,, &\text{(C.35)}\end{aligned}$$

where $dj_t = 0$ in the interior region and

$$\begin{aligned}\mu^w(w_t) &= (\zeta + r_V - g)w_t - (1-\theta_t - c(\theta_t))\,, &\text{(C.36)}\\ \sigma^{w,h}(w) &= (\phi^h + \psi_h w) = 0\,, &\text{(C.37)}\\ \sigma^{w,m}(w) &= (\phi^m + \psi_m w) = 0. &\text{(C.38)}\end{aligned}$$

Therefore, the w_t process evolves deterministically as:

$$\dot{w}_t = (\zeta + r_V - g)w_t - (1-\theta(b_t) - c(\theta(b_t)))\,. \tag{C.39}$$

Household promised value in default regime: $\widehat{w}$. In the default regime, the scaled promised value $\widehat{w}$ satisfies the following equation:

$$()\zeta + r)\widehat{w} = 1 - \gamma/\alpha - \kappa c(\gamma/\alpha) + (g - \rho\eta\sigma_Y)\widehat{w}\,, \tag{C.40}$$

which yields

$$\widehat{w} = \frac{1-\gamma/\alpha - \kappa c(\gamma/\alpha)}{\zeta + r_V - g}\,. \tag{C.41}$$

C.2 Equivalence of Primal and Dual Problems

The government's debt management problem (19) is equivalent to the planner's resource allocation problem (67) if and only if 1.) the credible debt capacity, $\overline{B}(Y)$, in the primal problem equals the planner's value when the limited-commitment constraint binds, $F(\underline{W}, Y)$ in the dual problem: $\overline{B}(Y) = F(\underline{W}, Y)$; 2.) the lumpy debt-issuance boundary, $\underline{B}(Y)$, equals the planner's value when the planner makes a lumpy payouts, $F(\underline{W}, Y)$ in the dual problem: $\underline{B}(Y) = F(\overline{W}, Y)$; 3.) the value function $P(B, Y)$ in the primal problem characterized by the HJB equation (40) and associated FOCs maps to the value function $F(W, Y)$ in the dual problem characterized by the HJB equation (74) and associated FOCs as follows: $P(B_t, Y_t) = W_t$ and $B_t = F(W_t, Y_t)$.

Using the homogeneity property, we obtain the following mapping for scaled variables and value functions:

$$b = f(w) \quad \text{and} \quad w = p(b). \tag{C.42}$$

Additionally, we have the following results at the boundaries:

$$\overline{b} = f(\underline{w})\,, \tag{C.43}$$

and

$$\underline{b} = f(\overline{w})\,. \tag{C.44}$$

Next, we demonstrate the equivalence between the two problems by showing that by substituting $b = f(w)$ into the ODE for $p(b)$, we obtain the ODE for $f(w)$, and vice versa. Substituting (C.42) and (C.43) into ODE (61) for $f(b)$, we obtain the ODE (79) for $p(w)$. Substituting (C.42) and (C.43) into the constraint (58) for $\overline{b}$ and ODE (64) for the default value $\widehat{f}$, we obtain the constraint (86) for w, and ODE (C.40) for the default value $\underline{w}$. Substituting (C.42) and (C.44) into the constraint (59) for $\underline{b}$, we obtain constraint (80) for $\overline{w}$. Substituting (C.42) into the optimal tax policy (56) in the government debt problem, we obtain the optimal tax policy (77) in the dual planner's problem.

D Calibration

We use the US annual debt-output ratio from 2000 to 2020 to estimate our model. US debt and GDP data are from FRED provided by St. Louis fed: https://fred.stlouisfed.org.

Let $\Xi = \{\varphi, \zeta, \kappa\}$. Our model asserts that the government debt-GDP ratio b_t grows deterministically at rate $\dot{b}_t \equiv \mu^b(b_t)$ given in (57). To account for measurement errors, we introduce a noise term into law of motion (57) for b_t and discretize the b_t process as follows:

$$b_{t_{i+1}} = b_{t_i} + \mu^b(b_{t_i};\Xi)(t_{i+1} - t_i) + \varepsilon_{i+1}\,, \quad i = 1, 2, \cdots\,, \tag{D.45}$$

where $\mu^b(b_{t_i};\Xi)$ makes explicit the dependence of the drift of b on Ξ and ε_{i+1} is a random variable that captures the effect of measurement errors. Let $h(\varepsilon_{i+1})$ denote the density function of ε_{i+1}:

$$h\left(b_{t_{i+1}} - b_{t_i} + \mu^b(b_{t_i};\Xi)(t_{i+1} - t_i)\right)\,. \tag{D.46}$$

Let $\{\widehat{b}_{t_i}, i = 1, \cdots, 21\}$, where $t_i = 1999 + i$, denote the annual US debt-to-GDP ratio from 2000 to 2020. Our estimate of Ξ is $\widehat{\Xi}$ where

$$\widehat{\Xi} = \arg\max_{\Xi} \quad \sum_{i=1}^{20} \ln h\left(\widehat{b}_{t_{i+1}} - \widehat{b}_{t_i} + \mu^b(\widehat{b}_{t_i};\Xi)\right)\,. \tag{D.47}$$