APPROXIMATIONS OF DYNAMIC, MULTILOCATION PRODUCTION AND INVENTORY PROBLEMS*

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Consider a central depot (or plant) which supplies several locations experiencing random demands. Orders are placed (or production is initiated) periodically by the depot. The order arrives after a fixed lead time, and is then allocated among the several locations. (The depot itself does not hold inventory.) The allocations are finally received at the demand points after another lag. Unfilled demand at each location is backordered. Linear costs are incurred at each location for holding inventory and for backorders. Also, costs are assessed for orders placed by the depot. The object is to minimize the expected total cost of the system over a finite number of time periods.

This system gives rise to a dynamic program with a state space of very large dimension. We show that this model can be systematically approximated by a single-location inventory problem. All the qualitative and quantitative results for such problems can then be applied. (INVENTORY AND PRODUCTION; APPROXIMATIONS; STOCHASTIC MODELS)

1. Introduction

Description of the System

This paper is concerned with a system consisting of a central depot which supplies J locations where exogenous, random demands for a single commodity must be filled. Inventories are reviewed and decisions taken periodically. In each time period the depot may place an *order* for the product. This order arrives at the depot after a lag of L periods. Then the order is *allocated* among the J demand points. These shipments reach the locations after a further delay of l periods.

Demands at the locations are assumed to be independent in different periods, but there may be dependence among demands at different locations in the same period. (For most of the paper we assume the demands in each period have a joint normal distribution. The results also apply to other distributions in certain special cases.)

Unfilled demand at each location is backordered. Linear costs are incurred at each location for holding inventory and for backorders. Also, costs are assessed for orders placed by the depot. *Any* structure may be assumed for the ordering cost functions; these may represent, for example, economies of scale, resulting from quantity discounts or fixed costs or smoothing costs (i.e., costs depending on the previous order as well as the current one).

Demands and costs need not be stationary. Holding and penalty costs may also differ among locations. The object is to minimize the expected total cost of the system over a finite number T of time periods.

Assumptions and Interpretations

An important restriction is the assumption that no inventory is carried at the central depot. This assumption is appropriate, for example, when the "depot" does not represent a physical location at all, but rather a centralized ordering function; here, directions for shipment to the ultimate destinations can be postponed until L periods

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after the bulk order is placed. Even if it does correspond to a certain place, the depot acts as a transshipment center, not a stocking point. See Eppen and Schrage (1981) for a fuller discussion of the assumption. Rosenfield and Pendrock (1980) refer to systems of this type as "coupled systems."

The model can also be viewed as representing a two-stage production process, involving several products instead of several locations. An order then corresponds to a decision to make blanks or some other intermediate product. This stage requires L periods. These blanks are later fabricated (in l periods) into the several finished products. The assumption of no central inventory implies that the intermediate product cannot be stored, perhaps because it is highly perishable or dangerous, e.g., a molten metal. In this context the freedom to specify any "ordering" cost function is important, since production smoothing costs may be significant.

The model may also represent a single-stage, multiproduct system, where the products are made from a common raw material with an *L*-period order lead time. With L = 0 we obtain the important multi-item inventory problem with joint purchase costs, cf., Peterson and Silver (1979, p. 495).

In general, centralized ordering as represented here may offer two distinct advantages. We mentioned above the possibility of economies of scale in ordering. Note also that one could choose to decide the future allocations at the same time as the order. Postponing the allocations permits one to observe the demands in the intervening Lperiods, and thus to make better informed allocations. The phrase "statistical economies of scale" has been used (Eppen and Schrage 1981) to describe this effect.

Summary of Results

The problem of determining optimal order sizes and allocations for this system can be formulated as a dynamic programming problem (§2). The state space of this problem has very large dimension, however, so standard numerical procedures for dynamic programs are not applicable. Our goal here is to obtain a good approximation to the problem which is computationally tractable.

A key concept in the simplification is that of *myopic allocation*: In period t we must divide up the order placed in period t - L, which has just arrived at the depot, among the J locations. The myopic allocation solves what we call the *myopic allocation problem*—it minimizes the expected costs in period t + l, when the allocation actually takes effect, ignoring costs in all subsequent periods.

We use a result of Zipkin (1982a) as part of a two-step procedure to approximate the cost of myopic allocation, as a function of state variables in each period (§3). This method, it turns out, can be applied recursively to the dynamic program (§4). (We confine ourselves to normal demand distributions; extensions to more general distributions are discussed in §6.) Two conclusions result: First, myopic allocations in fact are optimal in every period, up to the approximations. Second, what remains after the approximations are performed is a dynamic program with only a single state variable. The form of this model, moreover, is precisely that of a single-location dynamic inventory problem. The appropriate qualitative and quantitative results that have been developed for problems of this type (depending on the form of the original ordering cost function, which is inherited by the approximation) can thus be applied.

The optimal (or a good, heuristic) ordering policy for the reduced problem, together with myopic allocation, should thus comprise an effective overall policy for the original problem.

In §5, we present and discuss some computational results. For many important cases these confirm the analytical arguments supporting the approximation and myopic allocations.

In §6, we treat a number of special cases and extensions. First, we explore the single-cycle problem, there T = L + l + 1. This case represents, in effect, a one-time decision problem with a lag between the (single) order and the (single) allocation decision. Here, the approximation results in a single newsboy problem, whose cost function can be explicitly computed and easily minimized. This result is then used to reduce a three-stage stochastic transportation problem to a tractable two-stage model, thus extending earlier work by Zipkin (1982b). Next, we show how our results can be extended to other classes of demand distributions. Finally, we point out that systems with more than two levels can also be reduced to single-location problems. These results apply also to problems with both multiple products and multiple locations.

Related Literature

Clark (1972) surveys a variety of multi-echelon inventory models. Some more recent references are in Peterson and Silver (1979, Chapters 12, 13).

Considerable effort has been devoted to understanding single-period, multilocation models, often including complications not treated here such as redistribution or transshipment of inventories. Quite general qualitative results for such problems have been developed by Karmarkar and Patel (1975); see their references for earlier work in this area. More recent papers include those by Hoadley and Heyman (1977), Prastacos (1978) and Federgruen and Zipkin (1983).

Some of these qualitative results have been extended to multiperiod problems by Tan (1974), Karmarkar (1981) and Showers (1981a, 1981b). Laakso and Thomopoulos (1979) and Rosenbaum (1981) give heuristic analyses of certain aspects of such problems. See also the recent book edited by Schwarz (1981), especially Deuermeyer and Schwarz (1981) and Ehrhardt, Schultz and Wagner (1981).

The paper by Clark and Scarf (1960), best known for its treatment of systems with several levels in series, also contains a brief discussion of more general structures of the type studied here. Using an approximation which is essentially equivalent to our first step, they develop a parametric cost function for a problem analogous to our myopic allocation problem. This function, however, is expressed in terms of the optimal allocations, and thus is not computable in closed form.

The papers by Veinott (1965), Bessler and Veinott (1966) and Ignall and Veinott (1969) differ from ours mainly in the absence of an explicit depot; ordering and allocation occur at the same instant (L = 0), and there are no economies of scale. Their models allow joint constraints in inventory positions after allocations, while in our model the allocations themselves are constrained by previous orders. They show that myopic policies are optimal under various assumptions, and it is of interest that our (approximate) myopia results are reminiscent of theirs.

A recent paper of Federgruen, Groenevelt and Tijms considers a continuous-review problem similar to ours; they cite earlier work on this problem.

Our model and the results obtained extend those considered by Eppen and Schrage (1981). The major departures here are the following:

-Eppen and Schrage (1981) assume normal demands throughout. For some important special cases we allow other classes of distributions, including the exponentials and the gammas.

-Eppen and Schrage (1981) require holding and penalty costs to be identical across locations. No such restriction is imposed here. (The approximations, however, are "closer" the more nearly identical these costs are. See Zipkin 1982a.)

—Assuming demands and costs are stationary over time, Eppen and Schrage (1981) consider an infinite-horizon model, using the criterion of expected average cost per period. We allow nonstationary demands and costs and consider the finite-horizon case.

—Most importantly, Eppen and Schrage (1981) require that an ordering policy of a specific type (a stationary order-up-to, or critical-number policy) be followed. *Given* this restriction, and *assuming* myopic allocations, Eppen and Schrage (1981) show how to compute a critical number, which is optimal up to an approximation essentially equivalent to our first step (called the allocation assumption in Eppen and Schrage 1981). When there are fixed costs in ordering, Eppen and Schrage (1981) permit ordering every m periods, but the analysis requires still further approximations. Here, we permit the ordering policy to be determined by analysis of the approximate dynamic program, according to the actual nature of the ordering costs, and justify myopic allocations themselves as (approximately) optimal.

Many approximation schemes have been proposed to reduce the dimensions of general dynamic programs; see Morin (1979) for extensive references. Our aproach is somewhat similar in spirit to polynomial approximation (e.g., Bellman, Kalaba and Kotkin 1963), in that we fit a certain function by a simpler function of known form. We approximate one-period costs instead of the optimal cost function, however. The function we use is known to fit the true costs well, and this function has a special inventory-theoretic interpretation (cf. §3). Also similar in spirit is the approach of state-space relaxation (Christofides, Mingozzi and Toth 1981), although their problems and methods differ considerably from ours.

2. Formulation

Lags

L = time required for an order to arrive at the depot,

l = time required for allocations to arrive at the demand points.

Indices

j indexes demand locations, j = 1, ..., J, *t* indexes time periods, t = 1, ..., T (*s* is also used as a time index).

State and Action Variables

 x_{it} = inventory at location *j* at the beginning of period *t*,

 y_t = order placed by the depot in period t,

 z_{jt} = allocation to location j in period t, $z^{t} = (z_{jt})_{j=1}^{J}$.

We define a vector containing those orders already placed which have yet to arrive at the depot, and those (say, the last L') on which the cost of the current order depends:

$$\tilde{y}^{t} = (y_{s})_{s=t=N}^{t=1}$$
 where $N = \max\{L, L'\}.$

Our formulation will be simplified if we apply a transformation of variables (a standard one for problems with lags):

$$\hat{x}_{jt} = x_{jt} + \sum_{s=t-l}^{t-1} z_{js}$$

= initial inventory at location *j*, plus all previous allocations still in transit (analogous to the economic inventory at the beginning of period *t*), $\hat{x}^t = (\hat{x}_{jt})_{j=1}^J$.

The last relevant order is placed in period T - L - l, and the last allocation made in period T - l. For convenience we use the notation above for all periods, with zeroes inserted where appropriate.

The state of the system at the beginning of period t can thus be described by the

vector (\hat{x}^t, \tilde{y}^t) . The actions taken in period t must satisfy

$$y_t \ge 0, \qquad \sum_{j=1}^J z_{jt} = y_{t-L}, \qquad z^t \ge 0.$$
 (1)

Demands

 u_{jt} = demand at location j in period t, $u^t = (u_{jt})_{j=1}^J$,

 \vec{F}_{jt} = marginal cumulative distribution function (cdf) of u_{jt} .

For most of the paper we require that each u^{t} have a joint normal distribution. Thus, F_{ii} is characterized by the parameters

$$\mu_{jt} = E(u_{jt}), \qquad \sigma_{jt}^2 = \operatorname{Var}(u_{jt})$$

Also, let Φ denote the standard normal cdf. Other distributions are considered in §6, but only under some further assumptions. The u^t are assumed throughout to be independent random vectors.

We shall need to refer to the sum of the (independent) demands at location j over l+1 periods: $\hat{u}_{jt} = \sum_{s=t}^{t+1} u_{js}$. Letting \hat{F}_{jt} denote the cdf of \hat{u}_{jt} , the \hat{F}_{jt} are also normal cdf's with

$$\hat{\mu}_{jt} = E(\hat{u}_{jt}) = \sum_{s=t}^{t+1} \mu_{js}, \qquad (2)$$

$$\hat{\sigma}_{ji}^{2} = \operatorname{Var}(\hat{u}_{ji}) = \sum_{s=1}^{t+1} \sigma_{js}^{2}.$$
(3)

Costs

 $c_t(y_t, \tilde{y}^t) = \text{cost to order } y_t \text{ in period } t, \text{ possibly depending on earlier orders.}$

Since the allocation z^{t} is the last decision affecting holding and penalty costs in period t + l, we may count these costs as if they were incurred in period t. Thus,

 h_{jt} = unit cost of holding inventory at location j from period t + l to period t + l + 1,

 p_{ji} = unit penalty cost for demand backordered at location j from period t + l to t + l + 1.

The one period expected holding and penalty costs in period t + l, as viewed from period t, are $Q_t(z^t, \hat{x}^t) = \sum_{j=1}^{J} q_{ji}(z_{jt}, \hat{x}_{jt})$, where

$$q_{jt}(z_{jt}, \hat{x}_{jt}) = h_{jt} E[\hat{x}_{jt} + z_{jt} - \hat{u}_{jt}]^{+} + p_{jt} E[\hat{x}_{jt} + z_{jt} - \hat{u}_{jt}]^{-}.$$

By a standard argument,

$$q_{jt}(z_{jt}, \hat{x}_{jt}) = p_{jt} \big[\hat{\mu}_{jt} - (\hat{x}_{jt} + z_{jt}) \big] + \big(p_{jt} + h_{jt} \big) \int_{-\infty}^{\hat{x}_{jt} + z_{jt}} \hat{F}_{jt}(u) \, du.$$

We remark that linear shipping costs can be included also, by a suitable redefinition of p_{ji} and h_{ji} .

Dynamics and Recursive Equations

Given the initial state, the actions and the demands in period t, the state in the next period is determined as follows:

$$\hat{x}^{t+1} = \hat{x}^t + z^t - u^t, \qquad \tilde{y}^{t+1} = (y_{t-N+1}, \dots, y_{t-1}, y_t).$$

With the definitions above we can state the recursive functional equations whose

solution provides the optimal policy for the problem. Let

 $f_t(\hat{x}^t, \tilde{y}^t) = \text{minimum total expected ordering costs in periods } t$ through T, and penalty and holding costs in periods t + l through T, given the system starts period t in state (\hat{x}^t, \tilde{y}^t) .

Then, we may set $f_{T-l+1} = 0$. (Thus, f_1 in fact includes all costs which can be influenced by our decisions.) For periods t = 1, ..., T - l, we have

$$f_{t}(\hat{x}^{t}, \tilde{y}^{t}) = \min_{y_{t}, z^{t}} \{ c_{t}(y_{t}, \tilde{y}^{t}) + Q_{t}(z^{t}, \hat{x}^{t}) + Ef_{t+1}[(\hat{x}^{t} + z^{t} - u^{t}), (y_{t-N+1}, \dots, y_{t})] : y_{t}, z^{t} \text{ satisfying (1)} \}.$$
(4)

The state space of this dynamic program has dimension J + N, and there is no obvious way to decompose the problem into smaller ones. For any but the smallest values of J, N and T, clearly, exact solution of the equations is impractical.

More Aggregate Variables

We make use of a variety of aggregate-level variables below, so the following convention has been followed (more or less): Capital letters denote sums over j, a caret (^) denotes a sum over l + 1 periods, and a bar (⁻) denotes a sum over L periods. Thus, the notation \hat{x}^{t} and \hat{u}^{t} above.

Define

$$U_t = \sum_{j=1}^J u_{jt}, \qquad \hat{U}_t = \sum_{s=t}^{t+l} U_s, \qquad \overline{U}_t = \sum_{s=t}^{t+L-1} U_s.$$

These are all normal random variables, characterized by the parameters in Table 1. Also, let

$$X_{t} = \sum_{j=1}^{J} x_{jt},$$

$$\hat{X}_{t} = \sum_{j=1}^{J} \hat{x}_{jt} = X_{t} + \sum_{s=t-L-l}^{t-L-1} y_{s},$$

$$X_{t}^{\Delta} = \hat{X}_{t} + \sum_{s=t-L}^{t-1} y_{s} = X_{t} + \sum_{s=t-L-l}^{t-1} y_{s}.$$

Thus, X_t represents total inventory in the system, \hat{X}_t is system inventory plus all allocations yet to arrive at the locations, and X_t^{Δ} is system inventory plus outstanding allocations, plus all orders placed but not yet received at the depot (which we may describe as the total economic inventory in the system), all at the beginning of period t.

The symbol tilde (~) is used in several ways, and has no single meaning.

TABLE 1 Aggregate Demands					
	U _t	\hat{U}_t	\overline{U}_t		
cdf	G _t	\hat{G}_t	\overline{G}_t		
mean	$M_t = \sum_{j=1}^J \mu_{jt}$	$\hat{M}_{t} = \sum_{s=t}^{t+l} M_{s} = \sum_{j=1}^{J} \hat{\mu}_{jt}$	$\overline{M}_t = \sum_{s=t}^{t+L-1} M_s$		
variance	S_t^2	$\hat{S}_t^2 = \sum_{s=t}^{t+l} S_s^2$	$\overline{S}_t^2 = \sum_{s=t}^{t+L-1} S_s^2$		

3. Approximating the Myopic Allocation Problem

For any period $t \leq T - l$ the problem defining the myopic allocation is the following:

$$R_{t}(\hat{x}^{t}, y_{t-L}) = \min_{z^{t}} Q_{t}(z^{t}, \hat{x}^{t})$$

subject to $\sum_{j=1}^{J} z_{jt} = y_{t-L}, \quad z^{t} \ge 0.$ (5)

We now apply the methods of Zipkin (1982a) to approximate the function R_t . Later we shall discuss the effects of the approximation. (Note. Though we approximate (5) below in order to determine an order policy, it is (5) itself which constitutes the myopic allocation policy.)

The *first step* in the approximation is to relax the nonnegativity constraints in (5). Let $R_t^{-}(\hat{x}^t, y_{t-1})$ be the minimal cost of the remaining problem.

For the second step, the optimality conditions of the remaining problem can be manipulated (Zipkin 1982a) to yield the following (where λ is the Lagrange multiplier, and using $\hat{F}_{jt}^{-1}(u) = \hat{\mu}_{jt} + \hat{\sigma}_{jt}\Phi^{-1}(u)$, for all *u*):

$$\sum_{j=1}^{J} \hat{\sigma}_{jt} \Phi^{-1} \left(\frac{p_{jt} + \lambda}{p_{jt} + h_{jt}} \right) = \hat{X}_{t} + y_{t-L} - \hat{M}_{t}.$$
(6)

(The term in parentheses is the critical ratio that always appears in such constrained newsboy problems.) Expression (6) can be viewed as a parametric equation in λ . It is shown in Zipkin (1982a) that the solution to an equation of this form can be well approximated by the following function of its right-hand side:

$$\lambda \approx -p_{i}^{0} + \left(p_{i}^{0} + h_{i}^{0}\right) \Phi \left[(\hat{X}_{i} + y_{i-L} - \hat{M}_{i}) / \tilde{S}_{i} \right],$$
(7)

where $\tilde{S}_t = \sum_{j=1}^J \hat{\sigma}_{jt}$, and p_t^0 and h_t^0 are constants derived by any one among several interpolation methods. (When the costs are identical across locations, so $p_{jt} = p_t$ and $h_{jt} = h_t$, $j = 1, \ldots, J$, (7) is exact with $p_t^0 = p_t$, $h_t^0 = h_t$.) Note too that $\lambda = \partial R_t^- / \partial y_{t-L}$. Thus, an approximation $\tilde{R}_t \approx R_t^-$ can be obtained

by integrating (7): Letting

$$\tilde{G}_t(U) = \Phi\left[(U - \hat{M}_t) / \tilde{S}_t \right]$$

$$\tilde{R}_{t}(\hat{X}_{t}, y_{t-L}) \equiv p_{t}^{0} \left[\hat{M}_{t} - (\hat{X}_{t} + y_{t-L}) \right] + \left(p_{t}^{0} + h_{t}^{0} \right) \int_{-\infty}^{\hat{X}_{t} + y_{t-L}} \tilde{G}_{t}(U) \, dU.$$
(8)

(The constant term $p_t^0(\hat{M}_t - \hat{X}_t)$ can be shown to be a reasonable choice.) Observe that \tilde{R}_t has the form of a one-period expected holding-and-penalty cost function at a single location (which we may consider to be the depot). (In particular, $\tilde{R_t}$ is strictly convex in $\hat{X}_i + y_{t-L}$.) It depends only on the total (economic) inventory in the system \hat{X}_i , not its distribution among the locations.

Approximation of the Dynamic Program 4.

We are now prepared for the major result. We shall define recursively an approximation \tilde{f}_t to the function f_t , starting with $\tilde{f}_{T-l+1} = f_{T-l+1} = 0$. Suppose we can write the approximation for period t + 1 as $\tilde{f}_{t+1} = \tilde{f}_{t+1}(\hat{X}_{t+1}, \tilde{y}^{t+1})$ which, as the notation suggests, depends on \hat{x}^{t+1} only through the aggregate inventory \hat{X}_{t+1} . (This is certainly true for t = T - l.)

Given such an approximation, we would like to find the optimal policy for period t. That is, we wish to solve the following problem, derived from (4) by replacing f_{t+1} with \tilde{f}_{t+1} :

$$\min_{y_t, z^t} \{ c_t(y_t, \tilde{y}^t) + Q_t(z^t, \hat{x}^t) + E\tilde{f}_{t+1}[(\hat{X}_t + y_{t-L} - U_t), (y_{t-N+1}, \ldots, y_t)] \},\$$

subject to (1). But this is equivalent to

$$\min_{y_{t} \ge 0} \left\{ c_{t}(y_{t}, \tilde{y}') + E\tilde{f}_{t+1} \left[(\hat{X}_{t} + y_{t-L} - U_{t}), (y_{t-N+1}, \dots, y_{t}) \right] \right\}
+ \min_{z'} \left\{ Q_{t}(z', \hat{x}') : \sum_{j=1}^{J} z_{jt} = y_{t-L}, z' \ge 0 \right\}.$$
(9)

Thus, the minimizations over y_t and z^t separate. The minimization over z^t , moreover, is precisely problem (5). In sum, up to an approximation \tilde{f}_{t+1} of the form above, a myopic allocation is optimal in period t.

Further, suppose we define \tilde{f}_t by replacing the true minimum cost over z^t , namely R_t , by \tilde{R}_t in (9). Then, we may write

$$\tilde{f}_{t}(\hat{x}^{t}, \tilde{y}^{t}) = \tilde{f}_{t}(\hat{X}_{t}, \tilde{y}^{t})
= \tilde{R}_{t}(\hat{X}_{t}, y_{t-L})
+ \min_{y_{t} \geq 0} \left\{ c_{t}(y_{t}, \tilde{y}^{t}) + E\tilde{f}_{t+1} [(\hat{X}_{t} + y_{t-L} - U_{t}), (y_{t-N+1}, \dots, y_{t})] \right\}.$$
(10)

Thus, \tilde{f}_t has the form assumed above for \tilde{f}_{t+1} .

Straightforward induction, therefore, yields the following conclusion: If we approximate R_t by \tilde{R}_t for all t, then myopic allocations are optimal in all periods in the resulting problem.

In addition, the recursive equations (10) have precisely the form of a single-location dynamic inventory problem with delivery lag L, demands U_t and one period expected holding and penalty costs \tilde{R}_t .

This means that the same standard trick invoked in §2 can be applied to reduce the state space of (10). For now we assume that c_t depends only on the current order, that is, $c_t = c_t(y_t)$. We shall reformulate (10) by counting in period t the expected penalty and holding costs previously represented in period t + L. (Recalling the earlier transformation, these costs are actually incurred in period t + L + l.) Viewed from period t, the initial inventory in period t + L will be

$$\hat{X}_{t+L} = \hat{X}_t + \sum_{s=t-L}^{t-1} y_s - \sum_{s=t}^{t+L-1} U_s$$
$$= X_t^{\Delta} - \overline{U}_t.$$

The expected cost in period t + L is thus

$$P_t(X_t^{\Delta}, y_t) = E_{\overline{U}_t} \widetilde{R}_{t+L} (X_t^{\Delta} - \overline{U}_t, y_t)$$

= $\int \left\{ p_{t+L}^0 \left[\widehat{M}_{t+L} - (X_t^{\Delta} - \overline{U}_t + y_t) \right] + (p_{t+L}^0 + h_{t+L}^0) \int_{-\infty}^{X_t^{\Delta} - \overline{U}_t + y_t} \overline{G}_{t+L}(U) dU \right\} d\overline{G}_t(\overline{U}_t).$

Reversing the order of integration, we obtain

$$P_t(X_t^{\Delta}, y_t) = p_{t+L}^0 \Big[(\hat{M}_{t+L} + \overline{M}_t) - (X_t^{\Delta} + y_t) \Big] + (p_{t+L}^0 + h_{t+L}^0) \int_{-\infty}^{X_t^{\Delta} + y_t} H_t(U) \, dU,$$

where

$$H_t = \overline{G}_t * \widetilde{G}_{t+L}$$

We may now define

 $g_t(X_t^{\Delta}) =$ minimal total expected ordering costs in periods t through T, and penalty and holding costs represented in (10) in periods t + L through T, given the system starts period t in state X_t^{Δ} .

Then $g_{T-L-l+1} = 0$, and, for $t \leq T - L - l$,

$$g_t(X_t^{\Delta}) = \min_{y_t \ge 0} \{ c_t(y_t) + P_t(X_t^{\Delta}, y_t) + Eg_{t+1}(X_t^{\Delta} + y_t - U_t) \}.$$
(11)

The recursive equations (11) are equivalent in all essential respects to (10) and we have thus reduced the state space of the problem to a single dimension.

We now have a one-dimensional dynamic program whose form is that of an inventory problem with no lag, demands U_t , and (strictly) convex expected holding and penalty costs P_t . The many known results for problems of this type, depending on the form of c_t , can thus be applied. For example, if c_t is linear, a critical number (or order-up-to) policy is optimal (Karlin 1960). If c_t is linear with a fixed cost term, then an (s, S) policy is optimal (Scarf 1960). Clark (1981) recently solved numerous problems of this type, each with 72 periods and nonstationary data. On average these required $\frac{1}{2}$ CPU seconds on an Amdahl 7B. (This estimate is conservative.) For the most general functions c_t the codes described in Morin (1979) can be used.

Now suppose c_t depends also on y_{t-1} . A transformation similar to the above yields

$$g_t(X_t^{\Delta}, y_{t-1}) = \min_{y_t \ge 0} \left\{ c_t(y_t, y_{t-1}) + P_t(X_t^{\Delta}, y_t) + Eg_{t+1} \left[(X_t^{\Delta} + y_t - U_t), y_t \right] \right\}.$$

If c_t represents proportional smoothing costs, as in Sobel (1969), then the relatively simple type of policy derived in that paper is optimal for this dynamic program.

In general, if $c_t = c_t[y_t, (y_{t-L'}, \ldots, y_{t-1})]$ for some integer L', the program (10) can be transformed to one having an (L' + 1)-dimensional state space. For large L' computation of the true optimal policy for the approximate problem may still be a formidable task. The policy space has been considerably reduced, however, so it may be possible to use discretization and/or simulation approaches to locate good ordering policies, even when the size of the original problem would have defeated such techniques.

Observe that evaluation of P_t , like that of q_{jt} , requires integration of a normal cdf. It can be shown, however, that

$$\int_{-\infty}^{y} \Phi(u) \, du = y \Phi(y) + \phi(y),$$

where ϕ is the standard normal density. The integrals in the functions thus require no more work than evaluation of Φ , which is available in standard packages of scientific functions.

Form of the Reduced Model

Observe, the function P_t , like $\tilde{R_t}$ and the original q_{jt} , has the form of a one-period expected holding-and-penalty cost function derived from linear costs. The demand distribution H_t appearing in P_t is almost $\overline{G_t} * \hat{G_{t+L}}$, the distribution of system-wide demands in periods t through t + L + l: Both H_t and $\overline{G_t} * \hat{G_{t+L}}$ are normal distribu-

tions with mean

$$\overline{M}_t + \hat{M}_{t+L} = \sum_{s=t}^{t+L+l} M_s = \sum_{s=t}^{t+L+l} \sum_{s=t}^{J} \mu_{js}.$$

The variance of H_t , however, is

$$\overline{S}_{t}^{2} + \widetilde{S}_{t+L}^{2} = \sum_{s=t}^{t+L-1} S_{s}^{2} + \left[\sum_{j=1}^{J} \left(\sum_{s=t+L}^{t+L+l} \sigma_{js}^{2} \right)^{1/2} \right]^{2}$$

$$\geq \sum_{s=t}^{t+L+l} S_{s}^{2} = \overline{S}_{t}^{2} + \hat{S}_{t+L}^{2}$$

$$= \operatorname{Var}(\overline{G}_{t} * \hat{G}_{t+L}),$$

with equality holding if and only if $\hat{u}_{j,t+L}$, j = 1, ..., J, are perfectly correlated. In the special case where $\sigma_{jt} = \sigma$ for all j and t, and the u_{jt} are independent over j as well as t, we obtain

$$\operatorname{Var}(H_t) = \left[LJ + (l+1)J^2 \right] \sigma^2 > (L+l+1)J\sigma^2 = \operatorname{Var}\left(\overline{G}_t * \hat{G}_{l+L}\right).$$

For fixed L + l suppose L is large relative to l. Then (11) has costs as well as dynamics very similar to those of a completely centralized system (with larger variance, however, even when l = 0), where all demands are pooled at the depot. As l increases and L decreases, the variance of H_t increases, reflecting greater decentralization, in that allocations must be finalized relatively sooner.

5. Computational Results

To test the proposed method we compared the approximation with a simulation of the true system for each of several instances of the model of §2. Attention was restricted to relatively simple, but still interesting cases, specifically, L' = 0 (so N = L), stationary data, independent normal demands, and identical holding and penalty costs across locations. Also, the approximate program (11) was replaced by its infinitehorizon, average-cost analogue. The latter problem is computationally simpler, and the infinite-horizon case is of great practical interest. This represents, in effect, another approximation step. The simulations, of course, had finite (but very long) horizons.

Two types of order cost functions were considered, linear and fixed-plus-linear. (The systems tested are thus among those treated in Eppen and Schrage 1981.)

For each system an optimal (stationary) policy for the approximate program was computed. Five additional policies were evaluated, in order to test the robustness of the approximation and to search (albeit not very hard) for an improved policy. The cost of each policy according to the approximate program (referred to here as the "approximate cost") was then compared to the true average cost as estimated by simulation (the "estimated cost"); the difference is reported as percentage absolute error, that is

% error = 100 approximate cost-estimated cost / estimated cost.

Each simulation was run for about 8000 periods. (Some sampling error remains, but it is small.)

Note that, since the approximate program is derived by relaxing constraints only, its optimal cost is a lower bound on the true optimal cost of the (infinite-horizon) original problem; the estimated cost of the policy computed is (up to sampling error) an upper bound. Thus, the % error measures also the suboptimality of that policy.

	Results for System 1						
Critical number	Approximate cost	Estimated cost	Percent error				
260	27.840	27.757	0.299				
265	23.640	23.550	0.229				
267.23*	23.229	23.248	0.082				
268	23.269	23.282	0.056				
270	23.713	23.767	0.227				
275	26.425	26.436	0.042				

TABLE 2 Results for System I

* = optimal policy for approximate program.

Linear Order Costs

In this case a critical-number policy is optimal for the approximate program. (Because of the average-cost criterion, the order costs can be ignored, and were not included in the costs compared.) The optimal critical number is computed by globally minimizing the function $P_t = P$; this requires one inversion of Φ . The approximate cost of any policy of this type is computed by one evaluation of P at $X^{\Delta} + y = X^*$, where X^* is the critical number.

Seven systems were tested, numbered I-VII. Systems II-VII are variations on the "basic" system I. Characteristics of the systems are as follows:

I: J = 5, L = l = 2, p = 10, h = 1, demands at locations identical with mean $\mu = 10$, standard deviation $\sigma = 1.4$.

II: same as I, but p = 2.

III: same as I, but L = 3, l = 1.

IV: same as I, but L = 1, l = 3.

V: same as I, but J = 10.

VI: same as I, but means μ_j range from 5 to 25 in increments of 5, with coefficient of variation σ_i / μ_i held constant at 0.14.

VII: same as I, but standard deviations σ_i range from 0.1 to 3.0.

The results for system I are shown in Table 2. For the other systems the results are comparable; none exhibited markedly larger or smaller errors. Since there were 7 systems with 6 policies each, a total of 42 simulations were run. The largest error among these was 0.51%, and the average error was 0.14%. In no case was a policy found whose estimated cost was lower than that of the policy predicted to be optimal by the approximate program.

The approximations in this case are very close indeed.

Fixed-plus-Linear Order Cost

In this case an (s, S) policy is optimal for the approximate program. To evaluate the average cost of a given (s, S) policy the aggregate program was discretized on the integers; this represents another potential source of approximation error. (The simulations, however, were performed on the true, continuous problem.) The optimal policy for the discretized problem was found by the methods of Federgruen and Zipkin (to appear).

Each of the systems described above was tested, but with a fixed order cost K = 100. Also, System I was run with K = 50, 150 and 300.

The results for System I with K = 100 are shown in Table 3. Similar results were obtained for the other systems and other values of K, with the exception of System VII. Excluding VII, over the remaining 54 simulations, the maximum error was 4.35%, and the average was 1.77%. Although the correlation is not perfect, the error appears to increase with S - s, hence, with the average time between orders, as Table 3 suggests. (The relationship is clearer for other systems.)

		, , , , , , , , , , , , , , , , , , ,		
S	S	Approximate cost	Estimated cost	Percent error
243*	312*	94.294	96.405	2.190
253	312	94.373	96.340	2.042
263	312	128.783	125.196	2.865
253	322	98.486	99.305	0.825
263	322	98.608	99.430	0.827
220	400	115.393	119.690	3.590

TABLE 3 System I with K = 100

* = optimal policy for discretized, approximate program.

Note that the estimated cost of the second policy in Table 2 is lower than that of the approximately optimal policy. The difference, however, is only 0.07%, and this is the largest improvement found over all systems; in most cases no better policy was discovered.

While not as small as in the linear case, the errors here are still quite reasonable.

The results for System VII with K = 100 show errors ranging from 24% to 66%. Evidently, the approximation breaks down when coefficients of variation are unequal and there are many periods between orders. (Intuitively, the reason can be seen in an example with J = 2, $\sigma_1 = \sigma_2$, and $\mu_1 \gg \mu_2$. The key fact is that, in a period with nothing to allocate, the constraints $z_{jt} \ge 0$ are inessential—hence the approximation perfect—if and only if the quantities $(\hat{x}_{jt} - \hat{\mu}_j)/\hat{\sigma}_j$ are equal. It can be shown that, if a large amount is allocated myopically, these quantities are likely to be close in the *next* period; in *subsequent* periods with nothing to allocate, however, the quantities become increasingly unequal, hence the approximation deteriorates. The same argument suggests myopic allocation is a poor choice under these circumstances.)

Subsequent computations, not reported here, indicate that the results deteriorate slightly when the coefficients of variation are *larger* but *equal*; the approximations remain quite accurate, however.

6. Special Cases and Extensions

The "Single-Cycle" Problem

We first consider the special case where T = L + l + 1. The order placed in period 1 arrives at the beginning of period L + 1, and its allocation is received at the demand points in period T. There are thus only a single order and a single allocation decision which can affect costs; the only costs so affected, moreover, are the order cost in period 1 and the holding and penalty costs in period T. In this sense the system operates for a single cycle.

There are only three important epochs in this case, the beginning of period 1, the beginning of period L + 1, and the end of period T. In essence, therefore, the problem covers two "macro-periods" of different lengths, the first composed of periods 1 through L, and the second of periods L + 1 through T. The model can thus be viewed as a three-stage stochastic program.

The myopic allocation is certainly optimal in period L + 1 and suppose we approximate its cost by $\tilde{R} = \tilde{R}_{L+1}$. For simplicity assume $c = c_1 = c_1(y_1) = c(y)$, so the order cost depends only on $y = y_1$. Using (11), therefore, the problem for period 1 becomes the following newsboy problem:

$$\min_{y \ge 0} c(y) + P(X^{\Delta}, y) = c(y) + p^0 [(\hat{M} + \overline{M}) - (X^{\Delta} + y)] + (p^0 + h^0) \int_{-\infty}^{X^{\Delta} + y} H(U) dU.$$
(12)

(We have suppressed the now unnecessary time subscripts; in terms of earlier notation, p^0 , h^0 and \hat{M} would have subscript L + 1, and all other symbols subscript 1.) Here,

$$E(H) = \sum_{t=1}^{T} \sum_{j=1}^{J} \mu_{jt} = \hat{M} + \overline{M},$$

Var(H) = $\overline{S}_{1}^{2} + \tilde{S}_{L+1}^{2} = \sum_{t=1}^{L} S_{t}^{2} + \left[\sum_{j=1}^{J} \left(\sum_{t=L+1}^{T} \sigma_{jt}^{2} \right)^{1/2} \right]^{2}.$

The (approximately) optimal order is thus easily computed.

Application to Transportation Problems with Uncertain Demands

The closed-form cost function (12) can be embedded in a much more complex three-stage stochastic program: Suppose there are now K depots, and depot k serves the set of demand points T_k , k = 1, ..., K. There are also I sources of the commodity, where source i has supply a_i . At the beginning of period 1 we must decide the quantity w_{ik} to ship from source i to depot k (at constant unit cost \bar{c}_{ik}), and hence the total shipment y_k to depot k. From then on each depot faces the scenario described above: The shipment y_k arrives in period L + 1; it is then allocated among T_k , and the allocations reach the demand points in period T = L + l + 1.

The expected holding and penalty costs for the locations T_k , as viewed from period 1, can be approximated by $P_k(X_k^{\Delta}, y_k)$ of the form above. The entire problem, therefore, can be approximated by the following convex network-flow program:

$$\min \sum_{i=1}^{I} \sum_{k=1}^{K} \bar{c}_{ik} w_{ik} + \sum_{k=1}^{K} P_k (X_k^{\Delta}, y_k)$$

s.t.
$$\sum_{k=1}^{K} w_{ik} \leq a_i, \quad i = 1, \dots, I,$$

$$\sum_{i=1}^{I} w_{ik} - y_k = 0, \quad k = 1, \dots, K, \text{ all } w_{ik} \geq 0.$$
 (13)

This two-stage stochastic program has the form of a transportation problem with uncertain demand, and efficient methods for solving it have been developed, e.g., by Williams (1963).

Zipkin (1982b) considers the case L = 0, i.e., a completely decentralized system. Here, the second stage of the original problem effectively vanishes. Problem (13) is viewed in Zipkin (1982b) as an aggregation of a larger two-stage problem, where the sources ship directly to the demand points. In this case the subsets T_k do not correspond to any physical restrictions on supply relations; rather, they are chosen arbitrarily to reduce the size of the problem.

Let H_k denote the distribution appearing in P_k . With L = 0 we can write $Var(H_k) = \tilde{S}_k^2$ and $\hat{S}_k^2 =$ true variance of the sum of demands at locations T_k in periods 1 through T = l + 1. The fact that in general, $\tilde{S}^2 > \hat{S}^2$, it is argued in Zipkin (1982b), has implications for choosing the required total shipments to destinations in classical, deterministic transportation problems: If the destinations k represent aggregates of groups of smaller destinations T_k , $k = 1, \ldots, K$, one cannot set the shipment size for the whole subset T_k as if all demands occurred in one place; an estimate of \tilde{S}_k^2 must be used instead of an estimate of the true variance \hat{S}_k^2 . It is speculated (Zipkin 1982b), moreover, that this finding reflects the complete lack of centralization, and that in systems with some centralization the finding would be modified.

The results above, with the comments on the form of (11), confirm this speculation: As L increases from 0, with T = L + l + 1 fixed, the system becomes more centralized, and each $Var(H_k)$ decreases. As long as l > 0, however, $Var(H_k)$ remains larger than the true variance of demands at T_k over periods $1, \ldots, T$. Procedures for choosing the total shipment sizes, therefore, must reflect these larger $Var(H_k)$.

Linear Order Cost

We now return to the single-depot problem, and assume c(y) = cy for some scalar $c < p^0$. Also, suppose we can replace H(U) by $F[U - M^{\Delta})/S^{\Delta}]$ in (12), either exactly or approximately, where M^{Δ} and S^{Δ} are the mean and standard deviation of H. Note, $F = \Phi$ for normal demands. In this case straightforward analysis (cf., e.g., Eppen 1979 for the normal case) yields the following expression for the minimal cost of problem (12):

$$c(M^{\Delta} - X^{\Delta}) + \left[(c - p^{0})\gamma + (p^{0} + h^{0})\delta \right]S^{\Delta}, \quad \text{where}$$
(14)
$$\gamma = F^{-1} \left(\frac{p^{0} - c}{p^{0} + h^{0}} \right), \quad \delta = \int_{-\infty}^{\gamma} F(U) dU.$$

Expression (14) includes a constant term, a term linear in X^{Δ} , and a term linear in $S^{\Delta} = (\bar{S}_1^2 + \tilde{S}_{L+1}^2)^{1/2}$. Again, assuming independence and $\sigma_{jl} = \sigma$, $S^{\Delta} = \sigma [LJ + (l+1)J^2]^{1/2}$. Thus, not only problem (12) but also its minimal cost (14) reflect the degree of centralization of the system: The larger L is relative to l, the smaller S^{Δ} is, and (since $\delta > \gamma$) the smaller (14) is. Thus, greater centralization leads to reduced (approximate) cost. (This observation is similar to one in Eppen 1979.)

More General Distributions

Approximations for the $R_t(\cdot, \cdot)$ function, similar to (8), can be obtained for nonnormal demands under certain other assumptions. The crucial stipulation which must be satisfied is the following. Let \hat{F}_{jt} again be the cdf of \hat{a}_{jt} . Then, for each t, there exists an underlying distribution \hat{F}_t such that, for some positive constants $\hat{\mu}_{jt}$ and $\hat{\sigma}_{jt}$,

$$\hat{F}_{jt}(u) = \hat{F}_t\left(\frac{u-\hat{\mu}_{jt}}{\hat{\sigma}_{jt}}\right),$$

for all j and u. Moreover, \hat{F}_i must be continuous and strictly increasing where positive. In this sense, therefore, allocation-lead time demands at the various locations must be "of the same form".

The methods of Zipkin (1982a) apply when this more general condition holds, just as the normal case. The approximation to R_t is expression (8) above, with \hat{F}_t replacing Φ in the definition of \tilde{G}_t .

The assumption above is indeed satisfied for several important cases:

(a) l = 0, that is, the time required to deliver shipments after allocation decisions are made is negligible. Here, each $\hat{F}_{jt} = F_{jt}$, so we require that

$$F_{jt}(u) = F_t\left(\frac{u-\mu_{jt}}{\sigma_{jt}}\right),$$

for positive constants μ_{jt} and σ_{jt} , and a continuous cdf F_t which is increasing where positive.

This assumption is satisfied, for example, when u_{jt} is exponentially distributed, $j = 1, \ldots, J$: We can choose $F_t(u) = 1 - \exp(-u)$, $\mu_{jt} = 0$ and $\sigma_{jt} = E(u_{jt})$, j = 1, \ldots, J . The assumption holds also when the F_{jt} are all gamma, all Weibull or all Pareto distributions with the same shape parameter. By allowing positive μ_{jt} , translated distributions of these forms are also included. (b) Demands are stationary and all F_{jt} are exponential. In this case, each \hat{F}_t is a gamma distribution with shape parameter l + 1. More generally, if each F_{jt} is gamma with stationary scale parameters α_j and common shape parameters β_t , \hat{F}_t is also gamma with shape parameter $\sum_{s=t}^{t+l} \beta_s$. Again, translated distributions of these classes permit analogous results.

Moreover, where allocation-lead-time distributions are "of the same form", the reduction of the dynamic program can be performed precisely as above. The mean and variance of H_t , moreover, are the same as in the normal case, and hence the centralization/decentralization interpretation still holds.

In general, however, H_t may not have a simple form. (Even when l = 0 and the u_{jt} are independent exponentials, but with different means σ_{jt} , $j = 1, \ldots, J$, their sum over L periods has the complicated density given, e.g., in Feller 1971, p. 40.) To solve (12), therefore, it is probably necessary to approximate H_t by one of the tractable distributional forms, using its true mean and variance. Indeed, reasonable results can probably be obtained even when the assumptions of §3 are not strictly met; when l > 0 and the u_{jt} are general exponentials, for example, approximate the \hat{F}_{jt} and then the H_t by gamma distributions.

Systems with More than Two Levels

The fact that the one-period costs in the approximate problem have the same form as those of the individual locations implies that the methods here, repeated several times, can be used to reduce a system with more than two levels to a single-location problem. For this to work the system must have the form of an "arborescence" (each location has a unique supplier); also stock is held only at the lowest echelon.

The three-level model may be used to represent a multiproduct, multilocation system. An order at the highest level again represents production of an intermediate good or an order of a raw material. This resource is allocated among the several products at the second level, while at the third level the products are allocated among locations. (Thus, each "location" at the lowest level represents a product-location pair.) Alternatively, the second level might represent shipment of the intermediate product to locations, where it is then allocated to production of the various final goods.¹

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