

# Long-Run Risk through Consumption Smoothing

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## Abstract

We examine how long-run consumption risk arises endogenously in a standard production economy model where the representative agent has Epstein-Zin preferences. Even when technology growth is i.i.d., optimal consumption smoothing induces highly persistent time-variation in expected consumption growth (long-run risk). This increases the price of risk when investors prefer early resolution of uncertainty, and the model can then account for the low volatility of consumption growth and the high price of risk with a low coefficient of relative risk aversion. The asset price implications of endogenous long-run risk depends crucially on the persistence of technology shocks and investors preference for the timing of resolution of uncertainty.

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# 1 Introduction

Long-run consumption risk has recently been proposed as a mechanism for explaining important asset price moments such as the Sharpe ratio of equity market returns, the equity premium, the level and volatility of the risk free rate and the cross-section of stock returns (see Bansal and Yaron, 2004, Hansen, Heaton and Li, 2005, and Parker and Julliard, 2005). In this paper, we investigate how long-run consumption risk arises *endogenously* in a standard production economy framework and how this additional risk factor can help production economy models to jointly explain the dynamic behavior of consumption, investment and asset prices.<sup>1</sup>

We deviate from the standard production economy model by assuming that consumers have Epstein-Zin preferences.<sup>2</sup> Unlike in the power utility case, where risk is only associated with the shock to *realized* consumption growth, investors with Epstein-Zin preferences also demand a premium for holding assets correlated with shocks to *expected* consumption growth. The latter source of risk has been labelled "long-run risk" in previous literature (Bansal and Yaron, 2004). In production economy models, endogenous long-run risk arises because consumption smoothing induces highly persistent time-variation in expected consumption growth rates. We show that this endogenous long-run risk can substantially increase the price of risk in the economy. The production economy model with Epstein-Zin preferences can then generate a high Sharpe ratio of equity returns with a low volatility of consumption growth and a low coefficient of relative risk aversion.

We calibrate models with either transitory or permanent technology shocks to fit the relative volatility of consumption to output and analyze the equilibrium consumption dynamics. We consider both specifications of technology since both are commonly employed in the macro literature and since the long-run risk implications are contrary. In each case, the endogenous time-variation in expected consumption growth turns out to be a small, but highly persistent, fraction of realized consumption growth, similar to the exogenous processes that have been specified in the recent asset pricing literature (see, e.g., Bansal and Yaron,

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<sup>1</sup>For extensive discussions of the poor performance of standard production economy models in terms of jointly explaining asset prices and macroeconomic moments, refer to Rouwenhorst (1995), Lettau and Uhlig (2000), Uhlig (2004), and Cochrane (2005), amongst others.

<sup>2</sup>Epstein-Zin preferences provide a convenient separation of the elasticity of intertemporal substitution ( $\psi$ ) from the coefficient of relative risk aversion ( $\gamma$ ), which are forced to  $\gamma = \frac{1}{\psi}$  in the power utility case. If  $\gamma > \frac{1}{\psi}$ , investors prefer early resolution of uncertainty and are averse to time-varying expected consumption growth. If  $\gamma < \frac{1}{\psi}$ , investors prefer late resolution of uncertainty and *like* shocks to expected consumption growth.

2004). However, the endogenous correlation between shocks to realized consumption and long-run expected consumption growth is negative in the transitory shock case and positive in the permanent shock case. This correlation and whether the representative agent prefers early or late resolution of uncertainty are crucial for the asset pricing implications of long-run risk in the model.

Consider the case when agents have a preference for early resolution of uncertainty (the relative risk aversion is less than the reciprocal of the elasticity of intertemporal substitution). Then, permanent technology shocks lead to time-varying expected consumption growth that increases the price of risk in the economy, while transitory technology shocks lead to time-varying expected consumption growth that decreases the price of risk. The intuition for this is as follows. A permanent positive shock to productivity implies a permanently higher optimal level of capital. As a result, investors increase investment in order to build up a higher capital stock. High investment today implies low current consumption, but high future consumption. Thus, expected consumption growth is high. Since agents in this economy dislike negative shocks to future economic growth prospects, both shocks to expected consumption growth and realized consumption growth are risk factors. Furthermore, the shocks are positively correlated and thus reinforce each other. Therefore, endogenous consumption smoothing increases the price of risk in the economy if agents have a preference for early resolution of uncertainty and technology shocks are permanent.

If, on the other hand, shocks to technology are transitory, the endogenous long-run risk *decreases* the price of risk in the economy. A transitory, positive shock to technology implies that technology is expected to revert back to its long-run trend. Thus, if realized consumption growth is high, expected future long-run consumption growth is low as consumption also reverts to the long-run trend. The shock to expected future consumption growth is now *negatively* correlated with the shock to realized consumption growth, and the long-run risk component acts as a hedge for shocks to realized consumption.<sup>3</sup> The overall price of risk in the economy is then decreasing in the magnitude of long-run risk. In the case when agents have a preference for *late* resolution of uncertainty - i.e., the elasticity of intertemporal substitution is less than the reciprocal of the coefficient of relative risk aversion - agents *like*

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<sup>3</sup>This description is intentionally loose to emphasize the intuition. The consumption response to transitory technology shocks is often hump-shaped. Thus, a positive shock to realized consumption growth is followed by high expected consumption growth in the near term, but lower expected consumption growth in the long term - the negative correlation arises at lower frequencies. The low frequency effect dominates for standard values of the discount factor and leads to a lower price of risk unless the transitory shocks are extremely persistent.

long-run risk. Now endogenous long-run risk increases the price of risk when technology shocks are transitory and decrease the price of risk when technology shocks are permanent.

We evaluate the quantitative effects of transitory and permanent technology shocks on aggregate macroeconomic and financial moments with calibrated versions of our model. We identify two particularly interesting cases. First, we show that a model with *transitory* technology shocks and a *low* elasticity of intertemporal substitution can jointly explain the low volatility of consumption growth and the high price of risk with a low coefficient of relative risk aversion. This model has a high equity return volatility and risk premium, as in the data. However, the model generates too high volatility in the risk free rate. Second, we show that a model with *permanent* technology shocks and a relatively *high* elasticity of intertemporal substitution also can jointly explain the low volatility of consumption growth and the high price of risk with a low coefficient of relative risk aversion. Furthermore, this model has low volatility in the risk free rate, as in the data. The equity premium, however, is too low in this model. This is due to low capital adjustment costs and counter-cyclical dividends. Neither of the models deliver economically significant time-variation in the price of risk or predictability in excess equity returns. Thus, while endogenous long-run risk can help the standard real business cycle model jointly explain a high price of risk, a low level and volatility of the risk free rate, and a low relative volatility of consumption with a low coefficient of relative risk aversion, the simple benchmark model still has shortcomings.

A key parameter choice in the model relates to the persistence of the technology shocks. We evaluate a distinct prediction arising from this parameter choice regarding the time series properties of consumption growth. The model implies that the ratio of total factor productivity to consumption is a good proxy for the otherwise hard to measure expected consumption growth rate. If technology shocks are permanent, this ratio should forecast long-horizon consumption growth with a positive sign, while the sign should be negative if technology shocks are transitory. We find empirical support for the permanent shock model by showing that the ratio of log total factor productivity to consumption forecasts future consumption growth over long horizons with a positive sign in post-war U.S. data. The empirical evidence is significant at the 10% level, but not at the 5% level. We show that the marginal significance of the empirical evidence is consistent with the amount of consumption predictability implied by the permanent shock model, although this model on average implies more consumption predictability than what we find in the data. On the other hand, we can reject the pure transitory shock model at the 5% level based on the predictability regressions. This is in line with the evidence presented in Alvarez and Jermann (2005), who find that

permanent shocks are necessary to explain asset prices.

We suggest extensions of the model that increase the equity premium and generate excess return predictability, respectively. First, we note that the wage dynamics of the models are counter-factual and suggest that a more realistic interaction of labor and asset markets in these models is a fruitful avenue of future research. In particular, when we calibrate a more realistic, sticky wage process to the permanent shock model, dividends become more procyclical and the equity premium increases *by an order of magnitude*. In particular, including a reduced form specification for financial leverage, we achieve an annual equity premium of 4.6%. Second, we show how a model with *both* transitory and permanent technology shocks can be calibrated to deliver a time-varying price of risk and long-horizon predictability in equity returns even when aggregate technology shocks are homoskedastic. We further show that a model with asymmetric adjustment costs also can generate predictability of aggregate excess equity returns consistent with that found in the data.

We proceed as follows. We start by providing an overview of related literature. Then we develop and interpret the model. In Section 4 we calibrate and solve the model, demonstrate and interpret results, and provide intuition. In Section 5 we test an empirical implication of our model. Section 6 considers extensions of the model, while section 7 concludes.

## 2 Related Literature

This paper is mainly related to three strands of the literature: the literature on consumption smoothing, the literature on long-run risk, and the literature that aims to jointly explain macroeconomic aggregates and asset prices.

It is well-known that agents optimally smooth consumption over time (see, e.g., Friedman, 1957, and Hall, 1978). Time-variation in expected growth rates, arising from consumption smoothing in production economy models, has also been pointed out before. For example, Den Haan (1995) demonstrates that the risk free rate in production economy models is highly persistent (close to a random walk) even when the level of technology is i.i.d. Campbell (1994) solves a log-linear approximation to the standard real business cycle model with power utility preferences, which provides an analytical account of how the optimal consumption-savings decision induces time-varying expected consumption growth in this model. Relative to Campbell (1994), we show that the equilibrium level of time-variation in expected consumption growth induces large variation in the price of risk as we vary the

agent's preference for the timing of resolution of uncertainty. For instance, even though an agent with a preference for early resolution of uncertainty dislikes time-variation in expected consumption growth, such time-variation is still the equilibrium outcome. I.e., the agent does not engineer an equilibrium consumption process without long-run risk even though she would like to, all else equal. In Campbell (1994) the representative agent is indifferent to the timing of resolution of uncertainty, and therefore does not have this reason to engineer a consumption process with minimal time-variation in expected consumption growth.

Bansal and Yaron (2004) show that a small, persistent component of consumption growth can have quantitatively important implications for asset prices if the representative agent has Epstein-Zin preferences. Bansal and Yaron term this source of risk "long-run risk" and show that it can explain many aspects of asset prices. They specify exogenous processes for dividends and consumption with a slow-moving expected growth rate component and demonstrate that the ensuing long-run consumption risk greatly improves their model's performance with respect to asset prices without having to rely on, e.g., habit formation and the high relative risk aversion such preferences imply. We show that a consumption process similar to what Bansal and Yaron assume can be generated endogenously in a standard production economy model with Epstein-Zin preferences and the same preference parameters as in Bansal and Yaron (2004). The model thus provides a theoretical justification for the existence of long-run consumption risk, which it is difficult to establish empirically as pointed out by Harvey and Shepard (1990) and Hansen, Heaton and Li (2005).

A recent paper that generates interesting consumption dynamics is due to Panageas and Yu (2006). These authors focus on the impact of major technological innovations and real options on consumption and the cross-section of asset prices. These innovations are assumed to occur at a very low frequency (about 20 years), and are shown to carry over into a small, highly persistent component of aggregate consumption. In that sense, Panageas and Yu assume, contrary to us, the frequency of the predictable component of consumption growth. Moreover, time-variation in expected consumption growth (long-run risk) is not itself a priced risk factor in the Panageas and Yu model because the representative agent does not have Epstein-Zin preferences, but external ratio-habit as in Abel (1990). Panageas and Yu require that investment is irreversible, whereas we allow for a convex adjustment cost function. Also, since investment in their model means paying a "gardener" to plant a tree, their model does not have a clear separation of investment and labor income.

Parker and Julliard (2005) find that the Consumption CAPM can empirically explain

a large fraction of the cross-sectional dispersion in average excess stock returns only when consumption growth is measured over longer horizons. This is consistent with the presence of long-run risks. Bansal, Kiku and Yaron (2006) explicitly test and find considerable support for the long-run risk model in the cross-section of stock returns.

There are quite a few papers before Bansal and Yaron (2004) that emphasize a small, highly persistent component in the pricing kernel. An early example is Backus and Zin (1994) who use the yield curve to reverse-engineer the stochastic discount factor and find that it has high conditional volatility and a persistent, time-varying conditional mean with very low volatility. These dynamics are also highlighted in Cochrane and Hansen (1992). The model considered in this paper can generate such dynamics, and as such the paper complements the above earlier studies. The use of Epstein-Zin preferences provides a justification for why the small, slow-moving time-variation in expected consumption growth generates high volatility of the stochastic discount factor. These preferences have become increasingly popular in the asset pricing literature. By providing a convenient separation between the coefficient of relative risk aversion and the elasticity of intertemporal substitution, they help to jointly explain asset market data and aggregate consumption dynamics. An early implementation is Epstein and Zin (1991), while Malloy, Moskowitz and Vissing-Jorgensen (2005) and Yogo (2006) are more recent, successful examples.

This paper contributes to the literature Cochrane (2005) terms ‘production-based asset pricing’. This literature tries to jointly explain the behavior of macroeconomic time series, in particular aggregate consumption, and asset prices. The starting point of this literature is the standard production economy model (standard stochastic growth model) and the observation that this model, while being able to generate realistic processes for consumption and investment, fails markedly at explaining asset prices.<sup>4</sup>

Both Jermann (1998) and Boldrin, Christiano, Fisher (2001) augment the basic production economy framework with habit preferences in order to remedy its shortcomings. Boldrin, Christiano, Fisher also assume a two-sector economy with adjustment frictions across sectors and across time. Boldrin, Christiano, Fisher furthermore endogenize the labor-leisure decision, they assume however that labor can not be adjusted immediately in response to technology shocks. Jermann, and in particular Boldrin, Christiano, Fisher, succeed to a considerable extent to jointly explain with their models macroeconomic time series and as-

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<sup>4</sup>Cochrane (2005): "[Jermann (1998)] starts with a standard real business cycle (one sector stochastic growth) model and verifies that its asset-pricing implications are a disaster."

set prices. However, typical for standard internal habit specifications, both models display excessive volatility of the risk free rate and very high levels of risk aversion.

Tallarini (2000) proposes a model that is closely related to our setup. Tallarini restricts himself to a special case of our model with the elasticity of intertemporal substitution fixed at unity and no capital adjustment costs. By increasing the coefficient of relative risk aversion to very high levels, Tallarini manages to match some asset pricing moments such as the market price of risk (Sharpe ratio) as well as the level of the risk free rate, while equity premium and return volatilities in his model remain basically zero. We differ from Tallarini in that our focus is on changing the elasticity of intertemporal substitution and the implications for the existence and pricing of long-run risk. Relative to the Tallarini setup we show that (moderate) capital adjustment costs together with an elasticity of intertemporal substitution different from unity can dramatically improve the model's ability to match asset pricing moments. We confirm Tallarini's conclusion that the behavior of macroeconomic time series is driven by the elasticity of intertemporal substitution and largely unaffected by the coefficient of relative risk aversion. However, we do not confirm a "separation theorem" of quantity and price dynamics. When we change the elasticity of intertemporal substitution in our model, both macroeconomic quantity and asset price dynamics are greatly affected.

In recent research, Croce (2007) investigates the welfare implications of long-run risk in a general equilibrium production economy similar to the one we analyze. Finally, Campanale, Castro, and Clementi (2007) look at asset prices in general equilibrium production economies where the representative agent's preferences are in the Chew-Dekel class. Contrary to us, they do not consider the role of long-run risk.

### **3 The Model**

The model is a standard real business cycle model (Kydland and Prescott, 1982, and Long and Plosser, 1983). There is a representative firm with Cobb-Douglas production technology and capital adjustment costs, and a representative agent with Epstein-Zin preferences. Our objective is to demonstrate how standard production economy models endogenously give rise to long-run consumption risk and that this long-run risk can improve the performance of these models in explaining important moments of asset prices. To that end we keep both production technology as well as the process for total factor productivity as simple and as standard as possible. We describe the key components of our model in turn.



**The Representative Agent.** We assume a representative household whose preferences are in the recursive utility class of Epstein and Zin (1989):

$$U_t(C_t) = \left\{ (1 - \beta) C_t^{\frac{1-\gamma}{\theta}} + \beta (E_t [U_{t+1}^{1-\gamma}])^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}}, \quad (1)$$

where  $E_t$  denotes the expectation operator,  $C_t$  denotes aggregate consumption,  $\beta$  the discount factor, and  $\theta = \frac{1-\gamma}{1-1/\psi}$ . Epstein and Zin show that  $\gamma$  governs the coefficient of relative risk aversion and  $\psi$  the elasticity of intertemporal substitution. These preferences thus have the useful property that it is possible to separate the agent's relative risk aversion from the elasticity of intertemporal substitution, unlike the standard power utility case where  $\gamma = \frac{1}{\psi}$ . If  $\gamma \neq \frac{1}{\psi}$ , the utility function is no longer time-additive and agents care about the temporal distribution of risk - a feature that is central to our analysis. We discuss this property and its implications in more detail below.

**The Stochastic Discount Factor and Risk.** The stochastic discount factor,  $M_{t+1}$ , is the ratio of the representative agent's marginal utility between today and tomorrow:  $M_{t+1} = \frac{U'(C_{t+1})}{U'(C_t)}$ . Using a recursive argument, Epstein and Zin (1989) show that:

$$\ln M_{t+1} \equiv m_{t+1} = \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1 - \theta) r_{a,t+1}, \quad (2)$$

where  $\Delta c_{t+1} \equiv \ln \frac{C_{t+1}}{C_t}$  and  $r_{a,t+1} \equiv \ln \frac{A_{t+1} + C_{t+1}}{A_t}$  is the return on the total wealth portfolio with  $A_t$  denoting total wealth at time  $t$ .<sup>5</sup> If  $\gamma = \frac{1}{\psi}$ ,  $\theta = \frac{1-\gamma}{1-1/\psi} = 1$ , and the stochastic discount factor collapses to the familiar power utility case, where shocks to realized consumption growth are the only source of risk in the economy. However, if  $\gamma \neq \frac{1}{\psi}$ , the return on the wealth portfolio appears as a risk factor. Persistent time-variation in expected consumption growth (the expected "dividends" on the total wealth portfolio) induces higher volatility of asset returns (Barsky and DeLong, 1993). Thus, the return on any asset is a function of the dynamic behavior of realized *and* expected consumption growth (Bansal and Yaron, 2004). Depending on the sign of  $\theta$  and the covariance between realized consumption growth and the return on the total wealth portfolio, the volatility of the stochastic discount factor (i.e., the price of risk in the economy) can be higher or lower relative to the benchmark power utility case (see the appendix for further discussion). We show later how this covariance, and thus

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<sup>5</sup>Note that our representative household's total wealth portfolio is composed of the present value of future labor income in addition to the value of the firm.

the amount of long-run risk due to endogenous consumption smoothing, changes with the persistence of the technology shock.

**Technology.** There is a representative firm with a Cobb-Douglas production technology:

$$Y_t = (Z_t H_t)^{1-\alpha} K_t^\alpha, \quad (3)$$

where  $Y_t$  denotes output,  $K_t$  the firm's capital stock,  $H_t$  the number of hours worked, and  $Z_t$  denotes the (stochastic) level of aggregate technology. This constant returns to scale and decreasing marginal returns production technology is standard in the macroeconomic literature. Since we assume leisure not to enter the utility function, households incur no disutility of working and supply a constant amount of hours worked (as in, e.g., Jermann, 1998). We normalize  $H_t = 1$ . The productivity of capital and labor depends on the level of technology,  $Z_t$ , which is the exogenous driving process of the economy. We model log technology,  $z \equiv \ln(Z)$ , as:

$$z_t = \mu t + \tilde{z}_t, \quad (4)$$

$$\tilde{z}_t = \varphi \tilde{z}_{t-1} + \sigma_z \varepsilon_t, \quad (5)$$

$$\varepsilon_t \sim N(0, 1), \quad |\varphi| \leq 1. \quad (6)$$

Thus, (5) implies that technology shocks are permanent if  $\varphi = 1$  and transitory if  $\varphi < 1$ . Both specifications are commonly used in the literature.<sup>6</sup> We discuss these two cases separately because they have very different implications for asset prices and macroeconomic time series.

**Capital Accumulation and Adjustment Costs.** The agent can shift consumption from today to tomorrow by investing in capital. The firm accumulates capital according to the following law of motion:

$$K_{t+1} = (1 - \delta) K_t + \phi\left(\frac{I_t}{K_t}\right) K_t, \quad (7)$$

where  $I_t$  is aggregate investment and  $\phi(\cdot)$  is a positive, concave function, capturing the notion that adjusting the capital stock rapidly by a large amount is more costly than adjusting it

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<sup>6</sup>See, for example, Campbell (1994), who considers permanent and transitory, Cooley and Prescott (1995), transitory, Jermann (1998), permanent and transitory, Prescott (1986), permanent, Rouwenhorst (1995), permanent and transitory.

step by step. We follow Jermann (1998) and Boldrin, Christiano, Fisher (1999) and specify:

$$\phi(I_t/K_t) = \frac{\alpha_1}{1 - 1/\xi} \left( \frac{I_t}{K_t} \right)^{(1-1/\xi)} + \alpha_2, \quad (8)$$

where  $\alpha_1, \alpha_2$  are constants and  $\alpha_1 > 0$ . The adjustment cost specification implies that equilibrium aggregate investment is positive, and  $\alpha_1$  and  $\alpha_2$  are set so that the firm does not incur adjustment costs when investing at the steady state rate.<sup>7</sup> The parameter  $\xi$  is the elasticity of the investment-capital ratio with respect to Tobin's  $q$ . If  $\xi = \infty$  the capital accumulation equation reduces to the standard growth model accumulation equation without capital adjustment costs.

Each period the firm's output,  $Y_t$ , can be used for either consumption or investment. Investment increases the firm's capital stock, which in turn increases future output. High investment, however, means the agent must forego some consumption today ( $C_t = Y_t - I_t$ ).

**The Return on Investment and the Firm's Problem.** Let  $\Pi(K_t, Z_t; W_t)$  be the operating profit function of the firm, where  $W_t$  are equilibrium wages.<sup>8</sup> Firm dividends,  $D_t$ , equal operating profits minus investment:

$$D_t = \Pi(K_t, Z_t; W_t) - I_t. \quad (9)$$

The firm maximizes firm value. Let  $M_{t,t+1}$  denote the stochastic discount factor. The firm's problem is then:

$$\max_{\{I_t, K_{t+1}, H_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} M_{0,t} D_t, \quad (10)$$

where  $E_t$  denotes the expectation operator conditioning on information available up to time  $t$ . In the appendix, we demonstrate that the return on investment can be written as:

$$R_{t+1}^I = \phi' \left( \frac{I_t}{K_t} \right) \left( \Pi_K(K_{t+1}, Z_{t+1}; W_{t+1}) + \frac{1 - \delta + \phi \left( \frac{I_{t+1}}{K_{t+1}} \right)}{\phi' \left( \frac{I_{t+1}}{K_{t+1}} \right)} - \frac{I_{t+1}}{K_{t+1}} \right). \quad (11)$$

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<sup>7</sup>In particular, we set  $\alpha_1 = (\exp(\mu) - 1 + \delta)^{1/\xi}$  and  $\alpha_2 = \frac{1}{\xi-1} (1 - \delta - \exp(\mu))$ . It is straightforward to verify that  $\phi(\frac{I_t}{K_t}) > 0$  and  $\phi''(\frac{I_t}{K_t}) < 0$  for  $\xi > 0$  and  $\frac{I_t}{K_t} > 0$ . Furthermore,  $\phi(\frac{I}{K}) = \frac{I}{K}$  and  $\phi'(\frac{I}{K}) = 1$ , where  $\frac{I}{K} = (\exp(\mu) - 1 + \delta)$  is the steady state investment-capital ratio. Investment is always positive since the marginal cost of investing goes to infinity as investment goes to zero.

<sup>8</sup>Wages are in the first part of the paper assumed to be the marginal productivity of labor:  $W_t = (1 - \alpha) Y_t$ . Since  $C_t = D_t + W_t$ , it follows that  $D_t = \alpha Y_t - I_t$ .

**Table 1**  
**Calibration**

Table 1: Calibrated values of parameters that are constant across models.

Quarterly Model Calibration		
Parameter	Description	Value
$\alpha$	Elasticity of capital	0.36
$\delta$	Depreciation rate of capital	0.021
$\gamma$	Coefficient of relative risk aversion	5
$\mu$	Mean technology growth rate	0.4%
$\varphi$	Persistence of the technology shock	{0.95, 1}

This return to the firm’s investment is equivalent to the firm’s equity return in equilibrium,  $R_{t+1}^E \equiv \frac{D_{t+1} + P_{t+1}}{P_t}$ , where  $P_t$  denotes the net present value of a claim on all future dividends (see, e.g., Restoy and Rockinger, 1994, and Zhang, 2005).<sup>9</sup>

## 4 Results

The model generates macroeconomic aggregates such as output, investment, and consumption, in addition to the standard financial moments. In the first part of the analysis, we present two long-run risk calibrations of the model corresponding to transitory and permanent technology shocks, respectively. Then we explain the intuition for the endogenous long-run risk we document by analyzing the endogenous, dynamic behavior of consumption growth. Our discussion is centered around different values of the elasticity of intertemporal substitution and the two specifications of technology (permanent vs. transitory). We subsequently discuss the asset pricing implications of the model in more detail. We solve the model numerically by means of the value function iteration algorithm. Please refer to the appendix for a detailed discussion of our solution technique.

### 4.1 Calibration

We report calibrated values of model parameters that are constant across models in Table 1. The capital share ( $\alpha$ ), the depreciation rate ( $\delta$ ), and the mean technology growth rate

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<sup>9</sup>In particular, the production function and implied adjustment cost function satisfy Proposition 1 of Restoy and Rockinger (1994).

( $\mu$ ) are set to standard values for quarterly parametrizations (see, e.g., Boldrin, Christiano, and Fisher, 2001). We consider two values for the persistence of the technology shocks,  $\varphi \in \{0.95, 1\}$ , which are both commonly used in the literature.<sup>10</sup>

We set the coefficient of relative risk aversion ( $\gamma$ ) to 5 across all models in the main part of the paper. This value is in the middle of the range of reasonable values for the coefficient of relative risk aversion, as suggested by Mehra and Prescott (1985). The focus of our paper is on the role of endogenous long-run risk arising from optimal consumption smoothing, which in turn is largely determined by the elasticity of intertemporal substitution. Tallarini (2000) analyses a similar model (without capital adjustment costs), but instead assumes a fixed elasticity of intertemporal substitution and varies the level of relative risk aversion. He finds, and we confirm his finding (see appendix), that macroeconomic time series are almost unaffected by the level risk aversion, holding the elasticity of intertemporal substitution constant.

We vary the elasticity of intertemporal substitution ( $\psi$ ) across models and use the capital adjustment costs ( $\xi$ ) to fit (if possible) the relative volatility of consumption to output. The discount factor ( $\beta$ ) is set to match the level of the risk free rate. We vary the volatility of technology shocks ( $\sigma_z$ ) in order to fit the empirical consumption growth volatility with all models. We discuss the choice of specific parameter values for these variables as we go along.

## 4.2 Two Models with Long-Run Risk

We preview our results by showing two calibrations of the model which both, because of endogenous long-run risk, can match the low volatility of consumption growth and the high price of risk with a low coefficient of relative risk aversion. The key distinctions between the models are the difference in the elasticity of intertemporal substitution and the persistence of the technology shocks. The transitory technology shock model has a low (0.05) elasticity of intertemporal substitution, whereas the permanent shock model has a high (1.5) elasticity of intertemporal substitution.

Panel A of Table 2 shows that both models match the volatility of consumption, the relative volatility of consumption to output, the level of the risk free rate, and the Sharpe ratio of the aggregate claim to dividends. The latter fact is remarkable with a coefficient of relative risk aversion of only 5! With a consumption volatility of 2.72%, a power utility model would give a Sharpe ratio of only 0.14, whereas both calibrations of the Epstein-

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<sup>10</sup>See Prescott (1986) for a discussion of the empirical persistence of Solow residuals.

**Table 2**  
**Two Models with Long-Run Risk**

Table 2: This table reports key annualized moments for two calibrations of the stochastic growth model where the representative agent has Epstein-Zin preferences and there are adjustment costs to capital. The models have permanent and transitory technology shocks, respectively. The level of risk aversion ( $\gamma$ ) is 5 in both models, and the volatility of shocks to technology,  $\sigma_z$ , is the same for both models. The volatility of shocks to technology is calibrated such that the models fit the volatility of consumption growth. Both models are calibrated to match the relative volatility of consumption to output, the volatility of output, the level of the risk free rate, and the Sharpe ratio of equity returns. The equity returns in both models are for an unlevered claim on the endogenous, aggregate dividends. The equity premium due to "short-run" risk is defined as  $\gamma cov(\Delta c_t, R_t^E - R_{f,t})$ . The empirical moments are taken from the annual U.S. sample from 1929-1998, corresponding to the sample in Bansal and Yaron (2004).

Statistic	U.S. Data 1929 – 1998	Transitory shocks: $\varphi = 0.95$	Permanent shocks: $\varphi = 1$
		Long-Run Risk I $\psi = 0.05, \gamma = 5$ $\beta = 1.064, \xi = 0.52$	Long-Run Risk II $\psi = 1.5, \gamma = 5$ $\beta = 0.998, \xi = 18$
<b>Panel A - Calibrated Moments</b>			
<i>Volatility of Consumption Growth</i> $\sigma [\Delta c]$ (%)	2.72	2.72	2.72
<i>Relative Volatility of Consumption and Output (GDP)</i> $\sigma [\Delta c] / \sigma [\Delta y]$	0.52	0.52	0.52
<i>Level of Risk Free Rate</i> $E [R_f]$ (%)	0.86	0.85	0.82
<i>Sharpe ratio of Equity Returns</i> $E [R^E - R_f] / \sigma [R^E - R_f]$	0.33	0.34	0.36
<b>Panel B - Other Moments</b>			
<i>Volatility of the Risk Free Rate</i> $\sigma [R_f]$ (%)	0.97	4.60	0.45
<i>Equity Returns</i> $E [R^E - R_f]$ (%)	6.33	8.06	0.24
$\sigma [R^E - R_f]$ (%)	19.42	24.06	0.66
<i>Decomposing the Equity Premium (%)</i>			
	<i>Short-Run Risk</i>	3.27 (41%)	0.09 (38%)
	<i>Long-Run Risk</i>	4.79 (59%)	0.15 (62%)

Zin model match the sample annual Sharpe ratio of 0.33. Thus, both models generate a substantial amount of endogenous long-run risk in the consumption process, although the way in which they do so is quite different. In particular, in the model with a low EIS, the representative agent has a preference for late resolution of uncertainty ( $\gamma < \frac{1}{\psi}$ ), whereas in the model with a high EIS, the representative agent has a preference for early resolution of uncertainty ( $\gamma > \frac{1}{\psi}$ ). It is important to note that the transitory shock model matches the risk free rate by specifying a discount rate ( $\beta$ ) greater than one. Prices in this economy are still well-defined, however, since the economy is growing (see Kocherlakota, 1990). One may principally object to a value of  $\beta$  greater than one. If we were to restrict  $\beta < 1$ , the risk free rate in the transitory shock model would increase to over 25% on an annual basis (the risk free rate puzzle), since the EIS is low. The permanent shock model, however, has  $\beta = 0.998$ , so it is not subject to this problem.

Panel B shows financial moments the models were not calibrated to fit. The transitory shock model displays too high volatility of the risk free rate, since agents in this economy are very unwilling to substitute consumption across time. The equity claim is defined as the (unlevered) claim to aggregate dividends. The equity return volatility is higher than in the data and the equity premium is therefore also higher. This is quite the opposite of what we are used to from production economies, which are generally deemed to not be able to produce any kind of sizeable equity premium. The reason is that capital adjustment costs are set quite high in this economy to fit the relative volatility of consumption growth to output growth. Panel B reports that about 60% of the risk premium is due to long-run risk, where short-run risk is defined as  $\gamma Cov(R_t^E - R_{f,t}, \Delta c_t)$ .

In the permanent shock model, the risk free rate displays low volatility, as in the data, despite the high price of risk. This feature is a marked improvement over habit formation models in production economies, which can match the price of risk, but generate much too volatile risk free rates (see, e.g., Jermann, 1998, and Boldrin, Christiano, and Fisher, 2001). Since the reciprocal of the risk free rate is the conditional expectation of the stochastic discount factor, mismatching the risk free rate volatility implies mismatching the dynamic behavior of the stochastic discount factor. In this model, however, the equity return volatility and therefore the risk premium are too low. This is because capital adjustment costs must be very low in order to match the relative volatility of consumption growth when technology shocks are permanent. Again, about 60% of the risk premium is due to long-run risk.

In sum, *both* models generate substantial amount of long-run risk in the endogenous consumption process. Over the next sections, we analyze the mechanisms within the model

that give rise to these results. We also suggest a simple extension of the permanent shock model which increases the equity premium in this model by an order of magnitude.

### 4.3 The Endogenous Consumption Choice and The Price of Risk

Before we report moments from different calibrations of the model, it is useful to provide some general intuition for the endogenous consumption choice and how it is related to the persistence of the technology shocks and the price of risk in the economy. From the stochastic discount factor (see eq. (2)), we can see that there are two sources of risk in this economy. The first is the shock to realized consumption growth, which is the usual risk factor in the Consumption CAPM. The second risk factor is the shock to the return to total wealth. Total wealth is the sum of human and financial capital, and the dividend to total wealth is consumption. Assume for the moment that future expected consumption growth and returns are constant. Total wealth,  $A_t$ , is then given by:

$$A_t = \frac{C_t}{r_a - g_c}, \quad (12)$$

where  $r_a$  is the expected return to wealth and  $g_c$  is expected consumption growth: Total wealth is a function of both current and future expected consumption. Therefore, shocks to both realized and expected consumption growth translate into shocks to the realized return to wealth. This example illustrates how we can think of shocks to expected consumption growth as the second risk factor instead of the return to wealth.<sup>11</sup> Understanding the dynamic behavior of consumption growth is thus necessary in order to understand the asset pricing properties of the production economy model with Epstein-Zin preferences. In the following, we consider the consumption response to both transitory and permanent technology shocks.<sup>12</sup>

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<sup>11</sup>Following Bansal and Yaron (2004), we explicitly show this in the appendix through a log-linear approximation of the return to wealth.

<sup>12</sup>We make a strong distinction between transitory and permanent shocks in this section in order to provide clear intuition. As  $\varphi \rightarrow 1$ , the transitory shock specification (5) approaches the permanent shock specification (4). The dynamics of the model are in that case very similar for both specifications, so there is actually no discontinuity at  $\varphi = 1$  in terms of the model's asset pricing implications. However, the transitory shocks need to be extremely persistent for the transitory and permanent cases to be similar. At  $\varphi = 0.95$ , which is the case we consider in our calibration, the dynamic behavior of the model with permanent shocks is very different from the model with transitory shocks. The reader could therefore think of "transitory vs. permanent" shocks as "not extremely persistent vs. extremely persistent" shocks.



Figure 1 - Transitory and Permanent Shocks

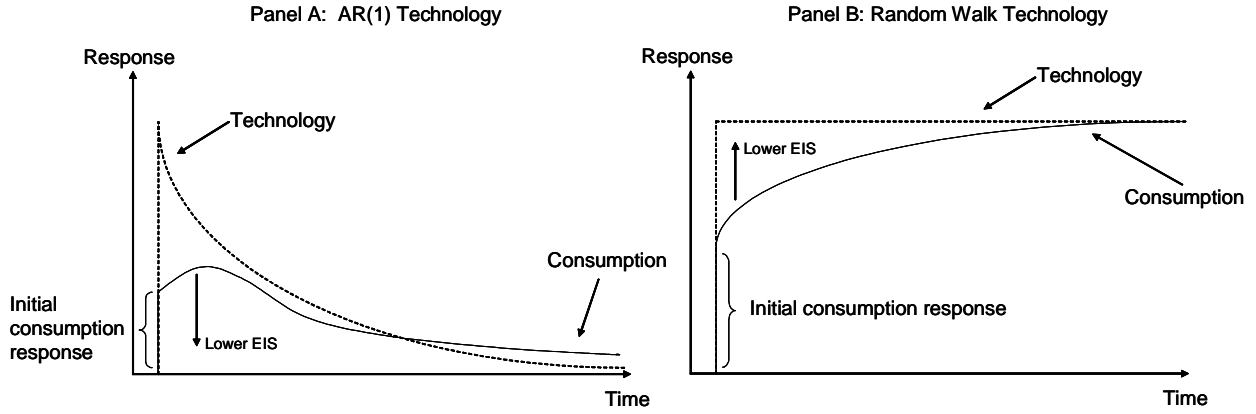


Figure 1: **Impulse-Responses for Technology and Consumption.** Panel A shows the impulse-response of technology and consumption to a transitory technology shock. Panel B shows the impulse-response of technology and consumption to a permanent technology shock. The arrows show the direction in which the optimal consumption response changes if the desire for a smoother consumption path increases (i.e., the elasticity of intertemporal substitution decreases).

**Transitory Technology Shocks.** Panel A of Figure 1 shows the impulse-response functions of technology and consumption to a transitory technology shock. Agents in this economy want to take advantage of the temporary increase in the productivity of capital due to the temporarily high level of technology. To do so, they invest immediately in capital at the expense of current consumption. As a result, the consumption response is hump-shaped. This figure illustrates how time-varying expected consumption growth arises endogenously in the production economy model: A *positive* shock to realized consumption growth (the initial consumption response) is associated with positive short-run expected consumption growth, but *negative* long-run expected consumption growth as consumption reverts back to the steady state. Thus, the shock to long-run expected consumption growth is negatively correlated with the shock to realized consumption growth.

**Permanent Technology Shocks.** With permanent technology shocks, long-run consumption risk has the opposite effect. Panel B of Figure 1 shows the impulse-response functions of technology and consumption to a permanent technology shock. Technology adjusts immediately to the new steady state, and the permanently higher productivity of capital implies that the optimal long-run levels of both capital and consumption are also higher. Agents invest immediately in order to build up capital at the expense of current

consumption, and consumption gradually increases towards the new steady state after the initial shock. Thus, a positive shock to realized consumption growth (the initial consumption response) is associated with *positive* long-run expected consumption growth. In this case, the two shocks are therefore positively correlated.

**The Elasticity of Intertemporal Substitution.** The elasticity of intertemporal substitution (*EIS*) is an important determinant of the dynamic behavior of consumption growth. A low *EIS* translates into a strong desire for intertemporally smooth consumption paths. In other words, agents strive to minimize the difference between their level of consumption today (after the shock) and future expected consumption levels. The arrows in Figure 1 indicate the directions in which the initial optimal consumption responses change if the desire for a smoother consumption path increases. As the elasticity of intertemporal substitution decreases, agents desire a "flatter" response curve. From the figure, we can conjecture that a lower *EIS* decreases the volatility of expected future consumption growth. A high *EIS*, on the other hand, implies a higher willingness to substitute consumption today for higher future consumption levels. Therefore, the higher the *EIS*, the higher the volatility of expected consumption growth.

**Capital Adjustment Costs.** Capital adjustment costs (*CAC*) make it more costly for firms to adjust investment. Therefore, higher *CAC* induce lower investment volatility. We can therefore use *CAC* in order to, as far as possible, match the empirical relative volatilities of consumption, investment, and output with each model.

**Implications for the Price of Risk.** The log return to wealth can be written as:

$$r_{a,t+1} = \Delta c_{t+1} + \tilde{r}_{a,t+1}, \quad (13)$$

where  $\tilde{r}_{a,t+1} = \log\left(1 + \frac{A_{t+1}}{C_{t+1}}\right) - \log\frac{A_t}{C_t}$ . Shocks to this "adjusted" wealth return reflect only updates in expectations about future consumption growth and discount rates (see, e.g.,

Campbell and Shiller, 1988). Now, write the stochastic discount factor as:

$$\begin{aligned}
m_{t+1} &= \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{a,t+1} \\
&= \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) \Delta c_{t+1} + (\theta - 1) \tilde{r}_{a,t+1} \\
&= \theta \ln \beta - \gamma \Delta c_{t+1} + (\theta - 1) \tilde{r}_{a,t+1}.
\end{aligned} \tag{14}$$

Since there is only one shock in this economy (the technology shock), we can write:

$$\Delta c_{t+1} = E_t [\Delta c_{t+1}] + \sigma_{\Delta c,t} \varepsilon_{t+1}, \tag{15}$$

and:

$$\tilde{r}_{a,t+1} = E_t [\tilde{r}_{a,t+1}] + \sigma_{\tilde{r},t} \varepsilon_{t+1}, \tag{16}$$

where  $\varepsilon_{t+1}$  is the technology shock. Note that the parameters  $\sigma_{\Delta c,t}$  and  $\sigma_{\tilde{r},t}$  multiplying the technology shock can be positive or negative, depending on the correlation. Innovations to the log stochastic discount factor are then:

$$m_{t+1} - E_t [m_{t+1}] = (-\gamma \sigma_{\Delta c,t} + (\theta - 1) \sigma_{\tilde{r},t}) \varepsilon_{t+1}. \tag{17}$$

Define the price of risk as the conditional volatility of the log stochastic discount factor:

$$\begin{aligned}
\Lambda_t &= \gamma \sigma_{\Delta c,t} + (1 - \theta) \sigma_{\tilde{r},t} \\
&= \gamma \sigma_{\Delta c,t} + (\gamma - 1/\psi) (1 - 1/\psi)^{-1} \sigma_{\tilde{r},t}.
\end{aligned} \tag{18}$$

In our calibrations, shocks to consumption growth are positively correlated with shocks to technology. Therefore, we have  $\sigma_{\Delta c,t} > 0$ . Below we consider the nature of the price of risk in this model for different attitudes to the resolution of uncertainty and different persistence of the technology shocks.

1. If  $\gamma = \frac{1}{\psi}$ , the preferences are standard additive expected utility:  $\theta = 1$  and  $\Lambda_t = \gamma \sigma_{\Delta c,t}$ .
2. If consumption growth is i.i.d. then, regardless of the preference parameters,  $\sigma_{\tilde{r},t} = 0$  and  $\Lambda_t = \gamma \sigma_{\Delta c,t}$ , as in the power utility model (in this case, both growth rates and discount rates are constant, so the wealth-consumption ratio is constant).
3. Now we turn to the relevant case where consumption growth is not i.i.d. and agents

are not indifferent to the timing of the resolution of uncertainty.

Whether the wealth-consumption ratio responds positively or negatively to technology shocks (i.e., whether  $\sigma_{\tilde{r},t} \leq 0$ ), depends on both the persistence of the shocks and on whether the substitution or the income effect dominates (i.e., whether  $\psi \geq 1$ ). Furthermore,  $(1 - 1/\psi)$  also changes sign depending on whether  $\psi \geq 1$ . Thus, the product  $(1 - 1/\psi)^{-1} \sigma_{\tilde{r},t}$  only depends on the persistence of the technology process. We use this observation to consider four general cases with different implications for the price of risk:

(a) Agents prefer early resolution of uncertainty ( $\gamma > 1/\psi$ ):

- i. **Permanent technology shocks:** In this case, a positive technology shock leads to a positive shock to expected consumption growth (see previous discussion and figure 1) and  $(1 - 1/\psi)^{-1} \sigma_{\tilde{r},t} > 0$ . Therefore, shocks to realized consumption growth and shocks to the adjusted wealth return reinforce each other and the price of risk is *higher* relative to the power utility case. As an example, consider the case where the substitution effect dominates, i.e.,  $(1 - 1/\psi)^{-1} > 0$ . Then  $\sigma_{\tilde{r},t} > 0$  since the shock to the wealth-consumption ratio is dominated by the positive shock to expected consumption growth.
- ii. **Transitory technology shocks:** In this case, long run expected consumption growth after a positive technology shock is *negative* as consumption must revert back to the trend and  $(1 - 1/\psi)^{-1} \sigma_{\tilde{r},t} < 0$ . Therefore, shocks to the adjusted wealth return and shocks to realized consumption growth hedge each other and the price of risk is *lower* relative to the power utility case. As an example, consider the case where the substitution effect dominates, (i.e.,  $(1 - 1/\psi)^{-1} > 0$ ). Then  $\sigma_{\tilde{r},t} < 0$  since the shock to the wealth-consumption ratio is dominated by the negative shock to expected consumption growth.

For agents who prefer early resolution of uncertainty, transitory technology shocks are less risky than permanent technology shocks.

(b) Agents prefer late resolution of uncertainty ( $\gamma < 1/\psi$ ):

- i. **Permanent technology shocks:** As in the permanent shock case above,  $(1 - 1/\psi)^{-1} \sigma_{\tilde{r},t} > 0$ . However, since  $\gamma < 1/\psi$ , shocks to the adjusted wealth return and shocks to realized consumption growth now hedge each other and the price of risk is lower relative the power utility case.

- ii. **Transitory technology shocks:** As in the transitory shock case above,  $(1 - 1/\psi)^{-1} \sigma_{\tilde{r},t} < 0$ . However, since  $\gamma < 1/\psi$ , shocks to the adjusted wealth return and shock to realized consumption growth now reinforce each other and the price of risk is *higher* relative to the power utility case.

For agents who prefer late resolution of uncertainty, transitory technology shocks are more risky than permanent technology shocks.

In order to generate a high price of risk, which is the empirically relevant case, we either need a preference for early resolution of uncertainty and permanent technology shocks, or a preference for late resolution of uncertainty and transitory technology shocks.

**What can agents do to endogenously decrease consumption risks?** Very little. While the agents will attempt to endogenously make the consumption risk as small as possible, they cannot easily get rid of it. Consider the permanent shock case displayed in figure 2. Agents can substitute consumption today for the future in order to decrease the volatility of realized consumption growth (dashed line). However, decreasing the shock to realized consumption growth increases the shock to expected consumption growth which with Epstein-Zin preferences also is a priced risk factor. Thus, unlike in the power utility model, the agents are caught in a Catch-22: Decreasing one risk increases another.<sup>13</sup>

In the following section, we show how the above developed intuition manifests itself quantitatively.

## 4.4 Results from Calibrated Models

Table 3 confirms the intuition from the impulse-responses in figures 1 and 2 by reporting relevant macroeconomic moments and the equilibrium price of risk for different model calibrations. The models have either transitory or permanent technology shocks and different levels of the elasticity of intertemporal substitution ( $\psi \in \{0.05, 1/\gamma = 0.2, 1.5\}$ ). Given these, we match the relative volatility of consumption and output growth (if possible) by changing the capital adjustment cost ( $\xi$ ). We match the volatility of consumption growth with all models by setting the volatility of the technology shocks,  $\sigma_z$ , appropriately. We re-calibrate

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<sup>13</sup>This is also part of the reason why the benchmark long-run risk models do not generate economically significant time-variation in the price of risk. There is some heteroskedasticity in shocks to both realized and expected consumption growth, but these approximately cancel in terms of the net effect on the price of risk per the intuition just given.

Table 3

Macroeconomic Moments and Consumption Dynamics

Table 3: This table reports relevant macroeconomic moments and consumption dynamics for models with either transitory ( $\varphi = 0.95$ ) or permanent ( $\varphi = 1.00$ ) technology shocks and different levels of the elasticity of intertemporal substitution ( $\psi$ ). The coefficient of relative risk aversion ( $\gamma$ ) is 5 across all models. We calibrate the discount factor ( $\beta$ ) for each model to match the level of the risk free rate. The volatility of the technology shock ( $\sigma_z$ ) is set so that the volatility of consumption growth is the same across models. Capital adjustment costs ( $\xi$ ) are set so that (if possible) the relative volatility of consumption to output growth is matched. We use annual U.S. data from 1929 to 1998 from the Bureau of Economic Analysis for the empirical moment values. The sample is the same as in Bansal and Yaron (2004).

Statistic	LRR I		LRR II	
	Model 2	Model 3	Model 4	Model 5
	Transitory Shocks $\varphi = 0.95, \gamma = 5$		Permanent Shocks $\varphi = 1.00, \gamma = 5$	
	$\psi = 0.05$	$\psi = 1.5$	$\psi = 0.05$	$\psi = \frac{1}{\gamma}$
	$\beta = 1.064$	$\beta = 0.998$	$\beta = 1.067$	$\beta = 1.012$
	$\xi = 0.70$	$\xi = 6.75$	$\xi = \infty$	$\xi = \infty$
	$\sigma_z = 4.05\%$	$\sigma_z = 4.05\%$	$\sigma_z = 1.61\%$	$\sigma_z = 1.88\%$
				$\sigma_z = 4.11\%$

Panel A: Macroeconomic Moments (Quarterly)

	U.S. Data 1929-1998	Model 2	Model 3	Model 4	Model 5
$\sigma[\Delta y]$ (%)	2.62	2.61	2.61	1.03	1.20
$\sigma[\Delta c]/\sigma[\Delta y]$	0.52	0.52	0.52	1.32	1.13
$\sigma[\Delta \tilde{v}]/\sigma[\Delta y]$	3.32	1.86	1.90	0.34	0.75
$AdjCosts/Output$ (%)	$n/a$	0.31	0.04	0.00	0.00
					2.64
					0.52
					1.83
					0.04

Panel B: Consumption Dynamics: (Quarterly)

	Bansal, Yaron Calibration	Model 2	Model 3	Model 4	Model 5
$\sigma[\Delta c_t]$ (%)	1.360	1.360	1.360	1.360	1.360
$\sigma[E_t[\Delta c_{t+1}]]$ (%)	0.172	0.144	0.130	0.013	0.040
$Autocorr(E_t[\Delta c_{t+1}])$	0.938	0.962	0.996	0.995	0.988
					1.362
					0.336
					0.970

Panel C: The Price of Risk and the Sharpe Ratio of the Equity Return (Annual)

	U.S. Data	Model 2	Model 3	Model 4	Model 5
$\sigma[M]/E[M]$	$n/a$	0.14	0.02	0.08	0.14
$SR[R^E]$	0.33	0.14	0.02	0.08	0.14
					0.36
					0.36

Figure 2 - The Catch-22 of Consumption Smoothing

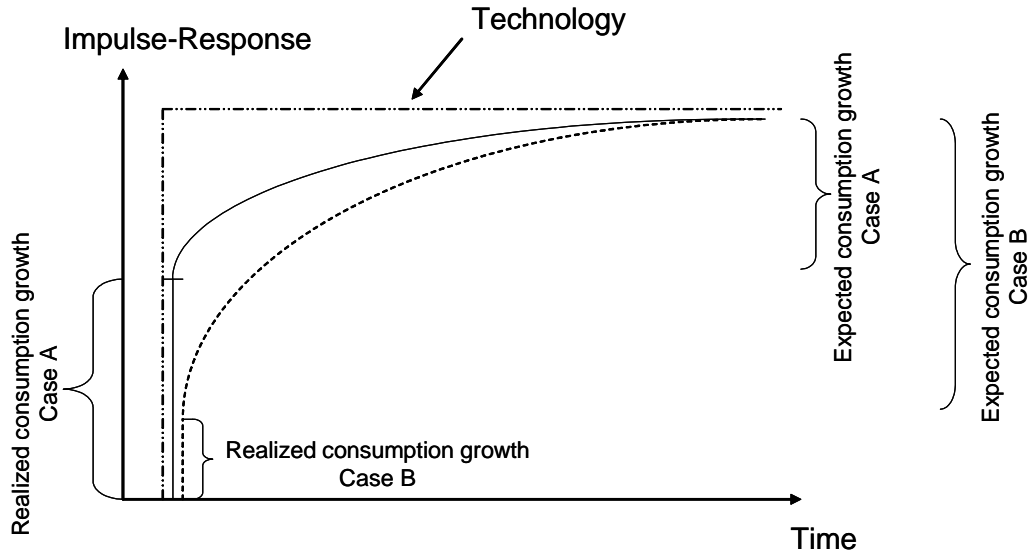


Figure 2: **Impulse-Responses for Technology and Consumption.** The dash-dotted line shows the impulse-response of the level of technology to a permanent technology shock. The solid line shows the impulse response of consumption when agents choose the initial consumption response to be large (Case A). The dashed line shows the consumption response when agents choose the initial consumption response to be small (Case B). The graph shows that by making the shock to realized consumption small, the shock to expected consumption growth becomes large.

the discount factor ( $\beta$ ) for each model to match the level of the risk free rate, as far as possible.<sup>14</sup> The coefficient of relative risk aversion ( $\gamma = 5$ ) is constant across models. We show in the appendix, confirming Tallarini (2000), that the level of  $\gamma$  has only second-order effects on the time series behavior of the macroeconomic variables.

#### 4.4.1 The Macroeconomic Moments.

All the models with transitory technology shocks match both the volatility of consumption and output growth. For low levels of the *EIS*, the agents strongly desire a smooth consumption path and therefore would like to decrease the consumption response to a transitory technology shock by investing more. To prevent this from happening, we increase the capital adjustment costs. The table reports that capital adjustment costs for the model with the lowest *EIS* are on average 0.88% of output. These high adjustment costs are the reason that

<sup>14</sup>We do not restrict  $\beta$  to be less than one, but if  $\beta$  becomes too large, prices are not well-defined (they are infinite).

asset prices are very volatile, as we show in the next section. The permanent shock model, on the other hand, cannot, even with no capital adjustment costs, match both the volatility of output and consumption growth unless the *EIS* is high (1.5). With permanent shocks, a strong desire to smooth consumption means agents want the initial consumption response to be *high* and close to its new, higher steady state level. Capital adjustment costs decrease investments, which otherwise would make the consumption response less strong. But, even in the case of no capital adjustment costs, the investment response is not strong enough to match the low volatility of consumption growth to relative to output growth unless the *EIS* is sufficiently high. The volatility of investment growth is about twice as high as the volatility of output growth in both models. Thus, as in the data, investment is substantially more volatile than output, but neither model quite captures the full magnitude of this fact.

**Expected Consumption Growth (Long-run Risk).** In Panel B of Table 3, we report both the volatility of consumption growth, the volatility of conditional expected consumption growth,  $\sigma [E_t [\Delta c_{t+1}]]$ , and the latter's first order quarterly autocorrelation ( $\rho$ ). These statistics illustrate the magnitude and nature of long-run risk in the models.<sup>15</sup> For comparison, Panel B also gives the corresponding values that Bansal and Yaron (2004) use in their calibration. The relative magnitudes of the volatility of realized and expected consumption growth show that the time-varying growth component is small. The average implied  $R^2$  for the predictability of quarterly consumption growth across models is around 1 – 2%, with a maximum  $R^2$  of 6% for the model with permanent shocks and *EIS* of 1.5 (LRR II). The persistence of the expected consumption growth rate ( $\rho$ ) is very high, which is important if risk associated with a small time-varying expected consumption growth rate component is to have quantitatively interesting asset pricing implications.

Figure 3 compares the model implied autocorrelations of annual consumption growth (time-averaged quarterly) from the model versus the empirical counter-part based on U.S. real, per capita aggregate log consumption growth 1948 - 2005, 47 observations. The empirical autocorrelations are given by the circles connected with the dotted line, while the average autocorrelation of similar samples generated from simulated data from the both the transitory (LRR I) and permanent shock (LRR II) models are given as solid lines with stars. The figure shows that the consumption growth is on average more highly autocorrelated

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<sup>15</sup>In the appendix, we show that these moments indeed capture most of the dynamics of consumption growth generated by the models and as such are meaningful moments to consider. There is some heteroskedasticity in both shocks to expected and realized consumption growth, but these effects are second order.



in the permanent shock model relative to the point estimates from the data. Thus, the autocorrelation structure we see in the data is not typical for the model. However, the autocorrelations in the data are within two standard deviation bounds, as depicted by the dashed lines. The average autocorrelations generated in the transitory shock model are overall close to the empirical sample estimates, although the one year autocorrelation here is too low. One may think a longer sample of consumption growth would tighten the standard error bounds sufficiently to reject one or both of the models, but this is not the case. Using the standard errors on autocorrelation estimates obtained from simulating 100 year samples from the models makes only the three year autocorrelation of the permanent shock model and the one year autocorrelation of the transitory shock model significantly different from that in the data. Further, the pre-WW2 consumption data is very noisy. Therefore, we conclude that neither of the models can be rejected based on the autocorrelation properties of the consumption process given the samples we have available.

**Figure 3 - Autocorrelation of Annual Consumption growth**

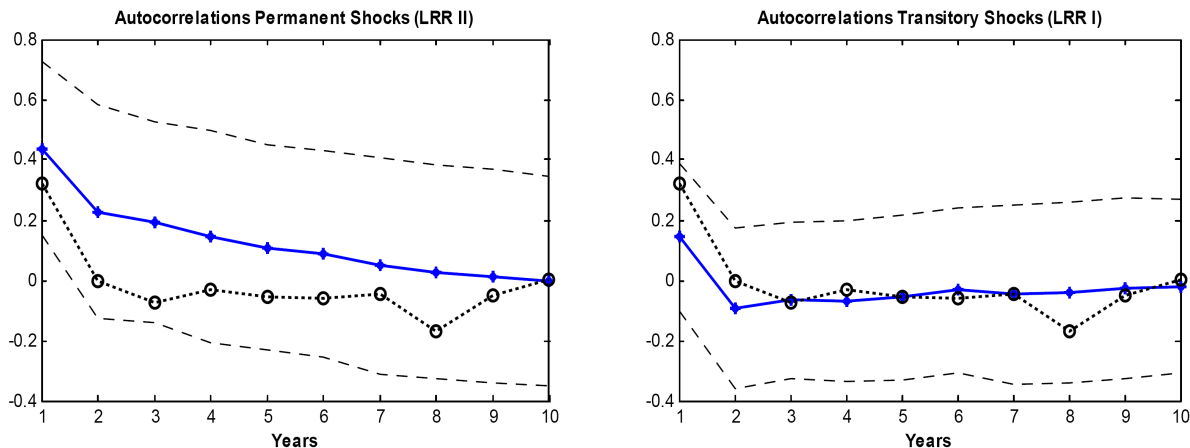


Figure 3: **Autocorrelations for Annual Consumption Growth.** The left graph shows the average autocorrelations of annual log consumption growth estimated from 57 year simulated samples from the permanent shock model (LRR II). The dashed lines are two standard deviation confidence bounds. The dotted line with circles give the empirical sample autocorrelations for annual consumption growth from 1947 - 2005 (57 observations). The right graph is similar, but for the model with transitory shocks (LRR I).

**The Price of Risk.** Even though the coefficient of relative risk aversion and the volatility of consumption growth are the same across all models, the price of risk varies from close to zero to 0.36. The power utility calibrations both give a price of risk of 0.14, and deviations

from this value are due to the effect of long-run risk in the model. In the case of transitory technology shocks, the price of risk is decreasing in the *EIS*. Holding  $\gamma$  constant and increasing the *EIS* increases the preference for early resolution of uncertainty, in which case agents dislike shocks to expected consumption growth: In Model 3, the price of risk is low because the two risk factors, shocks to realized and expected future consumption growth, are negatively correlated and therefore hedge each other. In the first Long-Run Risk model (LRR I), the agents instead prefer late resolution of uncertainty and therefore *like* shocks to expected consumption growth. For these agents, a world where shocks to realized consumption (which they dislike) and expected consumption (which they like) are negatively correlated, is a more risky world. That is why the price of risk in this case is high. The same logic applies for the case of permanent shocks, where the two shocks to consumption are positively correlated. In this case, it is the high *EIS* model (LRR II) that has a high price of risk.

**Figure 4 - Impulse Response 1**

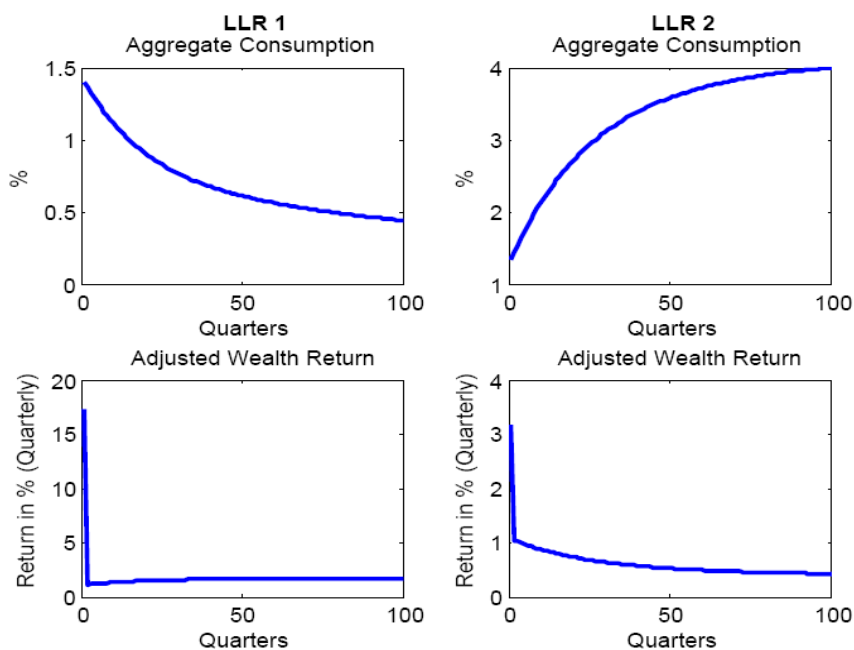


Figure 4: **Impulse-Responses for Consumption and the Adjusted Wealth Return.** The plots show the impulse-responses of consumption and the adjusted wealth return for the LLR I (transitory technology shocks) and the LLR II (permanent technology shocks).

This intuition is confirmed by figure 4, which shows the impulse-response of both con-

sumption and the adjusted return to wealth ( $\tilde{r}_{a,t} = \log\left(1 + \frac{A_{t+1}}{C_{t+1}}\right) - \log\frac{A_t}{C_t}$ ). Remember from the above discussion that the price of risk is:

$$\Lambda_t = \gamma\sigma_{\Delta c,t} + (\gamma - 1/\psi)(1 - 1/\psi)^{-1}\sigma_{\tilde{r},t}. \quad (19)$$

For both the long-run risk model with transitory technology shocks and low *EIS* ("LRR I") and the long-run risk model with permanent technology shocks and a high *EIS* ("LRR II"),  $(\gamma - 1/\psi)(1 - 1/\psi)^{-1} > 0$ . The figure shows that the response of the adjusted wealth return is the same. In the first case, the long-run expected consumption growth is negative, but since  $\psi < 1$ , the income effect dominates and the wealth-consumption ratio increases. In the second case, the long-run expected consumption growth is positive, and since  $\psi > 1$ , the substitution effect dominates. Thus, the wealth to consumption ratio increases here too and  $(\gamma - 1/\psi)(1 - 1/\psi)^{-1}\sigma_{\tilde{r}} > 0$ . It is worth noting, per this discussion, that the wealth-consumption ratio is pro-cyclical in both models, consistent with the data.

#### 4.4.2 Asset Pricing Implications

Table 4 presents key financial moments for the same models as in Table 3.

**The Risk Free Rate.** The level of the risk free rate is decreasing in the *EIS*, all else equal. A higher *EIS* increases the intertemporal substitution effect, which increases the demand for bonds in a growing economy.<sup>16</sup> For the transitory shock models this effect can be countered by a very high discount factor ( $\beta$ ), which we do not restrict to be less than one in this paper. If we impose this restriction ( $\beta < 1$ ), the risk free rate puzzle obtains. Note that we cannot simply increase  $\beta$  without bound, as the equilibrium prices then do not converge. While one can debate whether the magnitude of  $\beta$  is acceptable or not, the models with low *EIS* have in any case a too high volatility of the risk free rate. Since the risk free rate is the reciprocal of the conditional expected value of the stochastic discount factor, a misspecified risk free rate implies a misspecified stochastic discount factor. Habit formation models typically encounter this problem (see, e.g., Jermann (1998) or Boldrin, Christiano and Fisher (2001), as time-variation in the state variable "surplus consumption" induces much too volatile risk free rates when the models are calibrated to match empirical proxies for the price of risk (e.g., the equity Sharpe ratio). In contrast, the permanent shock

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<sup>16</sup>See eq. (43) in the appendix, for an approximate expression for the risk free rate.

**Table 4**  
**Financial Moments (Annual)**

Table 4: This table reports relevant financial moments for models with either transitory ( $\varphi = 0.95$ ) or permanent ( $\varphi = 1.00$ ) technology shocks and different levels of the elasticity of intertemporal substitution ( $\psi$ ). The coefficient of relative risk aversion ( $\gamma$ ) is 5 across all models. We calibrate the discount factor ( $\beta$ ) for each model to match the level of the risk free rate. The volatility of the technology shock ( $\sigma_z$ ) is set so that the volatility of consumption growth is the same across models. Capital adjustment costs ( $\xi$ ) are set so that (if possible) the relative volatility of consumption to output growth is matched. We use annual U.S. data from 1929 to 1998 from the Bureau of Economic Analysis for the empirical moment values. The sample is the same as in Bansal and Yaron (2004).

	<b>LRR I</b>	Model 2	Model 3	Model 4	Model 5	<b>LRR II</b>	
		Transitory Shocks $\varphi = 0.95, \gamma = 5$		Permanent Shocks $\varphi = 1.00, \gamma = 5$			
Statistic	U.S. data 1929-1998	$\psi = 0.05$ $\beta = 1.064$ $\xi = 0.70$ $\sigma_z = 4.05\%$	$\psi = \frac{1}{\gamma}$ $\beta = 1.016$ $\xi = 1.15$ $\sigma_z = 4.05\%$	$\psi = 1.5$ $\beta = 0.998$ $\xi = 6.75$ $\sigma_z = 4.05\%$	$\psi = 0.05$ $\beta = 1.067$ $\xi = \infty$ $\sigma_z = 1.61\%$	$\psi = \frac{1}{\gamma}$ $\beta = 1.012$ $\xi = \infty$ $\sigma_z = 1.88\%$	$\psi = 1.5$ $\beta = 0.998$ $\xi = 18.0$ $\sigma_z = 4.11\%$
<i>The Risk Free Rate</i>							
$E[R_f]$ (%)	0.86	0.85	0.90	1.84	2.93	2.34	0.82
$\sigma[R_f]$ (%)	0.97	4.60	1.40	0.17	0.56	0.41	0.45
<i>The Consumption Claim</i>							
$E[R^A - R_f]$ (%)	<i>n/a</i>	9.68	1.83	0.05	-0.38	-0.15	1.59
$\sigma[R^A - R_f]$ (%)	<i>n/a</i>	29.15	13.23	1.85	4.74	1.06	4.41
$SR[R^A]$	<i>n/a</i>	0.34	0.14	0.03	-0.08	-0.14	0.36
<i>The Dividend Claim</i>							
$E[R^E - R_f]$ (%)	6.33	8.06	1.18	0.04	0.00	0.01	0.24
$\sigma[R^E - R_f]$ (%)	19.42	24.06	8.58	1.59	0.06	0.07	0.66
$SR[R^E]$	0.33	0.34	0.14	0.03	0.08	0.14	0.36

model with high  $EIS$  (LRR II), can match both the level and a low volatility of the risk free rate, as well as a high price of risk.

**The Consumption Claim.** Aggregate wealth is the value of the claim to the aggregate consumption stream. The return volatility of the consumption claim is strongly increasing in capital adjustment costs. This is a well known feature of this friction in the case of the return to equity as it allows marginal  $q$  to deviate from 1. For model "LRR I" (the transitory shock case), the volatility of the consumption claim is very high at 29.15% per year. Model "LRR II", which has much lower capital adjustment costs, displays a return volatility of the consumption claim of 4.41%.

**The Dividend Claim.** The dividend claim is the claim to the aggregate dividend stream. This claim is unlevered, in contrast to what is the case for the empirical aggregate equity market statistics we report, and it is likely that the empirical volatility and risk premium of the unlevered equity return are substantially lower. Again, we see that the volatility of returns is increasing in the capital adjustment costs. For model "LRR I" (the transitory shock case), the volatility of the dividend claim is 24.06% per year. This is too high, especially in the light of these returns being unlevered. Model "LRR II", which has much lower capital adjustment costs, has a volatility of returns to wealth of only 0.66%. This gives an annual equity premium of 0.24%, which is too low to be explained by the lack of leverage.

The two long-run risk models presented here have too high and too low volatility of equity returns, respectively. However, the permanent shock model produces a stochastic discount factor which is in line with the data. Many papers define dividends as a levered claim to the consumption stream, in order to fit the volatility of dividend growth, the high equity return volatility and the equity risk premium. With a leverage factor of about 3 on the consumption claim, the resulting "equity" return premium for the "LRR II" Model would be around 4.5% with a return volatility of about 13%.

But why is it that the dividend claim has so low volatility in the permanent shock model when the consumption claim has a volatility that is an order of magnitude higher and dividends are the residual cash flow? Intuitively, we would expect the dividend claim to be more volatile. The answer lies with the dynamic behavior of dividends.

The production economy model generates dividends endogenously and the endogenous dividend process differs from the endogenous consumption process along important dimen-

sions: While equity dividends are given by  $D_t^E = \alpha Y_t - I_t$ , dividends to the wealth portfolio (i.e., aggregate consumption) are given by  $D_t^A = C_t = Y_t - I_t$ . Consider a permanent, positive shock to technology. If investors have higher  $EIS$ , this results in higher investment growth. Both equity dividends as well as consumption now respond less to a positive shock. However, equity dividends are much more sensitive to this effect since  $\alpha = 0.36 < 1$ , and may even decrease in response to a shock. So, while the price of the equity claim increases, the current dividend decreases, which dampens the total equity return response to technology shocks. The result is that the equity return volatility, and thus the equity premium, increase less with the  $EIS$  relative to the total asset return. Figure 5 confirms this argument as it shows the impulse response of the equity return and dividends to a positive shock to technology in both the long-run risk models.

**Figure 5 - Impulse Response 2**

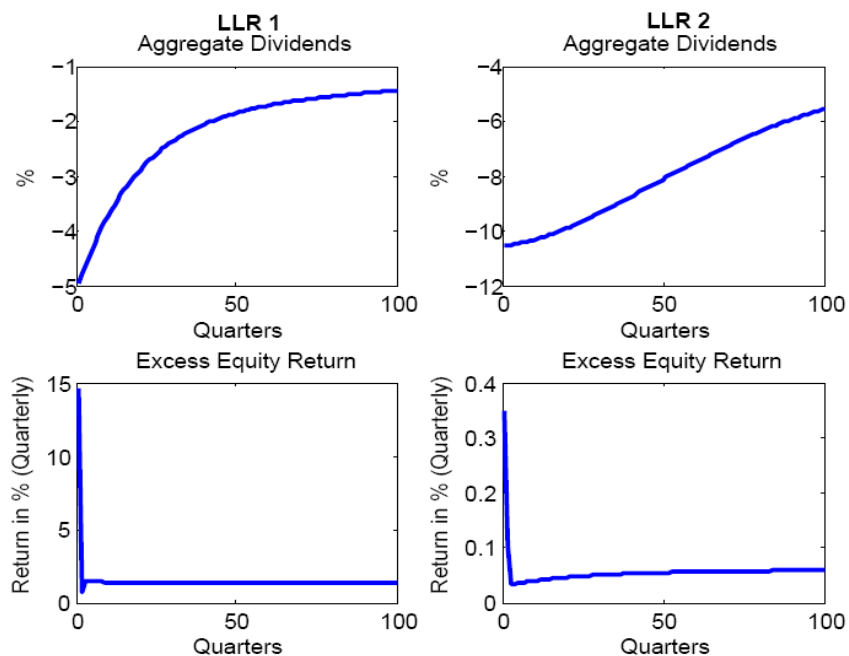


Figure 5: **Impulse-Responses for Dividends and Excess Equity Returns.** The plots show the impulse-responses of dividends and excess equity returns for the LLR I (transitory technology shocks) and the LLR II (permanent technology shocks). Note that the dividend response is counter-cyclical in both models, while the return response is pro-cyclical.

In an exchange economy, it is possible to exploit the fact that the claims to total wealth and equity have different dividend processes (i.e., consumption and dividends), and use this

as a degree of freedom to fit the asset pricing moments. Bansal and Yaron (2004), for instance, exogenously specify the dividend process such that expected dividend growth is very sensitive to shocks to expected consumption growth, which makes the equity claim risky and volatile. That way they are able to fit the equity return volatility, and thus the equity premium, with roughly the same (exogenous) consumption process and preference parameters as in the "LRR II" Model. The production economy model, on the other hand, restricts the joint dynamic behavior of aggregate consumption and dividends. Thus, while the general equilibrium framework considered so far in this paper provides a theoretical justification for a consumption process with long-run risk, it imposes constraints on dividends that are unfavorable in terms of matching the volatility of equity returns.

## 4.5 Alternative Specification for Wages

As discussed above, the models imply counter-cyclical dividends which decrease equity return volatility. This is especially a problem for the permanent shock model which generates too low return volatility. This counter-factual implication is related to the fact that wages in the model also do not correspond to those we observe in the data. In the model agents supply a constant amount of labor and wages are set such that it is optimal for the firm to demand exactly the same amount of labor: wages equal the marginal product of labor. The equilibrium total wages paid are then  $W_t = (1 - \alpha) Y_t$ . Thus, log wage growth is perfectly correlated with and as volatile as log output growth. In the data, however, wages are only weakly procyclical and less volatile than output. Thus, the labor share is counter-cyclical (see, e.g., King and Plosser, 2000; Donaldson and Danthine, 2002).

In this section, we argue that a promising avenue for future research is to carefully consider the mechanisms for labor supply and wages within the standard production economy model. We specify the wage process so as to match the empirical correlation of wages with output and the relative volatility of wages and output and show that this wage process, which is thus closer to what we observe in the data, allows the model to also generate a process for dividends that is much closer to the data. As a result, the equity premium increases by an order of magnitude.

In the recent labor market search literature, less volatile and less procyclical wages have been identified as an important avenue for making operating profits, firm value, and ultimately employment more volatile and more procyclical.<sup>17</sup> We essentially propose the same

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<sup>17</sup>In that literature, the counterfactually low volatility of employment, induced by too low volatility of firm

in order to alleviate the equity premium puzzle in our model. Instead of assuming that labor is paid its marginal product, we postulate a different wage process and calibrate that process to the data. In the presence of labor market frictions, for instance a search and matching friction, there is no reason to assume that labor is paid its marginal product. The search and matching literature assumes instead that wages are an outcome of a bargaining process between firm and worker. The wage process we propose is similar to the sticky wage rule Hall (2005) proposes. The consequence of less volatile wages are more volatile and more procyclical firm operating profits and firm dividends compared to the original model. This in turn leads to more volatile firm values and equity returns and to a higher equity risk premium. One way of viewing this is that we increase the operating leverage of the firm by introducing a less volatile and less procyclical "fixed-cost-component". Note that while the central planner problem is still well defined in this case and leads to the same equilibrium outcome, it no longer in general corresponds to the decentralized, competitive equilibrium outcome.

We specify the following wage process:

$$W_t^{adj} = \omega_0 (\omega_1 Y_t + (1 - \omega_1) Y_{t-1}), \quad (20)$$

so that wages are a weighted average of current and last period output. We calibrate this process to U.S. data from 1952 to 2004.<sup>18</sup> We solve the same social planner problem as before, and so the optimal aggregate investment (and thus consumption and output) is the same as in the original model. However, the division of wages and dividends differ, which implies that the dynamic behavior of the returns to the dividend claim also differ. With this assumed wage process, the solution to the social planner problem no longer corresponds directly to the de-centralized competitive equilibrium.

Table 5 reports asset pricing moments for the model with permanent technology shocks and a high EIS (LRR II). We report both the original equity return, that is the return of a claim on the original dividend process, as well as the adjusted equity return, that is a claim on the new dividend process ( $D_t^{adj} = Y_t - W_t^{adj} - I_t$ ). Note that the return on investment is no

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values, has been a long-standing puzzle, dubbed the "Shimer puzzle". For accounts of (and solutions for) the lack of movement in employment within the standard Mortensen and Pissarides (1994) labor market matching framework, see for example Den Haan, Ramey, Watson (2000), Hornstein, Krusell, Violante (2005), Shimer (2005), and Den Haan and Kaltenbrunner (2006). Making wages "stickier" has been originally promoted as an important contribution to the resolution of the Shimer puzzle by Hall (2005).

<sup>18</sup>We set  $\omega_1 = 0.28$  in order to match  $corr(\Delta w, \Delta y) = 0.38$ . As a result,  $\sigma(\Delta w) / \sigma(\Delta y) = 0.78$ , which turns out to be close to its empirical counter-part.



longer equal to the return on the equity claim in this case. Therefore, we solve numerically for the return to the equity claim.

From Table 5 (Column "LRR II") we can see that the standard production economy model has the potential to match both the level as well as the volatility of the equity premium. With our simple adjustment of the process for wages, the premium of the unlevered equity return increases from 0.24% to 1.46%. Because the equity return from the data is the return on a levered equity claim we add financial leverage to our model (the debt is assumed risk free and specified as in Jermann, 1998). We calibrate the average ratio debt to total firm value to 1/3, consistent with the average historical leverage in the U.S. Now the model is able to generate an equity premium of 2.02% with an equity return volatility of 6.45%. The resulting dividend growth in this model has a correlation of 0.5 with output, as opposed to the perfectly negative correlation in the original model. I.e., dividends are pro-cyclical, as in the data. Thus, the model now generates a more realistic dividend process.

**Matching the Equity Premium.** It is possible to further increase the equity premium in this model by increasing both the EIS,  $\psi$ , and the RRA,  $\gamma$ . Increasing the EIS further allows us to increase capital adjustment costs, which increase return volatility and thus the risk premium. Increasing the RRA increases the Sharpe ratio, while return volatility is, to a first-order, unaffected. This thereby increases the risk premium. The two right-most columns of Table 5 shows the permanent shock model (LLR II) with EIS increased to 2.5. This is an empirical upper bound as found by Campbell (1999). The two columns show the model output with  $\gamma = 5$  and  $\gamma = 10$ , respectively. The level of time-discounting,  $\beta$ , and adjustment costs,  $\xi$ , are calibrated for each model to match the level of the risk-free rate and the relative volatility of consumption growth. The model with  $\gamma = 5$  and  $\psi = 2.5$  (LRR III) has a risk-premium of 2.63% p.a. on the levered equity claim given the adjusted wage process. The increase from 2.02% is due to the higher adjustment costs. The risk premium of the original equity claim has increased even more in relative terms; from 0.24% to 0.82%. Increasing the relative risk aversion to 10 (LRR IV) has negligible impact on the macro moments, but doubles the Sharpe ratio. The risk premium on the levered claim given the adjusted wage process is now 4.61%, which is reasonably close to the historical average excess equity return of 6.33%. The standard deviation of equity returns, however, is still too low relative to the historical data. The original, unlevered equity claim now has a risk premium of 1.66%, which with leverage would be above 2% p.a.

Thus, a wage process closer to what we observe in the data yields a dividend process, and

**Table 5**

**Adjusted Wage Process — Asset Pricing Moments**

Table 5: This table reports asset pricing moments for the Long-Run Risk II - Model, which has permanent technology shocks, an elasticity of intertemporal substitution ( $\psi$ ) of 1.5, and relative risk aversion ( $\gamma$ ) of 5. We report both the original equity return, that is the return of a claim on the original dividend process, as well as the adjusted equity return, that is a claim on the adjusted dividend process. The two right-most columns (LRR III and LRR IV) report the same moments the same model but with EIS = 2.5 and RRA = 10. The data are taken from Bansal and Yaron (2004) who use U.S. financial markets data from 1929 to 1998. All values reported in the table are annual.

Statistic	Data	LRR II	LRR III	LRR IV
		$\gamma = 5, \psi = 1.5$ $\beta = 0.998, \xi = 18$	$\gamma = 5, \psi = 2.5$ $\beta = 0.996, \xi = 4.8$	$\gamma = 10, \psi = 2.5$ $\beta = 0.994, \xi = 4.9$
$\sigma[\Delta c]$ (%)	2.72	2.72	2.72	2.72
$\sigma[\Delta c] / \sigma[\Delta y]$	0.52	0.52	0.52	0.52
$\sigma[M] / E[M]$	<i>n/a</i>	0.36	0.36	0.71
$E[R_f]$ (%)	0.86	0.82	1.07	0.87
$\sigma[R_f]$ (%)	0.97	0.45	0.28	0.28
$SR[R^E]$	0.33	0.36	0.36	0.71
$E[R^E - R_f]$ (%)	6.33	0.24	0.82	1.66
$\sigma[R^E - R_f]$ (%)	19.42	0.66	2.25	2.34
$SR[R^{Eadj}]$	0.33	0.36	0.36	0.71
$E[R^{Eadj} - R_f]$ (%)	6.33	1.46	1.90	3.33
$\sigma[R^{Eadj} - R_f]$ (%)	19.42	4.06	5.23	4.69
<b>Financial Leverage:</b> (D/V = debt / value of firm)				
	$D/V \approx 0.33$	$D/V = 0.33$	$D/V = 0.33$	$D/V = 0.33$
$E[R^{Eadj} - R_f]$ (%)	6.33	2.02	2.63	4.61
$\sigma[R^{Eadj} - R_f]$ (%)	19.42	6.45	7.24	6.49

as a result equity returns, substantially closer to what we find in the data. An interesting avenue for future research is to endogenize labor and hours worked in the model presented in this paper.<sup>19</sup>

## 4.6 Discussion

We have demonstrated that a real business cycle model with capital adjustment costs can generate economically significant endogenous long-run risk as an outcome of the optimal consumption-savings decision. The predictability of consumption growth in the model is still relatively low, and we show in the next section that it can be hard to detect in small samples of data like the ones we have available. This is an important point, because empirically consumption growth is not highly predictable. The models presented here do not, however, generate economically significant time-variation in the price of risk or in the equity risk premium (see Appendix). Further, as can be seen from figures 4 and 5, dividends and consumption are negatively correlated at fairly long time-horizons, which is counter-factual. This shortcoming, however, is not particular real business cycle model with Epstein-Zin preferences and  $\gamma \neq \frac{1}{\psi}$ , but also arises in the special case of power utility preferences ( $\gamma = \frac{1}{\psi}$ ).

The low equity premium in the permanent shock model is not to a first order a result of low risk aversion, but of low adjustment costs. Increasing risk aversion will increase the risk premium and Sharpe ratio, but not substantially the equity return volatility. The problem is with the low capital adjustment costs and the counter-cyclical of dividends. The transitory shock model, which can generate a high equity premium, allows for much higher capital adjustment costs, but at the cost of too high volatility of the risk free rate. From a finance perspective, the high equity return volatility in this model is to a large extent due to a too volatile risk free rate. Thus, the high return volatility in this case arises from the wrong channel as the real risk free rate empirically is not very volatile.<sup>20</sup>

We propose an extension of the permanent shock model with a reduced form sticky wage formulation. The model is then able to generate an equity risk premium of the same order of magnitude as that in the data. It does so by inducing pro-cyclical dividends and by also adding financial leverage. By increasing the relative risk aversion to its economic upper

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<sup>19</sup>Kaltenbrunner (2006) incorporates a search and matching model into standard production economy models with habit preferences in order to jointly explain macroeconomic time series, including labor market series, and asset prices.

<sup>20</sup>This is a common problem with production economy models that match the volatility of equity returns (see, e.g., Jermann, 1998, and Boldrin, Christiano, and Fisher, 2001).

bound of 10 (following Mehra and Prescott, 1985) and the elasticity of substitution to 2.5, the permanent shock model with the adjusted wage process generates an equity risk premium of 4.6%, which is an impressive increase relative to the 0.24% of the original model.

## 5 Empirical Tests: Expected Consumption Growth and the Cross-Section of Stock Returns

A key variable in the model is the persistence of the technology shocks. In this section, we test a prediction of the model regarding the joint time series behavior of technology (total factor productivity) and consumption growth, which depend crucially on the persistence of the technology shocks.

In a recent paper, Bansal, Kiku, and Yaron (2006) test an exchange economy version of the model in this paper and find that forecasting variables such as lagged consumption growth, the default spread, and the market price-dividend ratio are significant predictors of future consumption growth. Furthermore, they show, using the cross-section of stock returns, that shocks to expected consumption growth are a positively priced risk factor, which is consistent with a model where agents prefer early resolution of uncertainty. Here, we therefore confine our empirical analysis to test a restriction that is particular to the production economy. We consider an instrument Bansal, Kiku, and Yaron do not use and which is related to the level of technology - the driving process of the production economy model.

The consumption data and data on Total Factor Productivity (TFP; the equivalent to "technology" in our model) are obtained from the Bureau of Economic Analysis and the Bureau of Labor Statistics, respectively. The return data are from Kenneth French's homepage.

### 5.1 Expected Consumption Growth

As highlighted by Harvey and Shepard (1990), Bansal and Yaron (2004) and Hansen, Heaton and Li (2005), amongst others, it is difficult to estimate long-run consumption growth dynamics from the relatively short samples of data we have available. In the production economy model, slow-moving expected consumption growth dynamics arise due to endogenous consumption smoothing, and the production economy model therefore identifies observable proxies for the otherwise unobservable expected consumption growth rate. Here, we consider

the ratio of the level of technology to the level of consumption as a forecasting variable for future consumption growth.

**Figure 6 - Z/C Ratio as Proxy for Expected Consumption Growth**

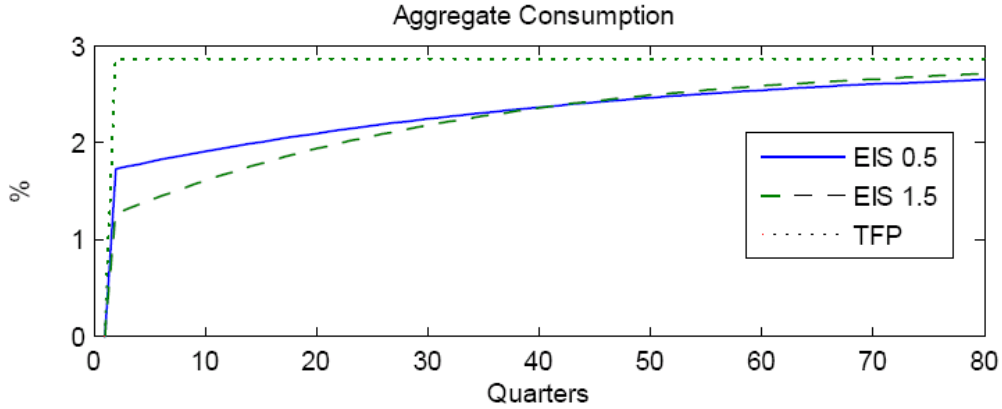


Figure 6: **Impulse-Response of Consumption** Impulse responses of consumption to a one standard deviation positive and permanent shock to technology for different levels of the EIS. The impulse-responses are for a model with EIS = 0.5 and the LRR II Model (EIS = 1.5), respectively, and illustrate how the ratio of technology to consumption is a proxy for expected consumption growth.

Define:

$$z_{c_t} \equiv \ln \left( \frac{Z_t}{C_t} \right). \quad (21)$$

If  $z_{c_t}$  is high (low) in the permanent technology shock case, consumption is expected to increase (decrease) towards a new steady-state level. Our model thus implies that  $z_{c_t}$  is a good instrument for the expected consumption growth rate and that  $z_{c_t}$  should predict future consumption growth with a positive sign in the permanent shock case. This intuition is confirmed in Figure 6, which shows the impulse-response of consumption to a one standard deviation permanent shock to technology (total factor productivity) for high and low levels of the *EIS*. For a transitory technology shock, the consumption response can be hump-shaped. However, the long-run consumption response is in this case always negative. Thus, with transitory technology shocks, the ratio  $z_{c_t}$  predicts long-horizon consumption growth with a negative sign.

In the model,  $z_{c_t}$  is stationary even though both  $Z_t$  and  $C_t$  are non-stationary. In particular, since the production technology is specified as:

$$Y_t = Z_t^{1-\alpha} K_t^\alpha N_t^{1-\alpha}, \quad (22)$$

all endogenous variables in the economy evolve around the (stochastic) trend  $Z_t$  (see Appendix 9.4). We get data on  $Z$  (TFP) from the Bureau of Labor Statistics (BLS). The BLS computes TFP as follows. For each time  $t$ , it collects data on  $Y$  (output), on  $K$  (capital input), and on  $N$  (labor input). Then the BLS estimates a value for the parameter  $\alpha$  and computes TFP as the Solow residual:

$$\ln \tilde{Z} = \ln Y - \alpha \ln K - (1 - \alpha) \ln N. \quad (23)$$

The BLS specifies the following production technology:

$$Y = \tilde{Z} K^\alpha N^{1-\alpha}. \quad (24)$$

It follows that we need to normalize:

$$Z = \tilde{Z}^{1/(1-\alpha)}. \quad (25)$$

We take as the value for  $\alpha$  the constant value we use in our model ( $\alpha = 0.34$ ). We check our results for robustness by assuming different values for  $\alpha \in [0.30, 0.40]$ , and find that our results are robust with respect to the choice of  $\alpha$ . We use the available annual data from 1948 to 2005, and the resulting time series for  $zc$  is plotted in the left panel of Figure 7. The log technology to consumption ratio is highly persistent with an annual autocorrelation of 0.98 and it is indeed not clear that it is stationary (we cannot reject a unit root). This may be because of the assumed value of  $\alpha$ . To address this concern, we also estimate the cointegration vector of  $z$  and  $c$  and find  $1.29z_t - c_t = zc_t^{coint}$  as the deviations from the estimated cointegration relationship. This variable is less persistent with an annual autocorrelation of 0.90 (see the right panel of Figure 7). We use both variables in the empirical analysis.

We suggest the following forecasting relationship:

$$\Delta c_{t,t+j} = \alpha + \beta zc_t + \varepsilon_{t,t+j}. \quad (26)$$

In the model the relation is not exactly linear, but when simulating data from our calibrated models (LRR I and LRR II) we find that  $zc_t$  accounts for more than 99% of the variation in expected consumption growth in a linear regression. We test this forecasting relationship both on real data from 1948 to 2005 and on simulated data generated by the transitory (LRR

**Figure 7 - The Historical Technology to Consumption Ratio**

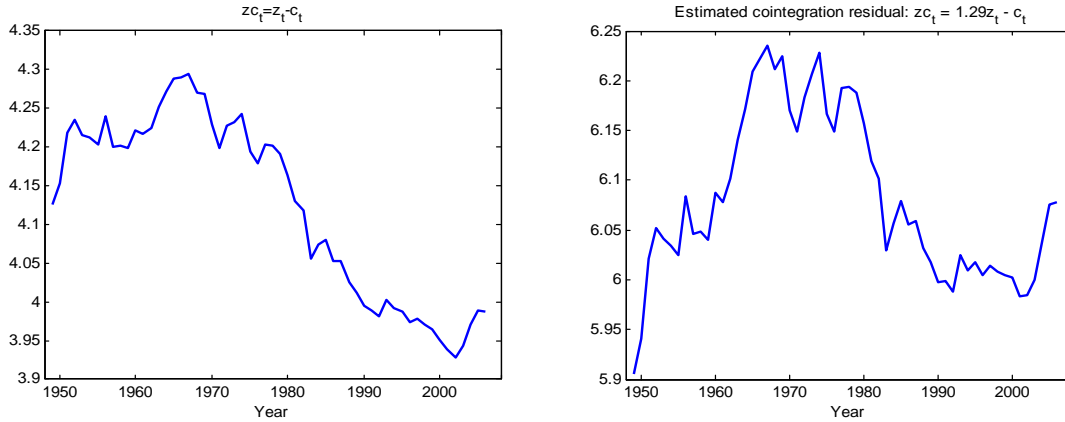


Figure 7: **The log Technology to Consumption Ratio.** The left graph shows the historical log technology to consumption ratio,  $z_{c_t}$ , using annual technology data from the Bureau of Labor Statistics and annual consumption data from the Bureau of Economic Analysis, 1948 - 2005. The right graphs shows the residual from the estimated cointegrating vector between log technology and consumption,  $z_{c_t}^{coint}$ .

I) and the permanent (LRR II) technology shock versions of the model for various forecasting horizons. The simulated quarterly data for consumption and technology are time-averaged to arrive at annual variables that correspond to their empirical counterparts. This is important as the time-averaging changes the autocorrelation and predictability patterns.<sup>21</sup> Panel A of Table 6 reports the results from running the regression in equation (26). The reported  $t$ -statistics (Newey-West) and regression coefficients from simulated data are sample averages of 1,500 samples with the same sample length as the historical sample (57 years).

The left half of Panel A of Table 6 shows that the empirical consumption growth forecasting regressions, using either  $z_{c_t}$  or  $z_{c_t}^{coint}$  as the predicting variable are significant at the 10% level for the 1 year to the 10 year forecasting horizons. The regression coefficients are in all cases positive and increasing in the forecasting horizon. Thus, we find evidence for predictability of consumption growth using the ratio of technology to consumption. However, the statistical significance is marginal and only for the 1 year forecasting regression using  $z_{c_t}^{coint}$  do we find significance at the 5% level.<sup>22</sup> However, the small size of the sample and the high persistence of the forecasting variable give us low power to reject the null of no predictability. In the data, shocks to consumption growth and the log technology to

<sup>21</sup>We thank Leonid Kogan for pointing this out.

<sup>22</sup>Note that we do not need to adjust the asymptotic  $t$ -statistics for the generated regressor problem when using  $z_{c_t}^{coint}$  since the estimate of the cointegrating vector is superconsistent.

**Table 6**

**Estimating Expected Consumption Growth**

Table 6: This table reports forecasting regressions of annual log nondurable- and services consumption growth on a lagged measure of expected consumption growth, the log TFP to Consumption ratio and lagged log technology growth (Panel B) in the left parts of Panel A and B, respectively.  $\hat{\beta}_{Data}$  corresponds to the regression coefficients from regressions using  $zc_t = z_t - c_t$ , while  $\hat{\beta}_{Data}^{Coint}$  corresponds to the regressions using the estimated cointegrating residual:  $zc_t = 1.29z_t - c_t$ . The consumption and TFP data are from the Bureau of Economic Analysis and the Bureau of Labor Statistics respectively. We use annual data from 1948 to 2005, resulting in 58 -  $j$  observations for a regression with a  $j$  year forecasting horizon: Multi-year forecasting regressions are overlapping at an annual frequency. The t-statistics (in parenthesis) are corrected for heteroskedasticity and overlapping observations using Newey-West (autocorrelation corrected with  $j - 1$  lags) standard errors. The right half of the table shows results from simulated data from the two benchmark models, LRR I and LRR II. Results for the models are based on 1,500 replications of sample size 58 years each. Numbers with two asterics indicate significance at the 5% level or more in a two-tailed t-test, while one asterisk indicates significance at the 10% level.

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**Panel A:** Regression  $\Delta c_{t,t+j} = \alpha + \beta zc_t + \varepsilon_{t,t+j}$  ( $j$  denotes forecasting horizon in years)

$j$	$\hat{\beta}_{Data}$	$R_{adj}^2$	$\hat{\beta}_{Data}^{Coint}$	$R_{adj}^2$	$\hat{\beta}_{LRR I}$	$R_{adj}^2$	$\hat{\beta}_{LRR II}$	$R_{adj}^2$
1	0.021* (1.88)	3.3%	0.040** (2.17)	7.7%	-0.022 (-0.50)	-1.0%	0.105** (3.58)	13.4%
3	0.060* (1.72)	7.6%	0.083* (1.66)	8.0%	-0.119 (-1.44)	5.2%	0.210** (2.42)	16.3%
5	0.084* (1.78)	13.8%	0.133* (1.72)	12.3%	-0.183* (-1.82)	10.6%	0.261* (1.79)	17.2%
10	0.233* (1.83)	30.0%	0.175* (1.82)	11.1%	-0.276** (-2.55)	14.5%	0.236 (0.76)	16.6%

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**Panel B:** Regression  $\Delta c_{t,t+1} = \alpha + \beta \Delta z_{t-1,t} + \varepsilon_{t,t+1}$

$\hat{\beta}_{Data}$	$R_{adj}^2$	$\hat{\beta}_{Data}^{Coint}$	$R_{adj}^2$	$\hat{\beta}_{LRR I}$	$R_{adj}^2$	$\hat{\beta}_{LRR II}$	$R_{adj}^2$
0.098* (1.68)	4.5%	0.076* (1.75)	4.5%	0.060 (1.46)	1.0%	0.150** (3.40)	3.7%

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consumption ratio are positively correlated, which means the Stambaugh (1999) bias is *negative* (about  $-0.010$  for the 1 year forecasting horizons in the  $zc_t$  case and  $-0.012$  for the  $zc_t^{coint}$  case). Also, technology, in particular, is measured with noise, which further biases the coefficients downward. We therefore view the  $t$ -statistics as conservative.

The right half of Panel A of Table 6 shows the average sample regression coefficients and  $t$ -statistics for the transitory (LRR I) and permanent (LRR II) shock models. As expected, the regression coefficients for the transitory shock model are negative, while the regression coefficients for the permanent shock model are positive. The transitory shock model gives insignificant average  $t$ -statistics at shorter forecasting horizons, whereas the permanent shock model has significant average  $t$ -statistics at shorter horizons, but not at longer horizons. The simulated data from the permanent shock models thus confirm the difficulty of detecting long-horizon predictability of consumption growth in small samples. The pattern in the  $\beta$ 's and the adjusted  $R^2$ 's in the data are closest to the permanent shock model, although the average consumption predictability is higher in this model than what we observe empirically. Figure 8 shows the distribution of the  $t$ -statistics at the annual consumption growth forecasting horizon for the transitory (LRR I) and permanent shock (LRR II) models, as well as the empirical  $t$ -statistic using the  $zc_t$  variable (the red vertical line). The graphs show that we based on this metric can reject the transitory model at the 5% level, while we cannot reject the permanent shock model (the empirical  $t$ -statistic is here at the 16<sup>th</sup> percentile). This conclusion is only strengthened if we consider the  $zc_t^{coint}$  variable.

Panel B of Table 6 shows results from a different forecasting regression:

$$\Delta c_{t+1} = \alpha + \beta \Delta z_t + \varepsilon_{t+1}, \tag{27}$$

where  $\Delta z_t = z_t - z_{t-1}$  denotes changes in log annual (time-averaged) technology growth. Since shocks to technology drive changes in expected consumption growth, this variable should also forecast consumption growth. Further, this forecasting variable is not subject to the near unit root issues that characterize the technology to consumption ratio. Panel B shows that changes in lagged technology do forecast annual consumption growth, although again only at the 10% significance level. The sign is positive, which is consistent with *both* models. The positive sign is expected for the permanent shock model, but for the transitory shock model the intuition is not as clear. The reason is both because of a slight, short-lived hump-shaped consumption response to technology shocks, and, more importantly in this case, the time-averaging of the technology and consumption data. If annual technology growth is high, it is

likely that technology is below trend, which means future growth rates in the economy also are high as the technology process is very persistent. This effect is magnified by the time-averaging. This time, the permanent shock model on average displays about the same level of consumption predictability as in the data as measured by the adjusted  $R^2$ . However, the average  $t$ -statistic from this model is substantially higher and the lower right plot of Figure 8 shows that the empirical  $t$ -statistic is not typical for the permanent shock model. But, again the permanent shock model cannot be rejected. As before, noise in the forecasting variable also biases the coefficient towards zero, which makes the test conservative. Since both models in this case predict a positive regression coefficient, these regressions are not as informative as the previous regression specification in terms of distinguishing between the models.

In sum, we find evidence that expected consumption growth is time-varying and related to the level and changes in aggregate technology, as measured by the Bureau of Labor Statistics. A relation is predicted by the standard production economy model. We show that the model with pure transitory shocks (LRR I) does not generate predictability patterns in consumption growth like those we observe in the data. The model with permanent shocks (LRR II), however, produces similar patterns in both the regression coefficients and the adjusted  $R^2$ , and the model cannot be rejected, although the average consumption predictability of this model is higher than what we find in the data. Thus, the evidence is in favor of a model with permanent technology shocks relative to a model with pure transitory shocks. This is consistent with the findings of Alvarez and Jermann (2005), who find that shocks to the stochastic discount factor (marginal utility of wealth) empirically are mainly permanent, with only a relatively small transitory component.

## 6 Extensions: Predictability

The benchmark model presented in section 3, does not yield economically significant time-variation in the equity premium. In the Appendix, we show that the time-variation in the equity premium in this model is very small, which is consistent with the findings in Campanale, Castro, and Clementi (2007). In this section, we illustrate how time-variation in the equity premium can arise in the model. In particular, we consider two avenues for generating predictability in aggregate equity returns as found in the data (e.g., Fama and French, 1989): Time-varying persistence of technology shocks and asymmetric adjustment costs. We focus on the original equity claim, i.e. the claim to dividends from the model

Figure 8

Small-Sample Distributions of  $t$ -statistics from Forecasting Regressions

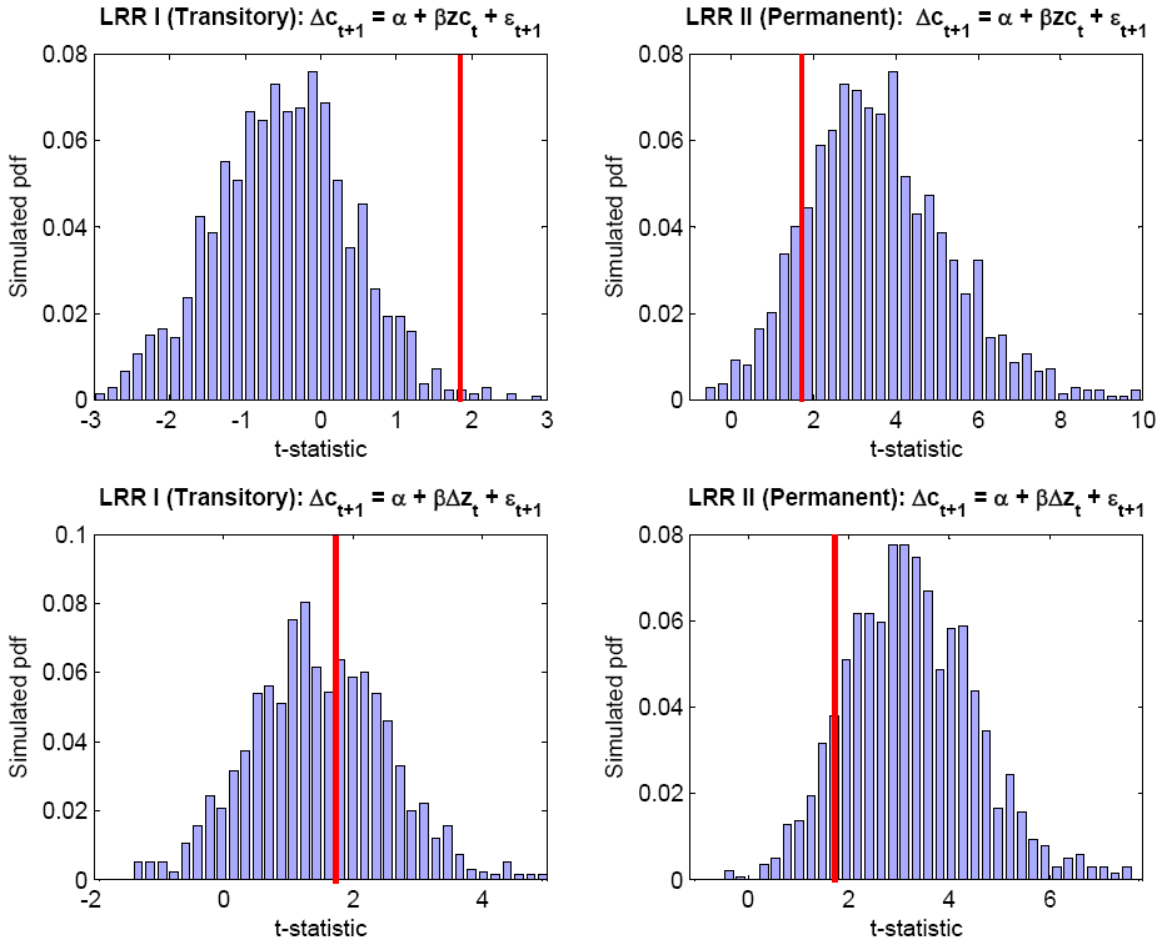


Figure 8: **Small-sample distribution of  $t$ -statistics.** This graph shows the small-sample distribution of  $t$ -statistics from the relevant forecasting regressions for the two benchmark models LRR I (transitory shock) and LRR II (permanent shock). The sample is annual, 57 observations, and the plots are based on 1,500 simulated samples. The annual consumption and technology data are time-averaged as in the data, based on the quarterly model output. The vertical line corresponds to the empirical  $t$ -statistic. All  $t$ -statistics are corrected for heteroskedasticity (White).

where wages are set equal to their marginal product.

## 6.1 A Model with both Transitory and Permanent Technology Shocks

Alvarez and Jermann (2000) decompose the stochastic discount factor into a permanent Martingale component and a transitory component. Using asset prices, they estimate that the permanent component is by far the most important and that the transitory component has an upper bound volatility of about 20% of the volatility of the stochastic discount factor, while the volatility of the permanent component is about as large as the volatility of the stochastic discount factor. Given this empirical evidence and the fact that the transitory shock model generates a too high equity premium, while the permanent shock model generates a too low equity premium, it is natural to ask whether a model with a mix of both types of shocks can fit the equity premium as well as the risk free rate volatility. The short answer is that a constant mix of the two types of shocks does not help much in terms of asset pricing implications. While it is possible to make the calibrations somewhat better with more degrees of freedom, the requirement to fit the relative volatility of consumption to output as well as the level of consumption volatility turns out to be a significant constraint. For instance, consider adding a negatively correlated transitory shock to the permanent shock model (LRR II). This leads to positively autocorrelated technology growth rates, which gives rise to more long-run risk. However, to fit the relative volatility of consumption to output, it is in this case necessary to reduce the capital adjustment costs, which decreases the return volatility. Also, consumption growth becomes too predictable. It is, however, possible for a two shock model to generate less autocorrelation in consumption growth, for the same level of long-run risk, but we do not report this here. Instead we present a model which generates time-variation in the price of risk.

### Predictability

The models presented so far do not generate economically significant time-variation in the equity risk premium (see Appendix). However, empirically there is predictability in excess stock market returns, which indicates that the equity risk premium is time-varying. The previous analysis shows that the price of risk depends on the persistence of the technology shocks. This suggests that a model where the technology shocks are more persistent in an expansion than a recession will generate a time-varying price of risk. In this section, we

calibrate such a model to illustrate this point.

Consider the following amended process for log technology:

$$\begin{aligned} z_t &= \mu t + x_t + \tilde{z}_t \\ x_t &= x_{t-1} + (1 - s_{t-1}) \varepsilon_t \\ \tilde{z}_t &= \varphi \tilde{z}_{t-1} + s_{t-1} \varepsilon_{t+1} \end{aligned}$$

where  $s_{t-1} = 0.3$  if  $\tilde{z}_{t-1} > 0$  and  $s_{t-1} = -0.3$  if  $\tilde{z}_{t-1} < 0$ . Here  $x_t$  is the random walk component of technology, while  $\tilde{z}_t$  is the stationary component. One may interpret  $\tilde{z}$  as a reduced form representation of business cycle fluctuations and let  $\tilde{z}$  be high in expansions and low in recessions. Thus, the technology process has relatively more permanent shocks in recessions than in expansions. If investors prefer early resolution of uncertainty, this leads to a higher price of risk in a recession than an expansion. Note, however, that the total shock to technology is homoskedastic. It turns out that the realized consumption growth is close to homoskedastic and so the time-variation in the price of risk is mainly due to time-varying endogenous long-run consumption risk. A higher absolute value of  $s_{t-1}$  gives more time-variation in the price of risk.

Again we focus on the model with EIS = 1.5 and RRA = 5, but now with the amended technology shock process. We call this model Long-Run Risk model V (LRR V). For comparison with the permanent shock model (LRR II), we keep all the parameters the same as in that model. Panel A of Table 7 shows that the calibrated model delivers about the same unconditional moments as model LRR II. However, the price of risk is in fact time-varying and the maximal, annualized Sharpe ratio in the model ranges between 0.26 to 0.41. These dynamics are inherited by the equity claim, which gives a time-varying equity premium. Since the business cycle variable is persistent, there is long-horizon predictability of excess equity returns, which is verified in Panel B of Table 7. Panel B gives the amount of predictability using a long (population) sample by regressing excess equity returns on the lagged business cycle variable,  $\tilde{z}_t$ . Both the regression coefficient and the  $R^2$  increase with the forecasting horizon. The latter is smaller than what we usually see empirically for long-horizon forecasting regressions, but these regressions suffer from substantial small-sample biases and the in sample  $R^2$ 's are likely to be overstated.

In sum, the model is able to deliver a time-varying price of risk and equity risk premium by specifying a time-varying combination of permanent and transitory technology shocks while keeping the net innovation to technology homoskedastic. This leads to time-variation

**Table 7**

**Moments from Model with Time-Varying Price of Risk**

Table 7: Panel A of this table reports annualized moments for the model with a time-varying combination of permanent and transitory technology shocks (LRR V) and for the model with asymmetric capital adjustment costs (LRR VI). For comparison, the corresponding moments from the data and the benchmark permanent shock model (LRR II) are also reported. The parameter values of the LRR V and model are the same as those of LRR II. That is,  $\gamma = 5$ ,  $\psi = 1.5$ ,  $\xi = 18$ ,  $\beta = 0.998$ ,  $\sigma_z = 0.0263$ . The model with asymmetric capital adjustment costs also has these preference parameters, but the adjustment costs parameters special to this model are  $a_K^- = 10$  and  $a_K^+ = 1$ . Panel B reports excess equity return forecasting regressions for different forecasting horizons. The forecasting variables are  $z_t$  for LRR V and the log consumption to aggregate wealth,  $\log(C/A)$ , for LRR VI, respectively.

<b>Panel A: Unconditional Moments</b>				
Statistic	Data	Benchmark model LRR II	Time-varying persistence <b>LRR V</b>	Asymmetric adj. costs <b>LRR VI</b>
$\sigma[\Delta c]$ (%)	2.72	2.72	2.83	2.75
$\sigma[\Delta c] / \sigma[\Delta y]$	0.52	0.52	0.54	0.52
$\sigma[M] / E[M]$	<i>n/a</i>	0.36	0.36	0.36
$E[R_f]$ (%)	0.86	0.82	0.81	0.80
$\sigma[R_f]$ (%)	0.97	0.45	0.45	0.47
$SR[R^E]$	0.33	0.36	0.36	0.36
$E[R^E - R_f]$ (%)	6.33	0.24	0.25	0.33
$\sigma[R^E - R_f]$ (%)	19.42	0.66	0.69	0.93

<b>Panel B: Predictability</b>				
Horizon	<b>LRR V:</b>		<b>LRR VI:</b>	
	$\beta$	$R^2$	$\beta$	$R^2$
1 <i>quarter</i>	-0.01	1.6%	0.01	0.5%
1 <i>year</i>	-0.04	5.3%	0.04	1.9%
4 <i>years</i>	-0.13	11 %	0.12	6.5%

in the amount of long-run risk.

## 6.2 Asymmetric Capital Adjustment Costs

Another way to generate predictability in models with production is to specify asymmetric capital adjustment costs (see, e.g., Kogan, 2001; Zhang, 2005). In this section, we will specify the function for adjustment *costs* directly and therefore change the notation from that previously used in the paper. In particular, the capital accumulation equation is given by

$$K_{t+1} = (1 - \delta)K_t + I_t, \quad (28)$$

where  $\delta$  is the capital depreciation rate and  $I$  is now investment *net* of adjustment costs (as opposed to gross previously). We specify adjustment costs that are proportional to capital and quadratic in the investment rate, as is relatively standard in the literature. Consistent with the adjustment costs specified earlier in the paper, there are zero adjustment costs at the steady state where  $I/K = \delta$ :

$$g_t = \frac{1}{2}a_{K,t} \left( \frac{I_t}{K_t} - \delta \right)^2 K_t. \quad (29)$$

We let adjustment costs be asymmetric as in Zhang (2005) by letting  $a_K$  be a step function in  $I/K$ . Different from Zhang, however, we let the asymmetry occur with respect to an investment rate above or below, not zero, but the steady state investment rate,  $\delta$ . In particular,

$$a_{K,t} = \begin{cases} a_K^+ & \text{if } I_t/K_t > \delta \\ a_K^- & \text{if } I_t/K_t \leq \delta \end{cases}, \quad (30)$$

where  $a_K^+ < a_K^-$ . Since aggregate investment is never negative in the data, it is necessary for the asymmetry to kick in at a higher point than zero for it to matter quantitatively in a well-calibrated model. We calibrate the permanent shock model with the same preference parameters as the LRR II model ( $\gamma = 5$ ,  $\psi = 1.5$ ), but with a relative strong asymmetry:  $a_K^+ = 1$  and  $a_K^- = 10$ . We call this model LRR VI. Panel A of Table 7 shows that this form of adjustment costs yield virtually the same unconditional moments as the benchmark LRR II model. However, Panel B shows that there now is significant predictability in aggregate excess equity returns. In line with the stylized facts (e.g., Lettau and Ludvigson, 2001), the aggregate consumption-wealth ratio predicts excess equity market returns with an increasing

$R^2$  and regression coefficient for increasing return horizons. While the  $R^2$ 's are smaller than those typically found in the data for both models, these are population  $R^2$  (i.e., not small-sample). In small-samples the  $R^2$  of long-horizon regressions are significantly biased upwards (e.g., Valkanov, 2003). Further, the actual predictability within the model is economically significant. With an annual equity return volatility of 19%, as in the data, an  $R^2$  of 1.9% in annual return forecasting regressions implies that the annual equity risk premium has a standard deviation of 2.6%, which is quite substantial.<sup>23</sup> The predictability arises in this model as higher adjustment costs in bad times imply higher volatility in the price of capital.

## 7 Conclusion

We analyze a standard stochastic growth model with capital adjustment costs and Epstein-Zin preferences and show how long-run risk arises endogenously as a consequence of the optimal consumption-savings decision. Consumption smoothing induces time-variation in expected consumption growth, even in the case when log technology follows a random walk. While previous research has shown (e.g., Campbell, 1994) that the standard production economy model with power utility preferences can generate time-variation in expected consumption growth, we show in this paper that such endogenous time-variation arises also in the case where agents have a preference for early resolution of uncertainty. I.e., even though agents dislike persistent shocks to expected consumption growth, such long-run consumption risk is optimal in equilibrium. Further, we show that this long-run risk has a substantial impact on the price of risk (the volatility of marginal utility) in the economy.

The implications of endogenous long-run risk depends crucially on the persistence of technology shocks. When the coefficient of relative risk aversion is greater than the reciprocal of the elasticity of intertemporal substitution, agents prefer early resolution of uncertainty and dislike shocks to future economic growth prospects. In this case, endogenous long-run risks increase the price of risk if technology shocks are permanent, but decrease the price of risk if technology shocks are transitory. If agents prefer late resolution of uncertainty, the opposite pattern occurs. Thus, long-run risk will generally be present in standard production based models with endogenously determined consumption where agents have a preference for the timing of the resolution of uncertainty.

The quantitative implications of long-run risk can be large when the standard real busi-

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<sup>23</sup>This follows since  $R^2 = \frac{\sigma^2(E_t[R_{t+1}^e])}{\sigma^2(R_{t+1}^e)} \iff \sigma(E_t[R_{t+1}^e]) = \sigma(R_{t+1}^e) \sqrt{R^2}$ .



ness cycle model is calibrated to match key macro economic moments. In fact, such risks help the model jointly explain a high price of risk, a low relative volatility of consumption growth with a low coefficient of risk aversion. As a priced risk factor, the presence of long-run risk is then an important consideration for welfare experiments in these models.

We find that the elasticity of intertemporal substitution, which strongly affects the dynamics of the macroeconomic variables, also strongly affects the price of risk and the Sharpe ratio of equity in all our calibrations of the model. The coefficient of relative risk aversion, however, mainly affects asset prices. Thus, the elasticity of substitution provides a tight link between quantity dynamics and asset prices in our implementation of the standard stochastic growth model. This is different from previous research on these models (e.g., Tallarini, 2000) that emphasize the role of high risk aversion.

The model provides a theoretical basis for a long-run risk component in aggregate consumption growth. This is of particular interest since it is very difficult to empirically distinguish a small predictable component of consumption growth from i.i.d. consumption growth given the short sample of data we have available. The production economy model identifies the ratio of technology to consumption as a proxy for the otherwise hard to estimate expected consumption growth. We test this link in the time-series of consumption growth and find evidence that consumption growth is predictable in a manner consistent with a model with permanent technology shocks.

The standard real business cycle model still has short-comings. In particular, the dividend and wage dynamics of the models in this paper are counter-factual, and the risk premium in the permanent shock model is too low. We suggest an extension with sticky wages to make both the wage and dividends more in line with what we observe in the data, which increases the risk premium by an order of magnitude. In general, the interaction of labor markets and asset markets in the presence of endogenous long-run risk is an interesting avenue for future research to improve and understand the performance of the standard real business cycle model.

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## 9 Appendix

### 9.1 Model Solution

**The Return to Investment and the Firm's Problem** The firm maximizes firm value. Let  $M_{t,t+1}$  denote the stochastic discount factor. The firm's problem is then:

$$\max_{\{I_t, K_{t+1}, H_t\}_{t=0}^{\infty}} E_0 \left[ \sum_{t=0}^{\infty} M_{0,t} \left\{ q_t \left( K_{t+1} - (1 - \delta) K_t - \phi \left( \frac{I_t}{K_t} \right) K_t \right) \right\} \right], \quad (31)$$

where  $q_t$  denotes the shadow price of the capital accumulation constraint, equivalent to marginal  $q$ : The expected present value of one marginal unit of capital. Maximizing over labor we obtain  $(1 - \alpha) Z_t^{1-\alpha} K_t^\alpha H_t^{-\alpha} = W_t$  and  $H_t = (1 - \alpha)^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}-1} W_t^{-\frac{1}{\alpha}} K_t$ . In other words, we assume an exogenous wage process such that it is optimal for the firm to always hire at full capacity ( $H_t = 1$ ), which is the same amount of labor as the representative agent is assumed to supply. In this case, total wages  $W_t H_t = W_t = (1 - \alpha) Y_t$ , so wages are pro-cyclical and have the same growth rate volatility as total output. The operating profit function of the firm follows as:

$$\begin{aligned} \Pi(K_t, Z_t; W_t) &= Z_t^{1-\alpha} \left[ (1 - \alpha)^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}-1} W_t^{-\frac{1}{\alpha}} K_t \right]^{1-\alpha} K_t^\alpha - W_t (1 - \alpha)^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}-1} W_t^{-\frac{1}{\alpha}} K_t \\ &= Z_t^{1-\alpha} \left[ (1 - \alpha)^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}-1} W_t^{-\frac{1}{\alpha}} \right]^{1-\alpha} K_t - (1 - \alpha)^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}-1} W_t^{1-\frac{1}{\alpha}} K_t \end{aligned} \quad (32)$$

$$= \left( (1 - \alpha)^{\frac{1}{\alpha}-1} Z_t^{\frac{1}{\alpha}-1} W_t^{1-\frac{1}{\alpha}} - (1 - \alpha)^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}-1} W_t^{1-\frac{1}{\alpha}} \right) K_t \quad (33)$$

$$= \left( \alpha (1 - \alpha)^{\frac{1}{\alpha}-1} Z_t^{\frac{1}{\alpha}-1} W_t^{1-\frac{1}{\alpha}} \right) K_t. \quad (34)$$

The operating profit function of the firm is thus linearly homogenous in capital. Substituting out equilibrium wages we obtain  $\Pi(K_t, Z_t; W_t) = \alpha Y_t$ . We re-state the firm's problem:

$$\max_{\{I_t, K_{t+1}\}_{t=0}^{\infty}} E_0 \left[ \sum_{t=0}^{\infty} M_{0,t} \left\{ \Pi(\cdot) - I_t - q_t \left( K_{t+1} - (1 - \delta) K_t - \phi \left( \frac{I_t}{K_t} \right) K_t \right) \right\} \right]. \quad (35)$$

Each period in time the firm decides how much to invest, taking marginal  $q$  as given. The first order conditions with respect to  $I_t$  and  $K_{t+1}$  are immediate:

$$0 = -1 + q_t \phi' \left( \frac{I_t}{K_t} \right), \quad (36)$$

and

$$0 = -q_t + E_t \left[ M_{t+1} \left\{ +q_{t+1} \left( (1 - \delta) - \phi' \left( \frac{I_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}} + \phi \left( \frac{I_{t+1}}{K_{t+1}} \right) \right) \right\} \right]. \quad (37)$$

Substituting out  $q_t$  and  $q_{t+1}$  in (37) yields:

$$\frac{1}{\phi' \left( \frac{I_t}{K_t} \right)} = E_t \left[ M_{t+1} \left\{ \Pi_K(\cdot) + \frac{(1 - \delta) - \phi' \left( \frac{I_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}} + \phi \left( \frac{I_{t+1}}{K_{t+1}} \right)}{\phi' \left( \frac{I_{t+1}}{K_{t+1}} \right)} \right\} \right], \quad (38)$$

$$1 = E_t \left[ M_{t+1} \left\{ \phi' \left( \frac{I_t}{K_t} \right) \left( \Pi_K(\cdot) + \frac{1 - \delta + \phi \left( \frac{I_{t+1}}{K_{t+1}} \right)}{\phi' \left( \frac{I_{t+1}}{K_{t+1}} \right)} - \frac{I_{t+1}}{K_{t+1}} \right) \right\} \right], \quad (39)$$

$$1 = E_t [M_{t+1} R_{t+1}^I]. \quad (40)$$

Equation (40) is the familiar law of one price, with the firm's return to investment:

$$R_{t+1}^I = \phi' \left( \frac{I_t}{K_t} \right) \left( \Pi_K(K_{t+1}, Z_{t+1}; W_{t+1}) + \frac{1 - \delta + \phi \left( \frac{I_{t+1}}{K_{t+1}} \right)}{\phi' \left( \frac{I_{t+1}}{K_{t+1}} \right)} - \frac{I_{t+1}}{K_{t+1}} \right). \quad (41)$$

## 9.2 Risk and the Dynamic Behavior of Consumption

Epstein-Zin preferences have been used with increasing success in the asset pricing literature over the last years (e.g., Bansal and Yaron, 2004, Hansen, Heaton and Li, 2005, Malloy, Moskowitz and Vissing-Jorgensen, 2005, Yogo, 2006). This is both due to their recursive nature, which allows time-varying growth rates to increase the volatility of the stochastic discount factor through the return on the wealth portfolio, as well as the fact that these preferences allow a convenient separation of the elasticity of intertemporal substitution from the coefficient of relative risk aversion.

Departing from time-separable power utility preferences with  $\gamma = \frac{1}{\psi}$  means agents care about the temporal distribution of risk. This is a key assumption of our analysis, because it is precisely this departure from the classic preference structure that renders time-varying expected consumption growth rates induced by optimal consumption smoothing behavior a priced risk factor in the economy.

### 9.2.1 Early Resolution of Uncertainty and Aversion to Time-Varying Growth Rates

To gain some intuition for why a preference for early resolution of uncertainty implies aversion to time-varying growth rates, we revisit an example put forward in Duffie and Epstein (1992). Consider a world where each period of time consumption can be either high or low. Next,

the consumer is given a choice between two consumption gambles,  $A$  and  $B$ . Gamble  $A$  entails eating  $C_0 \equiv \frac{1}{2}C^H + \frac{1}{2}C^L$  today, where  $C^H$  is a high consumption level and  $C^L$  is a low consumption level. Tomorrow you flip a fair coin. If the coin comes up heads, you will get  $C^H$  each period forever. If the coin comes up tails, you will get  $C^L$  each period forever. Gamble  $B$  entails eating  $C_0$  today, and then flip a fair coin *each* subsequent period  $t$ . If the coin comes up heads at time  $t$ , you get  $C^H$  at time  $t$ , and if it comes out tails, you get  $C^L$  at time  $t$ . Thus, in the first case uncertainty about future consumption is resolved early, while in the second case uncertainty is resolved gradually (late). If  $\gamma = \frac{1}{\psi}$  (power utility), the consumer is indifferent with respect to the timing of the resolution of uncertainty and thus indifferent between the two gambles. However, an agent who prefers early resolution of uncertainty (i.e., she likes to plan), prefers gamble  $A$ .

We can now also phrase this discussion in terms of *growth rates*. From this perspective, gamble  $A$  has constant expected consumption growth, while gamble  $B$  has a mean-reverting process for expected consumption growth. Thus, a preference for early resolution of uncertainty translates into an aversion of time-varying expected consumption growth.

Another, more mechanical, way to see this is by directly looking at the stochastic discount factor. It is well known, e.g. Rubinstein (1976), that the stochastic discount factor,  $M_{t+1}$ , is the ratio of the representative agent's marginal utility between today and tomorrow:  $M_{t+1} = \frac{U'(C_{t+1})}{U'(C_t)}$ . Using a recursive argument, Epstein and Zin (1989) show that:

$$\ln M_{t+1} \equiv m_{t+1} = \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1 - \theta) r_{a,t+1}, \quad (42)$$

where  $\Delta c_{t+1} \equiv \ln \frac{C_{t+1}}{C_t}$  and  $r_{a,t+1} \equiv \ln \frac{C_{t+1} + A_{t+1}}{A_t}$  is the return on the total wealth portfolio with  $A_t$  denoting total wealth at time  $t$ .<sup>24</sup> If  $\gamma = \frac{1}{\psi}$ ,  $\theta = \frac{1-\gamma}{1-1/\psi} = 1$ , and the stochastic discount factor collapses to the familiar power utility case. However, if the agent prefers early resolution of uncertainty, the return on the wealth portfolio appears as a risk factor. More time-variation in expected consumption growth (the expected "dividends" on the total wealth portfolio) induces higher volatility of asset returns, in turn resulting in a more volatile stochastic discount factor and thus a higher price of risk in the economy.<sup>25</sup>

The effect on the equity premium can be understood by considering a log-linear ap-

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<sup>24</sup>Note that our representative household's total wealth portfolio is composed of the present value of future labor income in addition to the value of the firm.

<sup>25</sup>This assumes that the correlation between the return on the wealth portfolio and consumption growth is non-negative, which it is for all parameter values we consider in this paper (and many more).



proximation (see Campbell, 1999) of returns and the pricing kernel, yielding the following expressions for the risk free rate and the equity premium:

$$r_{f,t+1} \approx -\log \beta + \frac{1}{\psi} E_t [\Delta c_{t+1}] - \frac{\theta}{2\psi^2} \sigma_{t,c}^2 + \frac{(\theta-1)}{2} \sigma_{t,r^A}^2, \quad (43)$$

$$E_t [r_{t+1}^E] - r_{f,t+1} \approx \frac{\theta}{\psi} \sigma_{t,r^E c} + (1-\theta) \sigma_{t,r^E r^A} - \frac{\sigma_{t,r^E}^2}{2}, \quad (44)$$

where  $E_t [\Delta c_{t+1}]$  is expected log consumption growth,  $\sigma_{t,c}$ ,  $\sigma_{t,r^A}$ ,  $\sigma_{t,r^E}$ , are the conditional standard deviations of log consumption growth, the log return on the total wealth portfolio, and the log equity return, and  $\sigma_{t,r^E c}$  and  $\sigma_{t,r^E r^A}$  are the conditional covariances of the log equity return with log consumption growth and the log return on the total wealth portfolio respectively. We can see how the level of the equity premium depends directly on the covariance of equity returns with returns on the wealth portfolio.

### 9.2.2 Predictability

The benchmark models presented in section 3 do not generate economically significant time-variation in the equity premium. The technology shocks are homoskedastic and risk preferences are constant, so any time-variation in the price of risk and/or equity premium must come from endogenous heteroskedasticity in the consumption (and/or dividend) process. Figure 9 shows the equity risk premium for the model with permanent technology shocks (LRR II) plotted against conditional expected consumption growth. When capital is low, relative to the level of technology, expected consumption growth is high in the permanent shock model as the marginal productivity of capital is high and agents therefore invest. Thus, these are good times. When expected consumption growth is low, investment is low, and we associate this with a recession. The figure shows that the equity risk premium is higher in recessions. However, the magnitude of this time-variation is too small to generate predictability regressions with the same  $R^2$ 's as in the data (not reported). There is some endogenous heteroskedasticity in shocks to both realized and expected consumption growth, but the two go in opposite directions. The result is a price of risk that is almost constant.

### 9.2.3 Technology and Risk Aversion

Standard production technologies do not allow agents to hedge the technology shock. Agents must in the aggregate hold the claim to the firm's dividends. Therefore, the only action

**Figure 9 - Conditional Moments**

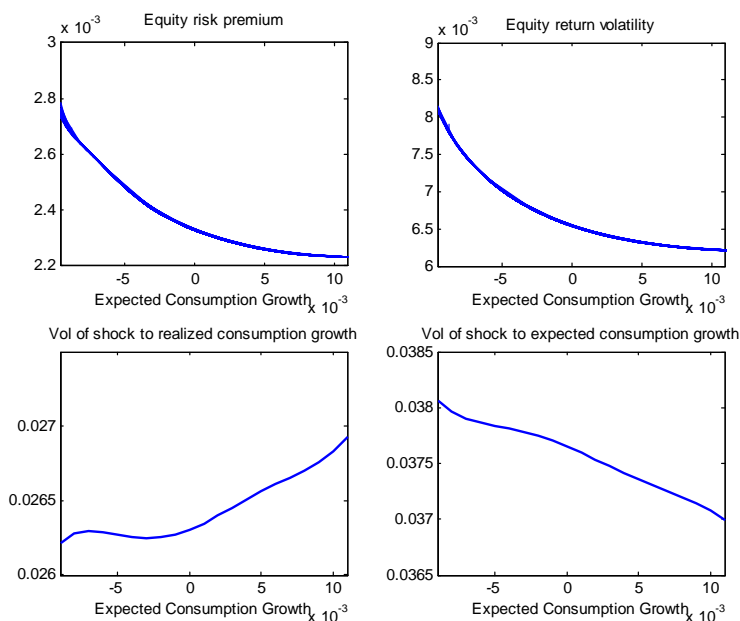


Figure 9: **Conditional Moments.** The plots show the conditional equity risk premium, return volatility, volatility of shocks to realized and expected consumption growth plotted against the conditional level of expected consumption growth. The latter is inversely related to the level of capital relative to the level of technology.

available to agents at time  $t$  in terms of hedging the shock at time  $t + 1$ , is to increase savings in order to increase wealth for time  $t + 1$ . The shock will still hit the agents at time  $t + 1$  though, no matter what. Wealth levels may be higher if a bad realization of the technology shock hits the agents, but wealth is also higher if a good realization of the technology shock occurs. The *difference* between the agents' utility for a good realization of the technology shock in period  $t + 1$  relative to their utility for a bad realization of the shock is thus (almost) unaffected. However, it is this utility *difference* the agents care about in terms of their risk aversion. Now, because the agents' utility function is concave, this is not quite true. A higher wealth level in both states of the world does decrease the difference between utility levels. Agents thus respond by building up what is referred to as "buffer-stock-savings". This is, however, a second-order effect. As a result, the dynamic behavior of consumption growth is largely unaffected by changing agents' coefficient of relative risk aversion. The fundamental consumption risk in the economy remains therefore (almost) the same when we increase risk aversion ( $\gamma$ ) while holding the *EIS* ( $\psi$ ) constant. Asset prices,

of course, respond as usual to higher levels of risk aversion.

Table 8 confirms this result for calibrations with both a coefficient of relative risk aversion ( $\gamma$ ) of 5, as well as versions of the models with a higher level of risk aversion ( $\gamma = 25$ ).

### 9.3 Accuracy of the Approximation of the Endogenous Consumption Process

In Section 8.4 we propose the following approximation for the dynamics of the endogenous process for consumption:

$$\Delta c_{t+1} = \mu + x_t + \sigma_\eta \eta_{t+1}, \quad (45)$$

$$x_{t+1} = \rho x_t + \sigma_e e_{t+1}, \quad (46)$$

$$\sigma_{\eta,e} = \text{corr}(\eta_{t+1}, e_{t+1}). \quad (47)$$

Here  $\Delta c_{t+1}$  is log realized consumption growth,  $x_t$  is the time-varying component of expected consumption growth, and  $\eta_t, e_t$  are zero mean, unit variance, and normally distributed disturbance terms with correlation  $\sigma_{\eta,e}$ . This functional form for log consumption growth is identical to the one assumed by Bansal and Yaron (2004) as driving process of their exchange economy model. Our results therefore provide a theoretical justification for their particular exogenous consumption growth process assumption. To evaluate whether the above specified process is a good approximation of the true consumption growth dynamics we first estimate the process from simulated data for a whole range of different model calibrations both with random walk- as well as with AR(1) technology processes. Then we compare the autocorrelation function obtained directly from the simulated data to the one implied by the above specified process which we have imposed on the data.

For the random walk technology the autocorrelation functions are virtually indistinguishable in all cases we have examined. Figure 10 shows this for the LRR II Model (permanent technology shocks).<sup>26</sup> For the AR(1) technology the approximation turns out to get worse the lower the persistence of the driving process. Figure 10 shows the autocorrelation functions for Model LRR I (transitory technology shocks). A look at Figure 1 makes clear why the above specified approximation for the dynamics of the endogenous process for consumption is worse for the case where technology shocks are transitory, because the impulse-response

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<sup>26</sup>We assume the disturbance terms  $\eta$  and  $e$  to be i.i.d. normally distributed. The shocks we obtain when we estimate our postulated process for consumption growth from simulated data turn out to be very close to normal. They display mild heteroscedasticity.

**Table 8**

**The Effect of Risk Aversion on Macroeconomic Time Series**

Table 8: This table reports relevant macroeconomic moments and consumption dynamics for the long-run risk models (LRR I and LRR II) with different levels of the coefficient of relative risk aversion. We estimate the following process for the consumption dynamics:  $\Delta c_{t+1} = \mu + x_t + \sigma_\eta \eta_{t+1}$ ,  $x_{t+1} = \rho x_t + \sigma_e e_{t+1}$ .  $\Delta x = \log(X_t) - \log(X_{t-1})$ , and  $\sigma[X]$  denotes the standard deviation of variable  $X$ . We use annual U.S. data from 1929 to 1998 from the Bureau of Economic Analysis. The sample is the same as in Bansal and Yaron (2004). Under Panel B we report the calibration of the exogenous consumption process Bansal and Yaron use. All values reported in the table are quarterly.

Statistic	Transitory Shocks		Permanent Shocks		
	$\varphi = 0.95$		$\varphi = 1.00$		
	$\psi = 0.05$	$\psi = 0.05$	$\psi = 1.5$	$\psi = 1.5$	
	$\beta = 1.064$	$\beta = 1.064$	$\beta = 0.998$	$\beta = 0.998$	
	$\xi = 0.70$	$\xi = 0.70$	$\xi = 18.0$	$\xi = 18.0$	
	$\sigma_z = 2.59\%$	$\sigma_z = 2.59\%$	$\sigma_z = 2.63\%$	$\sigma_z = 2.63\%$	
	$\gamma = \mathbf{5}$	$\gamma = \mathbf{25}$	$\gamma = \mathbf{5}$	$\gamma = \mathbf{25}$	
<b>Panel A: Macroeconomic Moments (Quarterly)</b>					
	U.S. Data 1929-1998				
$\sigma[\Delta y]$ (%)	2.62	2.61	2.61	2.64	2.64
$\sigma[\Delta c]/\sigma[\Delta y]$	0.52	0.52	0.45	0.52	0.43
$\sigma[\Delta i]/\sigma[\Delta y]$	3.32	2.36	2.12	1.83	1.60
<b>Panel B: Consumption Dynamics:</b> $\Delta c_{t+1} = \mu + x_t + \sigma_\eta \eta_{t+1}$ , $x_{t+1} = \rho x_t + \sigma_e e_{t+1}$ .					
	Bansal, Yaron Calibration				
$\sigma[\Delta c]$ (%)	1.360	1.360	1.129	1.362	1.123
$\sigma[x]$ (%)	0.172	0.126	0.120	0.336	0.342
$\rho$	0.938	0.959	0.959	0.970	0.974

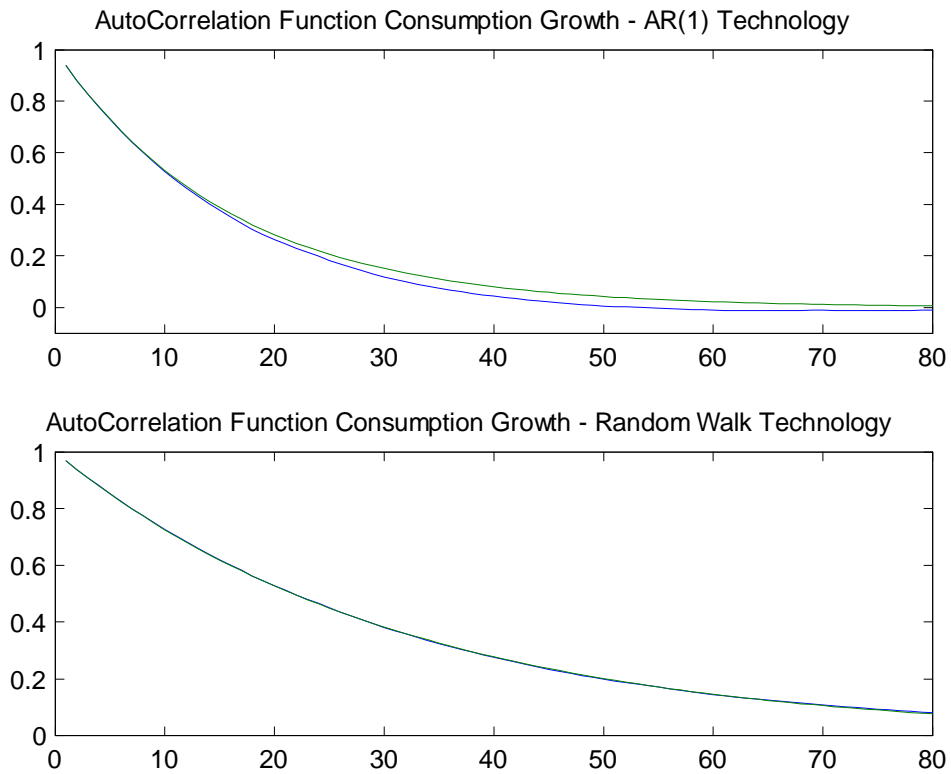


Figure 10: **Autocorrelation Functions Consumption Growth** Comparison of the autocorrelation function obtained directly from simulated data of models LRR I and LRR II to the autocorrelation function implied by the postulated process for expected consumption growth which we have estimated from the same simulated data from the respective models.

of consumption to technology shocks is sometimes "hump-shaped". We therefore conclude that our postulated process is a good representation of the endogenous consumption growth dynamics for models with highly persistent technology shocks.<sup>27</sup>

## 9.4 Numerical Solution

### 9.4.1 Solution Algorithm

We solve the following model:

$$V(K_t, Z_t) = \max_{C_t, K_{t+1}} \left\{ \left[ (1 - \beta) C_t^{\frac{1-\gamma}{\theta}} + \beta (E_t [V(K_{t+1}, Z_{t+1})^{1-\gamma}])^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \right\}, \quad (48)$$

$$K_{t+1} = (1 - \delta) K_t + \phi \left( \frac{I_t}{K_t} \right) K_t, \quad (49)$$

$$I_t = Y_t - C_t, \quad (50)$$

$$Y_t = Z_t^{(1-\alpha)} K_t^\alpha, \quad (51)$$

$$\ln Z_{t+1} = \varphi \ln Z_t + \varepsilon_{t+1}, \quad (52)$$

$$\varepsilon_t \sim N(\mu, \sigma_z). \quad (53)$$

We focus in this appendix on the case where  $\varphi = 1$ . Since then the process for productivity is non-stationary, we need to normalize the economy by  $Z_t$ , in order to be able to numerically solve the model. To be precise, we let  $\widehat{K}_t = \frac{K_t}{Z_t}$ ,  $\widehat{C}_t = \frac{C_t}{Z_t}$ ,  $\widehat{I}_t = \frac{I_t}{Z_t}$ , and substitute. In the so transformed model all variables are stationary. The only state variable of the normalized model is  $\widehat{K}$ .<sup>28</sup> We can work directly on the appropriately normalized set of equations and then re-normalize after having solved the model.<sup>29</sup>

<sup>27</sup>This conclusion relies on the assumption that the consumption process is covariance-stationary, which it is since the production function is constant returns to scale and preferences are homothetic. The autocorrelation function is then one of the fundamental time series representations. See, e.g., Hamilton (1994).

<sup>28</sup>Note that  $Z$  is *not* a state variable of the normalized model. This is due to the fact that we assume the autoregressive coefficient of the process for productivity  $\ln Z_{t+1} = \rho \ln Z_t + \varepsilon_{t+1}$  to be unity:  $\rho = 1$ . As a consequence,  $\Delta Z$  is serially uncorrelated.

<sup>29</sup>In the paper we also report results for models where  $\rho < 1$ . In this case we work directly on the above non-normalized set of equations. The state variables are then  $K$  and  $Z$ . The solution algorithm is identical to the case where  $\rho = 1$ .

The value function is given by:

$$\widehat{V}(\widehat{K}_t) = \max_{\widehat{C}_t, \widehat{K}_{t+1}} \left\{ \left[ (1 - \beta) \widehat{C}_t^{\frac{1-\gamma}{\theta}} + \beta \left( E_t \left[ (e^{\varepsilon_{t+1}})^{1-\gamma} \left( \widehat{V}(\widehat{K}_{t+1}) \right)^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \right\}. \quad (54)$$

We parameterize the value function with a 5th order Chebyshev orthogonal polynomial over a  $6 \times 1$  Chebyshev grid for the state variable  $\widehat{K}$ :

$$\Psi^A(\widehat{K}) = \widehat{V}(\widehat{K}). \quad (55)$$

We use the value function iteration algorithm. At each grid point for the state  $\widehat{K}$ , given a polynomial for the value function  $\Psi_i^A(\widehat{K})$ , we use a numerical optimizer to find the policy ( $\widehat{C}^*$ ) that maximizes the value function:

$$\widehat{K}_{t+1}^* e^{\varepsilon_{t+1}} = \widehat{Y}_t - \widehat{C}_t^* + (1 - \delta) \widehat{K}_t, \quad (56)$$

$$\widehat{V}^*(\widehat{K}_t) = \left[ (1 - \beta) \left( \widehat{C}_t^* \right)^{\frac{1-\gamma}{\theta}} + \beta \left( E_t \left[ (e^{\varepsilon_{t+1}})^{1-\gamma} \left( \Psi_i^A(\widehat{K}_{t+1}^*) \right)^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}}, \quad (57)$$

where Gauss-Hermite quadrature with 5 nodes is used to approximate the expectations operator. We use a regression of  $\widehat{V}^*$  on  $\widehat{K}$  in order to update the coefficients of the polynomial for the value function and so obtain  $\Psi_{i+1}^A(\widehat{K})$ .