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Min Dai
Cong Qin
Neng Wang

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ABSTRACT

We develop a model where a realization-utility investor (Barberis and Xiong, 2009, 2012; Ingersoll and Jin, 2013) optimally targets her liquid-illiquid wealth ratio at a constant w*. By saving in the risk-free asset (w* > 0), she makes smaller bets in the illiquid asset and realizes gains/losses more frequently. By leveraging (w* < 0), she makes bets larger than her equity and realizes gains/losses less frequently. For a discontinuous/jump-diffusion price process, the solution features four regions: loss-realization, gain-realization, and two disconnected (deep-loss and normal) holding regions. We generate a quantitatively significant non-monotonic propensity to realize losses consistent with evidence.

Min Dai
National University of Singapore
10, Lower Kent Ridge Road
Singapore 119076
Singapore
matdm@nus.edu.sg

Cong Qin
Center for Financial Engineering
Soochow University
Suzhou 215006
China
congqin@suda.edu.cn

Neng Wang
Columbia Business School
3022 Broadway, Uris Hall 812
New York, NY 10027
and NBER
nw2128@columbia.edu

1 Introduction

Much research in behavioral finance has shown that many investors derive utility from realizing gains and losses on the risky assets that they own. Barberis and Xiong (2012), henceforth BX (2012), develop a tractable model to analyze dynamic trading strategies for an investor with intertemporal piecewise linear realization utility. They show that the investor realizes a gain when it is sufficiently large but never realizes a loss. Ingersoll and Jin (2013), henceforth IJ (2013), generalize BX (2012) by incorporating reference-dependent S-shaped preferences over losses/gains from prospect theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992) into the intertemporal realization utility. IJ (2013) show that the investor also voluntarily realizes a loss, when it is sufficiently large. BX (2012) and IJ (2013) have significantly advanced our understanding on how realization-utility investors trade over time. For tractability and to derive economically intuitive and elegant formulas, both models assume "complete narrow framing" across assets and over time.[1] However this narrow-framing assumption restricts the investor to keep her entire wealth in the risky asset.

As investors with standard preferences, realization-utility investors have incentives to adjust their illiquid risky asset holdings and diversify risks over time. To capture this important real-world investment opportunity, we relax the narrow-framing assumption in the literature by allowing investors to invest in the risk-free asset. We propose a minimalistic model to focus on the economic mechanism underpinning portfolio rebalancing.[2]

Our model features an infinitely-lived investor who evaluates her utility burst solely based on realized gains and losses from trading a single risky asset. The price of this asset follows a geometric jump-diffusion process including the widely-used geometric Brownian motion (GBM) as a special case. Buying and selling the risky asset incurs proportional transaction costs. As in BX (2012), the reference level, which we use as the base to calculate realized gains and losses, grows at the risk-free rate. The investor dynamically rebalances her portfolio between the risky and risk-free asset subject to a leverage constraint. The investor may thus take either long or short positions. At each point in time, the investor decides between two options: a.) to hold on to her current

[1] Shefrin and Statman (1985) use "mental accounting" to justify investors' concentrating on specific separate incidents.

[2] An, Engelberg, Henriksson, Wang, and Williams (2021) find that the disposition effect at the portfolio level and suggest integrating the portfolio perspective into models with prospect theory and realization utility.

portfolio and let the value of her portfolio passively evolve; or b.) to obtain an immediate utility burst by selling the illiquid risky asset holdings to realize gains/losses and then rebalance her portfolio. In essence, our model integrates the realization-utility models of BX (2012) and IJ (2013) into the dynamic asset-allocation framework of Merton (1969). Importantly, we show that this integration generates novel conceptual insights and quantitively important predictions.

Despite keeping the setup minimalistic, it is inevitably a three-dimensional problem with the following state variables: liquid wealth W_t, illiquid wealth X_t, and the reference level B_t at time t. Using its homogeneity property, we solve our model by working with the following two state variables: 1.) scaled illiquid wealth $x_t = X_t/B_t$ as in BX (2012) and IJ (2013); and 2.) scaled liquid wealth $w_t = W_t/B_t$, which is the new (scaled) state variable in our model. The key new results are as follows.

First, the investor optimally sets the scaled liquid wealth (w) at a constant level (w^*) when rebalancing. Depending on preferences (e.g., the degree of loss aversion, the gain- and loss-sensitivity parameters) and/or the investment opportunity (e.g., the risk-free rate and the risky asset's expected return and volatility), the investor may choose to either save ($w^* > 0$) or borrow ($w^* < 0$).

Second, the optimal gain/loss realization strategies and the optimal portfolio policy (i.e., the optimal choice of w^*) are interdependent and interact with each other. By saving in the risk-free asset ($w^* > 0$), the investor has fewer dollars to bet on the risky asset and realizes gains more frequently as the transaction costs (in dollars) are smaller. Moreover, as losses (in dollars) are also smaller, the investor also realizes losses sooner as doing so is less painful. To sum up, an investor who saves ($w^* > 0$) makes smaller bets and realizes gains/losses more frequently. In contrast, an investor who uses leverage ($w^* < 0$) makes larger bets than her own equity and realizes gains/losses less frequently. This is because transaction costs (in dollars) are higher and the disutility of realizing losses is also larger, both of which discourage trading. Our model's prediction is consistent with the strong loss aversion observed in the housing market (Genesove and Mayer, 2001). The intuition is as follows. By using mortgage the household magnifies its bet on the house, which in turn amplifies its aversion to realize losses, generating a strong disposition effect.

Third, for the empirically plausible case where the illiquid asset price follows a jump-diffusion process, our model solution features *three* endogenous thresholds (*two* loss- and one gain-realization thresholds), which imply *four* endogenously determined regions. Importantly, these four regions include two disconnected holding regions: the standard holding region and the deep-loss holding region, where the investor has incurred large losses but is unwilling to realize losses. Separating these two holding regions is the loss-realization region. Our four-region solution differs from both the solution of the BX (2012) model, which features two regions (one gain-realization and one holding region), and that of the IJ (2013) model, which features three regions (one gain-realization region and one loss-realization region separated by the intermediate holding region).

The investor does not sell her asset at a loss when the loss is either too large (in the deep-loss holding region) or not large enough (in the normal holding region). Indeed, the investor only realizes losses in the loss-realization region that lies between the two holding regions. Taking these results together, we see that the investor's propensity to realize losses is non-monotonic in her losses. This non-monotonic propensity to realize losses implies that it is very hard to predict whether investors realize losses or not and how they feel about realizing losses versus keeping losses on paper. Our model prediction is consistent with the finding of Liu, Peng, Xiong, and Xiong (2022) in their survey that holding losers is more painful than selling them for some survey respondents while the opposite holds for others. As BX (2012) and IJ (2013), our model also predicts that the propensity to realize gains is monotonic, consistent with the finding that "more respondents say selling winners makes them happier" in Liu, Peng, Xiong, and Xiong (2022).

Moreover, our four-region solution also implies the following time-series prediction for the investor's trading behavior. While the investor does not realize losses when incurring a substantial loss and falling into the deep-loss holding region, she may choose to realize losses after a small rebound of the asset price that brings her loss from the deep-loss holding region into the loss-realization region.

We emphasize that the different mechanisms highlighted in BX (2012) and IJ (2013) are both important in reality and at work in our model. The BX (2012) mechanism gives rise to the deep-loss holding region in our model and the IJ (2013) mechanism results in the normal holding region. The two mechanisms complement each other and the interaction between them generates new and richer economic predictions: the (four-region) solution, a non-monotonic propensity to realize losses, and nonlinear trading strategies, as we discussed above.

Fourth, we make two technical contributions. We show that the standard double-barrier policy, which is based on the smooth-pasting conditions as typically done in the real-options literature (McDonald and Siegel, 1986; Dixit and Pindyck, 1994), no longer characterizes the optimal realization strategy and value functions for the jump-diffusion model. We must use the variational-inequality method to characterize our model solution. This is the technical explanation for why we have two disconnected holding regions and why the propensity to realize losses is non-monotonic in our model. Naively using the smooth-pasting conditions for our jump-diffusion model yields a wrong solution.

The second technical contribution is that regarding the diffusion model, we obtain closed-form solutions for the optimal portfolio and gain/loss realization strategies together with the value and payoff functions for our three-dimensional problem via a system of five nonlinear equations building on the insight of IJ (2013), who simplify their two-dimensional problem and obtain a closed-form solution given by a system of four nonlinear equations.

Fifth, we show that the economic value of being able to invest in the risk-free asset is large. Using the parameter values primarily based on IJ (2013), we find that the option to invest in the risk-free asset is worth about 21% of her wealth. This substantial value gain is due to the investor's option to make smaller and more frequent bets. When rebalancing, the investor allocates 36% of her total wealth to the risky asset by setting scaled liquid wealth w at $w^* = 1.76$. Moreover, the reduction of her exposure to the risky asset also makes her losses much smaller. As a result, the investor is more willing to realize losses as doing so is much less painful.

Consider two investors with the same budget of 100 dollars: investor A with and investor B without the option to invest in the risk-free asset. Let the risk-free rate be zero to ease exposition. Suppose that the asset price falls significantly after their purchases. Under this scenario, investor A who initially allocated 36 dollars to the risky asset would lose 11 dollars after the risky asset price falls by about 31% and rebalance her portfolio by allocating 32 dollars out of her total budget of 89 dollars to the risky asset. In contrast, investor B who initially allocated her entire budget of 100 dollars would wait longer to realize a loss of 45 dollars after the risky asset price falls by about 45% and then again reinvest the 55 dollars to the risky asset. The lifetime utility of investor A is about 21% higher than that of investor B as the former can make smaller bets and realize losses/gains more frequently to manage her risk-reward tradeoff.

Sixth, we show that the value of using leverage is high for investors who are less averse/less sensitive to losses or when investment opportunities are attractive. Consider the example where the annual volatility of the risky asset return is reduced to 20% from 30% in our preceding example (with all other parameter values unchanged). Rather than saving in the risk-free asset, the investor with a budget of 100 dollars invests 154 dollars in the risky asset by borrowing 54 dollars and the value of leveraging is 31 dollars. This is because the risky asset now offers a higher Sharpe ratio (30% versus 20% in the preceding savings case) making risk-taking much more attractive. Because of leverage, the investor realizes losses after losing 82 dollars, which is more than 53% of her 154 dollars investments in the risky asset, and almost 7.5 times the loss of 11 dollars in our preceding example when the Sharpe ratio is 20%.

We show that an investor with piecewise linear realization utility of BX (2012) always uses the maximally allowed leverage to buy the risky asset. The value of using leverage is high in our calibrated example. The intuition is as follows. BX (2012) show that the investor with piecewise linear realization utility never voluntarily realizes losses as the utility cost from realizing losses is too high for her to realize losses (due to constant marginal utility of realizing losses and loss aversion). Therefore, when using leverage is an option, the investor increases her exposure to the risky asset. As creditors have to break even, the investor is forced to realize losses after making sufficiently large losses. The investor's aversion to realizing losses makes her endogenously risk averse when she is in the paper loss region. The value function eventually turns convex when the investor is close to realizing gains.

This concave-then-convex value function is solely caused by leverage for a piecewise linear realization-utility investor and is fundamentally different from the globally convex value function in BX (2012). Using plausible parameter values, we show that when rebalancing, the investor with a budget of 100 dollars borrows about 52 dollars so as to invest 152 dollars in the risky asset. The investor's realization utility value increases by about 40%, measured in certainty-equivalent wealth. Also, unlike in BX (2012), the investor realizes losses after losing about 85 dollars, i.e., 56% of 152 dollars.

Finally, we show that the quantitative effects of empirically realistic jumps on dynamic rebalancing, value functions, and stationary distributions are very large. In the special case where the illiquid asset price is a diffusion process without jumps, the deep-loss holding region is never realized. In contrast, with jumps, the investor spends an economically meaningful fraction of her total time in the deep-loss region. As an example, in our calibrated example, the investor spends about 17% of her time in the deep-loss holding region and the expected duration between two consecutive rebalancing is about 113 days. In sum, fully taking into account the two disconnected holding regions, separated by a loss-realization region, is not only conceptually important but also quantitatively significant.

Related Literature. Our paper is related to a large body of work on prospect theory on realization utility in behavioral finance. Prospect theory, proposed by Kahneman and Tversky (1979) and generalized by Tversky and Kahneman (1992), posits that decision makers derive utility payoffs over gains and losses (e.g., the changes of their wealth over some reference levels) rather than the absolute levels of their wealth.[3] While this theory is considered one of the best descriptions of how individuals evaluate risks in experimental and real-world settings, implementing prospect theory in a dynamic trading environment requires additional model specifications (e.g., whether gains and losses are realized or not). Importantly, empirical predictions can critically depend on these specifications.

Barberis and Xiong (2009) show that the realization utility formulation of prospect theory where preferences are defined over *realized* gains and losses, reliably generates a disposition effect,[4] which is a robust empirical finding that individual investors have greater propensities to sell assets that have gone up in value since purchase than those that have gone down in value.[5] Since Barberis and Xiong (2009), realization utility has become a popular formulation of prospect theory for dynamic financial applications due to its empirical plausibility and analytical tractability. We provide a detailed comparison with BX (2012) and IJ (2013), the two most closely related papers, in Section 7.

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[3] The idea that individuals derive utility from gains and losses rather than the levels of wealth was first proposed by Markowitz (1952). Investors derive utility from fluctuations in their wealth (Benartzi and Thaler, 1995), from fluctuations in the value of their stock market holdings (Barberis, Huang, and Santos, 2001), or from changes in the value of specific stocks that they own (Barberis and Huang, 2001).

[4] In contrast, they show that the specification of prospect theory, defined over annual gains and losses, fails to predict a disposition effect consistently. Kyle, Ou-Yang, and Xiong (2006) and Henderson (2012) analyze one-time liquidation problems for a decision maker with prospect theory preferences but with no reinvestment options. Li and Yang (2013) develop a general-equilibrium model to examine the asset-pricing and trading-volume implications of prospect theory via the disposition effect.

[5] For evidence, among others, see Odean (1998) for individual investors' stock trading, Heath, Huddart, and Lang (1999) for executive stock option exercising, and Genesove and Mayer (2001) for the housing market. Using neural data, Frydman, Barberis, Camerer, Bossaerts, and Rangel (2014) show that investors experience direct positive utility when gains are realized. Using a numerical example in a static setting, Shefrin and Statman (1985) show that an investor who derives utility from realizing gains and losses and whose preference is concave over gains and convex over losses, exhibits a disposition effect. Grinblatt and Han (2005) develop a model with prospect theory and mental accounting generating a momentum effect for stock returns. Thaler (1999) states that a realized loss is more painful than a paper loss.

2 Model

An infinitely-lived investor evaluates her preferences using the present discounted value of realization utility, proposed by BX (2012) and IJ (2013). Importantly, in our model the investor can dynamically rebalance her portfolio between an illiquid risky asset and a risk-free asset. We show that the investor's opportunity to dynamically rebalancing her portfolio significantly alters her gain/loss realization strategies and enhances her welfare.

2.1 Investment Opportunity: Illiquid Risky Asset and Risk-free Asset

We assume that the price for a unit of the (homogeneous) illiquid risky asset, P_t, follows a geometric Brownian motion (GBM):

$$dP_t = \mu P_t dt + \sigma P_t d\mathcal{Z}_t \,, \quad t > 0 \,, \tag{1}$$

where $\mathcal{Z}_t$ is a standard Brownian motion, and μ and $\sigma > 0$ are constant parameters, representing the cum-dividend expected return and the return volatility, respectively.

As in BX (2012) and IJ (2013), we make two assumptions regarding transactions involving the illiquid asset. First, when rebalancing her portfolio, the investor is required to sell her entire position of the illiquid asset. Second, purchasing and selling the illiquid asset incurs proportional transaction costs. One example of the illiquid asset is housing. An investor changes her housing size typically by selling the house she owns when buying a new one. Let τ_i denote the i-th rebalancing time chosen by the investor. Let Q_t denote the investor's illiquid asset holdings and let X_t denote her wealth allocation to the illiquid asset in dollars (illiquid wealth) at time t: $X_t = P_t Q_t$. The investor is not allowed to short sell the illiquid risky asset, i.e., $Q_t \geq 0$ and hence $X_t \geq 0$ at all $t \geq 0$.

During the stochastic period between two consecutive rebalancing moments: $[\tau_i, \tau_{i+1})$, the illiquid asset holdings are constant: $Q_t = Q_{\tau_i}$. Therefore her illiquid wealth, X_t, follows the same GBM process as P_t does:

$$dX_t = \mu X_t dt + \sigma X_t d\mathcal{Z}_t \,, \quad \text{for } t \in [\tau_i, \tau_{i+1}) \,. \tag{2}$$

That is, both the drift and volatility parameters are the same as those for P_t in (1).

The risk-free asset pays interests at the constant risk-free rate $r > 0$. The wealth allocation to the risk-free asset (liquid wealth), W_t, grows exponentially and deterministically between two consecutive portfolio rebalancing:

$$dW_t = rW_t dt \,, \quad \text{for } t \in [\tau_i, \tau_{i+1}). \tag{3}$$

Following BX (2012) and IJ (2013), we assume that at each rebalancing time τ_i the investor incurs (purchase and sale) costs and adjusts her illiquid wealth from X_{τ_i-} to a new level, X_{τ_i}. Her liquid wealth after rebalancing at τ_i, W_{τ_i}, is given by

$$W_{\tau_i} = W_{\tau_i-} + (1-\theta_s)X_{\tau_i-} - (1+\theta_p)X_{\tau_i}, \tag{4}$$

where the second term on the right-hand side is the net proceeds from selling her illiquid asset holdings at τ_i- and the third term is the total out-of-pocket cost for acquiring her new illiquid asset at τ_i. The parameters θ_p and θ_s measure the proportional purchase and sale cost, respectively.[6]

Leverage Constraint. The investor can use the risky asset as collateral and borrow against it. That is, we assume that the following leverage constraint holds:

$$W_t \geq -\kappa X_t \ , \tag{5}$$

where κ is the maximal debt-X ratio satisfying

$$0 < \kappa \leq 1 - \theta_s \ . \tag{6}$$

When the leverage constraint (5) binds, the investor has to realize losses. This is necessary to ensure that creditors bear no credit risk and hence are willing to lend to the investor at the risk-free rate at all time. The no-borrowing case corresponds to $\kappa = 0$ and can be obtained as the limit by setting κ to zero.

2.2 Realization Utility

We model the investor's preferences using the realization utility proposed by BX (2012) and IJ (2013). An investor views her investing process as a series of separate episodes and that her utility payoff comes in a burst when realizing gains or losses from trading in the risky asset introduced earlier. That is, our model features (partial) narrow framing as the investor only evaluates gains and losses from her exposure to a single risky asset.

To calculate utility payoffs from realized gains and losses, we need a reference level against which the investor evaluates her realized gains and losses over time.[7] Let

[6] Grossman and Laroque (1990) develop an optimal portfolio choice problem where consumption services are generated by holding an illiquid durable good.

[7] Wang, Yan, and Yu (2017) argue that reference-dependent preferences is a promising explanation for why stock returns for firms with capital gain investors differ from stock return for firms with capital loss investors.

$B_t > 0$ denote the investor's reference level process. As in BX (2012), we assume that the reference level grows exponentially at the risk-free rate r between two consecutive endogenous rebalancing moments, τ_i and τ_{i+1}:

$$dB_t = rB_t dt \quad \text{for } t \in [\tau_i, \tau_{i+1}) \,. \tag{7}$$

Immediately after adjusting the illiquid asset holdings at moment τ_i, the reference level B_{τ_i} is adjusted to the new illiquid wealth X_{τ_i}:

$$B_{\tau_i} = X_{\tau_i} \,. \tag{8}$$

Let G_{τ_i} denote the investor's realized gain or loss at time τ_i after she sells her illiquid asset position and pays the proportional sales cost:[8]

$$G_{\tau_i} = (1 - \theta_s) X_{\tau_i -} - B_{\tau_i -} \,. \tag{9}$$

Anticipating that the homogeneity property plays an important role in our model solution, we define g_{τ_i} as the realized gain or loss, G_{τ_i}, scaled by the reference level $B_{\tau_i -}$:

$$g_{\tau_i} = G_{\tau_i} / B_{\tau_i -} \,. \tag{10}$$

As in IJ (2013), the investor derives the following utility burst when selling the illiquid asset and realizing a gain or loss at the rebalancing time τ_i:

$$U(G_{\tau_i}, B_{\tau_i -}) = B^{\beta}_{\tau_i -} u(G_{\tau_i} / B_{\tau_i -}) = B^{\beta}_{\tau_i -} u(g_{\tau_i}) \,, \tag{11}$$

where $\beta \in (0, 1]$ is a constant and $u(\,\cdot\,)$ is a function that depends on the scaled realized gain or loss, g_{τ_i}.[9] Next, we specify $u(\,\cdot\,)$ by adopting a reference-level-scaled version of the Cumulative Prospect Theory (CPT) utility of Tversky and Kahneman (1992):[10]

$$u(g) = \begin{cases} g^{\alpha_+} & \text{if } g \geq 0 \,, \\ -\lambda(-g)^{\alpha_-} & \text{if } g < 0 \,, \end{cases} \tag{12}$$

where $\lambda \geq 1$ and $\alpha_{\pm} \in (0, 1]$ are the three constant parameters describing $u(\,\cdot\,)$. In addition to the reference dependence property that we discussed above, $u(\,\cdot\,)$ inherits

[8] He and Yang (2019) develop a dynamic model where the reference point for the realization-utility component of the investor's preferences adapts asymmetrically to the stock's prior gains and losses. They do not analyze dynamic rebalancing between the risky and risk-free asset.

[9] The specification in (11) makes our model growth stationary and tractable in line with the finance tradition as noted by IJ (2013).

[10] The realization utility formulation of prospect theory ignores probability weighting, another key feature of CPT. Investors tend to overweight the tail outcomes of a probability distribution. Put differently, investors typically prefer lotteries and insurances compared with predictions of expected utility models.

two other key features of CPT. One is the diminishing sensitivity, which means that $u(\cdot)$ is concave ($0 < \alpha_+ \leq 1$) in the gain ($g \geq 0$) region but convex ($0 < \alpha_- \leq 1$) in the loss ($g < 0$) region. Note that the $g = 0$ point is a kink where $u(\cdot)$ is not differentiable. The other is loss aversion ($\lambda \geq 1$). A higher value of λ refers to a stronger loss aversion.

Finally, as in IJ (2013), we introduce the following condition:

$$\beta \leq \min\{\alpha_+, \alpha_-\}, \tag{13}$$

which ensures that $|U(G, B)|$ is decreasing in B for a fixed G. When $\beta < \min\{\alpha_+, \alpha_-\}$, the smaller the reference level, the greater utility impact of a realized gain or loss in the absolute value ($|G|$).[11] For the special piecewise linear realization utility case used in BX (2012) where $\beta = \alpha_+ = \alpha_- = 1$, $U(G, B)$ is independent of the level of B (for a given G) and does not feature the diminishing sensitivity property.

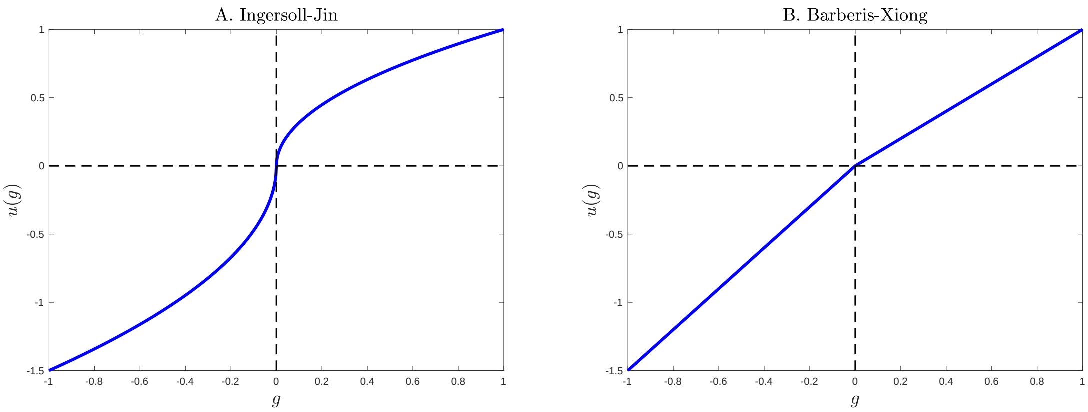

Figure 1: Scaled realization utility $u(\cdot)$. Panel A plots the $\alpha_\pm = 0.5$ case in IJ (2013). Panel B plots the piecewise linear $\alpha_\pm = 1$ case in BX (2012). In both panels, the loss aversion parameter is $\lambda = 1.5$ and $u(\cdot)$ is not differentiable at the kink point $x = 0$. In Panel A, $u(\cdot)$ is convex in the loss region and concave in the gain region. In Panel B, $u(\cdot)$ is globally concave.

Figure 1 plots the scaled realization utility $u(\cdot)$. Panel A plots the $\alpha_\pm = 0.5$ case used in IJ (2013). Panel B plots the $\alpha_\pm = 1$ case analyzed in BX (2012). In both panels, $\lambda = 1.5$. In Panel A, $u(\cdot)$ is S-shaped: convex in the loss region and concave in the gain region. In contrast, for the piecewise linear case, $u(\cdot)$ is globally concave (Panel B). In both panels, $u(\cdot)$ is not differentiable at $x = 0$.

2.3 The Investor's Optimization Problem

Let W_{0-} denote the investor's total wealth/budget to which she applies realization utility. The investor solves her problem via backward induction in two steps. First, suppose

[11] Here is an illustrative example from IJ (2013): "the gain or loss of \$10 is felt more strongly when the reference level is \$100 than when it is \$500."

that the investor has chosen the initial allocation X_0. That is, given (W_0, X_0, B_0) at $t = 0$, she chooses a sequence of optimal rebalancing times $\{\tau_i, i = 1, 2, \cdots\}$ and the illiquid asset allocation X_{τ_i} at each $\tau_i > 0$ to solve the following problem:

$$\sup_{\{\tau_i, X_{\tau_i}\}} \mathbb{E}\left[\sum_{i=1}^{\infty} e^{-\delta\tau_i} U(G_{\tau_i}, B_{\tau_i-})\right], \tag{14}$$

subject to the dynamics described by equations (2), (3), (4), (7), (8), and the leverage constraint (5). The parameter in (14), $\delta > 0$, is the investor's subjective discount rate. This optimization problem has three state variables: liquid wealth (W), illiquid wealth (X), and the reference level for the illiquid asset (B). Let $V(W, X, B)$ denote the value function for the problem defined in (14).

The second step is to solve for the initial allocation X_0. That is, given her initial wealth W_{0-}, the investor at $t = 0$ chooses her risky asset allocation X_0, pays the initial purchase cost $\theta_p X_0$, and allocates her remaining wealth

$$W_0 = W_{0-} - (1 + \theta_p)X_0 \tag{15}$$

to the risk-free asset. Mathematically, she chooses X_0 to maximize $V(W_0, X_0, B_0)$, where the initial reference level is $B_0 = X_0$, subject to (15).

Using this two-step procedure, we obtain the model solution. In IJ (2013), $W_t = 0$ at all t as the investor cannot save in the risk-free asset nor borrow. We show that the option to borrow and save fundamentally alters the investor's strategy and significantly improves her welfare.

3 Solution

Next, we solve the optimization problem (14) by using dynamic programming.

3.1 HJB Equation

Holding Region. In this region, holding the illiquid asset is optimal. By Itô's formula, the value function $V(W, X, B)$ satisfies a partial differential equation in this region:

$$\mathcal{L}V(W, X, B) = 0, \tag{16}$$

where

$$\mathcal{L}V = \frac{1}{2}\sigma^2 X^2 V_{XX} + \mu X V_X + rWV_W + rBV_B - \delta V. \tag{17}$$

Equation (16) is similar to the standard HJB equation in the waiting region in classic real-option models, e.g., in McDonald and Siegel (1986) and Dixit and Pindyck (1994).

Realization Region. In this region, the investor optimally realizes gains or losses, receives a direct utility burst, $U((1-\theta_s)X - B, B)$, and then rebalances her portfolio. The value function $V(W, X, B)$ equals the payoff function $F(W, X, B)$, defined as follows:

$$F(W,X,B) = U\big((1-\theta_s)X - B, B\big) + \max_{\widehat{X}} \; V\big(\widehat{W}, \widehat{X}, \widehat{X}\big)\,, \tag{18}$$

where $\widehat{X}$ is the investor's post-rebalancing illiquid wealth and $\widehat{W}$ is the investor's post-rebalancing liquid wealth given by

$$\widehat{W} = W + (1-\theta_s)X - (1+\theta_p)\widehat{X}\,. \tag{19}$$

Let X^* denote the optimal illiquid wealth, the maximand of (18) and let W^* denote the optimal liquid wealth immediately after rebalancing. The optimal X^* must be interior and satisfies the following first order condition (FOC):[12]

$$(1+\theta_p)V_W(W^*, X^*, X^*) - V_B(W^*, X^*, X^*) = V_X(W^*, X^*, X^*)\,. \tag{20}$$

Note that $B = X^*$ immediately after rebalancing and $V_B(W, X, B) < 0$.[13] The right-hand side of (20) is the marginal benefit of investing in the illiquid asset. To invest one unit in the illiquid asset, the investor not only forgoes an opportunity of investing $(1+\theta_p)$ units in the risk-free asset, but also bears the marginal cost of increasing reference level $(-V_B)$ as $B = X$ and $V_B < 0$. The investor optimally chooses X^* to equate the two sides of (20).

Next, we summarize our model solution by using the following variational inequality, which combines our analyses for the holding and realization regions.

Variational Inequality. The value function $V(W, X, B)$ satisfies the following variational inequality:

$$\max\Big\{\mathcal{L}V(W,X,B),\; F(W,X,B) - V(W,X,B)\Big\} = 0 \tag{21}$$

in the region where the leverage constraint (5) does not bind. Finally, when (5) binds, the investor has to realize losses in order to make debt payments. As a result, we have

$$V(W,X,B) = F(W,X,B)\,, \quad \text{when } \; W = -\kappa X < 0\,. \tag{22}$$

[12] If the solution were at the corner, i.e., $X^* = -W^*/\kappa$, the investor has to immediately realize losses so as to repay creditors. Doing so immediately incurs the rebalancing costs, which is suboptimal.
[13] Note that $V_B(W, X, B) < 0$ follows from the definition of V given in (14) and the $0 < \beta < \min(\alpha_+, \alpha_-)$ condition. This is because $U(G, B)$ defined in (11) is decreasing in B, as $G = (1-\theta_s)X - B$.

3.2 Using the Homogeneity Property to Simplify Solution

It is convenient to work with the following state variables scaled by the contemporaneous reference level B:

$$w = \frac{W}{B} \quad \text{and} \quad x = \frac{X}{B}. \tag{23}$$

Using Itô's formula, between two consecutive rebalancing moments, i.e., during each investment episode $(\tau_i, \tau_{i+1}]$, the scaled illiquid wealth x_t follows a geometric Brownian motion process with drift being the expected excess return $(\mu - r)$ and volatility σ:

$$dx_t = (\mu - r)x_t dt + \sigma x_t d\mathcal{Z}_t\,. \tag{24}$$

Additionally, the scaled liquid wealth w is constant over $(\tau_i, \tau_{i+1}]$:

$$dw_t = 0\,. \tag{25}$$

The investment opportunity and preferences together give rise to the homogeneity property in our model. Specifically, the value function $V(W, X, B)$ defined in (14) is homogeneous in (W, X, B) with degree β, so that $V(W, X, B) = B^{\beta} V(w, x, 1)$ for any level of $B > 0$. Rewriting this equation and letting $v(w, x) = V(w, x, 1)$, we have

$$v(w, x) = B^{-\beta} V(W, X, B)\,. \tag{26}$$

The payoff function defined in (18), $F(W, X, B)$, is also homogeneous with degree β in (W, X, B): $F(w, x, 1) = B^{-\beta} F(W, X, B)$. We thus write $f(w, x) = F(w, x, 1)$ where[14]

$$f(w, x) = u\big((1 - \theta_s)x - 1\big) + m(w^*)\,[w + (1 - \theta_s)x]^{\beta}\,. \tag{27}$$

In (27), $m(w)$ is given by

$$m(w) = \frac{v(w, 1)}{[w + (1 + \theta_p)]^{\beta}} \tag{28}$$

and $w^* = W^*/X^*$ is the optimal post-rebalancing scaled liquid wealth satisfying:

$$w^* = \underset{w \geq -\kappa}{\arg\max}\; m(w)\,. \tag{29}$$

[14] We obtain (27) using the homogeneity property. First, we write $f(w, x) = B^{-\beta} F(W, X, B)$, where $F(W, X, B) = U\big((1 - \theta_s)X - B, B\big) + \max_{\widehat{X}}\; V\big(\widehat{W}, \widehat{X}, \widehat{X}\big)$ and $\widehat{W} = W + (1 - \theta_s)X - (1 + \theta_p)\widehat{X}$. Second, we may write $B^{-\beta} V\big(\widehat{W}, \widehat{X}, \widehat{X}\big) = (\widehat{X}/B)^{\beta} v(\widehat{w}, 1)$, where $\widehat{w} = \widehat{W}/\widehat{X} = [W + (1 - \theta_s)X - (1 + \theta_p)\widehat{X}]/\widehat{X}$. We thus obtain $\widehat{X} = [W + (1 - \theta_s)X]/(\widehat{w} + 1 + \theta_p)$, which implies $\widehat{X}/B = [w + (1 - \theta_s)x]/[\widehat{w} + (1 + \theta_p)]$, where $w = W/B$ and $x = X/B$. Therefore, we obtain $B^{-\beta} V\big(\widehat{W}, \widehat{X}, \widehat{X}\big) = \left([w + (1 - \theta_s)x]/[\widehat{w} + (1 + \theta_P)]\right)^{\beta}\; v(\widehat{w}, 1)$. Using this equation, we obtain $f(w, x) = u\big((1 - \theta_s)x - 1\big) + \max_{\widehat{w}}\; m(\widehat{w})\,[w + (1 - \theta_s)x]^{\beta}$.

Using the homogeneity property, we simplify the HJB equation (21) to:

$$\max\{\mathscr{L}v(w,x),\ f(w,x)-v(w,x)\}=0 \quad \text{for} \quad (w,x)\in\mathcal{S}\,, \tag{30}$$

in the region $\mathcal{S}=\{x>0,\ w>-\kappa\, x\}$, where the leverage constraint (5) does not bind, and $\mathscr{L}(w,x)$ is the differential operator given by

$$\mathscr{L}v(w,x)=\frac{1}{2}\sigma^2x^2v_{xx}(w,x)+(\mu-r)xv_x(w,x)-(\delta-\beta r)v(w,x)\,. \tag{31}$$

There are two case in the region $\mathcal{S}$: the holding region $\mathcal{H}$ and the realization region $\mathcal{R}$, defined by

$$\mathcal{H}=\{(w,x)\in\mathcal{S}\mid v(w,x)>f(w,x)\} \ \text{ and } \ \mathcal{R}=\{(w,x)\in\mathcal{S}\mid v(w,x)=f(w,x)\}\,, \tag{32}$$

respectively. In the holding region $\mathcal{H}$, $\mathscr{L}v(w,x)=0$ holds.

When the leverage constraint (5) binds, the investor has to realize losses and hence

$$v(w,x)=f(w,x)\,, \quad \text{when} \quad w=-\kappa\, x\,. \tag{33}$$

3.3 Closed-Form Solutions

In the holding region $\mathcal{H}$, the value function $v(w,x)$, which solves $\mathscr{L}v(w,x)=0$, admits the following general solution:

$$v(w,x)=C_1(w)x^{\eta_1}+C_2(w)x^{\eta_2}, \tag{34}$$

where $\eta_1>0$ and $\eta_2<0$ are the two roots of the fundamental quadratic equation:

$$h(\eta)=\frac{\sigma^2}{2}\eta(\eta-1)+(\mu-r)\eta-(\delta-\beta r)\,. \tag{35}$$

In (34), w_t is time-invariant as $dw_t=0$ (see (25)) and $C_1(w)$ and $C_2(w)$ are two functions of w that we determine next. In Appendix A, we provide and verify a sufficient condition which ensures that the transversality condition holds.[15]

We show that the holding region $\mathcal{H}$ is characterized by two endogenous threshold functions of w, $\underline{x}(w)$ and $\overline{x}(w)$, which satisfy $x_L(w)\le\underline{x}(w)<1\le\overline{x}(w)$, where $x_L(w)$ is the involuntary liquidation point when the scaled liquid wealth is w:

$$x_L(w)=\begin{cases}0 & \text{if } w\ge 0,\\ -w/\kappa & \text{if } w<0.\end{cases} \tag{36}$$

[15] Different from the classical real-options literature, e.g., McDonald and Siegel (1986) and Dixit and Pindyck (1994), the positive root η_1 may be less than one, which means that the value function may not be globally convex in X. See Figure 3 for an example.

That is, the investor optimally keeps her scaled illiquid wealth constant at w provided that $x \in (\underline{x}(w), \overline{x}(w))$ and rebalances her portfolio only when $x = \underline{x}(w)$ or $x = \overline{x}(w)$. We refer to $\overline{x}(w)$ and $\underline{x}(w)$ as the optimal gain- and loss-realization boundary, respectively.

Using (27), (28), (34), and the result that $(w, 1)$ is always in the holding region, we obtain the scaled value function $f(w, x)$ in both the gain- and loss- realization regions:

$$f(w,x) = u\Big((1-\theta_s)x - 1\Big) + \frac{C_1(w^*) + C_2(w^*)}{[w^* + (1+\theta_p)]^\beta}\,[w + (1-\theta_s)x]^\beta\ , \tag{37}$$

where w^* is the optimal post-realization scaled liquid wealth given by:

$$w^* = \arg\max_{\widehat{w} \geq -\kappa} \frac{C_1(\widehat{w}) + C_2(\widehat{w})}{[\widehat{w} + (1+\theta_p)]^\beta}\,. \tag{38}$$

First consider the scenario where $\underline{x}(w) > x_L(w)$ for all w. We can infer that both boundaries are characterized by the value-matching and the smooth-pasting conditions for $v(w, x)$. That is, at the realization boundaries $\overline{x}(w)$ and $\underline{x}(w)$, we have the following four conditions:

$$C_1(w)[\overline{x}(w)]^{\eta_1} + C_2(w)[\overline{x}(w)]^{\eta_2} = f(w, \overline{x}(w)), \tag{39}$$

$$C_1(w)[\underline{x}(w)]^{\eta_1} + C_2(w)[\underline{x}(w)]^{\eta_2} = f(w, \underline{x}(w)), \tag{40}$$

$$C_1(w)\eta_1[\overline{x}(w)]^{\eta_1-1} + C_2(w)\eta_2[\overline{x}(w)]^{\eta_2-1} = f_x(w, \overline{x}(w)), \tag{41}$$

$$C_1(w)\eta_1[\underline{x}(w)]^{\eta_1-1} + C_2(w)\eta_2[\underline{x}(w)]^{\eta_2-1} = f_x(w, \underline{x}(w)), \tag{42}$$

where $f(w, x)$ is given in (37).

In sum, solving the system of five equations, (38)–(42), we obtain the optimal rebalancing target, w^*, and the four functions: $C_1(w)$, $C_2(w)$, $\overline{x}(w)$, and $\underline{x}(w)$, which jointly characterize the value functions $v(w, x)$ and $f(w, x)$.

For the other scenario where $\underline{x}(w) = x_L(w)$, the investor realizes losses only when the leverage constraint (5) binds. We refer readers to Appendix B for a complete solution procedure. We next analyze the model-implied transition dynamics.

3.4 Transition Dynamics to the Rebalancing Position $(x, w) = (1, w^*)$

We analyze the transition dynamics for an investor starting with a position given by (W, X, B). Let $\tau(x, w)$ denote the first time the investor realizes gains or losses starting from (x, w). Given the optimal double-barrier policy, $\overline{x}(w)$ and $\underline{x}(w)$, we have

$$\tau(x, w) = \inf\{t \geq 0 \mid x_0 = x,\ x_t \notin (\underline{x}(w), \overline{x}(w))\}\,. \tag{43}$$

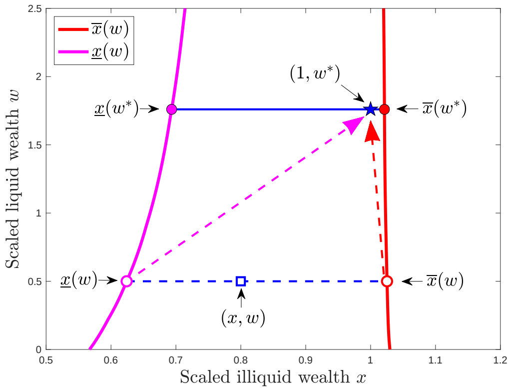

Figure 2: TRANSITION DYNAMICS OF (x_t, w_t).

Immediately after $\tau(x, w)$, the investor rebalances her portfolio to $(1, w^*)$. For all $t \geq \tau(x, w)$, $w_t = w^*$ and x_t follows (24) before the investor realizes the next gain or loss.

Figure 2 illustrates the transition dynamics for our baseline quantitative exercise that we conduct in the next section. The solid pink line depicts the (lower) loss-realization boundary $\underline{x}(w)$ and the solid red line depicts the (upper) gain-realization boundary $\overline{x}(w)$. Starting from the initial position (x, w) (the blue square), the investor's scaled illiquid wealth x_t moves stochastically along the horizontal dashed blue line in response to shocks reaching either the open pink circle (the intersection point with the $\underline{x}(w)$ line) on the left or the open red circle (the intersection point with the $\overline{x}(w)$ line) on the right at stochastic time $\tau(x, w)$. The investor realizes losses (gains) in the former (latter) case. In both cases, the investor rebalances her portfolio to the starred position $(1, w^*)$. Note the discrete change of her scaled liquid wealth from w to w^* at $\tau(x, w)$.

After $\tau(x, w)$, the investor reaches the steady state for her scaled liquid wealth, where $w_t = w^*$ at all time t, but her scaled illiquid wealth x continues to evolve stochastically along the horizontal solid blue line reaching either the loss-realization boundary $\underline{x}(w^*)$ or the gain-realization boundary $\overline{x}(w^*)$ on the right at stochastic time $\tau(1, w^*)$. Afterward, the investor again rebalances to $(1, w^*)$. The cycle repeats. Of course, the unscaled portfolio position (W_t, X_t) is stochastic due to shocks to the illiquid asset value.

4 Quantitative Analysis

In this section, we explore quantitative implications of our model. First, we propose a wealth-based measure for the option value of dynamic portfolio rebalancing.

4.1 Measuring the Value of Investing in the Risk-free Asset

To measure the value of investing in the risk-free asset, we first report the investor's value function in models of BX (2012) and IJ (2013), where the investor has no option to invest in the risk-free asset.

Benchmark: No Option to Invest in the Risk-free Asset (BX, 2012; IJ, 2013). As the investor can only hold the illiquid asset at all time t and hence $W_t = 0$, we have the following expression for the post-rebalancing illiquid wealth $\widehat{X}$:

$$\widehat{X} = \frac{1-\theta_s}{1+\theta_p} X \,, \tag{44}$$

where the multiple $\frac{1-\theta_s}{1+\theta_p}$ captures the costs of selling and purchasing the risky asset when realizing gains or losses. The value function, denoted by $V_N(X_t, B_t)$ where the subscript N refers to the "no-option" case, only depends on X_t and B_t. As $V_N(X, B)$ is homogeneous in X and B with degree β, we may work with the scaled value function $v_N(x) = V_N(x, 1)$:

$$v_N(x) = B^{-\beta} V_N(X, B) \,. \tag{45}$$

Similarly, let $f_N(x) = B^{-\beta} F_N(X, B)$ denote the corresponding scaled payoff function when the investor has no option to invest in the risk-free asset.

Comparing the Value Function $V(W, X, B)$ with $V_N(X, B)$ to Measure the Option Value of Dynamic Rebalancing. Consider an investor with an initial level of wealth W_{0-} and the realization utility preferences. How much wealth compensation (in dollars), denoted by $W_{0-}^{\Delta} - W_{0-}$, does she require for her to permanently forgo the option to invest in the risk-free asset? We use the percentage increase of *certainty equivalent wealth*, $\Delta = (W_{0-}^{\Delta} - W_{0-})/W_{0-}$, to measure the option value of dynamically rebalancing between the risky and risk-free assets. The following indifference condition defines Δ:

$$V(W_0, X_0, X_0) = V_N \left(\frac{(1+\Delta)W_{0-}}{1+\theta_p}, \frac{(1+\Delta)W_{0-}}{1+\theta_p} \right) , \tag{46}$$

where $W_0 = W_{0-} - (1+\theta_p)X_0$ is the liquid wealth after the investor purchases the illiquid asset at $t = 0$. Using the homogeneity property and the definition of $m(w)$ given in (28), we obtain the following expression for $\Delta(w)$:

$$\Delta(w) = \left(\frac{m(w)}{m_N} \right)^{1/\beta} - 1 \,, \tag{47}$$

where $w = W_0/X_0$ and $m_N = v_N(1)/(1+\theta_p)^\beta$. Since $\Delta(w)$ is increasing with $m(w)$ and $w = w^*$ is the maximand of $m(w)$, the welfare measure $\Delta(w)$ is also maximized when $w = w^*$. For notational convenience, we write $\underline{x}^* = \underline{x}(w^*)$, $\overline{x}^* = \overline{x}(w^*)$, and $x_L^* = x_L(w^*)$ for the remainder of the paper. To facilitate our economic interpretation, it is helpful to introduce the following risk attitude measure (RAM).

Risk Attitude Measure (RAM). We focus on the steady-state $w = w^*$ case and define RAM in the holding region where $x \in (\underline{x}^*, \overline{x}^*)$ as follows:[16]

$$\text{RAM} = -\frac{XV_{XX}}{V_X} = -\frac{xv_{xx}(w^*, x)}{v_x(w^*, x)} = \gamma(x)\,. \tag{48}$$

Because of the homogeneity property, RAM depends on x, which we denote by $\gamma(x)$. When $\gamma(x) > 0$, the investor is risk averse and $v(w^*, x)$ is concave in x. When $\gamma(x) < 0$, the investor is risk seeking and $v(w^*, x)$ is convex in x. In general, $v(w^*, x)$ can be concave and convex in different regions of x. Similarly, we define RAM in the holding region ($x \in (\underline{x}_N, \overline{x}_N)$) for $V_N(X, B)$ as $\gamma_N(x) = -xv_N''(x)/v_N'(x)$.

Next, we choose the parameter values for our quantitative analysis.

4.2 Parameter Choices

One period is one year in our analysis. Following IJ (2013), we set $\alpha_+ = \alpha_- = 0.5$, $\delta = 5\%$, $\beta = 0.3$, $\mu = 9\%$, $\sigma = 30\%$, and $\theta_s = \theta_p = 1\%$. We choose the loss aversion parameter, $\lambda = 1.5$, based on the estimate of Andersen, Badarinza, Liu, Marx, and Ramadorai (2021). To target the risk premium $(\mu - r)$ at 6%, a commonly used value for the US stock market and housing risk premia,[17] we set the risk-free rate at $r = 3\%$. Finally, we set $\kappa = 0.79$ so that the ratio of debt and liquidating net worth never exceeds 80%, a commonly used loan-to-value (LTV) ratio for housing.[18] As the illiquid asset is the collateral with 20% equity subordination that debt holders can seize, debt is risk

[16] This definition resembles but differs from that of relative risk aversion in the standard consumer theory, as we leave aside wealth W in this definition and also our investor has realization utility. This measure allows us to gauge the degree of the value function's curvature and is an easier number than V_{XX} to interpret.

[17] For example, see Hansen and Singleton (1982) and Mehra and Prescott (1985) for equity risk premium and Piazzesi and Schneider (2016) for housing risk premium.

[18] Since the liquidation value of the illiquid risky asset equals $(1-\theta_p)X_t$, the investor's net worth upon liquidating the illiquid asset is then $W + (1-\theta_p)X$. To target the maximal LTV at 80%, we require the ratio $(1-\theta_p)X/[W + (1-\theta_p)X]$ not to exceed $\phi = 1/(1-0.8) = 5$. Rewriting this leverage constraint yields $W \geq -\kappa X$, where $\kappa = (\phi-1)(1-\theta_p)/\phi = 0.79$, as $\phi = 5$ and $\theta_p = 1\%$ for our baseline quantitative analysis.

free. Table 1 summarizes these eleven parameter values for our baseline case. Next, we show that the value of dynamic portfolio rebalancing is high.

Table 1: PARAMETER VALUES FOR THE BASELINE CASE. The parameters (r, δ, μ, σ) are continuously compounded and annualized.

α_+	α_-	λ	β	r	δ	μ	σ	θ_s	θ_p	κ
0.5	0.5	1.5	0.3	3%	5%	9%	30%	1%	1%	0.79

4.3 Value of Saving in the Risk-free Asset

With our baseline parameters, the optimal steady-state value of w_t is $w^* = 1.76$. That is, only $1 - w^*/(1 + \theta_p + w^*) = 36\%$ of the investor's total wealth is allocated to the risky asset when she rebalances her portfolio. This result suggests that relaxing the required 100% allocation to the risky asset in BX (2012) and IJ (2013) can be quite valuable.

Value and Payoff Functions ($v(w^*, x)$ and $f(w^*, x)$) and Realization Boundaries ($\underline{x}^*$, $\overline{x}^*$). Panel A of Figure 3 plots the scaled value $v(w^*, x)$ (the solid red line) and the scaled payoff $f(w^*, x)$ (the dashed red line) against the scaled illiquid wealth x when $w = w^* = 1.76$. For comparison, we also plot the scaled value $v_N(x)$ (the solid blue line) and the scaled payoff $f_N(x)$ (the dashed blue line) for the IJ (2013) model.

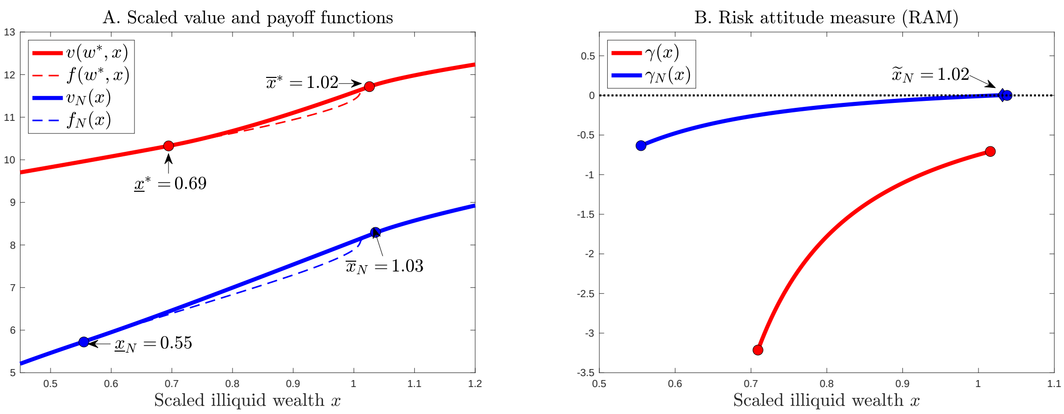

Figure 3: SCALED VALUE FUNCTIONS, PAYOFF FUNCTIONS, AND RISK ATTITUDE MEASURES (RAMS). The *baseline* parameter values given in Table 1 are used. The two roots of the fundamental quadratic equation (35) are $\eta_1 = 0.80$ and $\eta_2 = -1.14$.

Intuitively, $v(w^*, x) > f(w^*, x)$ in the holding region where $x \in (\underline{x}^*, \overline{x}^*) = (0.69, 1.02)$. That is, passively holding the portfolio is optimal in this region. Compared with the IJ

(2013) model, the holding region is significantly smaller in our baseline. This tightening mostly comes from the increase of the loss-realization threshold from $\underline{x}_N = 0.55$ in the IJ (2013) model to $\underline{x}^* = 0.69$ in our model. Intuitively, by saving 64% of her total wealth at the moment of rebalancing in the risk-free asset, the investor significantly reduces her gain/loss exposure to the risky asset and therefore is more willing to realize losses than in the IJ (2013) model so that she can realize gains sooner in the future. Nonetheless, the investor is still quite averse to realizing losses ($\underline{x}^* = 0.69$). In contrast, the investor frequently realizes gains. The upper boundary is $\overline{x}^* = 1.02$ in our model, which is slightly lower than $\underline{x}_N = 1.03$ in the IJ (2013) model.

Risk Attitude Measure (RAM). Panel B in Figure 3 shows that the RAM for our baseline model (the solid red line), $\gamma(x)$, is negative for all values of x in the holding region, which means that the value function is globally convex and the investor is endogenously risk seeking. Additionally, $\gamma(x)$ is lower than (more negative) the RAM $\gamma_N(x)$ in the IJ (2013) model (the solid blue line).[19] Next, we quantify the value of investing in the risk-free asset.

Quantifying the Value of Saving in the Risk-free Asset. Recall that the investor maximizes $m(w)$ given in (28), which we repeat below for ease of reference:

$$m(w) = v(w,1)[w + (1+\theta_p)]^{-\beta} \,. \tag{49}$$

Panel A of Figure 4 shows that $m(w)$ increases with w when $w \leq w^*$, reaches the maximum value at $w^* = 1.76$, and decreases with w for $w \geq w^*$. The investor optimally sets $w = w^*$ by trading off the marginal benefit of increasing w (via the $v(w,1)$ channel) against the marginal cost of increasing w (the multiplier $(w + 1 + \theta_p)^{-\beta}$ channel). The $v(w,1)$ term is analogous to price and the multiplier $(w + 1 + \theta_p)^{-\beta}$ is analogous to quantity in a monopoly problem.

Panel B of Figure 4 plots the value of dynamic rebalancing, measured by the percentage increase of the investor's certainty equivalent wealth, $\Delta(w)$. Note that $\Delta(w)$ has the same monotonicity property as $m(w)$ and also peaks at $w^* = 1.76$. To compensate for permanently giving up her option to invest in the risk-free asset, the investor demands an increase of her wealth by at least $\Delta(w^*) = 21\%$. This substantial

[19] For the IJ (2013) case, $v_N(x)$ is convex over $x \in (\underline{x}_N, \widetilde{x}_N) = (0.55, 1.02)$ and concave over $x \in (\widetilde{x}_N, \overline{x}_N) = (1.02, 1.03)$. The point $\widetilde{x}_N = 1.02$ is the inflection point at which $v''_N(x) = 0$.

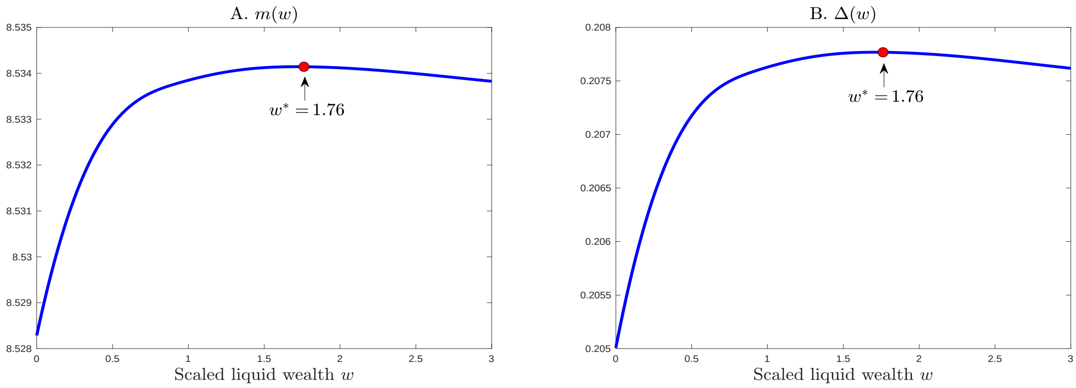

Figure 4: Quantifying the Value of Saving in the Risk-free Asset. Panel A plots $m(w)$ and Panel B plots the percentage increase of the investor's certainty-equivalent wealth due to savings, $\Delta(w)$. Both functions are hump-shaped and maximized at $w^* = 1.76$. The parameter values are reported in Table 1.

utility gain is because being able to make smaller bets and dynamically adjust her exposure to the risky asset over time is highly valuable for the investor.

In our baseline case, the investor never uses leverage and the value of investing in the risk-free asset solely comes from the option to save and make smaller bets over time. Next, we provide an example where the investor optimally uses leverage.

4.4 Value of Leverage

To ease exposition, we decrease the volatility parameter σ to 20% from 30% and keep all the other parameter values the same as in Table 1 used in Section 4.3. For this example, the value of using leverage is substantial: worth $\Delta = 31\%$ of the investor's wealth.

For the $\sigma = 20\%$ case, the investor with initial wealth $W_{0-} = 100$ allocates $X = W_{0-}/(1 + \theta_p + w^*) = 154$ to the risky asset by setting $w^* = -0.36 < 0$, i.e., financing $(X - W_{0-})/X = 35\%$ of her illiquid wealth via debt. This leveraged allocation to the risky asset is in sharp contrast with the much less aggressive allocation in our baseline $\sigma = 30\%$ case, where she sets $w^* = 1.76$ (which means allocating 36% of her wealth to the illiquid asset). Intuitively, the risky asset becomes so attractive as its volatility decreases from 30% to 20% that the investor increases her risky asset allocation by 154/36 =4.3 times.

Panel A of Figure 5 plots the scaled value $v(w^*, x)$ (the solid red line) and the scaled payoff $f(w^*, x)$ (the dashed red line), respectively. For comparison, we also plot the scaled value $v_N(x)$ (the solid blue line) and the scaled payoff $f_N(x)$ (the dashed blue line) for the IJ (2013) case where the investor can only hold the risky asset.

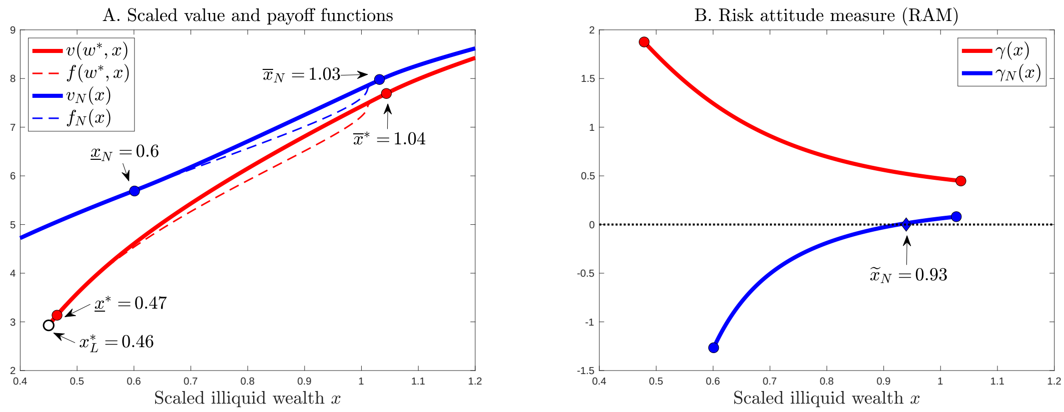

Figure 5: LEVERAGE EFFECT ON $v(w^*, x)$ AND $f(w^*, x)$. The steady state of w is $w^* = -0.36$. All the parameter values other than volatility σ, which we set to 20%, are given in Table 1. The two roots of the fundamental quadratic equation (35) are $\eta_1 = 0.75$ and $\eta_2 = -2.75$.

We find that the option to use leverage causes the investor to reduce her loss-realization threshold from $\underline{x}_N = 0.6$ to $\underline{x}^* = 0.47$ and slightly increase the gain-realization threshold from $\overline{x}_N = 1.03$ to $\overline{x}^* = 1.04$. The intuition is as follows. As leverage magnifies the investor's exposure to the risky asset and hence increases her reference level (the $X_0 = B_0$ channel), the investor becomes more reluctant to realize losses and gains. As a result, the holding region $(\underline{x}^*, \overline{x}^*)$ widens. Note that the investor still voluntarily realizes losses.[20]

Panel B provides additional insights about the curvatures of the value functions and risk attitudes. We see that the RAM $\gamma(x)$ in our leverage model is positive for all levels of x, which indicates that the investor is globally risk averse ($v(w^*, x)$ is globally concave). Moreover, $\gamma(x)$ decreases with x, which means that the investor becomes less risk-averse as paper losses decrease and liquidation risk caused by the leverage constraint decreases. In contrast, in the IJ (2013) model, the investor becomes less risk-seeking and even more risk-averse as x increases (the RAM γ_N increases with x, is negative for $x \in (0.6, 0.93)$, and positive for $x \in (0.93, 1.03)$).[21] Comparing $\gamma(x)$ for our leverage model with $\gamma_N(x)$ for the IJ (2013) model, we see that the option of using leverage significantly changes the investor's risk taking attitude (e.g., both the signs and monotonicity property of RAMs).

Next, we analyze the piecewise linear utility case analyzed in BX (2012), which provides additional important insights and quantitative results.

[20] The loss-realization threshold $\underline{x}^* = 0.47$ is slightly larger than the involuntary liquidation threshold $x_L^* = -w^*/\kappa = 0.36/0.79 \approx 0.46$ and hence the leverage constraint (5) does not bind.

[21] The corresponding scaled value function $v_N(x)$ is first convex and then concave with an inflection point at $\widetilde{x}_N = 0.93$.

4.5 Leverage for Piecewise Linear Realization Utility of BX (2012)

To ease comparison, we keep the parameter values the same as in our baseline model whenever feasible. Other than using the piecewise linear utility parameters, $\alpha_{\pm} = \beta = 1$, we set $\delta = 15\%$ in order to satisfy the transversality condition. We show that the investor highly values leverage and adopts very different gain/loss realization policies compared with the BX (2012) model.

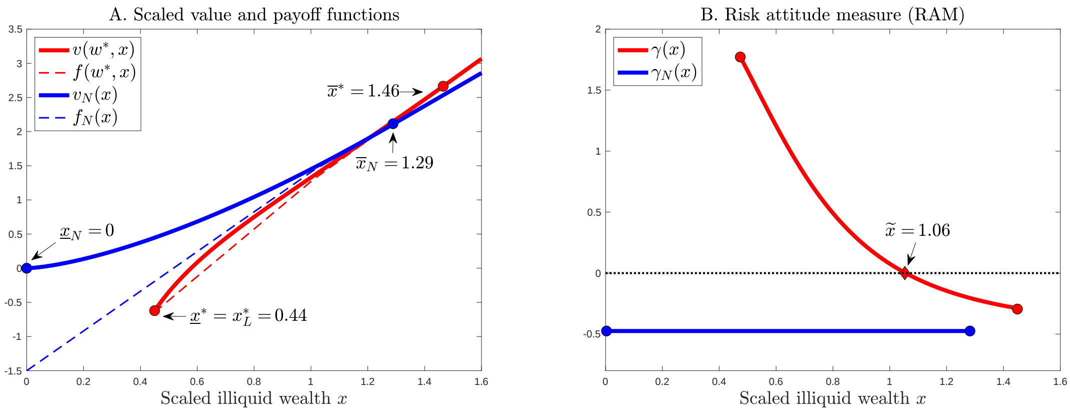

Figure 6: Introducing the Leverage Option into BX (2012). The steady state of w is $w^* = -0.35$. Other than $\alpha_{\pm} = \beta = 1$ (piecewise linear) and $\delta = 15\%$ (for convergence), all other parameter values are reported in Table 1. The two roots of the fundamental quadratic equation (35) are $\eta_1 = 1.47$ and $\eta_2 = -1.81$.

The optimal steady state of w is $w^* = -0.35$. That is, an investor with initial wealth $W_{0-} = 100$ allocates $X = W_{0-}/(1 + \theta_p + w^*) = 152$ to the risky asset by setting $w^* = -0.35 < 0$, i.e., financing $52/152 = 34\%$ of her total illiquid wealth via debt. This option value of leverage is worth $\Delta = 40\%$ of the investor's wealth.

Panel A of Figure 6 allows us to compare the optimal realization thresholds between our model and the BX (2012) model. Recall that in BX (2012), the investor never voluntarily realizes losses and the value function $v_N(x)$ is globally convex (the solid blue line). In our model, the investor does not voluntarily realize losses either. The loss-realization threshold is given by $\underline{x}^* = x_L^* = -w^*/\kappa \approx 0.44$, where x_L^* is the forced liquidation boundary at which the leverage constraint (5) binds. Moreover, with the leverage option, the optimal gain-realization threshold further increases to $\overline{x}^* = 1.46$ from $\overline{x}_N = 1.29$ in BX (2012).

Panel B of Figure 6 plots the corresponding investor's RAMs for both models with and without leverage. The value function $v(w^*, x)$ (the solid red line) is concave for $x \in (0.44, 1.06)$ but convex for $x \in (1.06, 1.46)$ where $\widetilde{x} = 1.06$ is the inflection point where $v_{xx}(w^*, x) = 0$. Therefore, the investor is risk averse when x is between the liquidation (also loss-realization) threshold $\underline{x}^* = x_L^* \approx 0.44$ and the inflection point $\widetilde{x} = 1.06$, but becomes risk-seeking as x exceeds $\widetilde{x}$.

This concave-then-convex shape of $v(w^*, x)$ is in sharp contrast with the globally convex $v_N(x)$ in the BX (2012) model, where the investor is globally risk seeking with a constant and negative RAM: $\gamma_N(x) \equiv 1 - \eta_1 = -0.47$ for all levels of x.

5 Comparative Statics

In this section, we conduct comparative statics analyses for optimal policies: the steady-state value w^* and realization strategies $(\underline{x}^*, \overline{x}^*)$. We focus on two sets of parameters: one for the investment opportunity and the other for the investor's preferences.

5.1 Investment Opportunity: (r, μ, σ)

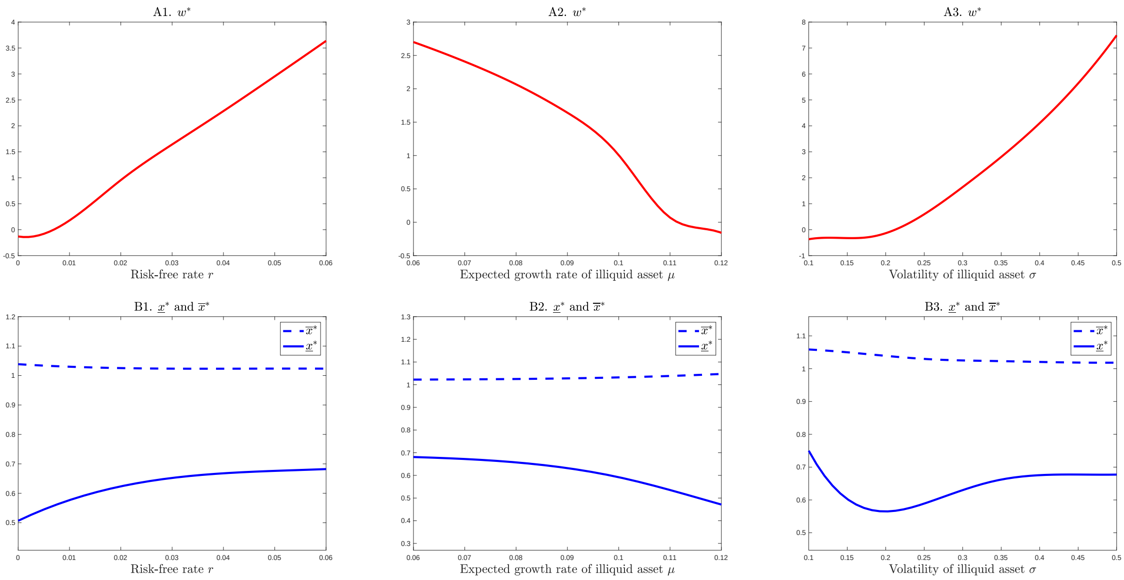

Figure 7: COMPARATIVE STATICS: EFFECTS OF CHANGING INVESTMENT OPPORTUNITY: (r, μ, σ). Panels A1, A2, and A3 plot the optimal w^* as we vary r, μ, and σ, respectively. Panels B1, B2, and B3 plot the optimal gain-realization threshold $\overline{x}^*$ (the dashed blue lines) and the optimal loss-realization threshold $\underline{x}^*$ (the solid blue lines), as we vary r, μ, and σ, respectively. All parameter values other than the one being studied are reported in Table 1.

Panels A1, A2, and A3 of Figure 7 plot the steady-state allocation between the risky and risk-free assets, w^*, as we vary r, μ, and σ, respectively. These panels show that the investor increases her exposure to the risk-free asset by decreasing w^*, as the risky asset becomes more attractive, i.e., as its Sharpe ratio $(\mu - r)/\sigma$ decreases.

Moreover, the quantitative effects of changing the investment opportunity within an economically plausible range are large. What is the effect of decreasing r from 3%

(our baseline) to 0% (as in our current low interest-rate world), increasing the risky asset's Sharpe ratio from 20% to 30%? As the cost of borrowing decreases to zero, the steady-state value of w^* decreases significantly from 1.76 to -0.14 (Panel A1), which means that the investor's risky asset allocation increases by 3.2 times. For example, with an initial wealth level of $W_{0-} = 100$ the investor significantly increases her risky asset allocation from 36 to 115.

We can also increase the risky asset's Sharpe ratio from 20% to 30% by either 1) increasing μ from 9% (baseline) to 12% or 2) decreasing σ from 30% (baseline) to 20%. An investor with $W_{0-} = 100$ increases her risky asset allocation by 3.2 times from 36 to 116 in the former case (Panel A2) and by 3.7 times from 36 to 132.

Panels B1, B2, and B3 of Figure 7 plot the gain-realization threshold $\overline{x}^*$ (dashed blue lines) and the loss-realization threshold $\underline{x}^*$ (solid blue lines), as we vary r, μ, and σ, respectively. Recall that as we increase μ or decrease r, the investor increases her dollar exposure to the risky asset (decreasing w^*), which in turn makes her more reluctant to realize losses and gains. As a result, the holding region $(\underline{x}^*, \overline{x}^*)$ widens as μ increases (Panel B1) or r decreases (Panel B2).

Surprisingly, the loss-realization threshold $\underline{x}^*$ is not monotonic in σ. First consider the "normal" scenario where $\sigma > 20\%$. Panel B3 shows that $\underline{x}^*$ increases and $\overline{x}^*$ decreases with σ, both of which cause the holding region $(\underline{x}^*, \overline{x}^*)$ to narrow. Intuitively, increasing σ causes the Sharpe ratio $(\mu - r)/\sigma$ to fall, making the risky asset less attractive, and as a result the investor reduces her risky asset holdings. Since her risky asset exposure is now smaller, the investor is less reluctant to realize losses. However, for $\sigma < 20\%$, the opposite result holds in that the loss-realization threshold $\underline{x}^*$ decreases with σ due to leverage constraints. After making large losses, the investor has to liquidate her risky asset holdings to satisfy the leverage constraint. In our analysis, for $\sigma \leq 15\%$ the loss-realization threshold $\underline{x}^*$ equals the mandatory liquidation threshold, i.e., $\underline{x}^* = x_L^* = -w^*/\kappa$.

Finally, the quantitative effects of changing r, μ, or σ on loss realization (the lower boundary) is much more significant than on gain realization (the upper boundary).

5.2 Realization Utility: $(\lambda, \alpha_-, \alpha_+)$

In Figure 8, we conduct a comparative static analysis for three key preference parameters: loss aversion λ, loss sensitivity α_-, and gain sensitivity α_+. We plot the steady-state scaled liquid wealth, w^*, as functions of λ, α_-, and α_+ in Panels A1, A2, and A3, respectively. Similarly, we plot the gain-realization threshold $\overline{x}^*$ (dashed blue lines) and the loss-realization threshold $\underline{x}^*$ (solid blue lines), as functions of λ, α_-, and α_+ in Panels B1, B2, and B3, respectively.

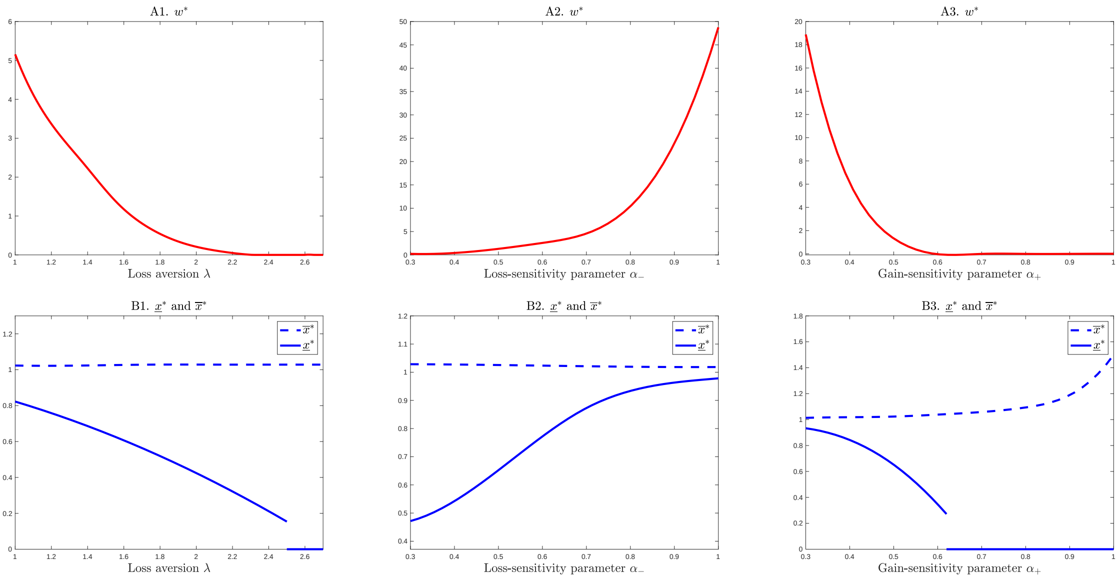

Figure 8: Comparative Statics: Effects of Changing Realization Utility: $(\lambda, \alpha_-, \alpha_+)$. Panels A1, A2, and A3 plot the optimal w^* as we vary λ, α_- and α_+, respectively. Panels B1, B2, and B3 plot the optimal gain-realization threshold $\overline{x}^*$ (the dashed blue lines) and the optimal loss-realization threshold $\underline{x}^*$ (the solid blue lines), as we vary λ, α_- and α_+, respectively. All parameter values other than the one being studied are reported in Table 1.

To ease comparison with IJ (2013), we first analyze the bottom row of Figure 8. The key takeaways are as follows. First, as the investor becomes more loss averse (higher λ), or more sensitive to losses (lower α_-), or less sensitive to gains (higher α_+), the loss-realization threshold $\underline{x}^*$ decreases and the gain-realization threshold $\overline{x}^*$ increases. These two forces reinforce each other widening the holding region. Intuitively, a more loss-averse or a more loss-sensitive investor is less willing to realize losses. Similarly, a less gain-sensitive (higher α_+) investor waits longer to realize gains. As a result, the option value of resetting the reference level decreases, which implies a lower loss-realization threshold $\underline{x}^*$.

Second, comparing the lower loss-realization threshold $\underline{x}^*$ (dashed lines) with the upper gain-realization threshold $\overline{x}^*$ (solid lines), we clearly see that the quantitative effects of changing these preference parameters on $\underline{x}^*$ are much more significant than on $\overline{x}^*$. For sufficiently large loss aversion λ (e.g., $\lambda \geq 2.5$) or sufficiently large gain sensitivity α_+ (e.g., $\alpha_+ \geq 0.62$), the investor never realize losses. These results are similar to those in IJ (2013).

The top row shows that the scaled liquid wealth w^* decreases with loss aversion λ (Panel A1), increases with α_- (Panel A2), and decreases with α_+ (Panel A3), reinforcing the results discussed above for the bottom row. Recall that as we decrease loss aversion (reducing λ), reduce loss sensitivity (increasing α_-), and increase gain sensitivity (decreasing α_+), the investor trades more frequently (a narrower holding region $(\underline{x}^*, \overline{x}^*)$). As a result, the benefit of placing more frequent and smaller bets on the risky asset increases, causing the scaled liquid wealth w^* to increase.

Quantitatively, the effects of changing these preference parameters within economically relevant ranges on w^* can be very large. For example, w^* decreases from 1.76 to zero and as a result the fraction of total wealth allocated to the risky asset upon rebalancing increases from 64% to 100% for any of the following three changes: a) loss aversion λ increases from 1.5 to 2.5, b) the parameter α_- decreases from 0.5 to 0.3, and c) the parameter α_+ increases from 0.5 to 0.62.

In sum, the option to invest in the risk-free asset significantly alters the investor's dynamic trading strategies and has large quantitative welfare effects. Moreover, varying the investment opportunity (r, μ, σ) and preferences $(\lambda, \alpha_-, \alpha_+)$ within economically relevant ranges can have very large quantitative effects.

6 A Generalized Jump Diffusion Model

In this section, we incorporate jumps into the illiquid asset price process. Importantly, unlike the diffusion model of Section 2, the solution features *four* mutually exclusive regions including two disconnected holding regions, a loss-realization region, and a gain-realization region. These four regions are divided by three thresholds: two loss-realization thresholds and one gain-realization threshold. While the investor does not realize losses when incurring a substantial loss and falling into the deep-loss holding region, she immediately realizes losses after a small rebound of the asset price that brings x from the deep-loss holding region into the loss-realization region. Moreover, we show that the quantitative effects of empirically realistic jumps on dynamic rebalancing and value functions are very large.

6.1 Model

Jump-Diffusion Process for the Illiquid Asset Price. We generalize the illiquid asset price process P_t by incorporating jumps into the GBM process given in (1):

$$\frac{dP_t}{P_{t-}} = \mu dt + \sigma d\mathcal{Z}_t - (1-Y)d\mathcal{J}_t\,, \quad P_0 > 0\,, \tag{50}$$

where $\mathcal{J}$ is a pure jump process with a constant arrival rate, ρ, and the random variable $Y \in [0,1]$ is drawn from a cumulative distribution function, $\Omega(Y)$. Let $\tau^{\mathcal{J}}$ denote the stochastic jump arrival time. If a jump occurs at t ($d\mathcal{J}_t = 1$), the price of the illiquid asset falls from P_{t-} to $P_t = YP_{t-}$. The expected percentage decrease of P conditional on a jump arrival is thus $1 - \mathbb{E}[Y]$, where $\mathbb{E}[\,\cdot\,]$ is the expectation operator associated with the distribution $\Omega(Y)$. The expected cum-dividend return of the illiquid asset, μ_P, is:

$$\mu_P = \mu - \rho(1 - \mathbb{E}[Y])\,, \tag{51}$$

where the term $\rho\,(1-\mathbb{E}[Y])$ captures the effect of jumps on μ_P. If a jump does not occur at t ($d\mathcal{J}_t = 0$), the price process is continuous and hence $P_t = P_{t-}$.

Value Function and HJB Equation. The key change from our baseline diffusion model of Section 2 is that the illiquid wealth X evolves as follows:

$$\frac{dX_t}{X_{t-}} = \mu dt + \sigma d\mathcal{Z}_t - (1-Y)d\mathcal{J}_t, \quad \text{for } t \in [\tau_i, \tau_{i+1})\,. \tag{52}$$

To ease exposition, we focus on the $W \geq 0$ case where the investor can save in the risk-free asset but cannot borrow.[22] The value function $V(W,X,B)$ satisfies the following HJB equation in the region where $W \geq 0$, $X > 0$, $B > 0$:

$$\max\{\mathcal{L}^{\mathcal{J}}V(W,X,B),\ F(W,X,B) - V(W,X,B)\} = 0, \tag{53}$$

where the payoff function $F(W,X,B)$ is defined in (18) and $\mathcal{L}^{\mathcal{J}}V(W,X,B)$ is the generator given by

$$\mathcal{L}^{\mathcal{J}}V(W,X,B) = \mathcal{L}V(W,X,B) + \rho\left(\mathbb{E}[V(W,YX,B)] - V(W,X,B)\right). \tag{54}$$

Recall that $\mathcal{L}V(W,X,B)$ on the right-hand side of (54) is the generator for diffusion models defined in (17). The effect of jumps is captured by the second term in (54).

[22] We can generalize our model to allow for leverage with a somewhat more involved analysis.

Using the homogeneity property, we obtain $dw_t = 0$ as in our diffusion model and the following process for x_t, which includes the effect of jumps:

$$dx_t/x_{t-} = (\mu - r)dt + \sigma d\mathcal{Z}_t - (1-Y)d\mathcal{J}_t \,. \tag{55}$$

Using the scaled variables, we simplify the variational inequality (53) as:

$$\max\{\mathscr{L}^{\mathcal{J}} v(w,x),\ f(w,x) - v(w,x)\} = 0, \tag{56}$$

where $f(w,x)$ is the scaled payoff defined in (27) and $\mathscr{L}^{\mathcal{J}}$ is the generator given by

$$\mathscr{L}^{\mathcal{J}} v(w,x) = \frac{\sigma^2 x^2}{2} v_{xx} + (\mu - r)x v_x - (\delta - \beta r)v + \rho\left(\mathbb{E}[v(w,Yx)] - v(w,x)\right). \tag{57}$$

As in our diffusion model of Section 2, there are two regions: one is the holding region $\mathcal{H} = \{(w,x) \in \mathcal{S} \mid v(w,x) > f(w,x)\}$, in which $\mathscr{L}^{\mathcal{J}} v(w,x) = 0$, and the other is the realization region $\mathcal{R} = \{(w,x) \in \mathcal{S} \mid v(w,x) = f(w,x)\}$. We show that in general there are two disconnected holding regions, $\mathcal{H}_1$ and $\mathcal{H}_2$, both of which are visited with strictly positive probability.[23] Let $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$. Relatedly, we show that the standard double-barrier policy, which is based on the smooth-pasting conditions as typically done in the real-options literature, no longer characterizes the optimal realization strategy and value functions. We have to directly solve the variational inequality (56) to which we next turn.

6.2 Solution, Economic Mechanism, and Quantitative Results

We choose parameter values, solve the model using variational inequality, inspect the model's mechanism, and then analyze the quantitative predictions of our model.[24]

Calibration and Parameter Choices. As in Section 4, we set $r = 3\%$ and the illiquid asset's risk premium at 6% as for the US stock market. Doing so yields $\mu = 10\%$, so that the expected risky asset return is $\mu_P = \mu - \rho\mathbb{E}[1-Y] = 9\%$ and $\mu_P - r = 6\%$. We set $\sigma = 20\%$ to target the Sharpe ratio for the illiquid asset at 30%. We use this baseline case to help us understand how a realization-utility investor trades the aggregate stock market index. We set the loss aversion parameter $\lambda = 2.7$, the loss sensitivity

[23] Mathematically, while there may also be two disconnected holding regions in IJ (2013) and our diffusion model, the deep-loss holding region $\mathcal{H}_2$ is never visited on the equilibrium path in diffusion models. We provide more detailed discussions on this point in Section 7.

[24] We use the penalty method (e.g., Dai and Zhong (2010)) to solve the variational inequality. It is important to note that the commonly-used solution method based on smooth-pasting conditions is invalid in our model.

parameter $\alpha_- = 0.8$, and the discount rate $\delta = 8\%$. Other parameter values are the same as in our baseline case reported in Table 1.

We specify the cumulative distribution function $\Omega(Y)$ for Y using the following widely used power law as in Barro (2006) and the rare-disaster literature:

$$\Omega(Y) = Y^{\psi}, \quad \text{for } Y \in [0, 1], \tag{58}$$

where $\psi > 0$ is a constant. The expected percentage decrease of the illiquid asset price is $1 - \mathbb{E}[Y] = \frac{1}{\psi+1}$. The lower the value of ψ the larger the percentage decrease of P upon a jump arrival. For our baseline jump case, we set the power-law parameter ψ at 6.3 as in the macro finance literature, e.g., Barro and Jin (2011), which corresponds to 14% decrease of the illiquid asset price ($\mathbb{E}[1 - Y] = 14\%$) conditional on a jump arrival. We set the jump arrival rate ρ at 73% per annum, which implies that a jump arrives once about 1.4 years on average.[25] Additionally, we purposefully choose our parameter values for our baseline jump-diffusion calculation so that $w^* = 0$, meaning that the investor always fully invests in the illiquid asset and the option value of investing in the risk-free asset is zero. We thus can clearly make the point that the four-region solution, a key result of our model, does not depend on the investor's access to the risk-free asset. In Section 7, we consider another setting with different parameter values where $w^* > 0$ and analyze the effect of diversification (over time) on trading strategies and value functions.

Value Functions and Optimal Rebalancing Policies. In Panel A of Figure 9, we plot the scaled value $v(w^*, x)$ (solid red line) and the scaled payoff $f(w^*, x)$ (dashed blue line) for our baseline jump-diffusion case.[26] The solution features *four* mutually exclusive regions for x on the positive real line, divided by three thresholds: *two* loss-realization thresholds, $\underline{x}_2^* = 0.47$ and $\underline{x}_1^* = 0.52$, which together define the loss-realization region $\mathcal{R}_2 = (\underline{x}_2^*, \underline{x}_1^*) = (0.47, 0.52)$, and one gain-realization threshold $\overline{x}^* = 1.03$, which defines the gain-realization region $\mathcal{R}_1 = (\overline{x}^*, \infty) = (1.03, \infty)$. In addition to the two realization regions ($\mathcal{R}_1$ and $\mathcal{R}_2$), there are two disconnected holding regions: the normal holding region $\mathcal{H}_1 = (\underline{x}_1^*, \overline{x}^*) = (0.52, 1.03)$ and the deep-loss holding region $\mathcal{H}_2 = (0, \underline{x}_2^*) = (0, 0.47)$.

In the two holding regions ($\mathcal{H}_1$ and $\mathcal{H}_2$), the investor's value function $v(w^*, x)$ is larger than her realization payoff $f(w^*, x)$ and as a result, the solid red line ($v(w^*, x)$) is above

[25] Our choice of the jump arrival rate λ is in line with empirical estimates in the asset pricing literature for individual stocks (see e.g., Huang and Huang (2012)).

[26] For this case, the investor puts all her wealth into the illiquid asset: $w^* = 0$.

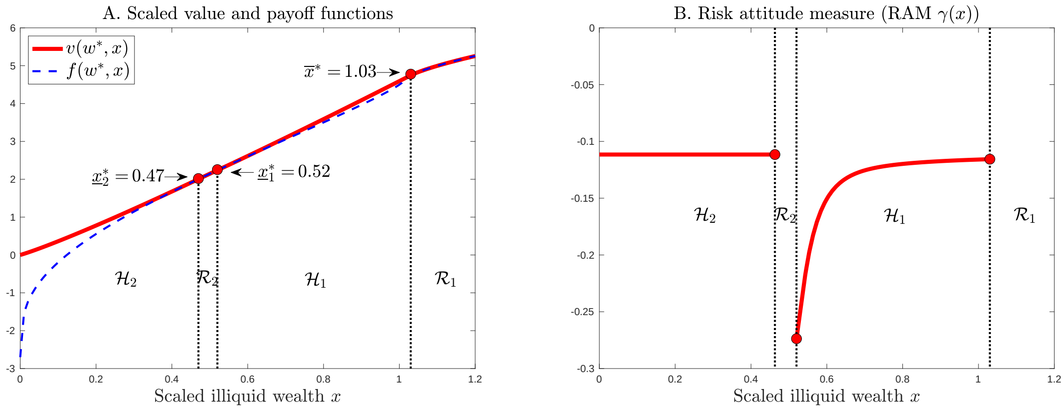

Figure 9: Scaled value function $v(w^*, x)$, scaled payoff function $f(w^*, x)$ and RAM $\gamma(x)$ for jump-diffusion models. Panel A plots $v(w^*, x)$ (solid red line) and $f(w^*, x)$ (dashed blue line). There are four regions: deep-loss holding $\mathcal{H}_2$, loss realization $\mathcal{R}_2$, normal holding $\mathcal{H}_1$, and gain realization $\mathcal{R}_1$ regions. Panel B plots the RAM: $\gamma(x)$. Parameter values are: $\mu_P = 9\%$, $\sigma = 20\%$, $\rho = 73\%$, $\psi = 6.3$, $r = 3\%$, $\theta_p = \theta_s = 1\%$, $\alpha_+ = 0.5, \alpha_- = 0.8, \lambda = 2.7$, $\beta = 0.3$, and $\delta = 8\%$.

the dash blue line $f(w^*, x)$, as seen from Panel A. In both the gain- and loss-realization regions ($\mathcal{R}_1$ and $\mathcal{R}_2$), $v(w^*, x) = f(w^*, x)$. This is why the solid red line and the dash blue line are on top of each other in $\mathcal{R}_1$ and $\mathcal{R}_2$.

First, consider the region where $x \in \mathcal{H}_2 = (0, \underline{x}_2^*) = (0, 0.47)$. The investor is in deep loss and passively holds onto her illiquid asset position as turning such a deep paper loss into reality is just too painful for her. Intuitively, the absolute value of the negative realization utility, $|U(G, B)|$, is larger than the investor's continuation utility (after realizing the loss) in this region. This can be seen from the positive gap between $v(w^*, x)$ (solid blue line) and $f(w^*, x)$ (dashed blue line) in the $\mathcal{H}_2$ region.

Second, when $x \in \mathcal{R}_1 = (\underline{x}_2^*, \underline{x}_1^*) = (0.47, 0.52)$, the investor realizes losses even though doing so is still quite painful (losing about half of the original investments, adjusted for time value) though less than in the $\mathcal{H}_2$ region. The intuition is as follows. The utility from immediate loss realization, $U(G, B)$ in the $\mathcal{R}_1$ region, is less negative than in the $\mathcal{H}_2$ region. The absolute value of the (immediate) negative realization utility, $|U(G, B)|$, is lower than the investor's continuation value (after loss realization). Therefore, it is optimal to realize losses and reset the reference level B, so that $x = 1$ which is much closer to the gain-realization threshold $\overline{x}^* = 1.03$, significantly increasing the likelihood of realizing gains.

Third, consider the region where $x \in \mathcal{H}_1 = (\underline{x}_1^*, \overline{x}^*) = (0.52, 1.03)$. The investor optimally holds onto her illiquid asset. This is because the likelihood for x to drift upward towards the gain-realization threshold $\overline{x}^*$ is large enough that it is optimal for the investor to keep her illiquid asset position unchanged, avoid paying the transactions costs, and wait for a rebound of the asset price. This $\mathcal{H}_1$ is the normal waiting region as in IJ (2013).

It is important to note that the investor's expectation in the deep-loss holding region $\mathcal{H}_2$ is different from that in the normal holding region $\mathcal{H}_1$. In the deep-loss holding region $\mathcal{H}_2$, the investor's hope is to reduce losses and the only path towards portfolio rebalancing and gain realizations later on is by reaching $\underline{x}_2^*$, the lower boundary of the loss-realization region $\mathcal{R}_2$. In the normal holding region $\mathcal{H}_1$ the investor hopes to realize gains by reaching $\overline{x}^*$, the gain-realization threshold.[27] The different expectations in the two disconnected holding regions $\mathcal{H}_2$ and $\mathcal{H}_1$ generate non-monotonic propensities to realize losses and the existence of the loss-realization region $\mathcal{R}_2$ separating $\mathcal{H}_2$ and $\mathcal{H}_1$.

Finally, the investor realizes gains whenever $x > \overline{x}^*$. In our example, the investor frequently realizes gains ($\overline{x}^* = 1.03$) as in BX (2012), IJ (2013), and our diffusion model.

Panel B plots the RAM measure $\gamma(x)$ defined in (48) for the two holding regions $\mathcal{H}_1$ and $\mathcal{H}_2$. The investor is risk-seeking and the value function is convex ($\gamma(x) < 0$) in both $\mathcal{H}_1$ and $\mathcal{H}_2$. In the normal holding region $\mathcal{H}_1$, the investor becomes more risk-seeking (more negative $\gamma(x)$) as x_t decreases. The existence of the $\mathcal{H}_1$ region implies the existence of the loss-realization region $\mathcal{R}_2$ explaining why the investor voluntarily realizes losses as emphasized in IJ (2013). In contrast, in the deep-loss holding region $\mathcal{H}_2$, the investor is unwilling to realize losses and moreover, the investor's RAM is constant ($\gamma(x) = -0.11$).[28] The mechanism for the existence of the $\mathcal{H}_2$ region is closely related to that driving the no-loss realization result in BX (2012).

In sum, our model generates both the no-loss realization insight of BX (2012) and the voluntary loss realization result of IJ (2013). Indeed, the different mechanisms in BX (2012) and IJ (2013) are both at work in our model. The BX mechanism results in a deep-loss holding region $\mathcal{H}_2$ and the IJ mechanism results in the normal holding region $\mathcal{H}_1$. To have these two disconnected holding regions, it is mathematically necessary and economically intuitive to have the loss realization region $\mathcal{R}_2$ in between them. We provide a detailed comparison with BX (2012) and IJ (2013) in Section 7.

[27] Of course, in $\mathcal{H}_1$, the investor may end up selling when x reaches the threshold $\underline{x}_1^*$, the upper boundary of the loss-realization region $\mathcal{R}_2$, or falling into the deep-loss holding region $\mathcal{H}_2$ should a significantly large downward jump arrive.

[28] As we have shown in Section 2, the implied RAM is also constant in the original BX (2012) diffusion model.

Non-monotonic Loss Realization Strategy and Dynamics of x. Our model solution cycles between holding and realization regions as we increase x from zero to ∞. That is, the investor does not sell her asset at a loss when the loss is either too large (in the deep-loss holding region $\mathcal{H}_2$) or not large enough (in the normal holding region $\mathcal{H}_1$). Indeed, the investor only realizes losses in the $\mathcal{R}_2 = (0.47, 0.52)$ region. Taking these results together, we see that the loss realization strategy is non-monotonic in x.

The investor reaches $\mathcal{R}_2$ via one of the following three ways: a.) a jump from within $\mathcal{H}_1$, b.) a negative diffusion shock hitting $\underline{x}_1^* = 0.52$, or c.) a positive diffusion shock hitting $\underline{x}_2^* = 0.47$ from within $\mathcal{H}_2$. Consider an investor starting with $x_0 = 1$ in $\mathcal{H}_1$. The following path can possibly arise from our model. Upon incurring a substantial loss which decreases x from within $\mathcal{H}_1$ to $\mathcal{H}_2$, the investor continues to hold onto the illiquid asset but then realizes her paper loss after a rebound of the asset price that brings x (from within $\mathcal{H}_2$) just to the lower boundary of $\mathcal{R}_2$: $\underline{x}_2^* = 0.47$. Upon realizing this loss, the investor resets her reference level, starts anew from $x^* = 1$ in $\mathcal{H}_1$, and repeats the process. This path captures the following prediction that would not have been possible in other realization-utility models in the literature: the investor while unwilling to sell her asset after incurring a large loss (in the deep-loss $\mathcal{H}_2$ region) is willing to sell after a small rebound. This path captures the behavior of some investors with loss aversion.

Our model predicts that investors' selling propensity is a V-shaped function of unrealized profits, i.e., selling probability increases as the gain or loss magnitude increases, reported in Ben-David and Hirshleifer (2012), Hartzmark (2015), and An (2016).

Next, we characterize the stationary distribution of x.

Stationary Distribution of x. Given the rebalancing policy, characterized by the optimal w^* and the four regions: $\mathcal{H}_2 = (0, \underline{x}_2^*)$, $\mathcal{R}_2 = [\underline{x}_2^*, \underline{x}_1^*]$, $\mathcal{H}_1 = (\underline{x}_1^*, \overline{x}^*)$, and $R_1 = [\overline{x}^*, \infty)$, where $0 \le \underline{x}_2^* \le \underline{x}_1^* < 1 < \overline{x}^*$, the density function $\varphi(\,\cdot\,)$ for the stationary distribution of x for our jump-diffusion model satisfies the Kolmogorov forward equation:

$$\mathcal{K}^*\varphi(x) = 0\,, \tag{59}$$

in the regions where $x \in \mathcal{H}\backslash\{1\} = \mathcal{H}_2 \cup \mathcal{H}_1\backslash\{1\}$ and $\mathcal{K}^*$ is the operator defined by

$$\mathcal{K}^*\varphi(x) = \frac{d^2}{dx^2}\Big(\frac{1}{2}\sigma^2 x^2 \varphi(x)\Big) - \frac{d}{dx}\Big((\mu - r)x\,\varphi(x)\Big) + \rho\left(\mathbb{E}\Big[\frac{\varphi(x/Y)}{Y}\Big] - \varphi(x)\right)\,. \tag{60}$$

Note that (59) holds for the two holding regions excluding the $x = 1$ point.[29] The last term in (60) captures the effect of jumps on φ. Next, we generalize the duration analysis for diffusion models in Ingersoll and Jin (2013) to allow for jumps.

Duration of Investment Episodes. Let τ denote the calendar time that the investor realizes the next trading gain or loss. Let $D(x_t)$ denote the expectation of $(\tau - t)$ conditional on the value of x_t at current time t, i.e., $D(x) = \mathbb{E}_t[(\tau - t)|x_t = x]$. The following result holds in the two holding regions where $x \in \mathcal{H} = \mathcal{H}_2 \cup \mathcal{H}_1$:

$$\mathcal{K}D(x) = -1\,, \tag{61}$$

where $\mathcal{K}$ is the operator, which is adjoint to $\mathcal{K}^*$ defined in (60), defined by

$$\mathcal{K}D(x) = \frac{1}{2}\sigma^2 x^2 D_{xx}(x) + (\mu - r)xD_x(x) + \rho\left(\mathbb{E}[D(Yx)] - D(x)\right). \tag{62}$$

By definition, $D(x) = 0$ if the investor immediately realizes gains or losses: $x \in \mathcal{R}_2 \cup \mathcal{R}_1$.

Let Φ_G denote the fractions of time that the illiquid asset has unrealized gains (on paper). Using the stationary distribution $\varphi(x)$, we obtain $\Phi_G = \int_1^{\overline{x}^*} \varphi(x)dx$. The fraction of time that the asset has unrealized losses is therefore given by $\Phi_L = 1 - \Phi_G$. The illiquid asset has paper losses either in the deep-loss holding region $\mathcal{H}_2$ or when $\underline{x}_1^* < x < 1$ in the normal holding region $\mathcal{H}_2$. Let $\Phi_{\mathcal{H}_2}$ denote the fraction of time that the investor is in the deep-loss holding region $\mathcal{H}_2$: $\Phi_{\mathcal{H}_2} = \int_0^{\overline{x}_2^*} \varphi(x)dx$. Therefore, $\Phi_L - \Phi_{\mathcal{H}_2}$ is the fraction of time that the investor incurs losses in the normal holding region $\mathcal{H}_1$ with $x \in (\underline{x}_1^*, 1)$.

In Figure 10, we plot the stationary density function $\varphi(x)$ for x_t for our baseline jump-diffusion case. The area under the left part of the stationary density function $\varphi(x)$ equals $\Phi_{\mathcal{H}_2} = 17\%$, which means that in the long run the investor spends about 17% of her time in the deep-loss holding region $\mathcal{H}_2$. The density function $\varphi(x)$ is single peaked near $x = 0$ indicating substantial loss on average conditional in the $\mathcal{H}_2$ region.

The area under the right part of the density function $\varphi(x)$ equals $1 - \Phi_{\mathcal{H}_2} = 83\%$, which means that in the long run the investor spends about 17% of her time in the deep-loss $\mathcal{H}_2$ region and the remaining 83% of her time in the normal holding region $\mathcal{H}_1$. Out of this 83% time spent in $\mathcal{H}_1$, the investor spends about 6% of her entire time in the paper gain region $x \in (1, \overline{x}^*) = (1, 1.03)$, as $\Phi_G = 6\%$, and the other 77% of

[29] The $x = 1$ point in $\mathcal{H}_1$ does not satisfy (59) as $\varphi(x)$ is not differentiable there. But $\varphi(x)$ is continuous at $x = 1$: $\lim_{x\to 1^-} \varphi(x) = \lim_{x\to 1^+} \varphi(x)$. Other conditions for $\varphi(x)$ are: a.) the fraction of time spent in the gain- and loss-realization regions is zero: $\varphi(x) = 0$ for $x \in \mathcal{R}_2 \cup \mathcal{R}_1$, as the investor immediately rebalances to $x = 1 \in \mathcal{H}_1$, and b.) the density function $\varphi(x)$ integrates to one: $\int_0^{\overline{x}^*} \varphi(x)dx = 1$.

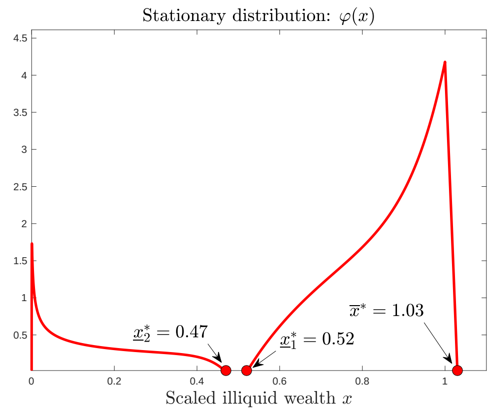

Figure 10: Stationary density function $\varphi(x)$: the baseline jump-diffusion case. The long-run probability for $x \in \mathcal{H}_2$ is 17% (the area under the left part of $\varphi(x)$). Parameter values are: $\mu_P = 9\%$, $\sigma = 20\%$, $\rho = 73\%$, $\psi = 6.3$, $r = 3\%$, $\theta_p = \theta_s = 1\%$, $\alpha_+ = 0.5, \alpha_- = 0.8, \lambda = 2.7$, and $\beta = 0.3$.

her time in the normal paper loss region $x \in (\underline{x}_1^*, 1) = (0.52, 1)$. In the normal holding region $\mathcal{H}_1$, $\varphi(x)$ is single peaked at $x = 1$, the post-rebalancing value of x. This is because the investor can only enter into $\mathcal{H}_1$ from either $\mathcal{R}_1$ (after realizing gains) or $\mathcal{R}_2$ (after realizing losses). For this reason, $\varphi(x)$ is not differentiable at $x = 1$. Finally, the expected duration between two consecutive rebalancing is about 113 days for our baseline jump-diffusion model ($D(1) = 0.31$).

In sum, fully taking into account the two disconnected holding regions $\mathcal{H}_1$ and $\mathcal{H}_2$, separated by a loss-realization region $\mathcal{R}_2$, is not only conceptually important but also quantitatively significant.

7 Inspecting Mechanism by Comparing to BX (2012) & IJ (2013)

In this section, we further inspect the mechanism of our jump-diffusion model (with the risk-free asset investment option) in two steps. First, we contrast the different mechanisms in BX (2012) and IJ (2013) using two special cases of our model. Second, we compare our model with BX (2012) and IJ (2013) highlighting three key differences.

7.1 Highlighting Different Mechanisms in BX (2012) and IJ (2013)

First, we show that the key result in BX (2012) that the investor with piecewise linear realization utility never realizes losses continues to hold in jump-diffusion settings where the investor can save in the risk-free asset. We summarize this generalized BX (2012) result in the following proposition and relegate its proof to Appendix C.

Proposition 1. *For an investor with piecewise linear realization utility* $U(G, B)$*, i.e.,* $\alpha_{\pm} = \beta = 1$*, it is never optimal to voluntarily realize losses and hence there are only two regions: the holding region* $\mathcal{H}$ *and the gain-realization region* $\mathcal{R}_1$*. Therefore, it is optimal to invest her entire wealth in the illiquid asset at all* t*:* $w_t = w^* = 0$*. The optimal scaled value function* $v(w^*, x) = v(0, x)$ *is globally convex in* x *with* $v(0, 0) = 0$*. The closed-form expressions for* $v(0, x)$ *and the scaled payoff function* $f(0, x)$ *are reported in Appendix C.*

Intuitively, an investor with piecewise linear realization utility $U(G, B)$ is unwilling to realize losses due to loss aversion and $\alpha_- = 1$. Since the investor never realizes losses, the loss-realization region $\mathcal{R}_2$ disappears, which implies that the optimal lower boundary is zero ($\underline{x}_1^* = \underline{x}_2^* = 0$) and there is only one holding region $\mathcal{H}$, as there is no $\mathcal{R}_2$ dividing the holding region into two. (That is, the two holding regions $\mathcal{H}_1$ and $\mathcal{H}_2$ become connected as a single region $\mathcal{H}$.)

Panels A1 and B1 of Figure 11 reproduce these key results in BX (2012). First, we verify that the optimal w is $w^* = 0$ and the solution indeed features two regions as in BX (2012): the gain-realization region $\mathcal{R}_1$ and the holding region $\mathcal{H}$. Second, the scaled value function is increasing and convex in the holding region $\mathcal{H} = (0, \overline{x}^*) = (0, 1.27)$: $v(w^*, x) = v(0, x) = C_1^* x^\eta$ where $\eta > 1$ is a root of (A.15) and $C_1^* > 0$ is a constant given in (C.15). As a result, the RAM is constant: $\gamma(x) = 1 - \eta = -0.11 < 0$ in the $\mathcal{H}$ region. In the gain-realization region $\mathcal{R}_1 = [\overline{x}^*, \infty) = [1.27, \infty)$, the value function is linear and given by $v(w^*, x) = [1 + C_1^*/(1 + \theta_p)](1 - \theta_s)x - 1$.

Panels A2 and B2 of Figure 11 reproduce the key mechanism of IJ (2013) in our generalized setting. First, we note that the optimal w is $w^* = 0.65$. That is, the investor optimally spreads her bets over time by allocating 61% of her total wealth to the risky asset and the remaining 39% in the risk-free asset for future betting when rebalancing. To reproduce the key mechanism in IJ (2013), we choose parameter values so that the solution features three regions: the loss-realization region $\mathcal{R}_2 = (\underline{x}_2^*, \underline{x}_1^*) = (0, 0.65)$, the holding region $\mathcal{H} = (\underline{x}_1^*, \overline{x}^*) = (0.65, 1.03)$, and the gain-realization region $\mathcal{R}_1 = (\overline{x}^*, \infty) = (1.03, \infty)$. In this example, there is no deep-loss holding region $\mathcal{H}_2$ as $f(w^*, x)$ is positive even at $x = 0$. The reason for $f(w^*, x) > 0$ is that realizing a total loss of her bet on the risky asset is not too costly due to her savings on the side, i.e., $u(g)$ is not too negative even when $g = -1$. Therefore, the solution features the double-barrier policy as in IJ (2013).

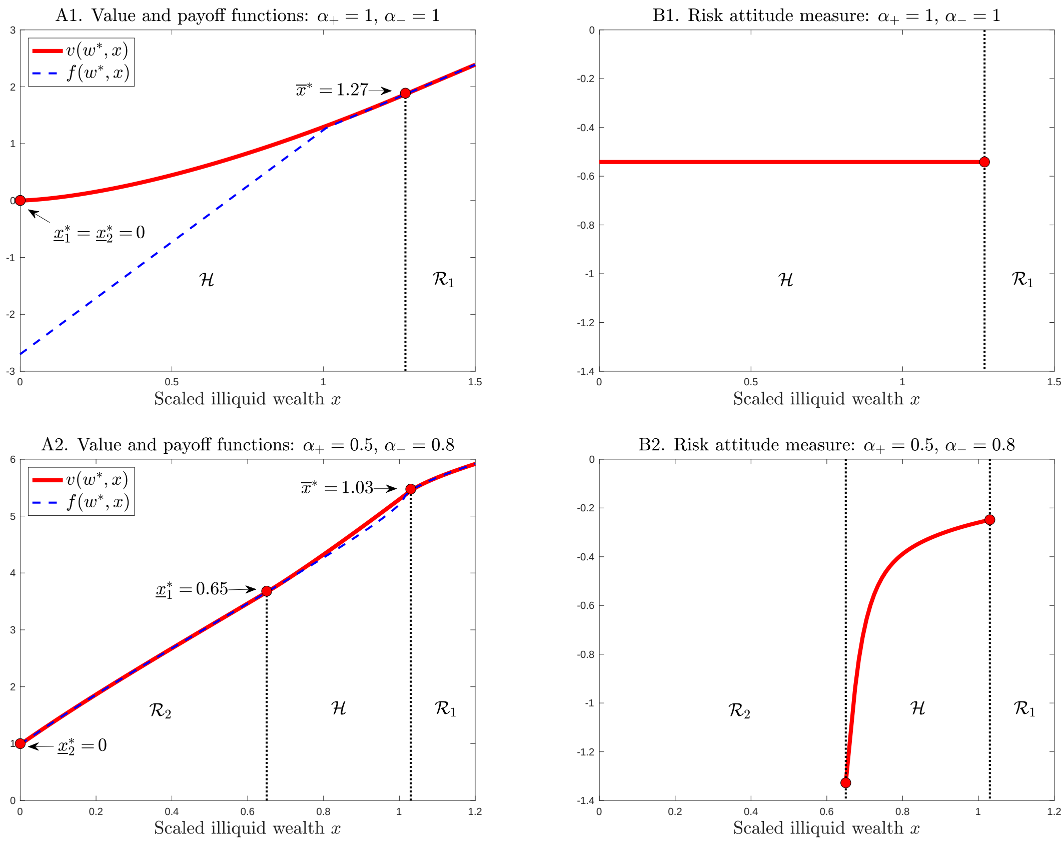

Figure 11: Special cases of our jump-diffusion models with two-region and three-region solutions. The top panels reproduce the key mechanism in BX (2012) and the bottom panels reproduce the key mechanism in IJ (2013). Panels A1 and B1 show that the optimal w is $w^* = 0$ and the solution features two regions as in BX (2012): the gain-realization region $\mathcal{R}_1$ and the holding region $\mathcal{H}$. The parameter values are $\alpha_+ = 1$, $\alpha_- = 1$, $\delta = 15\%$, and $\rho = 73\%$. Panels A2 and B2 show that the optimal w is $w^* = 0.65$ and the solution features three regions as in IJ (2013): the gain-realization region $\mathcal{R}_1$, the holding region $\mathcal{H}$, and the loss-realization region $\mathcal{R}_2$. The parameter values are $\alpha_+ = 0.5$, $\alpha_- = 0.8$, $\delta = 8\%$, and $\rho = 150\%$. Other parameter values are the same for both cases: $\mu_P = 9\%$, $\sigma = 20\%$, $\psi = 6.3$, $r = 3\%$, $\beta = 0.3$, and $\lambda = 2.7$.

Having reproduced the different mechanisms in BX (2012) and IJ (2013) using numerical examples of our jump-diffusion models, we next highlight the key differences of our general model from theirs.

7.2 Comparing with BX (2012) and IJ (2013)

First, the investor in our model can adjust the size of her illiquid wealth over time. As we have shown, the value of investing in the risk-free asset is large for an empirically plausible range of parameter values. The investor either spreads her investments over time by making smaller bets in the illiquid asset ($w^* > 0$) or increases her exposure to the illiquid asset using leverage ($w^* < 0$). Technically, this generalization introduces an additional state variable. Despite this increase of dimensionality, our model remains highly tractable and the economic mechanism is transparent and intuitive.

Second, we introduce jumps into the illiquid asset price process. This generalization is not only empirically and quantitatively relevant, but also importantly allows us to show that the solution features *three* endogenous thresholds (*two* loss- and one gain-realization thresholds), which imply four endogenously determined regions. Importantly, these four regions include two disconnected holding regions: the standard holding region $\mathcal{H}_1$ as in IJ (2013) and the deep-loss holding region $\mathcal{H}_2$, where the investor has incurred large losses and is unwilling to realize losses. Separating these two holding regions is the loss-realization region $\mathcal{R}_2$. Finally, as in BX (2012) and IJ (2013), our model also has the gain-realization region $\mathcal{R}_1$. The investor stochastically moves across these four regions.

Recall that for the special BX (2012) case with piecewise linear realization utility, the investor never realizes losses and thus the loss-realization region $\mathcal{R}_2$, which lies between $\mathcal{H}_1$ and $\mathcal{H}_2$ for the general case, disappears. As a result, the two holding regions $\mathcal{H}_1$ and $\mathcal{H}_2$ in our general case are then connected and combined into a single holding region $\mathcal{H}$. The smooth-pasting condition characterizes the optimal gain-realization threshold $\overline{x}^*$ as in BX (2012): the investor realizes gains when $x \in \mathcal{R}_1 = (\overline{x}^*, \infty)$ and passively holds on to the illiquid asset when $x \in \mathcal{H} = (0, \overline{x}^*)$. In sum, BX (2012) is a special case where the solution features one holding region and one gain-realization region.

The IJ (2013) model features three regions, the loss-realization region $\mathcal{R}_2$ and the gain-realization region $\mathcal{R}_1$ separated by the standard holding region $\mathcal{H}_1$, but does not have the deep-loss $\mathcal{H}_2$ region.[30] The standard smooth-pasting conditions for the gain- and loss-realization thresholds characterize the investor's optimal trading strategy.[31]

One implication of the four-region solution for our jump-diffusion model is the following time series dynamics, which we can relate to in the real-world trading practice. While an investor may choose not to realize losses after incurring a substantial loss (the

[30] The IJ (2013) model also includes cases where the investor never realizes losses, e.g., BX (2012). The discussions in the preceding paragraph applies to these cases where voluntary loss realizations are not optimal and the solution features two regions.

[31] Mathematically, the solution for the IJ (2013) diffusion model also typically features four regions. However, since the illiquid asset price follows a diffusion process in IJ (2013), the investor never reaches the deep-loss holding region $\mathcal{H}_2$. This is because the investor always starts with $x = 1$ and realizes losses whenever reaching $\underline{x}_1^*$, the upper boundary of the loss-realization region $\mathcal{R}_2$. As a result, the only point in the loss-realization region $\mathcal{R}_2$ that the investor can reach is $\underline{x}_1^*$. The investor is thus never in the deep-loss holding region $\mathcal{H}_2$ as $x \le \underline{x}_2^* \le \underline{x}_1^*$ for all $x \in \mathcal{H}_2$. Therefore, IJ (2013) can use value-matching and smooth-pasting conditions to correctly obtain the gain-realization threshold $\overline{x}^*$ and the upper loss-realization threshold $\underline{x}_1^*$, which define the three regions, $\mathcal{R}_2$, $\mathcal{H}_1$, and $\mathcal{R}_1$. Unlike in our jump-diffusion model, the investor in the IJ (2013) diffusion model can ignore the deep-loss holding region $\mathcal{H}_2$ and still correctly characterize the equilibrium path.

illiquid asset price jumps into the deep-loss holding region $\mathcal{H}_2$), she may choose to realize losses after a small rebound of the asset price that brings x into the loss realization region $\mathcal{R}_2$.

Third, because we have two disconnected holding regions, the standard threshold-based solution methods used in the real options literature, e.g., McDonald and Siegel (1986) and Dixit and Pindyck (1994), which use the value-matching and smooth-pasting conditions to pin down the thresholds, do not work in our model. This is because the investor's tendency to realize losses is *non-monotonic*. The tendency to realize losses is obviously the largest in the intermediate range $\mathcal{R}_2 \in (\underline{x}_2^*, \underline{x}_1^*) = (0.47, 0.52)$. Using the standard double-barrier method yields a wrong solution. It is necessary and sufficient to use the variational inequality approach to obtain the correct solution.[32]

In sum, we show that the different and complementary mechanisms in BX (2012) and IJ (2013) are important for realization-utility investors. The interaction of these two mechanisms generates highly nonlinear and non-monotonic dynamic portfolio rebalancing policies. We show that the investor is often willing to realize losses in the intermediate range of losses (the $\mathcal{R}_2$ region) implying non-monotonic propensities to realize losses. The nonlinearity and non-monotonicity properties generate rich empirical predictions consistent with evidence (Ben-David and Hirshleifer, 2012; Liao, Peng, and Zhu, 2022).

8 Conclusion

Building on BX (2012) and IJ (2013), we develop a dynamic tractable jump-diffusion model where the investor with realization utility can dynamically rebalance her portfolio between the risky asset and the risk-free asset.

First, we show that the investor's realization strategies crucially depend on whether she can invest in the risk-free asset. The option to save allows the investor to lower her risky asset exposure by making smaller and more frequent bets and hence diversifying realized gains and losses over time. The investor thus becomes less averse to realize losses and more willing to realize gains. In contrast, should the investor choose to take a position larger than her wealth via leverage, she will be less willing to realize losses. The

[32] To the best of our knowledge, this is among the first models in the finance literature, where the commonly used double-barrier policy method fails to produce the correct solution. A technical takeaway from this result is that while a threshold-based policy might be optimal based on heuristic reasoning, one should verify the optimality of the conjectured barrier policy for problems described by variational inequalities.

dynamic feedback between gain/loss realizations and (risky and risk-free) asset allocation is at the core of our model mechanism.

Second, we show that for widely used jump-diffusion models, our model solution features *four* regions including two disconnected holding regions, a loss-realization region, and a gain-realization region. These four regions are divided by three thresholds: two loss-realization thresholds and one gain-realization threshold. While choosing not to realize losses after incurring a substantial jump loss and falling into the deep-loss holding region (a crash event), the investor nonetheless may immediately realize her losses after a small rebound of the post-crash asset price. Interestingly, the investor in our model has non-monotonic propensities to realize losses. Related to this result, we also show that the standard double-barrier policy yields an incorrect solution for realization-utility investors. We must use the variational-inequality method to characterize our model solution.

Quantitatively, we find that having access to the risk-free asset is worth about 20-40% of the investor's total wealth in our calibrated diffusion models. Incorporating jumps further significantly enhances the quantitative importance of our model mechanism.

In our model, while the investor can allocate her wealth between the risky and risk-free assets, she only derives utility from realizing gains and losses by trading a single risky asset. In reality, investors hold and trade multiple risky assets, and different assets have different histories of gains and losses. We plan to analyze the investor's optimal strategy when there are two risky assets from which she derives realization utility. This richer setting may allow us to better capture how diversification considerations across risky assets influence dynamic asset allocation and gain/loss realization strategies for investors with realization utility.

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Appendices

A Transversality Conditions

In this appendix, we first provide a transversality condition which ensures that the value function defined in (14) is finite for our diffusion model in Section 2 and then a transversality condition for our jump-diffusion model in Section 7.

A.1 Diffusion Models

For our diffusion model, we impose the following transversality condition:

$$\delta > \beta r + \max\Big(0,\ \frac{1}{2}\sigma^2\alpha_+(\alpha_+ - 1) + (\mu - r)\alpha_+\Big). \tag{A.1}$$

We verify this condition as follows. Under (A.1), the value function $V(W, X, B)$ defined in (14) satisfies

$$V(W, X, B) \le B^\beta\Big[C_0 + C_1\Big(\frac{X}{B}\Big)^{\alpha_+} + C_2\Big(\frac{W}{B} + 1 + \theta_p\Big)^\beta\Big], \tag{A.2}$$

where C_0, C_1, and C_2 are sufficiently large constants. The homogeneity property allows us to equivalently express (A.2) as the following condition:

$$v(w, x) \le \overline{v}(w, x) = C_0 + C_1x^{\alpha_+} + C_2(w + 1 + \theta_p)^\beta. \tag{A.3}$$

To show (A.3), it is sufficient to show that $\overline{v}(w, x)$ is a supersolution of the HJB equation (30), i.e., the following inequality holds:

$$\max\big\{\mathscr{L}\overline{v}(w, x),\ \overline{f}(w, x) - \overline{v}(w, x)\big\} \le 0, \tag{A.4}$$

where the differential operator $\mathscr{L}$ and the function $\overline{f}(w, x)$ are respectively given by

$$\mathscr{L}\overline{v}(w, x) = \frac{\sigma^2}{2}x^2\overline{v}_{xx}(w, x) + (\mu - r)x\overline{v}_x(w, x) - (\delta - \beta r)\overline{v}(w, x), \tag{A.5}$$

$$\overline{f}(w, x) = u((1 - \theta_s)x - 1) + \max_{\widehat{w} \ge -\kappa} \frac{\overline{v}(\widehat{w}, 1)}{[\widehat{w} + 1 + \theta_p]^\beta}\,[w + (1 - \theta_s)x]^\beta. \tag{A.6}$$

Next, we derive (A.4) in two steps.

First, we verify $\overline{f}(w, x) \le \overline{v}(w, x)$ as follows:

$$\begin{aligned}
\overline{f}(w, x) &= u((1 - \theta_s)x - 1) + \max_{\widehat{w} \ge -\kappa} \frac{\overline{v}(\widehat{w}, 1)}{[\widehat{w} + 1 + \theta_p]^\beta}\,[w + (1 - \theta_s)x]^\beta && \text{(A.7)}\\
&\le [(1 - \theta_s)x]^{\alpha_+} + C_2[w + (1 - \theta_s)x]^\beta && \text{(A.8)}\\
&\le C_0 + C_1x^{\alpha_+} + C_2(w + 1 + \theta_p)^\beta = \overline{v}(w, x), && \text{(A.9)}
\end{aligned}$$

where the first line is given in (A.6), the inequality (A.8) follows from (A.3), and the inequality (A.9) follows from the condition $0 < \beta \leq \alpha_+$. Second, substituting the expression given in (A.3) for $\overline{v}(w,x)$ into (A.5), we obtain:

$$\mathscr{L}\overline{v}(w,x) \;=\; h(\alpha_+)C_1x^{\alpha_+} - (\delta - \beta r)[C_0 + C_2(w+1+\theta_p)^\beta] < 0, \qquad \text{(A.10)}$$

where the inequality follows from the result that (A.1) implies $h(\alpha_+) < 0$, where $h(\,\cdot\,)$ is given in (35). We thus have verified that (A.4) holds.

A.2 Jump-diffusion Models

For our jump-diffusion model, we impose the following transversality condition:

$$\delta > \beta r + \max\left(0,\ \frac{1}{2}\sigma^2\alpha_+(\alpha_+ - 1) + (\mu - r)\alpha_+ + \rho\,\mathbb{E}[Y^{\alpha_+} - 1]\right) \qquad \text{(A.11)}$$

by generalizing the transversality condition (A.1) for our diffusion model to account for jump effects. Following essentially the same procedure used to verify the transversality condition for our diffusion model, it is sufficient to show that $\overline{v}(w,x)$ satisfies

$$\max\left\{\mathscr{L}^{\mathcal{J}}\overline{v}(w,x),\ \overline{f}(w,x) - \overline{v}(w,x)\right\} \leq 0, \qquad \text{(A.12)}$$

where $\overline{f}(w,x)$ is given by (A.6) and the differential operator $\mathscr{L}^{\mathcal{J}}$ in (A.12) is related to the differential operator $\mathscr{L}$ defined in (A.5) as follows:

$$\mathscr{L}^{\mathcal{J}}\overline{v}(w,x) = \mathscr{L}\overline{v}(w,x) + \rho\,\mathbb{E}(\overline{v}(Yx) - \overline{v}(x)). \qquad \text{(A.13)}$$

We verify (A.12) as follows. First, using the same argument as we did for the diffusion model, we can show $\overline{f}(w,x) \leq \overline{v}(w,x)$. Second, using substituting the expression given in (A.3) for $\overline{v}(w,x)$ into (A.13), we obtain:

$$\mathscr{L}^{\mathcal{J}}\overline{v}(w,x) = h^{\mathcal{J}}(\alpha_+)C_1x^{\alpha_+} - (\delta - \beta r)[C_0 + C_2(w+1+\theta_p)^\beta] < 0, \qquad \text{(A.14)}$$

where the function $h^{\mathcal{J}}(\,\cdot\,)$ is defined by

$$h^{\mathcal{J}}(\eta) = \frac{1}{2}\sigma^2\eta(\eta-1) + (\mu - r)\eta - (\delta - \beta r + \rho) + \rho\,\mathbb{E}[Y^\eta]. \qquad \text{(A.15)}$$

The inequality in (A.15) follows from the result that (A.11) implies $h^{\mathcal{J}}(\alpha_+) < 0$. We thus have verified that (A.12) holds.

B Procedure for Finding Analytical Solutions in Section 3

We first analyze the $\kappa > 0$ case and then the $\kappa = 0$ case.

B.1 The $\kappa > 0$ Case

We proceed in three steps.

Step 1: Showing $w^* > -\kappa$. The optimal allocation to the risk-free asset w^* upon rebalancing must be in the interior region of w: $w^* > -\kappa$. Otherwise, immediately after rebalancing, the investor has to again incur rebalancing costs, which is suboptimal. Therefore, the following first-order condition for (38) implies:

$$[w^* + 1 + \theta_p][C_1'(w^*) + C_2'(w^*)] - \beta[C_1(w^*) + C_2(w^*)] = 0\,. \tag{B.1}$$

We also check the second-order condition. Recall that $x_L(w)$ is the involuntary loss-realization threshold given by (36), which we use below. We obtain the solution for $\{w^*, C_1(w), C_2(w), \underline{x}(w), \overline{x}(w)\}$ in two steps.

Step 2: Finding the solution for $w = w^*$. Recall that $w_s = w^*$ for all $s \geq t$ where $\{t : \inf_u w_u = w^*\}$. The solution for this case boils down to a one-dimensional problem as we will show. There are two possible scenarios.

- Scenario (i) where the leverage constraint does not bind: $\underline{x}(w^*) > x_L(w^*)$. In this case, the value-matching and smooth-pasting conditions hold at both $x = \underline{x}(w^*)$ and $x = \overline{x}(w^*)$. We thus obtain a candidate solution $\{w^*, C_1(w^*), C_2(w^*), \underline{x}(w^*), \overline{x}(w^*)\}$ by solving a system of five equations: (B.1) and (39)-(42).
- Scenario (ii) where the leverage constraint binds: $\underline{x}(w^*) = x_L(w^*)$. We thus obtain a candidate solution for $\{w^*, C_1(w^*), C_2(w^*), \overline{x}(w^*)\}$ by solving a system of four equations: (B.1) and (39)-(40), as $\underline{x}(w^*) = x_L(w^*)$.

Comparing the candidate solutions for the two scenarios and choosing the one with the larger $v(w^*, 1)$, we obtain the optimal solution.

Step 3: Finding the solution for $w \neq w^*$. For this case, there are two possible scenarios.

For scenario (i) where the leverage constraint does not bind ($\underline{x}(w) > x_L(w)$), using the value-matching and smooth-pasting conditions at $x = \underline{x}(w)$ and $x = \overline{x}(w)$, we derive a candidate solution $\{C_1(w), C_2(w), \underline{x}(w), \overline{x}(w)\}$ by solving a system of four equations: (39)-(42). (This is because we can use the explicit expression for the payoff function $f(w, x)$ obtain from our analysis for $w = w^*$ from Step 2.) For scenario

(ii) where the leverage constraint binds ($\underline{x}(w) = x_L(w)$), we obtain the solution for $\{C_1(w), C_2(w), \overline{x}(w)\}$ by solving a system of three equations (39)-(40). Again, the larger solution for these two scenarios corresponds to the true solution for $w \neq w^*$.

We now have described the procedure for obtaining the optimal solution for the $\kappa > 0$ case where the investor can borrow. Next, we consider the no-borrowing case: $\kappa = 0$.

B.2 The $\kappa = 0$ Case

In this case, since the investor cannot borrow, there is no forced loss realization, which implies $x_L(w) \equiv 0$ and $\mathcal{S} = \{x > 0,\ w > 0\}$. Since it may be optimal to invest all her wealth in the risky asset, there are two possible cases for $w = w^*$: (i) the $w^* > 0$ case where the first-order condition (B.1) holds and (ii) the $w^* = 0$ case. For both cases, we have two scenarios depending on whether the investor is willing to voluntarily realize losses: (a) $\underline{x}(w^*) > 0$ and (b) $\underline{x}(w^*) = x_L(w^*) = 0$. Using the same argument as in the $\kappa > 0$ case, we can obtain the true solution at w^*.

Similarly, for $w \neq w^*$, using essentially the same procedure, we can derive the true solution by comparing the candidate value functions for the two scenarios: (i) $\underline{x}(w) > x_L(w) = 0$ and (ii) $\underline{x}(w) = x_L(w) = 0$.

C Proof of Proposition 1 (Piecewise Linear Realization Utility)

Proof. We prove this proposition for our jump-diffusion model where the investor can save in three steps. First, we show that the investor never realizes losses as in BX (2012): $\underline{x}(w^*) = 0$. Second, we show that the investor has no incentives to save in the risk-free asset as in BX (2012), i.e., $w^* = 0$, despite having the option to do so. Finally, we report the explicit expressions for our solution.

Step 1: Proving $\underline{x}(w^*) = 0$. For piecewise linear $u(\cdot)$ where $\alpha_\pm = \beta = 1$, $f(w^*, x)$ is given by

$$f(w^*, x) = \begin{cases} (1-\theta_s)x - 1 + \frac{v(w^*,1)}{w^*+1+\theta_p}[w^* + (1-\theta_s)x] & \text{if } x \in [1/(1-\theta_s), \infty) \\ -\lambda[1-(1-\theta_s)x] + \frac{v(w^*,1)}{w^*+1+\theta_p}[w^* + (1-\theta_s)x] & \text{if } x \in [0, 1/(1-\theta_s)). \end{cases} \tag{C.1}$$

Note that $f(w^*, x)$ is linear in x with slope $\left[\frac{v(w^*,1)}{w^*+1+\theta_p} + \lambda\right](1-\theta_s)$ in the region where $x \in [1/(1-\theta_s), \infty)$ and $f(w^*, x)$ is linear in x with slope $\left[\frac{v(w^*,1)}{w^*+1+\theta_p} + 1\right](1-\theta_s)$ in the region where $x \in [0, 1/(1-\theta_s))$. Because $\lambda > 1$, the slope for the $x \in [0, 1/(1-\theta_s))$

region is larger than for the $x \in [1/(1-\theta_s), \infty)$ region, $f(w^*, x)$ is increasing and globally concave in x.

Now we prove $\underline{x}(w^*) = 0$ by contradiction. Suppose $0 < \underline{x}(w^*) < \overline{x}(w^*)$. Then the smooth-pasting conditions apply at both $\underline{x}(w^*)$ and $\overline{x}(w^*)$. Using these conditions together with the concavity of $f(w^*, x)$ in x, we obtain

$$0 < v_x(w^*, \overline{x}(w^*)) = f_x(w^*, \overline{x}(w^*)) < f_x(w^*, \underline{x}(w^*)) = v_x(w^*, \underline{x}(w^*)). \tag{C.2}$$

Using the above smooth-pasting conditions at the two boundaries and the result that $v > f$ in the holding region, we know that v_x must attain a local maximum at a point denoted by $\breve{x}$ between the two boundaries: $\breve{x} \in (\underline{x}(w^*), \overline{x}(w^*))$. That is, $v_{xx}(w^*, \breve{x}) = 0 \geq v_{xxx}(w^*, \breve{x})$. As $\mathscr{L}^{\mathcal{J}} v(w^*, x) = 0$ holds in the holding region, differentiating $\mathscr{L}^{\mathcal{J}} v(w^*, x) = 0$ with respect to x where $\mathscr{L}^{\mathcal{J}}$ is given in (57), we obtain the following contradiction at $(w, x) = (w^*, \breve{x})$:

$$\begin{aligned} 0 &= \frac{\sigma^2 x^2}{2} v_{xxx} + (\mu - r + \sigma^2) x v_{xx} - (\delta - \mu + \rho) v_x + \rho\, \mathbb{E}[v_x(w^*, Yx) Y] && \text{(C.3)} \\ &\leq -(\delta - \mu + \rho) v_x + \rho\, \mathbb{E}[Y] v_x && \text{(C.4)} \\ &= -(\delta - \mu + \rho(1 - \mathbb{E}[Y])) v_x < 0\,. && \text{(C.5)} \end{aligned}$$

The inequality (C.4) uses the result that $v_x(w^*, \breve{x})$ is a local maximum, $v_{xxx}(w^*, \breve{x}) \leq 0$, and the last line follows from $v_x \geq 0$ and the following necessary condition:

$$\delta - \mu + \rho(1 - \mathbb{E}[Y]) > 0\,, \tag{C.6}$$

which is also implied by the sufficient condition (A.11).

We thus conclude $\underline{x}(w^*) = 0$, which means that the investor never voluntarily realizes losses.

Step 2: Proving $w^* = 0$. Using the HJB equation $\mathscr{L}^{\mathcal{J}} v(w, x) = 0$ in the holding region, we obtain the following form for the general solution:

$$v(w, x) = C_1(w) x^{\eta_1} + C_2(w) x^{\eta_2} \tag{C.7}$$

where $C_1(\cdot)$ and $C_2(\cdot)$ are two functions of w to be determined, and $\eta_1 > 0$ and $\eta_2 < 0$ are the two roots of the following equation:[33]

$$h^{\mathcal{J}}(\eta) = \frac{1}{2}\sigma^2 \eta(\eta - 1) + (\mu - r)\eta - (\delta - \beta r) + \rho\,(\mathbb{E}[Y^\eta] - 1) = 0. \tag{C.8}$$

[33] For expositional simplicity, in this appendix, we assume that there are two roots for the $h^{\mathcal{J}}(\eta) = 0$ equation. Absent the jump term, $h^{\mathcal{J}}(\eta) = 0$ becomes a quadratic equation with at most two roots. However, in general with jumps, there may be more than two roots. We can solve the model using a method similar to the one here. For our numerical example, we have two roots and our results here apply.

Recall that in Appendix A, we have shown that the smaller root of (C.8) is negative: $\eta_2 < 0$. Next we prove by contradiction. Suppose that $w^* > 0$. Using $\underline{x}(w^*) = 0$ obtained from Step 1 and $\eta_2 < 0$, we obtain $C_2(w) = 0$. Using the first-order condition (B.1) for w^*, we obtain

$$C_1'(w^*)(w^* + 1 + \theta_p) = C_1(w^*)\,. \tag{C.9}$$

The value-matching and smooth-pasting conditions at $x = \overline{x}(w^*)$ imply the following:

$$C_1'(w^*)[\overline{x}(w^*)]^{\eta_1} = \frac{C_1(w^*)}{w^* + 1 + \theta_p}. \tag{C.10}$$

Combining (C.9) and (C.10), we obtain the following contradiction:

$$1 = \overline{x}(w^*) > 1\,, \tag{C.11}$$

We thus have shown $w^* = 0$.

Step 3: Deriving closed-form solutions for $v(0, x)$ and $f(0, x)$. Using the value-matching and smooth-pasting conditions at the optimal gain realization threshold $\overline{x}^*$, we obtain the following results:

$$C_1^*[\overline{x}^*]^{\eta_1} \;=\; (1-\theta_s)\overline{x}^* - 1 + \frac{1-\theta_s}{1+\theta_p}C_1^*\overline{x}^*, \tag{C.12}$$

$$C_1^*\eta_1[\overline{x}^*]^{\eta_1 - 1} \;=\; 1 - \theta_s + \frac{1-\theta_s}{1+\theta_p}C_1^*\,, \tag{C.13}$$

where $\overline{x}^* \equiv \overline{x}(0)$ and $C_1^* \equiv C_1(0)$. Solving (C.12) and (C.13) yields the following equation for the unique root $\overline{x}^* \in (1/(1-\theta_s), \infty)$:

$$(\eta_1 - 1)(\overline{x}^*)^{\eta_1} - \frac{\eta_1}{1-\theta_s}(\overline{x}^*)^{\eta_1 - 1} + \frac{1}{1+\theta_p} = 0 \tag{C.14}$$

and the following expression for the constant C_1^*:

$$C_1^* = \frac{(1-\theta_s)\overline{x}^* - 1}{(\overline{x}^*)^{\eta_1} - (1+\theta_p)^{-1}(1-\theta_s)\overline{x}^*}. \tag{C.15}$$

The scaled value function $v(w, x)$ is thus given by

$$v(w,x) = v(0,x) = \begin{cases} C_1^* x^{\eta_1} & \text{if } x \in [0,\, \overline{x}^*) \\ f(0,x) & \text{if } x \in [\overline{x}^*,\, \infty) \end{cases} \tag{C.16}$$

and the scaled realization payoff function $f(w, x)$ is given by

$$f(w,x) = f(0,x) = \begin{cases} -\lambda + \left(\lambda + \frac{C_1^*}{1+\theta_p}\right)(1-\theta_s)x & \text{if } x \in [0, (1-\theta_s)^{-1}) \\ -1 + \left(1 + \frac{C_1^*}{1+\theta_p}\right)(1-\theta_s)x & \text{if } x \in [(1-\theta_s)^{-1}, \infty) \end{cases}. \tag{C.17}$$

□