THE GREEDY PROCEDURE FOR RESOURCE ALLOCATION PROBLEMS: NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMALITY

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In many resource allocation problems, the objective is to allocate discrete resource units to a set of activities so as to maximize a concave objective function subject to upper bounds on the total amounts allotted to certain groups of activities. If the constraints determine a polymatroid and the objective is linear, it is well known that the greedy procedure results in an optimal solution. In this paper we extend this result to objectives that are "weakly concave," a property generalizing separable concavity. We exhibit large classes of models for which the set of feasible solutions is a polymatroid and for which efficient implementations of the greedy procedure can be given.

In many resource allocation problems, the objective is to allocate discrete resource units to a set of activities so as to maximize a *concave* objective function subject to upper bounds on the total amount allotted to certain groups of activities. These problems can be formulated as integer programs of the following type:

$$\begin{aligned} &\text{maximize} \quad r(z) \\ &\text{subject to} \quad \sum_{i \in S} z_i \leq V(S), \qquad \text{(P)} \\ &\qquad\qquad S \in A \quad \text{and} \quad z_i \geq 0 \quad \text{and integer.} \end{aligned}$$

In this model, A is a class of subsets of a finite set E and $V(\cdot)$ is a given function defined on subsets S of E.

The *greedy* or *marginal allocation* procedure assigns available units sequentially to the activity that benefits most from an additional allocation among all activities whose allotment can be increased without creating infeasibilities. It terminates as soon as no such activity can be found.

In the simplest case, $r(\cdot)$ is separable and $A = \{E\}$ (so the model contains a single budget constraint), and as is well known, the *greedy* procedure results in an optimal solution (Gross 1956 and the references cited later in this section). Tamir (1980) extended this result to models with a *nested* set of constraints ($A = \{S_i$ for $i = 1, \ldots, n\}$ with $S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n = E$). Brucker (1982) established a further generalization to *tree-structured* models that are structured so that, for each pair S, $T \in A$, either $S \subset T$ or $T \subset S$ or $S \cup T = \varnothing$. (see also Mjelde 1983).

We recently developed an optimization model for an investment company that deals in oil and gas ventures (Federgruen and Groenevelt 1986). The model determines which of the company's clients should apply for a lease on each of the parcels offered by the U.S. government in its bimonthly special drawings. The model can be formulated as a special instance of the class P and the greedy procedure can be shown to result in an optimal solution in spite of its failing to have a tree-structure. On the other hand, the transportation problem with non-positive cost coefficients is a special case of the problem class P; yet here, the greedy procedure may fail to generate an optimal solution. Also, the set-covering problem can be formulated as a special case of our class of models, and this problem is known to be notoriously hard; in fact, it is strongly NP-complete (Karp 1972, Garey and Johnson 1979 and Section 6).

For linear objectives, as is well known (see Edmonds 1970) the greedy procedure results in an optimal solution if and only if the constraints determine (the independence polytope of) a *polymatroid*. The intent of this paper is to extend this result to objectives that are specified by a so-called *weakly concave* complete order on R^E. This class of objectives includes all orders generated by separable concave functions, as well as other important nonseparable cases. (Megiddo 1974

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and Fujishige 1980, for example, consider the problem of constructing a maximal flow in a capacitated network with multiple sinks while (in ascending order) lexicographically maximizing *each* of the amounts supplied to individual sinks. Such an objective corresponds with a weakly concave order; the set of possible supply vectors is a polymatroid, and an optimal flow can be found by the greedy procedure.)

We also identify a set of *local* optimality conditions that imply *global* optimality, provided the set of feasible solutions is a polymatroid and the objective satisfies a second, slightly stronger concavity property.

There are two problems associated with the practicality of these optimality results:

(a) it is often difficult to verify whether a feasible region is a polymatroid, and

(b) each iteration of the greedy procedure involves multiple checks as to whether a particular component of the current solution can feasibly be incremented by one unit. This feasibility test can be extremely complicated for general polymatroids and is related to the well-known "membership problem" (Bixby, Cunningham and Topkis 1985, Topkis 1983b and Grötschel, Lovasz and Schrijver 1981).

A second major purpose of this paper is, therefore, to enumerate important and easily verifiable cases that satisfy the polymatroid condition and for which an efficient implementation of the feasibility test can be given.

Gross' (1956) initial optimality result for models with a single budget constraint was refined by Fox (1966) and Veinott (1964). The result was later rediscovered by many others, e.g., Einbu (1977), Hartley (1976), Kao (1976), Mjelde (1975), Proll (1976) and Shih (1974); see also Ibaraki (1980). This single constraint model (or its continuous version) has numerous applications (see, for example, Zipkin 1980). Asymptotically more efficient algorithms were proposed by Katoh, Ibaraki and Mine (1979), Galil and Megiddo (1979), Ibaraki, and Frederickson and Johnson (1982). As mentioned previously, researchers also extended the optimality result to *nested* models (Tamir, and Galperin and Wacksman 1981) and then to tree-structured models (Brucker, and Section 6). Federgruen and Zipkin (1983) deal with models that include generalized upper bounds in addition to the budget constraint, which is a model with a tree-structure as described in Section 4.

Polymatroids, a generalization of matroids, are associated with a special class of polyhedra introduced by Edmonds (1970). They, as well as the associated submodular set functions (see Section 1), play a central role in the modern theory of combinatorial optimization (see, for example, Lovasz 1982). Important applications are mentioned in Edmonds and Giles (1977), Frank (1982) and Schrijver (1982); see also Bixby, Cunningham and Topkis, and Topkis (1983b) and the references they cite. Edmonds and Giles, Hassin (1978) and Lawler and Martel (1982) introduced generalizations of the network flow problem in which restrictions on net flows into individual vertices or capacities on individual arcs are generalized to polymatroidal restrictions on net flows into *subsets* of vertices and capacities on *subsets* of arcs respectively. An application of this model to machine scheduling problems can be found in Martel (1982).

We start (Section 1) with some notation and preliminary results, such as necessary and sufficient conditions for the feasible region of P to be (the independence polytope of) a polymatroid. Section 2 introduces and discusses concavity notions, and Section 3 analyses the behavior of the greedy procedure as well as equivalence between local and global optimality. The next two sections enumerate a number of classes of models for which the feasible region of P is easily verified to be a polymatroid (Section 4) and for which an efficient implementation of the feasibility test can be given (Section 5). Section 6, finally, makes some concluding remarks with respect to related problems.

1. Notation and Preliminaries

Let N denote the set of nonnegative integers and let e^i for $i \in E$ be the ith unit basis vector of $\mathbf{N}^E$. For $x, y \in \mathbf{N}^E$, we write $x < y$ if $x \leqslant y$ and $x \neq y$.

For a finite set A, let $|\mathbf{A}|$ denote the number of elements of A, and let $2^A = \{S : S \subseteq A\}$ be its power set. For $z \in \mathbf{N}^E$, $S \subseteq E$ we will write $z(S) = \sum_{i \in S} z_i$; for $\mathbf{B} \subseteq 2^E$, we write $z[\mathbf{B}] = \sum_{S \in \mathbf{B}} z(S)$. Likewise, for $\mathbf{B} \subseteq \mathbf{A} \subseteq 2^E$ and V a set function defined on $\mathbf{A}$, we write $V(\mathbf{B}) = \sum_{S \in \mathbf{B}} V(S)$.

For $\mathbf{A} \subseteq 2^E$ with $\cup\, \mathbf{A} = \cup_{S \in \mathbf{A}} S = E$, and V a nonnegative set function defined on $\mathbf{A}$, let $F(\mathbf{A}, V) = \{z \in \mathbf{N}^E : z(S) \leqslant V(S) \text{ for all } S \in \mathbf{A}\}$. Let $\geqslant_R$ denote a complete order on $\mathbf{N}^E$. We write $x >_R y$ if $x \geqslant_R y$ and $y \not\geqslant_R x$. Some orders on $\mathbf{N}^E$ are *induced* by a real-valued function f. In this case, $x \geqslant_R y$ if and only if $f(x) \geqslant f(y)$ $(x, y \in \mathbf{N}^E)$.

Let $F \subseteq \mathbf{N}^E$ and R a complete order on $\mathbf{N}^E$. In this paper we consider certain types of integer problems.

Find $z \in F \subseteq \mathbf{N}^E$ which is maximal with respect to R. $\quad P(R, F)$

The set F is called the *feasible region* of $P(R, F)$. A point $z \in F$ is a (*global*) *optimum* for $P(R, F)$ if

$z \geqslant_R z'$ for all $z' \in F$. A point $z \in F$ will be called a *local optimum* if

(L1) $z \geqslant_R z + e^i \quad (i \in E: z + e^i \in F)$,
(L2) $z \geqslant_R z - e^j \quad (j \in E: z - e^j \in F)$, and
(L3) $z \geqslant_R z + e^i - e^j \quad (i, j \in E: z + e^i - e^j \in F)$.

Of particular interest in this paper are certain special sets F, the so-called independence polytopes of polymatroids. We first define a *rank function* (with *groundset* E) as a set function V, defined on 2^E with the properties:

(V1) V is *normalized*: $V(\phi) = 0$,
(V2) V is *nondecreasing*: $V(S) \leqslant V(T)$ $(S \subset T \subseteq E)$, and
(V3) V is *submodular*: $V(S) + V(T) \geqslant V(S \cup T) + V(S \cap T)$ for all $S, T \subseteq E$.

Let $G \subseteq \mathbf{N}^E$. We call G a *polymatroid* (with ground set E) if $G = F(2^E, V)$ for some rank function V with groundset E. (In the standard literature, G is usually referred to as the independence polytope of a polymatroid.) Several equivalent definitions for polymatroids can be found in the literature; see, e.g., Edmonds, and Dunstan and Welsh (1973) and Welsh (1976). In this paper we need the following "properties":

Lemma 1. *Let F be a polymatroid with groundset E and rank function V, i.e., $F = \{z \in \mathbf{N}^E : z(S) \leqslant V(S), S \subseteq E\}$. Then F satisfies*

(F1) $0 \in F$;
(F2) *if $z \in F$, $y \in \mathbf{N}^E$ and $y < z$ then $y \in F$;*
(F3) *if $z \in F$, $y \in \mathbf{N}^E$ and $j \in E$ satisfy*
 (i) $y < z$,
 (ii) $y_j = z_j$ *and*
 (iii) $y + e^j \in F$,
 then there exists some $l \neq j$ such that $z + e^j - e^l \in F$ and $z_l > y_l$.

Proof. (F1) and (F2) are well-known and immediate properties (see, for example, Welsh p. 336). To show (F3), let $\mathbf{B} = \{S \subseteq E \mid z(S) = V(S) \text{ and } j \in S\}$. If $\mathbf{B} = \varnothing$, then $z + e^j \in F$, and (i) and (ii) imply that there exists an $l \neq j$ with $z_l > y_l$. So $z + e^j - e^l \in F$, in view of (F2). If $\mathbf{B} \neq \varnothing$, we first show that $\mathbf{B}$ is closed under intersection. Let $S, T \in \mathbf{B}$. Then $j \in S \cap T$ and $z(S \cap T) = z(S) + z(T) - z(S \cup T) = V(S) + V(T) - z(S \cup T) \geqslant V(S) + V(T) - V(S \cup T) \geqslant V(S \cap T)$. So $z(S \cap T) = V(S \cap T)$ and $S \cap T \in \mathbf{B}$. Let $H = \cap\, \mathbf{B}$. Then $y(H) < V(H)$ by (iii) and $V(H) = z(H)$. So there exists an $l \in H \backslash \{j\}$ satisfying $y_l < z_l$. But then $z + e^j - e^l \in F$ and $z_l > y_l$.

Remark. The fact that B in this proof is closed under intersection is well known; see, e.g., Lemma 2.3 in Fujishige.

Later on, we will see that (F1), (F2) and (F3) are indeed sufficient as well as necessary for $F \subseteq \mathbf{N}^E$ to be a polymatroid.

Lemma 2. *Assume $F \subseteq \mathbf{N}^E$ satisfies* (F1), (F2) *and* (F3) *and suppose $x \in F$. Define $Fx = \{z \in \mathbf{N}^E : z + x \in F\}$. Then Fx satisfies* (F1), (F2) *and* (F3).

Proof. $0 \in Fx$ since $0 + x = x \in F$. If $z \in Fx$, $y \in \mathbf{N}^E$ and $y \leqslant z$, then $y + x \leqslant z + x$. Since $z + x \in F$, $y + x \in F$ in view of F satisfying (F2). Thus $y \in Fx$, showing Fx satisfies (F2). If $z \in Fx$, $y \in \mathbf{N}^E$ and $k \in E$ satisfy (i), (ii) and (iii) of (F3) with respect to Fx, then $z' = z + x \in F$ and $y' = y + x \in \mathbf{N}^E$ satisfy (i), (ii), and (iii) of (F3) with respect to F. Hence there exists $l \neq k$ such that $z' + e^k - e^l \in F$, $z'_l > y'_l$ and $z' + e^k - e^l \geqslant y'$. But then $z + e^k - e^l \in Fx$.

Federgruen and Groenevelt (1984) specify convenient ways to check if a set $F(\mathbf{A}, V)$ (feasible region of P) is a polymatroid by extending the set function V from $\mathbf{A}$ to 2^E.

In Section 3 we characterize the behavior of the following greedy algorithm to solve $P(R, F)$.

Greedy or Marginal Allocation Algorithm

Step 0. $z := 0$;
Step 1. find $i \in E$ with $z + e^i \in F$, $z + e^i \geqslant_R z$ and $z + e^i \geqslant_R z + e^j$ $(j \in E: z + e^j \in F)$,
Step 2. if no such $i \in E$ exists, stop,
Step 3. $z := z + e^i$ and go to Step 1.

2. Concave Orders

In this section we introduce and discuss three increasingly stronger concavity properties. A complete order R is called *concave* if it satisfies

(R1) if $y \geqslant x$, $x \geqslant_R x + e^i$, then $y \geqslant_R y + e^i$, $i \in E$;
(R2) if $y \geqslant x$, $x >_R x + e^i$, then $y >_R y + e^i$, $i \in E$;
(R3) if $y \geqslant x$, $x_i = y_i$ and $x + e^i \geqslant_R x + e^j$, then $y + e^i \geqslant_R y + e^j$, $i, j \in E$;
(R4) if $y \geqslant x$, $x_i = y_i$ and $x + e^i >_R x + e^j$, then $y + e^i >_R y + e^j$, $i, j \in E$.

A complete order is called *weakly concave* if it satisfies (R1) and (R3). Similarly, it is called *strongly concave* if it is concave and satisfies

(R5) if $y \geqslant x$, $x_i = y_i$, $x \leqslant_R x + e^i$, then $y \leqslant_R y + e^i$,
(R6) if $y \geqslant x$, $x_i = y_i$, $x <_R x + e^i$, then $y <_R y + e^i$.

Observe first that the order induced by any separable concave (real-valued) function is strongly concave. In addition, various nonseparable optimization problems can be handled by an appropriate choice of a concave order; see the examples given later in this section. The following three lemmas show that concavity is maintained through translation over a given vector, through lexicographic combinations of several concave orders, as well as through extensions of strongly concave orders to a larger groundset.

Lemma 3 (Translation). *Let R be a complete order on $\mathbf{N}^E$. For $b \in \mathbf{N}^E$, define Rb by $x \geqslant_{Rb} y$ if and only if $x + b \geqslant_R y + b$. If R satisfies (Ri) $(i = 1, \ldots, 6)$, then so does Rb.*

Proof. The proof is trivial.

Lemma 4 (Lexicographic combinations). *Let R_1, R_2 be complete orders on $\mathbf{N}^E$. Define the order R on $\mathbf{N}^E$ by $x \geqslant_R y$ if and only if $x >_{R_1} y$, or $x =_{R_1} y$ and $x \geqslant_{R_2} y$.*

(a) *If R_1 is concave and R_2 satisfies property (R_i), then so does R $(i = 1, 2, 3, 4)$.*
(b) *If R_1 is strongly concave and R_2 satisfies property (Ri) $(i = 1, \ldots, 6)$, then so does R.*

Proof

(a) Assume R_2 satisfies (R1): Let $y \geqslant x$, $x + e^i \leqslant_{R_1} x$, $i \in E$.

Case 1: $x + e^i <_{R_1} x$. Then $y + e^i <_{R_1} y$ by (R2) of R_1, so $y + e^i <_R y$.

Case 2: $x + e^i =_{R_1} x$, $(x + e^i) \leqslant_{R_2} x$. Then $y + e^i \leqslant_{R_1} y$ and $y + e^i \leqslant_{R_2} y$ by (R1) of R_1 and R_2. So, $y + e^i \leqslant_R y$.

Assume R_2 satisfies (R2): verified as in the proof for (R1).

Assume R_2 satisfies (R3): Let $y \geqslant x$, $y_i = x_i$ and $x + e^i \geqslant_R x + e^j$ *for* $i, j \in E$.

Case 1: $x + e^i >_{R_1} x + e^j$. Then $y + e^i >_{R_1} y + e^j$ by (R4) of R_1, and $y + e^i >_R y + e^j$.

Case 2: $x + e^i =_{R_1} x + e^j$, $x + e^i \geqslant_{R_2} x + e^j$. Then $y + e^i \geqslant_{R_1} y + e^j$ and $y + e^i \geqslant_{R_2} y + e^j$ by (R3) of R_1 and R_2, so $y + e^i \geqslant_R y + e^j$.

Assume R_2 satisfies (R4): verified as in the previous proofs.

(b) Assume R_2 satisfies (R5): Let $y \geqslant x$, $y_i = x_i$ and $x + e^i \geqslant_R x$, $i \in E$.

Case 1: $x + e^i >_{R_1} x$. Then $y + e^i >_{R_1} y$ by (R6) of R_1, so $y + e^i >_R y$.

Case 2: $x + e^i =_{R_1} x$ and $x + e^i \geqslant_{R_2} x$. Then $y + e^i \geqslant_{R_1} y$ and $y + e^i \geqslant_{R_2} y$ by (R5) of R_1 and R_2 and so $y + e^i \geqslant_R y$.

Assume R_2 satisfies (R6): verified as in the previous proofs.

Lemma 5 (Extension of the groundset). *Let R_1 be a strongly concave order on $\mathbf{N}^{E_1}$ with $E_1 \subseteq E$. For $x \in \mathbf{N}^E$, write $x1 = (x_i)_{i \in E_1}$. The order R, defined on $\mathbf{N}^E$ by $x \geqslant_R y$ if and only if $x1 \geqslant_{R_1} y1$, is strongly concave.*

Proof. Immediate.

Remark. Lemma 5 fails to hold when we replace strong concavity by concavity.

Example 1. The following class of nonseparable objectives was considered by, for example, Megiddo (1974), Fujishige, and Ichimori, Ishii and Nishida (1982): For a given $w \in \mathbf{R}^E$, let $T(x)$ denote the vector $(w_i x_i)_{i \in E}$ ranked in ascending order of its components $(x \in \mathbf{N}^E)$. Define the order R on $\mathbf{N}^E$ by $x \geqslant_{\mathbf{R}} y$ if and only if $T(x)$ is lexicographically larger than $T(y)$. It is easy to verify that R is a strongly concave order if $w \geqslant 0$. More generally, R remains strongly concave if $T(x)$ denotes the vector $(f_i(x_i))_{i \in E}$, ranked in ascending order, when $f_i(\cdot)$ for $i \in E$ are arbitrary nondecreasing functions. These criteria are sometimes referred to as "the sharing problem," see Brown (1979a,b) and Ichimori, Ishii and Nishida.

An ingenious proof in Fujishige shows that when Tx is the vector $(w_i x_i)_{i \in E}$, the order may, for purposes of optimization over polymatroids, be replaced by another separable objective (i.e., an order induced by a separable function). As in Fujishige, the results in this paper apply to concave nonseparable objectives directly; in other words, there is no need to identify and prove a potential equivalency with a separable objective.

Note also that the order R induced by the simpler criterion $\min_{i \in E}(x_i)$ fails to be concave, since (R3) fails to hold. (Let $x = (0, 1, 2)$ and $y = (2, 1, 2)$; $(0, 1, 3) \geqslant_R (0, 2, 2)$ but $(2, 1, 3) \not\geqslant_R (2, 2, 2)$.)

Example 2. Let $c \in \mathbf{N}$, and let $\bar{R}$ be the complete order on $\mathbf{N}^E$ defined by $x \geqslant_{\bar{R}} y$ if and only if $y(E) \leqslant x(E) \leqslant c$ or $\min\{y(E), x(E)\} \geqslant c$. It is easy to verify that $\bar{R}$ is concave, though not strongly concave. Also, if $c > 0$, $\bar{R}$ *cannot* be induced by a separable function. The concavity of $\bar{R}$ has important implications for optimization problems for which a (strongly) concave order R_1 is to be optimized over a region $\bar{F} = F \cap \{z \in \mathbf{N}^E : z(E) \geqslant c\}$, with F a polymatroid. Let R be defined by $x \geqslant_R y$ if and only if $x \geqslant_{\bar{R}} y$, or $x =_{\bar{R}} y$ and $x \geqslant_{R_1} y$. In view of the concavity of $\bar{R}$ and Lemma 4, the problem is equivalent to the maximization of R over F, with R concave.

3. Optimizing a Concave Order Over a Polymatroid

We first characterize the behavior of the Marginal Allocation Algorithm (MAA):

Theorem 1. *Let R be a complete order satisfying* (R3) *and let $F \subseteq \mathbf{N}^E$ satisfy* (F1) *and* (F2). *Then the MAA results in a local optimum for $P(R, F)$.*

Proof. Let y be the solution obtained by the MAA. In view of (F1), $0 \in F$. Hence, $y \in F$, by induction and in view of Step 1. Step 1 and Step 2 also imply (L1). Fix $i \in E$ with $y_i > 0$. Let x be the last point generated by the algorithm with $x_i = y_i - 1$, and let $y - e^j$ be the next to last point generated. Hence, $y - e^j \geqslant x$, or $y \geqslant x + e^j$ and in view of (F2), $x + e^j \in F$. Since $x + e^i \geqslant_R x + e^j$, we have by (R3) and Step 1,

$$y \geqslant_R y - e^j = (y - e^j - e^i) + e^i$$

$$\geqslant_R (y - e^j - e^i) + e^j = y - e^i.$$

This result proves (L2). To show (L3), let $j \in E$ with $y + e^j - e^i \in F$. Since $x + e^j \leqslant y - e^i + e^j$, we have $x + e^j \in F$ in view of (F2). Thus, $x + e^i \geqslant_R x + e^j$ which implies, by (R3), $y = (y - e^i) + e^i \geqslant_R y - e^i + e^j$.

Lemma 6. *Let R be a complete order on $\mathbf{N}^E$ satisfying* (R1), *and let F satisfy* (F1) *and* (F2). *If 0 is a local optimum of $P(R, F)$, then 0 is a global optimum.*

Proof. Let $z \in F$ with $z_i > 0$ for some $i \in E$. In view of (F2), $e^i \in F$ and hence $0 \geqslant_R e^i$ by (L2). Applying (R1) with $x = 0$ and $y = z - e^i$ then gives $z - e^i \geqslant_R z$. The lemma follows by repeated application of this argument.

Theorem 2. (Sufficient condition for optimality of MAA). *Let R be a weakly concave order on $\mathbf{N}^E$, and let $F \subseteq \mathbf{N}^E$ satisfy* (F1), (F2) *and* (F3). *The MAA solves $P(R, F)$.*

Proof. By induction on $m(F) = \max\{z(E): z \in F\}$. Let x be the solution found by the MAA. If $m(F) = 0$, the theorem is true, so assume it holds whenever $m(\cdot) \leqslant k - 1$ with $k \geqslant 1$ and let $m(F) = k$. If the algorithm terminates at the first iteration, $x = 0$ is a local optimum by Theorem 1 and hence x is optimal by Lemma 6. In the remaining case, let $i \in E$ be the index found during the first pass through Step 1. By Lemma 3, the order Re^i is weakly concave and by Lemma 2, Fe^i satisfies (F1), (F2) and (F3).

Since $x - e^i$ is a solution that the MAA could attain for the problem $P(Re^i, Fe^i)$ and $m(Fe^i) < k$, we have $x \geqslant_R z$ $(z \in F: z_i > 0)$, by the induction assumption. To complete the optimality proof of x, let $z \in F$ with $z \neq 0$ and $z_i = 0$. By (F3) with $y = 0$, there is a $j \in E$ with $z' = z + e^i - e^j \in F$ and $z_j > 0$. By (R3) and $e^i \geqslant_R e^j$, we have $z' \geqslant_R z$. Since $z'_i > 0$, we have $x \geqslant_R z'$ and hence $x \geqslant_R z$. This result concludes the induction proof.

Theorem 2 generalizes Edmonds' classical result for *linear* objectives. For the special case of *separable* objective functions, Theorem 2 may be established by exhibiting an equivalence between $P(R, F)$ and a *linear* optimization problem with a matroid as its feasible region, invoking Edmonds' classical results. For the general case (with nonseparable functions), no such transformation is possible (see also Girlich and Kowaljow 1981, remark 2.66, p. 181).

Corollary 1 (Main result). *Let $A \subseteq 2^E$, let V be a nonnegative, integer-valued set function on A, and let $F = F(A, V)$. The MAA results in an optimal solution for every weakly concave order R. The following statements are equivalent*:

(i) *F is a polymatroid.*
(ii) *F satisfies* (F1), (F2), *and* (F3).
(iii) *MAA results in an optimal solution for every weakly concave order R.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follows from Lemma 1 and Theorem 2. If F is not a polymatroid, a linear objective exists for which MAA fails to generate an optimal solution (see Edmonds, showing (iii) $\Rightarrow$ (i)).

Remark. If the feasible region F is not a polymatroid, MAA may fail to generate an optimal solution for separable and strictly concave, as well as nonseparable, concave objectives (see Federgruen and Groenevelt 1984).

Problems $P(R, F)$ with R concave and F a polymatroid have the additional property that every *local* optimum is a global optimum. This property will be established by Theorem 3. (Note that the property may fail to hold for weakly concave orders.) We first prove the following lemma:

Lemma 7. *Consider $P(R, F)$ with R concave and F a polymatroid. Let $z \neq 0$ be a local optimum. For some $i \in E$ with $z_i > 0$, a global optimum z^* can be found with $z^*_i > 0$.*

Proof. Let $l \in E$ with $e^l \in F$ and $e^l \geqslant_R e^k$ ($k \in E$: $e^k \in F$). If $z_l > 0$, choose $i = l$. Otherwise, by (F3) with $y = 0$, there is a $j \in E$ satisfying $z + e^l - e^j \in F$ and $z_j > 0$; in this case, choose $i = j$. By (L3), $z \geqslant_R z + e^l - e^j$. So by (R4) with $x = 0$ and $y = z - e^j$, we have $e^j \geqslant_R e^l$. Hence, in either case, $e^i \in F$, and $e^i \geqslant_R e^k$ ($k \in E: e^k \in F$). Since $z_i > 0$, $z \geqslant_R z - e^i$ by (L2) and hence by (R2) with $x = 0$ and $y = z - e^i$, we have $e^i \geqslant_R 0$. Thus $i \in E$ could be chosen in Step 1 of the first pass of the MAA, and in view of Theorem 2 there exists a global optimum z^* with $z_i^* > 0$.

Theorem 3. *Consider $P(R, F)$ with R concave and F a polymatroid. Every local optimum is a global optimum.*

Proof. By induction on $m(F) = \max\{z(E): z \in F\}$. The theorem is trivially true if $m(F) = 0$, so assume it holds whenever $m(F) < k$ with $k \geqslant 1$. Let $m(F) = k$. If 0 is a local optimum, it is a global optimum by Lemma 6. Thus, let $z \neq 0$ be a local optimum and let $i \in E$ satisfy the requirements of Lemma 7. By Lemma 2, Fe^i is a polymatroid and by Lemma 3, Re^i is a concave order. Then $z - e^i$ is a local optimum for $P(Re^i, Fe^i)$ and $m(Fe^i) < k$, so by the induction assumption, $z - e^i$ is a global optimum of $P(Re^i, Fe^i)$. By Lemma 7, there is a global optimum z^* of $P(R, F)$ with $z_i^* > 0$. Hence $z^* - e^i \in Fe^i$ and $z - e^i \geqslant_{Re^i} z^* - e^i$. But this conclusion implies $z \geqslant_R z^*$, so z is a global optimum.

4. Polymatroid Feasible Regions

In this section we enumerate several classes of models for which the feasible region is a polymatroid.

4.1. Feasible Regions Specified by Upper Bounds on Sets of Variables

Let $F = \{z \in \mathbf{N}^E: z(S) \leqslant V(S),\ S \in \mathbf{A}\}$ with $A \subseteq 2^E$.

4.1.1. Ring families

If $S, T \in \mathbf{A}$ then $(S \cap T) \in \mathbf{A}$ *and* $(S \cup T) \in \mathbf{A}$ *and* $V(\cdot)$ *is submodular on* $\mathbf{A}$ (see Edmonds).

4.1.2. Intersecting families

If $S, T \in \mathbf{A}$ and $(S \cap T) \neq \varnothing$, then $(S \cap T) \in \mathbf{A}$ and $(S \cup T) \in \mathbf{A}$ *and* $V(\cdot)$ *is submodular on* S and T (see Lawler 1982).

An important subclass is the class of *tree-structured* models: $\mathbf{A}$ has a tree-structure if for all $S, T \in \mathbf{A}$, $S \cap T \neq \varnothing$ implies $T \subset S$ or $S \subset T$. Note that tree-structured models are intersecting families for *all* set functions $V(\cdot)$ on $\mathbf{A}$.

As pointed out in the introduction, the class of tree-structured models contains many important cases: (1) a single resource constraint: $\mathbf{A}_1 = \{E\}$; (2) a single resource constraint with simple upper bounds: $\mathbf{A}_2 = \{E\} \cup \{S: S \subset E, |S| = 1\}$; (3) a single resource constraint with simple and generalized upper bounds: for some partition $\{E_k: k \in K\}$ of E, $A_3 = A_2 \cup \{E_1, \ldots, E_{|K|}\}$, (4) nested constraints: $\mathbf{A}_4 = \cup_{i=1} \{S_i\}$ with $S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n = E$.

4.1.3. Crossing families

If $S, T \in \mathbf{A}$ and $(S \cap T) \neq \varnothing$, and $(S \cup T) \neq E$, then $(S \cap T) \in \mathbf{A}$ and $(S \cup T) \in \mathbf{A}$ *and* $V(\cdot)$ *is submodular on* S *and* T (see Lawler).

4.1.4. Generalized symmetric models

$\mathbf{A} = 2^E$ and $V(S) = f(w(S))$, $S \subseteq E$ with $f(0) = 0$ and $f(\cdot)$ a nondecreasing concave function, $w \in \mathbf{N}^E$ and $w > 0$. It is easy to verify that $V(\cdot)$ is a rank function. Researchers refer to the special case with $w_i = 1$ for $i \in E$ as the symmetric case, (see, for example, Lawler and Martel, and Topkis (1983a,b).

Generalized symmetric models arise, for example, in the analysis of dynamic priority scheduling rules for multiclass queueing systems (Kleinrock 1976, Wood and Sargent 1984, Gelenbe and Mitrani, Chapter 6, 1980; and the references they cite). Each priority rule implies an average waiting time for each customer class, and the feasible region of waiting time vectors is a generalized symmetric polymatroid with $f(t) = t/(1 + t - w(E))$ and an appropriate choice of w. Designing a dynamic priority rule often amounts to optimizing an aggregate performance measure stated as a concave order on this feasible region of waiting time vectors; see the previous references.

We note that the classes 4.1.1, 4.1.2 and 4.1.3 are nested in increasing order of generality.

4.2. Network-based Models

Let $G = (N, E)$ be a connected network with node set N and arc set E. Let $S \subseteq N$ be the set of sources, and $T \subseteq N \backslash S$ the set of sinks. For each $i \in S$, let b_i denote the net capacity of source i. Also, u_{ij} denotes the (integer) capacity of arc $(i, j) \in E$. We define the variables

x_{ij} = flow on arc $(i, j) \in E$;

z_i = net supply to node i, $\quad i \in T$; $\quad z = (z_i)_{i \in T}$.

The network flow model has constraints:

$$0 \leqslant \sum_{l:(i,l)\in E} x_{il} - \sum_{l:(l,i)\in E} x_{li} \leqslant b_i, \quad i \in S$$

$$\sum_{l:(i,l)\in E} x_{li} - \sum_{l:(l,i)\in E} x_{il} = z_i, \quad i \in T \qquad (*)$$

$$\sum_{l:(i,l)\in E} x_{il} - \sum_{l:(l,i)\in E} x_{li} = 0, \quad i \in N\backslash S\backslash T$$

$$0 \leqslant x_{ij} \leqslant u_{ij}; \quad (i,j) \in E; \quad x_{ij} \text{ integer};$$

$$z_i \geqslant 0, \quad i \in T.$$

The set $\{z \in \mathbf{N}^T : (z, x)$ satisfies (*) for some $x\}$ is a polymatroid (Megiddo). We refer to Federgruen and Groenevelt (1984) for a survey of applications of this model.

5. Efficient Implementations of the MAA

The computational requirements of the MAA depend almost entirely on the possibility of implementing Step 1 efficiently. For general polymatroids, the feasibility check $(z + e^j \in F)$ is a special form of the general polymatroid membership problem. Grötschel, Lovasz and Schrijver's ellipsoid method performs the feasibility test in a polynomial amount of time, assuming V can be evaluated in polynomial time. Efficient combinatorial algorithms have been established only for matroids (Cunningham 1981) and for testing membership in the case of a structured convex game (Topkins 1983a,b). In this section, we describe efficient implementations of the MAA for several of the polymatroid classes enumerated in Section 4.

5.1. Tree-structured Models

The feasibility tests are simple, since each $i \in E$ can be contained in at most $|E|$ sets in **A**. Brucker presents a greedy procedure for tree-structured models and nondecreasing, separable concave objective functions. The procedure requires $0(|E|\,V(E))$ steps and is an immediate implementation of the MAA. In addition, he gives a polynomial $0(|E|^2\log\,(V(E))$ algorithm. This algorithm uses as a subroutine an $0(|E|\,\log\,(V(E)))$ procedure by Galil and Megiddo for the single constraint problem. For problems with a nested set of constraints (i.e., $\mathbf{A} = \cup_{i=1}^n \{S_i\}$ with $S_1 \subset S_2 \subset \ldots \subset S_n = E$) an $0(|E|^2 \log(V(E)/|E|))$ algorithm is given by Galperin and Waksman and Tamir.

5.2. Network-based Models

The feasibility check of Step 1 of MAA is easily performed by an augmenting path algorithm (Federgruen and Groenevelt 1986).

5.3. Generalized Symmetric Models

The following lemma implies an efficient membership test.

Lemma 9. *Let V be a generalized symmetric rank function with $V(S) = f(w(S))$. Let $x \in \mathbf{N}^E$ and $E = \{i_1, \ldots, i_{|E|}\}$ and assume $x_{i_1}/w_{i_1} \geqslant x_{i_2}/w_{i_2} \geqslant \ldots \geqslant x_{i_{|E|}}/w_{i_{|E|}}$. Let $E_n = \{i_1, \ldots, i_n\}$, $n = 1, \ldots, |E|$. $x \in F$ if and only if $x(E_n) \leqslant V(E_n)$ for $n = 1, \ldots, |E|$.*

Proof. Consider the collection $W = \{(w(S), x(S)) : S \subset E\}$ of points in $R \times R$. Then $x \in F$ if and only if the region $\{(s, t) : t \leqslant f(s), s \geqslant 0\}$ contains W or, more precisely, the set of vertices V on the "upper-left" part of the convex hull of W.

For fixed x, consider the parametric programming problem

$$\text{maximize} \sum_i b_i(x_i - \lambda w_i)$$
$$\text{subject to} \quad 0 \leqslant b_i \leqslant 1, \quad i \in E \quad \text{for all} \quad \lambda \geqslant 0. \qquad \mathrm{P}(\lambda)$$

Note that the largest solution $b(\lambda)$ to $\mathrm{P}(\lambda)$ has $b(\lambda)_i = 1$ if $x_i \geqslant \lambda w_i$, and $b(\lambda)_i = 0$, otherwise $(i \in E)$. Define the scalars $x(\lambda) = \sum_i b(\lambda)_i x_i$ and $w(\lambda) = \sum_i b(\lambda)_i x_i$ and note that $V = \{(w(\lambda), x(\lambda)) : \lambda \geqslant 0\} = \{(w(E_n), x(E_n)) : n = 1, \ldots, |E|\}$. The lemma follows from the properties of f.

The following proposition facilities the feasibility test of Step 1 of MAA.

Proposition 2. *Let $V(S) = f(w(S))$ be a generalized symmetric rank function. Let $x \in F$, $E = \{i_1, \ldots, i_{|E|}\}$ and assume $x_{i_1}/w_{i_1} \geqslant x_{i_2}/w_{i_2} \geqslant \ldots \geqslant x_{i_{|E|}}/w_{i_{|E|}}$. Let $E_n = \{i_1, \ldots, i_n\}$, $n = 1, \ldots, |E|$. Then $x + e^{i_n} \in F$ if and only if $x(E_m) < V(E_m)$ $(m = n, n+1, \ldots, |E|)$.*

Proof. Assume $x(E_m) = V(E_m)$ for some $m \geqslant n$. Then $(x + e^{i_n})(E_m) = x(E_m) + 1 > V(E_m)$ so $x + e^{i_n} \notin F$. This conclusion proves the "only if" part of the proposition. Next, assume $x(E_m) < V(E_m)$ for all $m = n, \ldots, |E|$. Let $k = 0$ if $[(x_{i_n} + 1)/w_{i_n}] > [x_{i_1}/w_{i_1}]$; otherwise, let $k = \max\{l : x_{i_l}/w_{i_l} \geqslant (x_{i_n} + 1)/w_{i_n}\}$. In view of Lemma 9, verification of $(x + e^{i_n}) \in F$ requires merely showing that $x(E_l \cup \{i_n\}) < V(E_l \cup \{i_n\})$ for $l = k, \ldots, n-2$. Now assume that $x(E_l \cup \{i_n\}) = V(E_l \cup \{i_n\})$ for some l with $k \leqslant l < n - 1$. We will show that this assumption leads to a contradiction. Since $x(E_l) \leqslant V(E_l)$, we have

$$x(e^{i_n}) = x(E_l \cup \{i_n\}) - x(E_l) \geqslant V(E_l \cup \{i_n\}) - V(E_l)$$
$$= f(w(E_l \cup \{i_n\})) - f(w(E_l)).$$

Hence,

$$x(E_{n-1} \setminus E_l)/w(E_{n-1} \setminus E_l)$$
$$\geq x_{i_n}/w_{i_n}$$
$$\geq [f(w(E_l \cup \{i_n\})) - f(w(E_l))]/w_{i_n}$$
$$\geq [f(w(E_n)) - f(w(E_l \cup \{i_n\}))]/w(E_{n-1} \setminus E_l).$$

Multiplying both sides of this inequality by $w(E_{n-1} \setminus E_l)$ yields

$$x(E_{n-1} \setminus E_l) \geq f(w(E_n)) - f(w(E_l \cup \{i_n\}))$$
$$= V(E_n) - V(E_l \cup \{i_n\})$$

and hence

$$x(E_n) = x(E_{n-1} \setminus E_l) + x(E_l \cup \{i_n\})$$
$$\geq V(E_n) - V(E_l \cup \{i_n\})$$
$$+ V(E_l \cup \{i_n\}) = V(E_n),$$

a contradiction.

The following efficient implementation of the MAA follows from Proposition 2.

Algorithm (Marginal Allocation for Generalized Symmetric Polymatroids)

Step 0. $z := 0$;
Step 1a. $n := |E|$; $y := z$; let $E = \{i_1, \ldots, i_{|E|}\}$ where $z_{i_1}/w_{i_1} \geq z_{i_2}/w_{i_2} \geq \ldots \geq z_{i_{|E|}}/w_{i_{|E|}}$, and $E_m = \{i_1, \ldots, i_m\}$ $(m = 1, \ldots, |E|)$;
Step 1b. While $z(E_n) < N(E_n)$ and $n > 0$ do
Begin
if $z + e^{i_n} >_R y$ then $y := z + e^{i_n}$;
$n := n - 1$
End;
Step 2. if $y = z$, then stop;
Step 3. $z := y$; *go to* Step 1a.

6. Concluding Remarks and Related Problems

Problem P, considered in the Introduction, may be viewed as a general integer problem with constraint set $Az \leq b$, $0 \leq z \leq u$, defined by a binary matrix A. If the objective function is linear, the problem is equivalent to the multiple set covering problem (Van Slyke 1982).

$$\begin{aligned} &\text{Minimize} \quad \sum_{i \in E} c_i z_i \\ &\text{subject to} \quad \sum_{i \in S} z_i \geq V'(S), \\ &\qquad S \in A \quad \text{and} \quad 0 \leq z_i \leq u_i, \\ &\qquad i \in E; \quad z \text{ integer.} \end{aligned} \qquad \text{(MSC)}$$

(Transform MSC into P as follows: write $z_i = u_i - z_i'$, $0 \leq z_i' \leq u_i$ and substitute z_i' for z_i in all constraints. The same substitution transforms P into MSC provided the objective function is linear.) Thus the multiple set covering problem can be solved exactly by the greedy procedure if and only if the feasible region of the transformed problem is a polymatroid.

The multiple set covering problem was introduced as a generalization of the well-known (unit) set covering problem specified by the data $u_i = 1$ for $i \in E$ and $V'(S) = 1$ for $S \in \mathbf{A}$. In general this problem is notoriously hard; in fact, it is strongly NP-complete since the minimum cover problem is NP-complete (Karp, and Garey and Johnson). Consequently, the class P is strongly NP-complete even for linear objective functions, $V'(\cdot)$ *symmetric on* $\mathbf{A}$, i.e., $V'(S) = V'(T)$ for all $S, T \in \mathbf{A}$ with $|S| = |T|$, and even if for every $i \in E$ there are at most 3 sets $S \in \mathbf{A}$ that contain i (Garey and Johnson p. 222). Chvatal (1979) has shown that a similar greedy procedure may, at worst, result in a solution whose value is inferior to the optimal value by a factor that is logarithmic in $d = \max_{i \in E} d_i$ defined by $d_i = |\{S \in \mathbf{A} : i \in S\}|$. This worst case behavior applies to the multiple set covering problem as well (Dobson 1982), and the worst case bound has been shown to be tight.

We note that, in general, our results cannot be extended to objective functions that can be viewed as restrictions to $\mathbf{N}^E$ of a concave function on $\mathbf{R}^E$. In fact, the problem is strongly NP-complete for such objective functions, even if $\mathbf{A} = \{\{i\} : i \in E\}$. To verify this statement, consider the "exact cover by 3-sets problem" which is known to be NP-complete (Karp, and Garey and Johnson, p. 222). The problem is to determine whether the set of equalities

$$\sum_{j=1}^{n} a_{ij} z_j = 1,$$
$$i = 1, \ldots, n \quad \text{with} \quad a_{ij} = 0, 1 \quad \text{and} \quad \sum_i a_{ij} = 3,$$

for all $j = 1, \ldots, n$ has a zero-one solution. Note that this problem is equivalent to

$$\text{minimize } r(z) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} z_j - 1 \right)^2$$
$$\text{subject to} \quad z_i = 0, 1,$$

where $r(\cdot)$ is concave (as a function on $\mathbf{R}^E$). The same reduction shows that our results cannot be extended to general supermodular functions (Topkis 1978).

References

BIXBY, R., W. CUNNINGHAM AND D. TOPKIS. 1985. The Partial Order of a Polymatroid Extreme Point. *Math. Opns. Res.* **10,** 367–379.

BROWN, J. 1979a. The Sharing Problem. *Opns. Res.* **27,** 324–340.

BROWN, J. 1979b. The Knapsack Sharing Problem. *Opns. Res.* **27,** 341–355.

BRUCKER, P. 1982. Network Flows in Trees and Knapsack Problems with Nested Constraints. In *Proceedings of the 8th Conference on Graph Theoretic Concepts in Computer Science*, pp. 25–35, H. Schneider and H. Gottler (eds.). Hanser, Munchen.

CHVATAL, V. 1979. A Greedy Heuristic for the Set-covering Problem. *Math. Opns. Res.* **4,** 233–235.

CUNNINGHAM, W. 1981. Testing Membership in Matroid Polyhedra. Report 81207—OR. Universität Bonn, Bonn.

DOBSON, G. 1982. Worst Case Analysis of Greedy Heuristics for Integer Programming with Non-negative Data. *Math. Opns Res.* **7,** 515–531.

DUNSTAN, F. D. J., AND D. J. A. WELSH. 1973. A Greedy Algorithm for Solving a Certain Class of Linear Programs. *Math. Program.* **5,** 338–353.

EDMONDS, J. 1970. Submodular Functions, Matroids, and Certain Polyhedra. In *Combinational Structures and Their Applications*, pp. 69–87. R. Guy et al. (eds.). Gordon & Breach, New York.

EDMONDS, J., AND F. GILES. 1977. A Min-Max Relation for Submodular Functions on Graphs. *Ann. Discret. Math.* **1,** 185–204.

EINBU, J. 1977. On Shih's Incremental Method in Resource Allocations. *Opnl. Res. Quart.* **27,** 759–763.

FEDERGRUEN, A., AND H. GROENEVELT. 1984. The Greedy Procedure for Resource Allocation Problems: Necessary and Sufficient Conditions for Optimality. Columbia University Graduate School of Business Working Paper (unabridged version of this paper).

FEDERGRUEN, A., AND H. GROENEVELT. 1986. Optimal Flows in Networks with Multiple Sources and Sinks, with Applications to Oil and Gas Lease Investment Programs. *Opns. Res.* **34,** 218–225.

FEDERGRUEN, A., AND P. ZIPKIN. 1983. Solution Techniques for Some Allocation Problems. *Math. Program.* **25,** 12–24.

FOX, B. 1966. Discrete Optimization via Marginal Analysis. *Mgmt. Sci.* **4,** 136–153.

FRANK, A. 1982. An Algorithm for Submodular Functions on Graphs. *Ann. Discrete Math.* **16,** 189–122.

FREDERICKSON, E., AND D. JOHNSON. 1982. The Complexity of Selection and Ranking in $X + Y$ and Matrices with Sorted Columns. *J. Comput. Syst. Sci.* **24,** 197–209.

FUJISHIGE, S. 1980. Lexicographically Optimal Base of a Polymatroid with Respect to a Weight Vector. *Math. Opns. Res.* **5,** 186–196.

GALIL, Z., AND N. MEGIDDO. 1979. A Fast Selection Algorithm and the Problem of Optimum Distribution of Efforts. *J. Assoc. Comput. Mach.* **26,** 58–64.

GALPERIN, A., AND Z. WACKSMAN. 1981. More on Maximization of Separable Concave Functions Subject to Nested Constraints. Unpublished paper.

GAREY, M., AND D. JOHNSON. 1979. *Computers and Intractibility: A Guide to the Theory of NP-Completeness.* Freeman, San Francisco.

GELENBE, E., AND I. MITRANI. 1980. *Analysis and Synthesis of Computer Systems.* Academic Press, New York.

GIRLICH, E., AND M. KOWALJOW. 1981. *Nichtlineare diskrete Optimiering*, Vol. 6. Akademie-Verlag, Berlin.

GROSS, O. 1956. A Class of Discrete-Type Minimization Problems. Working Paper RM-1655-PR, the Rand Corporation. Santa Monica, Calif.

GRÖTSCHEL, M., L. LOVASZ AND A. SCHRIJVER. 1981. The Ellipsoid Method and Its Consequences in Combinational Optimization. *Combinatorica* **1,** 169–197.

HARTLEY, R. 1976. On an Algorithm Proposed by Shih. *Opnl. Res. Quart.* **27,** 765–767.

HASSIN, R. 1978. On Network Flows. Ph.D. dissertation, Yale University.

IBARAKI, T. 1980. Resource Allocation with a Convex Objective Function and Its Generalizations. Working Paper, Faculty of Engineering, Kyoto University, Kyoto, Japan.

ICHIMORI, T., M. ISHII AND T. NISHIDA. 1982. Optimal Sharing. *Math. Program.* **23,** 341–348.

KAO, E. 1976. On Incremental Analysis in Resource Allocations. *Opnl. Res. Quart.* **27,** 759–763.

KARP, R. 1972. Reducibility among Combinatorial Algorithms. In *Complexity of Computer Computations*, R. Miller and J. Thatcher (eds.). Plenum Press, New York.

KATOH, N., T. IBARAKI AND H. MINE. 1979. A Polynomial Time Algorithm for the Resource Allocation Problem with a Convex Objective Function. *Opnl. Res. Quart.* **30,** 449–455.

KLEINROCK, L. 1976. *Queueing Systems, Vol. 2: Computer Applications.* Wiley Interscience, New York.

LAWLER, E. 1982. Generalizations of the Polymatroid Network Flow Model. Mathematical Centre Working Paper BW 158/82, Amsterdam, Netherlands.

LAWLER, E., AND C. MARTEL. 1982. Computing Maximal Polymatroidal Network Flows. *Math. Opns. Res.* **7,** 334–348.

LOVASZ, L. 1982. Submodular Functions and Convexity. In *Proceedings of the XI International Symposium on Mathematical Programming.* Bonn, West Germany.

MARTEL, C. 1982. Scheduling Uniform Machines with Release Times, Deadlines and Due Times. *J. Assoc. Comput. Mach.* **29,** 812–829.

MEGIDDO, N. 1974. Optimal Flows in Networks with Multiple Sources and Sinks. *Math. Program* **7,** 97–107.

MJELDE, K. 1975. The Optimality of an Incremental Solution of a Problem Related to Distribution of Effort. *Opnl. Res. Quart.* **26,** 867–870.

MJELDE, K. 1983. Resource Allocation with Tree Constraints. *Opns. Res.* **31,** 881–891.

PROLL, L. 1976. Marginal Analysis Revisited. *Opnl. Res. Quart.* **27,** 765–767.

SCHRIJVER, A. 1982. Total Dual Integrality from Directed Graphs, Crossing Families and Sub- and Supermodular Functions. In *Proceedings of the Silver Jubilee Conference on Combinatorics*, University of Waterloo, Waterloo, Canada.

SHIH, W. 1974. A New Application of Incremental Analysis in Resource Allocation. *Opnl. Res. Quart.* **25,** 587–597.

TAMIR, A. 1980. Efficient Algorithm for a Selection Problem with Nested Constraints and Its Application to a Production-Sales Planning Model. *SIAM J. Contr. Opt.* **18,** 282–287.

TOPKIS, D. 1978. Minimising a Submodular Function on a Lattice. *Opns. Res.* **26,** 305–321.

TOPKIS, D. 1983a. Activity Selection Games and the Minimum-Cost Problem. *Networks* **13,** 93–105.

TOPKIS, D. 1983b. Adjacency on Polymatroids. Working Paper, Bell Laboratories, Holmdel, N.J.

VAN SLYKE, R. 1982. Redundant Set-Covering in Telecommunication Networks. In *IEEE Large Scale Systems Symposium*, Virgina Beach, Va.

VEINOTT, A. 1964. Production Planning with Convex Costs: A Parametric Study *Mgmt. Sci.* **3,** 441–460.

WELSH, D. 1976. *Matroid Theory*. Academic Press, London.

WOOD, D., AND R. SARGENT. 1984. The Synthesis of Multiclass Single Server Queueing Systems. Working paper, Syracuse University, Syracuse, N.Y.

ZIPKIN, P. 1980. Simple Ranking Methods for Allocation of Resources. *Mgmt. Sci.* **26,** 34–43.