Cost Formulas for Continuous Review Inventory Models with Fixed Delivery Lags

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(Received August 1981; accepted January 1983)

In continuous review models with a fixed delivery lag T, the state of the system is conveniently described by the *net inventory position* = (inventory on hand) plus (outstanding orders), in spite of most cost components depending on the *actual inventory on hand.* To relate these two inventory concepts one observes that the distribution of the inventory on hand at time $t + T$ is determined by the inventory position at time t. This explains the standard convention of charging the expected costs incurred in $[S_n + T, S_{n+1} + T)$ to the decision made at time S_n, where S_n denotes the nth decision epoch. This paper derives simple expressions for the expected costs in $[S_n + T, S_{n+1} + T)$ as a function of the inventory position just after decision epoch S_n.

In CONTINUOUS review single-item inventory models, the stochastic demand is conveniently characterized by the renewal reward process $\{(X_n, Y_n); n = 1, 2, \cdots\}$. The renewal process $X_1, X_2, \cdots$ describes the interarrival times of customers. ($S_n = \sum_{i=1}^{n} X_i$ with $S_0 = 0$ represents the arrival epoch of the nth customer.) The random variable Y_n ($n = 1, 2, \cdots$) indicates the change in inventory resulting from the demand D_n of the nth customer (i.e., $Y_n = -D_n$) and may possibly depend on X_n; however, we suppose that the pairs (X_n, Y_n), $n = 1, 2, \cdots$ are i.i.d. having a joint distribution function $F(x, y)$.

Beckmann [1961], Finch [1961], Rubalskiy [1972a, 1972b], Sivazlian [1974], Sahin [1979], Tijms [1972] have all studied models of this type. Other contributors to the literature consider mainly the special case where demands are described by a compound Poisson process, i.e., where all X_n, $n \geq 1$, are i.i.d. exponentials, and where X_n and Y_n for $n \geq 1$ are independent.

When excess demands are backlogged, and delivery of an order takes a fixed time T, the inventory control problem can be represented by a semi-Markov decision model with a one-dimensional state space. This formulation chooses the demand points as decision epochs and the state of the system is given by the *net inventory position* = (inventory on hand) + (outstanding orders) − (backlog), in spite of the fact that holding and

Subject classification: 118 semi-Markov processes, 362 stochastic inventory models.

Operations Research 0030-364X/83/3105-0957 $01.25
Vol. 31, No. 5, September-October 1983

shortage costs depend on the *actual inventory on hand.* To relate these two inventory concepts, one observes that the inventory on hand at time $t + T$ is distributed as the inventory position at time t minus the total demand during $[t, t + T)$. The inventory position just after decision epoch S_n thus unambiguously determines the distribution of the inventory on hand at time $S_n + T$. This explains the standard convention of charging expected holding and shortage costs incurred in $[S_n + T, S_{n+1} + T)$ to the decision made at epoch S_n (with appropriate discounting if the objective is to minimize the expected present value of current and future costs). The holding and shortage costs in $[0, T)$ are omitted since no decision can take effect before time T. This note derives simple expressions for the expected (discounted) holding and shortage costs in $[S_n + T, S_{n+1} + T)$ for $n \geq 1$ as a function of the inventory position just after decision epoch S_n. These are needed to compute optimal inventory control policies as well as to characterize the structure of an optimal control rule. For example, under appropriate assumptions (such as convexity of the holding and shortage cost function) these formulae show that an (s, S) policy is optimal.

Our development considerably simplifies existing expressions such as those in Beckmann [1961] for general demand processes. We achieve further simplification for the case of compound Poisson processes, thereby providing a substantially shorter and more direct derivation of the results in Archibald [1976] (see also Archibald and Silver [1978]). Our results are related to Sahin's who derived the steady state distribution of the inventory position and on-hand inventory, assuming inventories are controlled via an (s, S) policy. Whereas the steady state distribution enables the evaluation of average costs per unit time for arbitrary cost *rate* functions (see Section 1), it is insufficient to handle fixed penalties, e.g. for running out of stock (Section 2). Also, the formulas in Sahin are hard to evaluate, inappropriate when computing total discounted costs, and reflect the costs only under an assumed (s, S) replenishment policy.

Section 1 derives the expected (discounted) value of an arbitrary cost rate function in $[S_n + T, S_{n+1} + T)$ and discusses some special cases. Section 2 discusses the treatment of *fixed* penalty costs incurred for any requested unit or demand order that cannot (in part or in total) be delivered from current inventory. Finally Section 3 shows how similar cost expressions arise in multi-item models with compound Poisson demand processes.

1. EXPECTED (DISCOUNTED) HOLDING AND SHORTAGE COSTS IN $[S_n + T, S_{n+1} + T)$

Assume the system incurs a holding (shortage) cost rate $h(y)$ per unit of time during which the inventory on hand equals y units. Let

$\mathbf{C}_n(y_0, \beta)$, $n \geq 0$ represent the total costs in the interval $[S_n + T, S_{n+1} + T)$, discounted back to S_n, when the inventory position at time S_n equals y_0 by using a (continuously compounded) interest rate β. Note that the random variables $\mathbf{C}_n(y_0, \beta)$ are i.i.d. Hence it suffices to characterize $\mathbf{C}_0(y_0, \beta)$, the expected discounted costs in $[T, T + X_1)$. In this section, we derive an expression for $c(y_0, \beta) = E\mathbf{C}_0(y_0, \beta)$. We first introduce the following notation:

$N(t) = \sup\{n: S_n \leq t\}$.
$Y(t) = y_0 + \sum_{n=1}^{N(t)} Y_n$.
$G(x) = F(x, \infty)$ is the distribution function of the interarrival times.
$\tilde{G}(\cdot)$ denotes its Laplace-Stieltjes transform (L.S.T.) and $\mu = \int x dG(x) < \infty$.
$H(y) = F(\infty, y)$ is the distribution function of the change in inventory resulting from the demand of a single customer.
$F_z(x, y) = \text{Prob}[X_1 \leq x + z,\ Y_1 \leq y \mid X_1\ z]$
$= [F(x + z, y) - F(z, y)][1 - G(z)]^{-1}$
(the joint distribution function of the next arrival time and demand given the last customer arrived z time units ago).

Note that in the relevant interval $[T, T + X_1)$, $Y(t)$ for $t \in [T, T + X_1)$ represents the inventory on hand.

With the above notation, we can express $c(y_0, \beta)$ as

$$c(y_0, \beta) = E\left[\int_T^{T+X_1} h(Y(t))e^{-\beta t}dt \mid Y(0) = y_0\right]. \tag{1}$$

Lemma 1 demonstrates that $c(y_0, \beta)$ can be obtained by evaluating the expectation of the $h(\cdot)$-value of the inventory level at time T that would arise if $\{(X_n, Y_n)\}_{n=1}^{\infty}$ were generated by a *delayed* renewal reward process (cf. Karlin and Taylor [1974], Ross [1970]). In this delayed renewal process, the c.d.f. of (X_n, Y_n), $n \geq 2$ is given by $F(x, y)$. The c.d.f. of (X_1, Y_1), however, is given by a distribution

$$F_\beta{}^D(x, y) = \begin{cases} \displaystyle\int_0^\infty F_z(x, y)\beta e^{-\beta z}(1 - G(z))(1 - \tilde{G}(\beta))^{-1}dz, & \beta > 0 \\ \displaystyle\int_0^\infty F_z(x, y)\mu^{-1}(1 - G(x))dz, & \beta = 0. \end{cases}$$

Thus $F_\beta{}^D$ is merely a mixture of $\{F_z\}$ with respect to the density

$$\begin{cases} \beta e^{-\beta z}(1 - G(z))(1 - \tilde{G}(\beta))^{-1}, & \beta > 0 \\ \mu^{-1}(1 - G(z)), & \beta = 0. \end{cases}$$

(The fact that $\beta[1 - \tilde{G}(\beta)]^{-1}e^{-\beta z}(1 - \tilde{G}(z))$ is a proper density function can be verified by integrating $\int_0^\infty e^{-\beta z}(1 - G(z))dz$ by parts.) We will use

$E_{(y_0,z)}[E^D_{(y_0,\beta)}]$ to denote the expectation of a measurable function under the delayed renewal reward process with $Y_0 = y_0$ and $F_z(x, y)\cdot[F_\beta{}^D(x, y)]$ as the c.d.f. of (X_1, Y_1). We also use E_{y_0} as an abbreviation for $E_{(y_0,0)}$. (Note that

$$E^D_{(y_0,\beta)}[\cdot] = \textstyle\int_0^\infty E_{(y_0,z)}[\cdot]\beta e^{-\beta z}(1 - G(z))\ (1 - \tilde{G}(\beta))^{-1}dz$$

and in particular $E^D_{(y_0,0)}[\cdot] = \int_0^\infty E_{(y_0,z)}[\cdot]\mu^{-1}(1 - G(z))dz$.)

LEMMA 1.

(a) *For any* $\beta > 0$, $c(y_0, \beta) = e^{-\beta T}\beta^{-1}[1 - \tilde{G}(\beta)]E^D_{(y_0,\beta)}[h(Y(T))]$.

(b) *For* $\beta = 0$, $c(y_0, 0) = \mu E^D_{(y_0,0)}[h(Y(T))]$.

Proof. (a) Let $I(t) = 0$ or 1 according to whether $X_1 \le t$ or $X_1 > t$ respectively; then

$$c(y_0, \beta) = E_{y_0}\left[\int_T^\infty h(Y(t))e^{-\beta t}I(t - T)dt\right]$$

$$= e^{-\beta T}E_{y_0}\left[\int_0^\infty h(Y(t + T))e^{-\beta t}I(t)dt\right]$$

which by Fubini's theorem

$$= e^{-\beta T}\int_0^\infty e^{-\beta t}E_{y_0}[h(Y(t + T))I(t)]dt.$$

Now:

$$E_{y_0}[h(Y(t + T))I(t)] = E_{y_0}[h(Y(t + T)\,|\,X_1 > t](1 - G(t))$$

$$= E_{(y_0,t)}[h(Y(T))]((1 - G(t)).$$

Hence,

$$c(y_0, \beta) = e^{-\beta T}\beta^{-1}[1 - \tilde{G}(\beta)]\int_0^\infty E_{(y_0,t)}[h(Y(T))]$$

$$\cdot\{\beta[1 - \tilde{G}(\beta)]^{-1}e^{-\beta t}(1 - G(t))\}dt$$

$$= e^{-\beta T}\beta^{-1}[1 - \tilde{G}(\beta)]E^D_{(y_0,\beta)}[h(Y(T))].$$

(b) Follows in a similar way.

Observe that the result for $\beta = 0$ can be obtained by letting $\beta \downarrow 0$. Also, when X and Y are independent, $F_0{}^D(x, y) = H(y)\mu^{-1}\int_0^x (1 - G(t))dt$, the product of $H(\cdot)$ and the equilibrium distribution of $G(\cdot)$.

The significance of Lemma 1 is that $c(y_0, \beta)$ can be expressed in the form

$$c(y_0, \beta) = \begin{cases} e^{-\beta T}\beta^{-1}[1 - \tilde{G}(\beta)] \int_0^\infty h(y_0 - u)dR(u), & 0 < \beta \\ \mu \int_0^\infty h(y_0 - u)dR(u), & \beta = 0 \end{cases} \tag{2}$$

where $R(u)$ is the c.d.f. of the demand in $[0, T)$ as generated by the delayed renewal process specified by $F_\beta{}^D(x, y)$ and $F(x, y)$.

We next discuss a few special cases in which further simplifications can be achieved. For simplicity, assume that demands have a discrete distribution $\{\phi(j); j \geq 0\}$.

I. The Compound Poisson Case

The analysis of the compound Poisson-demand process (with the interarrival times $\{X_n\}_{n\geq 1}$ i.i.d. exponentials, independent of $\{Y_n\}_{n\geq 1}$) is immediate from Lemma 1:

COROLLARY. *If (X_n, Y_n) is compound Poisson with $G(t) = 1 - e^{-\lambda t}$, then*
(a) *For $\beta > 0$, $c(y_0, \beta) = e^{-\beta T}(\lambda + \beta)^{-1}E_{y_0}[h(Y(T))]$, and*
(b) *For $\beta = 0$, $c(y_0, 0) = \lambda^{-1}E_{y_0}[h(Y(T))]$.*

The corollary shows that in the compound Poisson case, the expected costs in $[T, T + X_1)$ equal the expected costs in an interval starting at time T and terminating at the first arrival after T.

$$c(y_0, \beta) = E_{y_0}\left[\int_T^{S_{N(T)+1}} h(Y(t))e^{-\beta t}dt\right], \qquad \beta \geq 0. \tag{3}$$

Let $r(j)$ represent the probability that the number of units demanded in a period of length T equals j, $j = 0, 1, \cdots$. Then,

$$E_{y_0}[h(Y(T))] = \textstyle\sum_{j=0}^\infty h(y_0 - j)r(j).$$

In particular, when

$$h(y) = \begin{cases} h^+y, & y > 0 \\ -h^-y, & y \leq 0 \end{cases}$$

we have after some algebraic manipulations and using $\sum_j jr(j) = \lambda TE[Y_1]$,

$$\begin{aligned} E_{y_0}[h(Y(T))] &= h^+\textstyle\sum_{j=0}^{y_0} (y_0 - j)r(j) + h^- \sum_{j=1}^\infty jr(j + y_0) \\ &= h^-(\lambda TE[Y_1] - y_0) + (h^+ + h^-) \textstyle\sum_{j=0}^{y_0} (y_0 - j)r(j). \end{aligned}$$

The numbers $r(j)$ can be computed from the stable recursive scheme (cf. Adelson [1966]):

$$
\begin{aligned}
r(0) &= e^{-\lambda T(1-\phi(0))}, \\
r(j) &= (\lambda T/j) \textstyle\sum_{k=1}^{j} k\phi(k) r(j-k), \qquad j \geq 1.
\end{aligned} \tag{4}
$$

II. Erlang Distributed Interarrival Times

Let $F(t) = \int_0^T [(\lambda u)^{m-1} \lambda e^{-\lambda u}/(m-1)!]du$ for m integer. Let X_n and Y_n for $n \geq 1$ be independent. Let $\phi^{(n)}(\cdot)$ be the n-fold convolution of $\{\phi(\cdot)\}$. As before, let $r(i)$ represent the probability of i units of demand in $[0, T)$ where the demand process is a delayed renewal process, and the first renewal time is distributed as the equilibrium distribution of G, i.e., the probability density function of the first renewal epoch t is given by

$$\lambda m^{-1} \textstyle\sum_{l=0}^{m-1} e^{-\lambda t} \lambda^l t^l / l!.$$

Let $P(i, t)$ be the probability of i units of demand during an interval of length t immediately following a demand. Conditioning upon the first renewal time t, and using the formulas on p. 55 in Beckmann, we obtain after some algebra for $i \geq 1$

$$
\begin{aligned}
r(i) &= \lambda m^{-1} \textstyle\sum_{j=0}^{i} \phi(j) \displaystyle\int_0^T P(i-j, T-t) \textstyle\sum_{l=0}^{m-1} [e^{-\lambda t}(\lambda t)^l / l!]dt \\
&= \lambda m^{-1} \textstyle\sum_{j=0}^{i} \phi(j) \sum_{n=0}^{\infty} \phi^{(n)}(i-j) \\
&\qquad \cdot \textstyle\sum_{r=mn}^{mn+m-1} \sum_{l=0}^{m-1} \\
&\qquad \cdot \displaystyle\int_0^T [e^{-\lambda T}(\lambda t)^l (\lambda(T-t))^r / l! r!]dt \\
&= m^{-1} \textstyle\sum_{n=1}^{\infty} \phi^{(n)}(i) \sum_{r=mn}^{mn+m-1} \sum_{l=0}^{m-1} e^{-\lambda T}(\lambda T)^{l+r+1}/(l+r+1)! \\
&= m^{-1} \textstyle\sum_{n=1}^{\infty} \phi^{(n)}(i) \sum_{l=mn+1}^{mn+2m-1} (e^{-\lambda T}(\lambda T)^l / l!) \\
&\qquad \cdot \min(mn+2m-l,\, l-mn)
\end{aligned}
$$

and

$$
\begin{aligned}
r(0) &= (\lambda/m) \displaystyle\int_T^{\infty} \textstyle\sum_{l=0}^{m-1} [e^{-\lambda t}(\lambda t)^l / l!]dt + m^{-1} \sum_{n=1}^{\infty} [\phi(0)]^n \\
&\qquad \cdot \textstyle\sum_{l=mn+1}^{mn+2m-1} (e^{-\lambda t}(\lambda t)^l / l!) \min(mn+2m-l,\, l-mn) \\
&= m^{-1} \textstyle\sum_{r=0}^{m-1} (m-r) e^{-\lambda T}(\lambda T)^r / r! + m^{-1} \sum_{n=1}^{\infty} [\phi(0)]^n \\
&\qquad \cdot \textstyle\sum_{l=mn+1}^{mn+2m-1} (e^{-\lambda T}(\lambda T)^l / l!) \min(mn+2m-l,\, l-mn).
\end{aligned}
$$

The numbers $r(i)$, $i \geq 1$, enable us to evaluate

$$E^D_{(y_0,0)}[h(Y(T))] = \textstyle\sum_{j=0}^{\infty} h(y_0 - j) r(j).$$

2. FIXED (DISCOUNTED) PENALTY COSTS

Assume that (in additioin to the previously considered cost components), the system incurs a fixed penalty $\psi \geq 0$ for any *demand* that cannot be satisfied (in part or in total) from current inventory and a penalty cost $\pi \geq 0$ for any requested *unit* that has to be backlogged. If we let $D(t) = \sum_{n=1}^{N(t)} D_n$ be the total demand by time t, then the expected discounted cost in $[T, T + X_1)$ is given by:

$$q(y_0, \beta) = E\left[\pi \int_T^{T+X_1} e^{-\beta t}[D_{N(t^-)+1} - y(t^-)^+]^+ dN(t) \right. \\ \left. + \psi \int_T^{T+X_1} e^{-\beta t} 1\{D_{N(t^-)+1} + D(t^-) > y_0\} dN(t)\right] \tag{5}$$

where $x^+ = \max(0, x)$ and $1\{\cdot\}$ is the indicator function of $\{\cdot\}$. Lemma 2 below shows that at least in the compound Poisson case, we can obtain a simple expression for $q(y_0, \beta)$.

LEMMA 2. *Let* (X_n, Y_n) *be a compound Poisson process with rate* $\lambda > 0$ *and let* $\beta \geq 0$ *then*:

$$q(y_0, \beta) = e^{-\beta T}\lambda(\lambda + \beta)^{-1}\{(\psi + \pi E[D_1]) \textstyle\sum_{j>y_0} r(j) \\ + \textstyle\sum_{j=0}^{y_0} r(j) \sum_{n=y_0-j+1}^{\infty} \phi(n)(\psi + \pi(n - y_0 + j))\} \tag{6}$$

Proof. Let $I(t)$ be as in Lemma 1. (5) can be rewritten as

$$q(y_0, \beta) = e^{-\beta T} E \int_0^{\infty} e^{-\beta t} I(t)\{\pi(D_{N(T+t^-)+1} - [Y(T + t^-)]^+)^+ \\ + \psi 1\{D_{N(T+t^-)+1} + D(T + t^-) > y_0\}\} dN(t + T)$$

$$= e^{-\beta T} \int_0^{\infty} \lambda e^{-(\lambda+\beta)t} E\{\pi(D_{N(T+t^-)+1} - [Y(T + t^-)]^+)^+ \\ + \psi 1\{D_{N(T+t^-)+1} + D(T + t^-) > y_0\} \mid X_1 > t\} dt.$$

Interchanging the order of integration and using $E[dN(t)] = \lambda dt$, we obtain

$$q(y_0, \beta) = e^{-\beta T}\lambda(\lambda + \beta)^{-1} E\{\pi(D_{N(T)+1} - [Y(T)]^+)^+ \\ + \psi 1\{D_{N(T)+1} + D(T) > y_0\}\}.$$

After rearranging terms, we see that this expression leads to (6).

Once again we note from (6) that the total (discounted) expected penalty costs in $[T, T + X_1)$ equal the expected costs in an interval starting at time T and terminating at the first arrival after T.

$$q(y_0, \beta) = E\left[\pi \int_T^{S_{N(T)+1}} e^{-\beta t}[D_{N(t^-)+1} - Y(t^-)^+]^+ dN(t) + \psi \int_T^{S_{N(T)+1}} e^{-\beta t} 1\{D_{N(t^-)+1} + D(t^-) > y_0\} dN(t)\right]. \quad (7)$$

Lemma 2 can be extended to any penalty cost structure for which the cost in $[t, t + dt)$ depends merely upon $D(t^-)$ and the size of a single demand in $[t, t + dt)$. Note that the evaluation of (6) requires the computation of $r(0), \cdots, r(y_0)$ (via (4)) only.

3. MULTI-ITEM INVENTORY SYSTEMS WITH COMPOUND POISSON DEMANDS

Silver [1974], Thompstone and Silver [1975] and Federgruen et al. [1983] consider continuous review multi-item inventory systems where the demand processes for the items are independent compound Poisson processes; excess demands are backlogged and replenishments have constant lead times. There is a *major* setup cost associated with a replenishment for the family, and a *minor* setup cost for any item included in the order. In addition, the cost structure consists of holding-penalty and variable order costs.

The methods in the above references decompose the coordinated control problem into a single-item problem for each item in the family. Each single-item problem has "*normal*" replenishment opportunities at the major setup cost occurring at the demand epochs for this item and "*special*" replenishment opportunities at *reduced* setup costs at epochs generated by a Poisson process which is an approximation to the superposition of the ordering processes triggered by the other items. For a given item, let $F(t) = 1 - e^{-\lambda t}$, be the c.d.f. of the interarrival times and let μ be the rate of the Poisson process describing the special replenishment opportunities. The normal and the special replenishment opportunities together constitute the decision epochs. Hence X_i, $i = 1, \cdots, n$ ae i.i.d. exponentials with rate $(\lambda + \mu)$.

To evaluate $c(y_0, 0)$ and $q(y_0, 0)$, note that the process can be described as a renewal reward process with for all $n \geq 1$, $\text{Prob}[Y_n = -l] = \lambda(\lambda + \mu)^{-1}\phi(l)$ for $l > 0$ and $\text{Prob}[Y_n = 0] = \lambda(\lambda + \mu)^{-1}\phi(0) + \mu(\lambda + \mu)^{-1}$. Applying Lemmas 1 and 2 we obtain, cf. (2.2) in Federgruen et al.:

$$c(y_0, 0) + q(y_0, 0) = (\lambda + \mu)^{-1} \textstyle\sum_{j=0}^{\infty} h(y_0 - j)r(j) + \lambda(\lambda + \mu)^{-1}\{\textstyle\sum_{j=0}^{y_0} r(j) \sum_{k=y_0-j}^{\infty}[\psi + \pi(k - y_0 + jj)]\phi(k) + (\psi + \pi E[D_1]) \textstyle\sum_{j=y_0+1}^{\infty} r(j)\}.$$

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