AN ALLOCATION AND DISTRIBUTION MODEL FOR PERISHABLE PRODUCTS

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This paper presents an allocation model for a perishable product, distributed from a regional center to a given set of locations with random demands. We consider the combined problem of allocating the available inventory at the center while deciding how these deliveries should be performed. Two types of delivery patterns are analyzed: the first pattern assumes that all demand points receive individual deliveries; the second pattern subsumes the frequently occurring case in which deliveries are combined in multistop routes traveled by a fleet of vehicles. Computational experience is reported.

In a distribution system for a perishable product, the supply to various locations in a particular geographic or administrative region is often coordinated by a regional center. Among the difficult operational issues such a center must resolve are the problems of allocating the available inventory at the center among the delivery points, each experiencing random demands, while deciding how these deliveries should be performed.

Although they are often treated as such, these problems are not independent. Existing allocation models concentrate on shortage and outdating costs, without explicitly accounting for the costs of transportation. We consider two types of delivery patterns: the *first* pattern assumes that all demand points receive individual deliveries, and that delivery costs are linear; the *second* pattern subsumes the frequently occurring case in which deliveries are combined in multistop routes traveled by a fleet of vehicles. Here the transportation costs alone lead to a complex combinatorial optimization problem, the vehicle routing problem (VRP). Separate treatments of allocation and distribution can result in poor performance of the system as a whole: "optimizing" the allocations can force the use of awkward, costly delivery patterns and even increase the number of vehicles required. On the other hand, an "optimal" delivery pattern may lead to unacceptable shortages and/or waste.

This paper attempts to integrate the inventory allocation and the transportation planning problems in a single model. We present an efficient algorithm for each of the two delivery patterns. The first case leads to a relatively simple convex inventory allocation problem (referred to below as problem IA). For the second pattern, the integrated approach uses a similar procedure repeatedly as a subroutine within a routing algorithm. We compare the performance of this integrated procedure with the "separate" approach that solves an inventory allocation problem to minimize shortage and outdating costs only; the resulting allocations are used as (deterministic) delivery requirements within a standard code for the VRP.

We use the term perishable to refer to a product that has a fixed lifetime during which it can be used and after which it must be discarded. Common examples of perishable products are human blood, food and medical drugs.

The operating cost of the system consists of three components: (a) *shortage costs* paid for units that are demanded but unavailable in a particular location; (b) *out-of-date costs* paid for every unit that reaches the maximum age in inventory without being used, and which therefore must be discarded; (c) *transportation costs* associated with the deliveries. (The measurement of the shortage and outdate costs is discussed in Nahmias 1982; see also the Appendix.)

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For the sake of precision, we now describe the scenario in somewhat greater detail: the center acquires or produces periodically a fresh quantity of the perishable product; the time interval between two consecutive production points is constant and defines one period; the lifetime of the product is a constant (integer) M periods; at the beginning of each period the starting inventory in each location is known to the center, and this information is used to determine the deliveries (and, for the second pattern, the vehicle routes) for that period. After the deliveries are made, the demands occur, and out-of-date and shortage costs are incurred at each location, proportional to the end-of-period inventory levels.

Ours is thus a *one-period* model in the spirit of Prastacos (1978) as discussed later. (This approach has been shown to work very well for certain special cases; see Prastacos 1981). Under this scenario, only units of age $M - 1$ at the beginning of the period may become outdated by the end of the period if they remain unused. Hence, it suffices to distinguish two age classes, the "old" units of age $M - 1$ and all other "fresh" units of age $0, \ldots, M - 2$.

We remark that a variety of overall distribution policies are seen in practice; our model accommodates at least the following:

1. A *rotation policy* that removes all "still usable" product from the individual locations' inventories at the end of every period and returns it to the center for redistribution, together with the fresh quantity. In this case, starting inventories at the delivery points are always zero;
2. A *retention policy* that maintains product received by each location at that location until it is used or outdated;
3. A combination of (1) and (2).

Previous allocation models for perishable goods include those studied by Prastacos (1978) and Yen (1975). Prastacos (1978) treats the first pattern of deliveries under the additional assumption of *identical* per-unit costs for all locations. His results have been the basis of the blood distribution system implemented in Long Island, cf., Brodheim and Prastacos (1979) (see Section 1 for further details). Gregor, Forthofer and Kopadia (1982) develop a simulation model for the Gulf Coast Regional Blood Center in Houston, Texas, analyzing different inventory allocation and routing policies. Similar studies are reported in Graf, Katz and Morse (1972) and Yahnke et al. (1972). Yen examines a decentralized inventory system for perishables in which each location follows a critical number (order-up-to) policy. He derives structural results for the ordering and issuing policies of the center. (For a complete review of perishable inventory models, see Nahmias.)

This paper permits the per-unit costs to differ among locations and also examines the allocation problem in combination with the distribution/routing problem. For the first pattern (Section 1) that makes deliveries on an individual basis, we present a relatively simple solution procedure (Section 2). This method draws upon earlier results for simpler allocation problems, requiring no distinction between age classes, as described by Luss and Gupta (1975), Zipkin (1980) and Federgruen and Zipkin (1983).

For the second pattern (Section 3) that serves locations in multistop routes, we describe computational approaches for the combined inventory allocation and vehicle routing problem. These methods generalize those of Federgruen and Zipkin (1984) by allowing for an explicit distinction between age classes. They can be viewed as modifications of efficient techniques for the classical VRP with deterministic, predetermined delivery sizes. (For reviews of the VRP literature refer to Christofides 1976, Fisher and Jaikumar 1978, 1981, and Golden, Magnanti and Nguyen 1977. An application of vehicle scheduling to the distribution of blood is described by Or 1976, assuming fixed delivery sizes.)

These algorithms repeatedly require the solution of an inventory allocation problem, each time with only minor changes in the data. This allocation problem is closely related to the one presented in Section 1 for the first pattern. We thus emphasize efficient procedures to recover optimality from one problem instance to the next.

Section 4 presents numerical tests of one of the algorithms described in Section 3. The results show, as in Federgruen and Zipkin (1984), that the complicating factors of random demands and inventory allocations require a rather modest amount of extra computation, even with the additional complication of multiple age classes.

An Appendix describes an alternative cost structure that may be more realistic in some applications.

1. The Allocation Problem for Systems with Individual Deliveries

Let $Y = \{1, \ldots, n\}$ represent the set of delivery points in the region. $F_i(\cdot)$ denotes the cumulative distribution function of one-period demand in location i for $i = 1, \ldots, n$, and is assumed continuous and strictly increasing. Each location i has a per-unit shortage cost h_i^-, a per-unit outdate cost h_i^+, and a per-unit transportation cost γ_i from the center. Let A and B represent the amounts of fresh and old product available at the center, A_i and B_i the starting inventories at location i, and $T_i = A_i + B_i$ for $i = 1, \ldots, n$. The center faces the following allocation problem: determine fresh and old shipments v_i and w_i ($i = 1, \ldots, n$) in order to:

$$\text{(IA)} \quad \text{minimize} \sum_{i=1}^{n} [p_i(v_i + w_i) + r_i(w_i)] \tag{1}$$

$$\text{subject to} \quad \sum_{i=1}^{n} v_i \leqslant A, \qquad v \geqslant 0, \tag{2}$$

$$\sum_{i=1}^{n} w_i = B, \qquad w \geqslant 0. \tag{3}$$

The $p_i(\cdot)$, $r_i(\cdot)$ and their derivatives are given by

$$p_i(z) = \int_{T_i+z}^{\infty} h_i^-(\xi - T_i - z)\, dF_i(\xi) + \gamma_i z \tag{4}$$

$$r_i(w_i) = \int_0^{A_i+w_i} h_i^+(A_i + w_i - \xi)\, dF_i(\xi) \tag{5}$$

$$Dp_i(z) = h_i^-[F_i(T_i + z) - 1] + \gamma_i, \tag{6}$$

$$Dr_i(w_i) = h_i^+ F_i(A_i + w_i), \qquad i \in Y, \tag{7}$$

where, if $f(\cdot)$ is a real valued function, then $Df(\cdot)$ indicates its first derivative, $D^2f(\cdot)$ its second derivative, and $D^+f(\cdot)$ and $D^-f(\cdot)$ its right-hand and left-hand derivatives respectively.

The term $p_i(v_i + w_i)$ in (1) represents expected shortage and transportation costs and the term $r_i(w_i)$ the expected out-of-date costs. Note from (6) and (7) that $p_i(\cdot)$ and $r_i(\cdot)$ for $i \in Y$ are strictly convex and C^1. Constraint (3) is stated as an equality rather than an inequality since all of the old units must be distributed to demand locations to enable their consumption before out-dating.

For the special case in which all cost parameters are independent of the location, i.e.,

$$h_i^+ = h^+; \qquad h_i^- = h^-; \qquad \gamma_i = \gamma; \qquad i \in Y, \tag{8}$$

and every location receives a positive allocation of old units, Prastacos (1978) describes a simple two-stage procedure to solve (IA): (a) first allocate old units so as to equalize the probability that a unit outdates among the locations; and (b) allocate the fresh stock so as to equalize the probability of a stockout at any location. The optimal allocation is independent of the unit costs h^+, h^- and γ and minimizes both the expected shortage and out-of-date costs in the next period, not just the sum of the two. (The transportation costs reduce to a constant in this case, independent of the allocation.)

Unfortunately, this procedure may not solve (IA) and the properties may not hold when costs are location-dependent or when some locations have adequate initial inventories of old units. The next section presents a solution procedure for the general model (IA).

2. Solution of the Allocation Problem with General Costs

Our proposed procedure uses a Lagrangean dualization approach. The Lagrangean relaxations are shown to reduce to single-resource allocation problems for which special efficient methods have been developed. The efficiency of the procedure is based on quick updates of the optimal solution of the relaxed problem as the Lagrangean multiplier is varied.

Outline of the Algorithm

Let $z_i = v_i + w_i$ for $i \in Y$ represent the total amount allocated to location i. Rewriting (IA) in terms of the variables z_i, we obtain:

$$\text{(IA}'\text{)} \quad \zeta^* = \text{minimize} \sum_{i=1}^{n} [p_i(z_i) + r_i(w_i)] \tag{9}$$

$$\text{subject to} \quad \sum_{i=1}^{n} z_i \leqslant (A + B) \tag{10}$$

$$\sum_{i=1}^{n} w_i = B \tag{11}$$

$$z_i \geqslant w_i \geqslant 0, \qquad i \in Y. \tag{12}$$

(10) replaces the sum of (2) and (3). The inequalities $z_i \geqslant w_i$ in (12) guarantee $v_i = z_i - w_i \geq 0$ for $i \in Y$. Hence (IA′) and (IA) are equivalent.

To solve (IA′) we dualize (11), i.e., we define

$$L(\lambda) = \text{minimize}\left\{ \sum_{i=1}^{n} [p_i(z_i) + r_i(w_i)] + \lambda \sum_{i=1}^{n} w_i - \lambda B, \quad \text{subject to (10), (12)} \right\}. \tag{13}$$

Note that the objective function (9) is convex, and (10)–(12) describe a convex polyhedral set. Hence there is no duality gap, i.e.,

$$\zeta^* = \max_\lambda L(\lambda). \tag{14}$$

(See Theorem 5.4 in Shapiro 1979). Finally, $L(\lambda)$ is a concave function. (See Theorem 5.1 in Shapiro.) Therefore, given an efficient procedure to evaluate

$L(\cdot)$, we can solve (14) via standard unconstrained maximization techniques.

Computational efficiency is further enhanced by an efficient search procedure for (14), including a method to recover the optimal solution of (13) when changing the value of λ to a neighboring one. (Details are specified in an earlier version of this paper, which is available from the authors.)

Evaluation of $L(\lambda)$

Project problem (13) onto the variables z_i for $i = 1, \ldots, n$, by defining:

$$q_i^{\lambda}(z_i) = \min_{0 \leq w_i \leq z_i} \{p_i(z_i) + r_i(w_i) + \lambda w_i\}, \quad z_i \geq 0. \tag{15}$$

The minimization problem in (13) is thus equivalent in an obvious sense to the following problem:

$$L(\lambda) = \text{minimize} \quad \sum_{i=1}^{n} q_i^{\lambda}(z_i) \tag{16a}$$

$$\text{subject to} \quad \sum_{i=1}^{n} z_i \leq A + B \tag{16b}$$

$$z_i \geq 0, \qquad i \in Y. \tag{16c}$$

Problem (16) is a single-resource allocation problem for which special, efficient methods have been developed by Luss and Gupta, Zipkin, and Federgruen and Zipkin (1983), among others. While the implicit definition in (15) requires some careful attention, the difficulties can be handled (as shown in an earlier version of this paper, available from the authors). For example, these algorithms require the quantities $Dq_i^{\lambda}(0)$ and $(Dq_i^{\lambda})^{-1}(\mu)$, which can be shown to be equal to

$$Dq_i(0) = Dp_i(0) + \min\{Dr_i(0) + \lambda, 0\},$$

$$(Dq_i)^{-1}(\mu) = \max\{(Dp_i + Dr_i)^{-1}(\mu - \lambda), (Dp_i)^{-1}(\mu)\}.$$

Also, a variety of observations can be exploited to facilitate resolution of (16), following a change in λ.

Furthermore, the function $L(\lambda)$ can be shown to be C^1, and

$$DL(\lambda) = \sum_{i \in Y} w_i^*(\lambda) - B,$$

where $w^*(\lambda)$ is the optimal value of w in (13); thus, the search for the optimal λ in (14) is fairly simple.

3. A Combined Routing and Inventory Allocation Problem

In this section we consider the case in which deliveries are combined in routes with multiple stops. The problem as a whole can be formulated as a complex, nonlinear, mixed-integer program. Rather than present the full model (which is closely analogous to that given in Federgruen and Zipkin 1984), we outline only the fundamental ideas needed to understand the computational approach.

Define

K = number of vehicles
b_k = capacity of vehicle k, $k = 1, \ldots, K$.

Let $k = 0$ denote a dummy route including the locations receiving no shipments at all, with $b_0 = 0$. Among the major decisions required is the assignment of locations to routes. This decision can be represented by the *assignment variables* y_{ik} for $i \in Y$ and $k = 0, \ldots, K$, where

$$y_{ik} = \begin{cases} 1, & \text{if delivery point } i \text{ is assigned to route } k \\ 0, & \text{otherwise.} \end{cases}$$

These must satisfy $\sum_{k=0}^{K} y_{ik} = 1$ for $i \in Y$. Let y denote all the variables y_{ik}.

The key to the computational approach is the following observation: suppose y is fixed. The remainder of the problem is to determine the sequence of deliveries within each route and to allocate the fresh and old product among the locations. *These two decisions, sequencing and allocation, are entirely separate.* That is, once the assignment is fixed, allocations do not affect travel costs, and the delivery sequence does not affect shortage and outdating costs.

The problem of determining the *best* allocation, given y, can be expressed as an optimization model as follows: the assignment y determines a partition $\{Y_k : k = 0, \ldots, K\}$ of the set of locations Y, where $Y_k = \{i \in Y : y_{ik} = 1\}$. (Thus, Y_k for $k = 1, \ldots, K$ is the subset of locations served by vehicle k, and Y_0 consists of locations not visited.) The problem is then

$$\text{(IAB)} \quad \zeta^* = \text{minimize} \sum_{i \in Y} [p_i(z_i) + r_i(w_i)] \tag{17}$$

$$\text{subject to} \sum_{i \in Y} z_i \leq A + B \tag{18}$$

$$\sum_{i \in Y} w_i = B \tag{19}$$

$$\sum_{i \in Y_k} z_i \leq b_k, \quad k = 0, \ldots, K \tag{20}$$

$$z_i \geq w_i \geq 0, \qquad i \in Y. \tag{21}$$

(This allocation subproblem is described further in our subsequent discussion.)

Precisely the same kind of separation occurs in the work of Federgruen and Zipkin (1984), who consider only one age class, and use the separation as the basis of their computational methods. When y is fixed, the allocation problem yields a model similar to (IAB), involving the subsets Y_k, but simpler in other respects.

We focus on the simplest methods described in Federgruen and Zipkin (1984), which are based on *interchange heuristics*, a class of algorithms for the deterministic VRP. (This includes the methods of Russell 1977, Christofides and Eilon 1969, Wren and Holliday 1972 and Cassidy and Bennett 1972, and Lin and Kernighan 1973. We remark that another approach described in Federgruen and Zipkin 1984, based on generalized Benders' decomposition (Geoffrion 1972), can also be adapted to the current problem.) The overall logic of these methods for the VRP is as follows: start with a given solution, that is, an assignment y and the delivery sequence for each route. Evaluate the cost effects of many small changes in this solution. Perform the best such change. Continue evaluating and then performing small changes until no further improvement is possible.

The adaptation in Federgruen and Zipkin (1984) follows the same logic. The algorithm evaluates changes in y and the delivery sequences; for each y considered, the cost of the optimal allocation is computed or estimated. Some potential changes may affect only the sequencing, not y; such changes do not affect allocation costs, and thus can be evaluated as in deterministic models. When a potential change includes a change in y, however, it is necessary to evaluate the resulting change in shortage and outdating costs, either exactly or approximately, in addition to the effects on travel costs.

Also, when a change in y is actually performed, the new total shortage and outdating costs must be recorded, to set the stage for evaluation of subsequent changes. All such additional cost effects can be calculated by recovering (or approximating) the solution to the single-age-class analogue of (IAB), following changes in the subsets Y_k. Computational methods for that allocation subproblem are thus embedded as subroutines within the VRP algorithm.

It is clear from this discussion that *exactly the same approach can be applied to our current problem.* The only difference is that the more complex subproblem (IAB) replaces the corresponding single-age-class model.

Details concerning the mechanics of interchange heuristics in general and a code implementing one such algorithm are given in Federgruen and Zipkin (1984). The code used for our numerical experiments (Section 4) follows the same structure.

We now outline an algorithm to solve (IAB) exactly and a quicker heuristic technique that works well empirically. The exact method is used in the code only for the initial and final values of y. The heuristic is used when changes are actually performed in intermediate iterations. (This replacement of the exact method with the heuristic reduced computation times considerably with only slight degradation of performance.) To evaluate potential switches, a simple, closed-form approximation of the change in cost was used (analogous to the formula for ΔIA in Federgruen and Zipkin 1984).

Exact Solution of (IAB)

Note that (IAB) is just (IA) with additional generalized-upper-bound constraints (20). As in Section 1, the approach is based on solving the Lagrangean dual with respect to (19). Evaluation of the dual again uses methods for simpler allocation problems.

Define

$$L(\lambda) = \text{minimize} \sum_{i=1}^{n} q_i^{\lambda}(z_i),$$

$$\text{subject to (18), (20) and } z_i \geq 0, \quad i \in Y, \quad (22)$$

where functions $q_i^{\lambda}(\cdot)$ are defined by (15). As is the case of (13), $L(\cdot)$ is a concave function, and

$$\zeta^* = \max_{\lambda} L(\lambda). \quad (23)$$

To evaluate $L(\lambda)$, we project problem (22) onto new variables $Z_k = \sum_{i \in Y_k} z_i$. That is, for each $k = 1, \ldots, K$ define

$$Q_k^{\lambda}(Z_k) = \text{minimize} \sum_{i \in Y_k} q_i(z_i)$$

$$\text{subject to} \sum_{i \in Y_k} z_i \leq Z_k$$

$$z_i \geq 0, \quad i \in Y_k. \quad (24)$$

(22) is equivalent in an obvious sense to the following problem:

$$\text{minimize} \sum_{k=1}^{K} Q_k^{\lambda}(Z_k)$$

$$\text{subject to} \sum_{k=1}^{K} Z_k \leq A + B$$

$$0 \leq Z_k \leq b_k, \quad k = 1, \ldots, K. \quad (25)$$

Federgruen and Zipkin (1983) describe an efficient method to solve problems of form (25) with cost functions defined implicitly by subproblems of form (24). This procedure can be adapted easily to handle functions $q_i^{\lambda}(\cdot)$ defined as in (15); we omit the details.

Heuristic Solution Method for (IAB)

This method follows a simple two-phase procedure, similar to the one suggested by Prastacos (1978) for (IA) described in Section 1:

Phase I. Solve

(IAB 1) minimize $\sum_{i \in Y} r_i(w_i)$

subject to $\sum_{i \in Y_k} w_i \leq b_k, \quad k = 0, \ldots, K$

$\sum_{i \in Y} w_i = B; \quad w_i \geq 0, \quad i \in Y.$

Let $\bar{w}$ denote the optimal solution of (IAB 1).

Phase II. Set

$$b_k' = b_k - \sum_{i \in Y_k} \bar{w}_i, \quad k = 0, \ldots, K$$

$$T_i' = T_i + \bar{w}_i = A_i + B_i + \bar{w}_i, \quad i \in Y,$$

and solve

(IAB 2) minimize $\sum_i p_i'(v_i)$

subject to $\sum_{i \in Y_k} v_i \leq b_k', \quad k = 0, \ldots, K$

$\sum_{i \in Y} v_i \leq A.$

where $p_i'(\cdot)$ is defined as in (4), replacing T_i by T_i', $i \in Y$. Let $\bar{v}$ denote the optimal solution of (IAB 2).

The vectors $\bar{w}$ and $\bar{z} = \bar{v} + \bar{w}$ constitute a feasible solution for (IAB). Both (IAB 1) and (IAB 2) have the same form as (22) and can thus be solved by the method described above for the latter problem.

4. Computational Results

We begin with a brief summary of our experience in solving numerous instances of problems IA and IAB exactly using the algorithms described above. We solved problems with $n = 50$ and $n = 75$. (These were subproblems within larger routing/allocation models: the following discussion gives more details about the problems.) For $n = 50$, all instances of (IA) were solved within 0.7 seconds and (IAB) within 1.0 second. For $n = 75$, all instances of (IA) were solved within 0.9 seconds and (IAB) within 1.7 seconds. (Our computer times refer to virtual CPU seconds on an IBM 4341.)

We now turn to the combined vehicle-routing-inventory-allocation problem described in Section 3. As in Federgruen and Zipkin (1984), we experimented with problems adapted from deterministic VRPs (problems 8 and 9 in Chapter 9 of Eilon, Watson-Gandy and Christofides 1971), having $n = 50$ and $n = 75$. The starting inventories A_i and B_i for $i \in Y$ were generated independently from a uniform distribution on the interval [0, 10]. The demand distributions were all normals, with coefficients of variation equal to one; the means were chosen to make the problems roughly similar to their original deterministic counterparts. The costs h_i^+ and h_i^- were set equal for all values of i, and the capacities b_k equal for all values of k (to h^+, h^- and b respectively). The parameters K, h^+, h^-, A and B were varied to yield a variety of problems.

As a base of comparison, we also tried an algorithm that, while still quite sophisticated, treats the allocation and routing decisions separately in traditional fashion. Problem IA is solved, and the optimal z_i are used as (deterministic) delivery requirements within a standard code for the VRP. For the latter step we used the heuristic algorithm of Gillett and Miller (1974). (When some $z_i = 0$, location i is not included in the VRP; in effect it is assigned to the dummy vehicle $k = 0$). We shall refer to this strategy as the *separate approach*, as distinguished from the *combined approach* developed in this paper. Table I compares the performance of the two approaches. For each problem instance (described by the parameters in the first six columns) the table shows the cost of the solution, broken down into inventory (expected shortage and outdating) costs and variable travel costs. The code used for the separate approach determines the minimal number of vehicles required to make all deliveries, so only that value of K appears in the table. Two values of K were tried using the combined approach for each setting of the other parameters.

The combined approach produces a total variable cost that is only slightly less than that of the separate approach. It is clear, however, that this reflects the particular choice of parameter scales, with inventory-related costs dominating travel costs. The key observation from the table is that travel costs are substantially less using the combined approach, with a very minor reduction in inventory performance. Furthermore, in the 75-location problems the combined approach can achieve these results with one fewer vehicle (9 versus 10)!

The table also shows what happens when fewer vehicles are used (which in the separate approach is simply infeasible). Whether or not the corresponding increases in variable costs are tolerable depends, of course, on the specifics of the problem setting (blood versus milk, urban versus rural versus international, etc.).

In our implementations, the combined approach

TABLE I
Computational Results

Parameters						Separate Approach					Combined Approach									
							Costs			CPU time		Costs			CPU time		Costs			CPU time
n	b	A	B	h^+	h^-	K	Inventory	Travel	Total		K	Inventory	Travel	Total		K	Inventory	Travel	Total	
50	225	500	500	1.0	5	5	1238	516	1754	5.8	5	1245	473	1718	6.4	4	1415	429	1843	7.5
				2.5	5	5	1753	516	2269	5.9	5	1760	473	2233	6.9	4	1931	429	2360	7.4
				2.5	10	5	2648	516	3164	5.8	5	2661	473	3135	7.1	4	2971	458	3429	4.5
				5.0	20	5	5296	516	5812	5.9	5	5307	482	5789	8.2	4	5927	468	6395	5.6
50	225	750	250	1.0	5	5	1139	516	1655	5.9	5	1146	473	1620	6.6	4	1315	429	1744	7.2
				2.5	5	5	1507	516	2023	5.9	5	1513	473	1989	7.0	4	1688	430	2117	9.2
				2.5	10	5	2401	516	2917	5.9	5	2415	473	2888	7.2	4	2724	458	3182	4.5
				5.0	20	5	4803	516	5319	6.0	5	4815	482	5296	8.1	4	5434	468	5902	5.3
75	200	900	900	1.0	5	10	1901	835	2736	6.6	9	1928	764	2692	10.2	8	2269	666	2935	13.3
				2.5	5	10	2783	835	3618	6.6	9	2811	764	3575	13.2	8	3157	666	3822	14.4
				2.5	10	10	4096	835	4931	6.6	9	4112	787	4900	18.4	8	4700	739	5440	21.3
				5.0	20	10	8192	835	9027	6.7	9	8216	804	9020	13.0	8	9389	766	10154	14.6
75	200	1350	450	1.0	5	10	1719	835	2554	6.6	9	1746	764	2510	11.5	8	2085	666	2751	14.5
				2.5	5	10	2330	826	3156	6.3	9	2351	765	3116	18.9	8	2697	666	3363	15.4
				2.5	10	10	3642	835	4477	6.6	9	3658	787	4446	20.0	8	4246	739	4985	22.7
				5.0	20	10	7284	835	8119	6.6	9	7307	804	8111	14.5	8	8480	766	9245	15.6

requires about 75% more computer time than the separate approach. We expect the combined approach to remain feasible for many applications.

Appendix: An Alternative Specification of the Shortage Costs

For products like blood, shortages are resolved by emergency deliveries, the costs of which tend to be independent of the size of the delivery. In such cases, the shortage cost function is more appropriately represented by

$$p_i(z_i) = E_i(1 - F_i(T_i + z_i)), \qquad i = 1, \ldots, n, \qquad \text{(A6)}$$

where E_i represents the fixed cost associated with an emergency delivery to location i. Assuming the demands all have unimodal densities, let $m_i =^{\text{def}}$ mode of the demand density in location i, and $\mu_i =^{\text{def}} \int_o^\infty u\, dF_i(u)$. (A6) is (strictly) convex only for values of z_i for which

$$T_i + z_i \geqslant m_i, \qquad i = 1, \ldots, n.$$

We propose, therefore, to append the lower bounds

$$Z_i \geqslant [m_i - T_i]^+, \qquad i = 1, \ldots n,$$

to (IA) and (IAB); by a suitable translation of the z-variables, this approach results in problems of the same form as (IA) and (IAB).

In most practical cases, the imposition of these bounds can be justified: when $m_i = 0$ (as for Weibull distributions with shape parameter $\leqslant 1$, in particular for exponential distributions), (A6) is strictly convex for all $z_i \geqslant 0$ for $i = 1, \ldots, n$. Even when $m_i > 0$, many standard distributions have $m_i \leqslant \mu_i$, e.g., the gammas and the normals. In many applications, therefore, there is little lost by requiring (as the lower bounds do) that each inventory *after* replenishment be at least as large as m_i.

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