OPTIMAL FLOWS IN NETWORKS WITH MULTIPLE SOURCES AND SINKS, WITH APPLICATIONS TO OIL AND GAS LEASE INVESTMENT PROGRAMS

AWI FEDERGRUEN

Columbia University, New York, New York

HENRY GROENEVELT

University of Rochester, Rochester, New York

(Received March 1984; revisions received July 1984, February 1985; accepted February 1985)

In the classical maximal flow problem, the objective is to maximize the supply to a single sink in a capacitated network. In this paper we consider general capacitated networks with *multiple* sinks: the objective is to optimize a general "concave" preference relation on the set of feasible supply vectors. We show that an optimal solution can be obtained by a marginal allocation procedure. An efficient implementation results in an adaptation of the augmenting path algorithm. We also discuss an application of the procedure for an investment company that deals in oil and gas ventures.

In the classical maximal flow problem (Ford and Fulkerson 1962), the objective is to maximize the supply to a single sink in a capacitated network. In this paper, we consider general capacitated networks with *multiple* sinks and an objective of optimizing a general preference relation on the set of feasible supply vectors. (These preference relations are assumed to have certain concavity properties, to be defined subsequently.)

We show that an optimal integer solution can be obtained by a (greedy) marginal allocation procedure. (The continuous case requires the use of different methods; see Groenevelt 1984, 1985.) An efficient implementation of this procedure results in an adaptation of the classical augmenting path algorithm of Ford and Fulkerson. We also discuss alternative implementations that apply to special classes of networks. Our results are obtained by showing that the set of feasible supply vectors define the independence polytope of a polymatroid (see, for example, Welsh 1975) and by applying the results in Federgruen and Groenevelt (1986).

This paper was motivated by a special case of our class of models, namely, an optimization model we recently developed and implemented for an investment company that deals in oil and gas ventures. The model determines which (if any) of the company's clients should apply for a lease on land parcels offered by the U.S. government in bimonthly special drawings. Section 4 contains a detailed discussion of this application.

Many other resource allocation problems can be represented as special cases of our model. Megiddo (1974) and Fujishige (1980) consider a general network and the special objective of lexicographic maximization (in ascending order) of *each* of the sinks' supplies. Gross (1956), Fox (1966), Veinott (1964), Einbu (1977), Hartley (1976), Kao (1976), Mjelde (1975, 1976, 1983), Proll (1976), Shih (1974), Ibaraki (1980), Katoh, Ibaraki and Mine (1979), Galil and Megiddo (1979), Fredrickson and Johnson (1982), Tamir (1980), Galperin and Wacksman (1981), Brucker (1982), and Federgruen and Zipkin (1983) all consider resource allocation problems of the following type:

$$\begin{aligned} &\text{maximize} \quad r(z) \\ &\text{subject to} \quad \sum_{i\in S} z_i \le N(S), \quad S \in \mathbf{A}; \\ &\qquad\qquad z \text{ integer}, \end{aligned} \tag{1}$$

where $r(\cdot)$ is a concave function, $N(\cdot)$ an arbitrary set function and A a tree-structured collection of sets, i.e., if $S, T \in \mathbf{A}$, then (i) $S \subseteq T$, or (ii) $T \subseteq S$, or (iii) $S \cap T = \varnothing$. (Zipkin 1980 discusses numerous applications of these models.) Such problems can be represented as capacitated *tree-structured* networks with the sinks' supplies denoted by the vector z.

Luss and Gupta (1975), Danskin (1967) and Einbu (1978, 1983, 1984) consider bipartite networks to model budgeting, portfolio and marketing problems as well as assignments of weapons of various types to

Subject classification: 482 network applications, 484 flow algorithms, 662 nonlinear programming algorithms.

0030-364X/86/3402-0218 $01.25

a collection of targets. The objective is to maximize a separable concave function of supplies to the sinks:

$$\text{(P)}\quad \text{maximize} \quad \sum_{j\in J} r_j(z_j) \tag{2}$$

$$\text{subject to} \quad \sum_{i\in I} x_{ij} = z_j;\quad j\in J \tag{3}$$

$$\sum_{j\in J} x_{ij} \leqslant a_i;\quad i\in I \tag{4}$$

$$x_{ij} \leqslant u_{ij};\quad i\in I,\quad j\in J \tag{5}$$

$$x_{ij} \geqslant 0 \quad \text{and integer.} \tag{6}$$

(Luss and Gupta, and Einbu 1978, 1983, 1984 also consider generalizations of (P) with (3) replaced by $\sum_{i\in I} e_{ij}x_{ij} = z_j$ with $e_{ij} \geqslant 0$.)

Our model can be viewed as a special case of the convex cost network flow problem, which has an objective that may depend on the flows on *all* arcs and which requires more complex algorithms (see, for example, Hu 1966 and Kennington and Helgason 1980). Our model also bears at least some similarity to the polymatroidal network flow model considered by Lawler and Martel (1982) and Hassin (1978). The latter considers the problem of maximizing the supply to a single sink when the flows in the network are constrained by the capacities of *sets* of arcs (rather than capacities of individual arcs only).

In Section 1 we derive our basic algorithm. Section 2 discusses alternative implementations for special cases. Section 3 exhibits an efficient adaptation of the basic algorithm for problems with parametric objective functions. This extension was needed in the oil and gas lease investment problem described in Section 4. Section 4 also reports on our computational experience.

1. Model and Algorithms

Let $G = (N, E)$ be a connected network with node set N and arc set E. Let $S \subset N$ be the set of sources, and $T \subset N\backslash S$ the set of sinks. For each $i \in S$, let b_i denote the net capacity of source i. Also, u_{ij} denotes the (integer) capacity of arc $(i, j) \in E$. We define the variables

x_{ij} = flow on arc $(i, j) \in E$;

z_i = net supply to node i, $\quad i \in T$; $\quad z = (z_i)_{i\in T}$.

The network flow model has constraints

$$0 \leqslant \sum_{l:(i,l)\in E} x_{il} - \sum_{l:(l,i)\in E} x_{li} \leqslant b_i,\quad i\in S$$

$$\sum_{l:(l,i)\in E} x_{li} - \sum_{l:(i,l)\in E} x_{il} = z_i,\quad i\in T \tag{7}$$

$$\sum_{l:(i,l)\in E} x_{il} - \sum_{l:(l,i)\in E} x_{li} = 0,\quad i\in N\backslash(S\cup T)$$

$$0 \leqslant x_{ij} \leqslant u_{ij};\quad (i,j)\in E;$$

$$x_{ij}\ \text{integer};\quad z_i \geqslant 0,\quad i\in T. \tag{8}$$

In the classical maximal flow model, the objective is to maximize $\sum_{i\in T} z_i$. We consider a general objective expressed by a complete order $\geqslant_R$ on $\mathbf{N}^T$ that satisfies two "concavity" properties (R1) and (R2). (Let e^j for $j \in T$ be the jth unit basis vector in $\mathbf{N}^T$; we write $x <_R y$ if $x \leqslant_R y$ and $y \nleqslant_R x$.) For all $x, y \in \mathbf{N}^T$:

(R1) if $y \geqslant x$, $x \geqslant_R x + e^i$, then $y \geqslant_R y + e^i$, $i \in T$.

(R2) if $y \geqslant x$, $x_i = y_i$, and $x + e^i \geqslant_R x + e^j$ then $y + e^i \geqslant_R y + e^j$; $i, j \in T$.

These properties are satisfied, for example, by order relations induced by separable concave functions in z, as well as the objectives of sink optimality and weighted sink optimality introduced by Megiddo and Fujishige, respectively. (To define the weighted sink optimality, let $w = (w_i)_{i\in T}$ be a given vector of positive weights; let $T(z)$ denote the $|T|$-tuple of numbers $\{z_i/w_i : i \in T\}$ arranged in ascending order; z^* is called sink-optimal with respect to the weight vector w if $T(z^*)$ is lexicographically larger than $T(z)$ for all feasible z.) These criteria are sometimes referred to as "the sharing problem," see Brown (1979a, 1979b) and Ichimori, Ishii and Nishida (1982). Section 3 of Federgruen and Groenevelt contains additional examples.

We first observe that the network can be transformed into a single source network by appending a new node s^* to N and, for each node $i \in S$, an arc from s^* to i with capacity b_i. Hence, without loss of generality, the network is assumed to have a single source s, i.e., $|S| = 1$. It is also possible to show that an equivalent problem arises when $(b_i,\ i \in S)$ are variables and $(z_i,\ i \in T)$ are known parameters.

Define, on 2^T, the set function $v(\cdot)$ by

$$v(A) = \text{minimize}\left\{\sum_{i\in X}\sum_{j\notin X} u_{ij} : s^* \in X, A \subset N\backslash X\right\},\quad A \subset T;$$

i.e., $v(A)$ is the minimum capacity of a cut separating A from the source. In view of the max-flow, min-cut theorem (see, for example, Ford and Fulkerson), $v(A)$ also represents the maximal flow into set A. A supply vector $z \in \mathbf{N}^T$ is called feasible if (x, z) for some $x = (x_{ij}: (i, j) \in E)$ satisfies (7) and (8). Let Z denote the set of feasible supply vectors. Z is described by the following inequalities (see Megiddo, Lemma 4.1):

$$\sum_{t \in A} z_t \leq v(A), \quad A \subset T. \tag{9}$$

Moreover, the set function $v(\cdot)$ is a rank function (see Megiddo, Lemma 3.2):

$$v(\varnothing) = 0; \tag{10}$$

$$A \subset B \Rightarrow v(A) \leq v(B)\ (monotonicity); \tag{11}$$

$$v(A \cup B) + v(A \cap B) \leq v(A) + v(B) \quad (submodularity). \tag{12}$$

The set of feasible supply vectors is thus the independence polytope of a polymatroid (Welsh). Federgruen and Groenevelt show that, as a consequence, an optimum supply vector can be found by the following marginal allocation procedure:

Algorithm I (Marginal Allocation Algorithm):

1. *for* $t \in T$ *do* $z_t := 0$;
2. *while* $\sum_{t \in T} z_t < v(T)$ *do*
   begin
3. find t such that $z + e^t \in Z$ and $z + e^t \geq_R z + e^{t'}$ for all t' with $z + e^{t'} \in Z$.
4. *if* (no such t exists) *or* $z + e^t <_R z$ *then* stop;
5. $z_t := z_t + 1$;
   end;

Theorem 1. (Federgruen and Groenevelt, Theorem 2). *Let R satisfy* ($R1$) *and* ($R2$). *The Marginal Allocation Algorithm finds an optimal solution.*

We call a supply vector $z \in Z$ a local optimum if

(i) $z \geq_R z - e^t$, for all $t \in T$ with $z - e^t \in Z$;
(ii) $z \geq_R z + e^t$, for all $t \in T$ with $z + e^t \in Z$; (13)
(iii) $z \geq_R z + e^t - e^{t'}$ for all $t, t' \in T$ with $z + e^t - e^{t'} \in Z$.

Federgruen and Groenevelt show that every local optimum in Z is a global optimum provided the order R *satisfies* ($R1$), ($R2$) *and*

(R1′) if $y \geq x$, $x >_R x + e^i$ then $y >_R y + e^i$, $i \in T$;
(R2′) if $y \geq x$, $x_i = y_i$, and $x + e^i >_R x + e^j$ then $y + e^j >_R y + e^j$; $i, j \in T$.

Theorem 2. (Federgruen and Groenevelt, Theorem 4). *Let R satisfy* ($R1$), ($R2$), ($R1'$), *and* ($R2'$). *Every local optimum in Z is a global optimum.*

The computational requirements of Algorithm I depend almost entirely on the possibility of implementing Step 3 efficiently. For general polymatroids this feasibility check may be rather cumbersome and is related to the general polymatroid membership problem (Grötschel, Lovasz and Schryver 1981, Cunningham 1981 and Topkis 1983).

In our context, however, the feasibility test is equivalent to verifying the existence of an augmenting path from the source to a specific sink. The following implementation of Algorithm I thus results in a generalization of the well-known augmenting path algorithm: in each iteration, labels are given to nodes of the form i^+ or i^-. (Only node s has a special label $-$.) A label i^* $[i^-]$ indicates that there exists a unit-size augmenting path from the source to node j in question, and that (i, j) $[(j, i)]$ is the last arc in this path. For any given $t \in T$ and $z \in Z$, $z + e^t \in Z$ if and only if the labeling procedure succeeds in labeling node $t \in T$.

Algorithm II (Augmenting Path Algorithm):

1. *for* $t \in T$ *do* $z_t := 0$; *for* $(i, j) \in E$ *do* $x_{ij} := 0$;
2. *while* $\sum_{t \in T} z_t < v(T)$ *do*
   begin
3. Give node s a special label $-$.
4. If all labeled nodes have been scanned, go to Step 6.
5. Fix a labeled but unscanned node i and scan it as follows: if $(i, j) \in E$, $x_{ij} < u_{ij}$ and j unlabeled give j the label i^+; if $(j, i) \in E$, $x_{ji} > 0$ and j unlabeled, give j the label i^-. Go to Step 4.
6. Find $t \in T$ such that t is labeled and $z + e^t \geq_R z + e^{t'}$ for all labeled $t' \in T$.
7. *if* (no such t exists) *or* $z + e^t <_R z$ *then* stop.
8. Starting at node t, backtrack an augmenting path; for a node j on this path with label i^+ (i^-), increase (decrease) x_{ij} (x_{ji}) by one; set $z_t := z_t + 1$; erase all labels.
   end;

Assume a ranking subroutine is available to perform the test $x \geq_R y$ (for any $x, y \in N^T$) with constant running time. Algorithm II requires up to $v(T)$ iterations through Steps 2–7 and each iteration requires scanning of no more than $2\,|E|$ arcs as well as $0(|T|)$ calls to the ranking subroutine. The overall running time is thus $0(v(T)\,|E|)$.

In the next section we show how more efficient implementations or faster alternative procedures may be used in certain special cases.

2. Special Cases

Trees

We first show that all resource allocation problems of type (1) with a tree-structured collection of sets

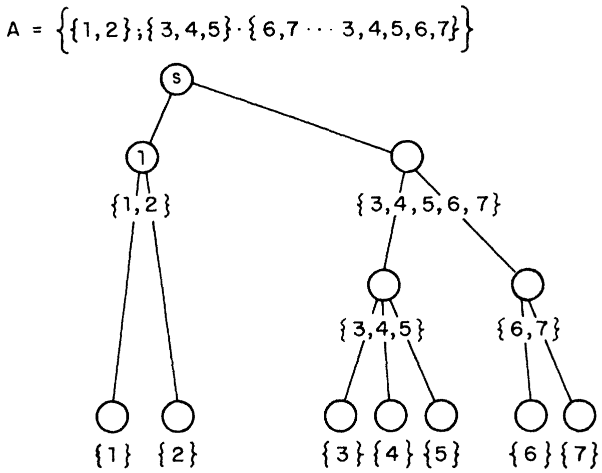

Figure 1. Network representation of the structures.

are special cases of the general model treated in Section 1. As pointed out in the introduction, problem (1) contains many important cases: (i) a single resource constraint: $\mathbf{A}_1 = \{E\}$; (ii) a single resource constraint with simple upper bounds: $\mathbf{A}_2 = \{E\} \cup \{S: S \subset E, |S| = 1\}$; (iii) a single resource constraint with simple and generalized upper bounds: there exists a partition $\{E_k: k \in K\}$ of E such that $\mathbf{A}_3 = \mathbf{A}_2 \cup \{\mathrm{E}_1, \ldots, \mathrm{E}_{|K|}\}$; (iv) nested constraints: $\mathbf{A}_4 = \cup_{i=1}^{n} \{S_i\}$ with $S_1 \subset S_2 \subset \cdots \subset S_n = E$.

Define a network (see Figure 1) in which each element of $\mathbf{A}$ and $\cup\, \mathbf{A}$ is represented by a node. In addition, append a source node s. If $S \in \mathbf{A}$, *either* no $S' \in \mathbf{A}$ has $S' \supset S$ *or* there exists a smallest set S' with $S' \supset S$. In the former case, introduce an arc with capacity $N(S)$ connecting s with the node representing S. In the latter case, introduce an arc of capacity $N(S)$, connecting the node representing S' with the node representing S. Let $\cup\, \mathbf{A} = \{i \in E: i \in S, S \in \mathbf{A}\}$. For each $i \in \cup\, \mathbf{A}$, connect its corresponding node with the node representing the smallest $S \in \mathbf{A}$ containing i (see Figure 1). This network is a tree and has $\cup\, A$ as its set of sinks T. Since every $t \in \cup\, \mathbf{A}$ is connected to the source s through a unique path, existence of an augmenting path is trivial to verify, thus simplifying Steps 3 and 4 in Algorithm II. For this class of models, our algorithm reduces to Algorithm I of Brucker. We also conclude that for resource allocation problems of type (1), with A a tree-structured collection of sets, the set of feasible solutions defines the independence polytope of a polymatroid. (See Theorem 5 in Federgruen and Groenevelt for an alternative proof of this result.)

Bipartite Graphs

Next consider a bipartite graph with $N = \{s\} \cup I \cup J$, with s the unique source, J the set of sinks and arcs going from s to I and from I to J (only). Let a_i be the capacity on the arc connecting s with $i \in I$, and u_{ij} the capacity on the arc connecting $i \in I$ with $j \in J$. This network represents the feasible region of the optimization problem (P) (see (3)–(6)). The set of feasible supply vectors z is described by

$$\sum_{j \in A} z_t \leqslant \sum_{i \in I} \min\left(\sum_{j \in A} u_{ij},\, a_i\right), \quad A \subset J, \tag{14}$$

as follows from (9). (We verify the identity $v(A) = \sum_{i \in I} \min(\sum_{j \in A} u_{ij}, a_i)$ as follows: let $X = \{s\} \cup I_1 \cup J_1$ where $I_1 \subset I$, $J_1 \cup (J \backslash A)$. The cut separating X from $N \backslash X$ has capacity $\sum_{i \in I_1} \sum_{j \in A} u_{ij} + \sum_{i \in I \backslash I_1} a_i \geqslant \sum_{i \in I} \min(\sum_{j \in A} u_{ij}, a_i)$ and there exists a cut whose capacity equals the right-hand-side expression.)

If $u_{ij} = u_i$ for all $i \in I$, $j \in J$ (a property satisfied by the oil and gas investment problem in Section 4) then (14) simplifies to $\sum_{j \in A} z_j \leqslant \sum_{i \in I} \min(|A|\, u_i, a_i)$, i.e., $v(A)$ depends on A only through $|A|$. A polymatroid whose rank function satisfies this property is called *symmetric*. For symmetric polymatroids, an efficient implementation of the feasibility test in Step 3 of Algorithm I can be achieved without using the underlying network structure (see also Proposition 2 in Federgruen and Groenevelt): for a given flow vector z, let $z(k)$ be the sum of the largest k components of z. Writing $v(A) = v(|A|)$, note that $z \in Z$ if and only if $z(k) \leqslant v(k)$ for all k. An index i is said to be tight if $z(i) = v(i)$. Assume the indices are relabeled so that $z_1 \geqslant \ldots \geqslant z_{|J|}$, and for each $k = 1, \ldots, |J|$ define $\mathrm{FIRST}(k) = \min(i: z_i = z_k)$ and $\mathrm{LAST}(k) = \max(i: z_i = z_k)$.

Lemma 1. *Consider (P) with $u_{ij} = u_i$ for all $i \in I$, $j \in J$. Let z be a feasible supply vector.*

(a) *$z + e^j$ is feasible for $j \in J$ if and only if no index i with $j \leqslant i \leqslant |J|$ is tight.*

(b) *There exists an index j^* $(1 \leqslant j^* \leqslant |J| + 1)$ such that $\{j: z + e^j \text{ feasible}\} = \{j^*, \ldots, |J|\}$. (If $j^* = |J| + 1$, $\{j: z + e^j \text{ is feasible}\} = \varnothing$.)*

Proof. (a) Let $z' = z + e^j$. Note $z'(i) = z(i)$ for $i < \mathrm{FIRST}(j)$ and $z'(i) = z(i) + 1$ for $i \geqslant \mathrm{FIRST}(j)$. Also, if i is tight for some i and if $\mathrm{FIRST}(j) \leqslant i < j$, then $z_{i+1} = z_i = z(i) - z(i-1) \geqslant v(i) - v(i-1) \geqslant v(i+1) - v(i)$ by the submodularity of the $v(\cdot)$ function, and an induction shows that j is tight.

The "if" part thus follows from $z(i) < v(i)$, $i \geqslant \mathrm{FIRST}(j)$ and the "only if" part is immediate from our first observation. Part (b) follows from part (a).

Reordering the indices so that $z_1 \geqslant \ldots \geqslant z_{|J|}$ takes $O(\log |J|)$ steps per iteration of Algorithm I. Computing $z(k)$ for $k = 1, \ldots, |J|$ takes $O(|J|)$ steps.

The index j^* defined in Lemma 1 may thus be determined by starting with $k = |J|$ and decreasing k by unit steps until the first tight index is found. If no such index is found, $j^* = 1$. The search requires $O(|J|)$ steps, which implies that Step 3 of Algorithm I requires $O(|J|)$ operations and evaluations of the order $\geqslant_R$. Since the algorithm clearly terminates in at most $v(J) = \sum_{i \in I} \min(\sum_j u_{ij}, a_i) \leqslant \sum_i a_i$ iterations, an optimal supply vector z^* may thus be obtained in $O(v(J)|J|)$ time, which compares favorably with the bound $O(v(J)|I||J|)$ for Algorithm II. (The number of arcs in the bipartite graph is $|I||J|$.)

Once an optimal supply vector z^* is found, a corresponding (optimal) vector x may be obtained by applying the initialization phase (Phase I) of any primal network flow code, or more specifically, a primal algorithm for capacitated transportation problems (see, for example, Langley, Kennington and Shetty 1974). Alternatively, exploiting the bipartite network structure, x may be obtained via the algorithm of Gusfield, Martel and Fernandez-Baca (1985) and can thus be solved in $O(\min(|I|^2|J|, |I||J|^2))$ steps.

3. Parametric Programming

In this section we show that the marginal allocation procedure is ideally suited for parametric programming, provided the order R is induced by a real-valued objective function. The bidding model for oil and gas ventures, discussed in Section 4, uses parametric programming for a systematic trade-off analysis between two competing performance measures.

Thus, suppose two real-valued objective functions $r(\cdot)$ and $q(\cdot)$ are specified, and assume both induce order relations on $\mathbf{N}^T$ that satisfy the concavity properties ($R1$), ($R2$), ($R1'$) and ($R2'$). Also, to facilitate the presentation and proofs, we assume $r(z)$ and $q(z)$ are nondecreasing in z. (Extensions to the general case are straightforward.) For all $0 \leqslant \lambda \leqslant 1$, let $s(\lambda; z) = (1-\lambda)r(z) + \lambda q(z)$, assume $s(\lambda, \cdot)$ satisfies ($R1$), ($R1'$), ($R2$) and ($R2'$) for all relevant λ, and consider the family of problems

$Q(\lambda)$: maximize $s(\lambda; z)$
subject to (7) and (8).

The following procedure determines a (finite) sequence of optimal solutions. At each stage, a range is computed on the parameter λ for which the same solution remains optimal. The variable transition on the boundary of these ranges is easily determined, and a simple interchange of one unit determines the solution in the adjacent range.

Multiplier Search Algorithm (MSA)

0. Solve $Q(0)$ using Algorithm I or II and denote the optimal solution by $z^{(0)}$. Set $\lambda^{(0)} = 0$, $n = 0$.
1. Find $\lambda^{(n+1)} = \inf\{\lambda > \lambda^{(n)}: s(\lambda;\, z^{(n)} + e^l - e^i) > s(\lambda; z^{(n)})$ and $(z^{(n)} + e^l - e^i) \in Z$ for some $i, l \in T\}$. Let $i^*, l^* \in T$ be the indices for which this infimum is attained. If $\lambda^{(n+1)} > 1$, stop.
2. Set $z^{(n+1)} := z^{(n)} + e^{l^*} - e^{i^*}$; $n := n + 1$; go to Step 1.

Proposition 1. *Let $z^{(n)}$, $\lambda^{(n)}$, for $n \geqslant 1$ be specified by the MSA. $z^{(n)}$ is optimal for $Q(\lambda)$ with $\lambda^{(n)} \leqslant \lambda \leqslant min(1, \lambda^{(n+1)})$.*

Proof. In view of Theorem 2 in Federgruen and Groenevelt, every local optimum of $Q(\lambda)$ is a global optimum for $0 \leqslant \lambda \leqslant 1$. It thus suffices to show that $z^{(n)}$ is a local optimum for $\lambda^{(n)} \leqslant \lambda \leqslant \lambda^{(n+1)}$. We do so by induction. Suppose $z^{(n)}$ is a local optimum for $Q(\lambda^{(n)})$. Since $s(\lambda; z)$ is nondecreasing in z, we have $\sum_{j \in T} z_j = v(T)$, so there is no $t \in T$ for which $z^{(n)} + e^t$ is feasible or $z^{(n)} - e^t$ strictly better than $z^{(n)}$ for any $\lambda \geqslant 0$. But then, by step (1) and (13), $z^{(n)}$ is a local optimum for $\lambda^{(n)} \leqslant \lambda \leqslant \lambda^{(n+1)}$. By continuity of s as a function of λ, we have $s(\lambda^{(n+1)}; z^{(n)}) = s(\lambda^{(n+1)}; z^{(n+1)})$, so $z^{(n+1)}$ is an optimal solution of $Q(\lambda^{(n+1)})$. Step 0 of the MSA establishes the basis of the induction.

We now specify implementations of Step 1 of the MSA. First we need the following lemma:

Lemma 2. *Let z be an optimal solution of $Q(\lambda)$ and let $i, l \in T$, $i \neq j$. Then there is a $\lambda' > \lambda$ such that $\delta(\lambda') = s(\lambda', z + e^l - e^i) - s(\lambda'; z) > 0$ if and only if*

$$dis \stackrel{def}{=} q(z + e^l - e^i) - q(z) - r(z + e^l - e^i) + r(z) > 0. \quad (15)$$

Under (15),

$$inf\{\lambda' > \lambda:\ \delta(\lambda') > 0\} = -[r(z + e^l - e^i) - r(z)]/dis. \quad (16)$$

Proof. Note that

$$\delta(\lambda') = r(z + e^l - e^i) - r(z) + \lambda'\text{dis}. \quad (17)$$

The "if" part of the lemma follows by letting $\lambda' \to \infty$ in (17).

Since z is optimal for $Q(\lambda)$, $\delta(\lambda) \leqslant 0$. Also, if $\delta(\lambda') > 0$ for some $\lambda' > \lambda$, $0 < \delta(\lambda') - \delta(\lambda) = (\lambda' - \lambda)\text{dis}$. Hence (15) follows from $\lambda' - \lambda > 0$.

(16) follows immediately from (17).

The infimum in Step 1 can thus be obtained as the minimum of the zeroes of at most $|T|$ $(|T| - 1)$

known linear functions. The only remaining problem is the feasibility test ($z + e^l - e^i \in Z$?), given $z \in Z$. Assume (x, z) is a feasible solution of (7) and (8). Observe that for fixed $i \in T$, $\{l \in T: z + e^l - e^i \in Z\} = \{l \in T:$ there exists an augmenting path from i to $l\}$. The latter set may thus be determined by the classical labeling procedure with node i as starting point (see Steps 3–5 in Algorithm II).

For problem (P) with $u_{ij} = u_i$ for all $i \in I; j \in J$, the sets $\{l \in J: z + e^l - e^i \in Z\}$ (for fixed $i \in J$) can again be obtained without using the underlying network structure.

Lemma 3. *Let $z \in Z$ and assume $z_1 \geqslant z_2 \geqslant \cdots \geqslant z_{|J|}$. Assume $z_i > 0$. $z + e^l - e^i$ (with $l \neq i$) $\in Z$ if and only if there is no tight index k with $FIRST(l) \leqslant k \leqslant LAST(i) - 1$.*

Proof. Let $z' = z + e^l - e^i$. Note $z' \geqslant 0$ since $z_i > 0$. Thus z' is feasible if and only if $z'(k) \leqslant v(k)$ for all $k = 1, \ldots, |J|$. Note that for some ordering $\{j_1, \ldots, j_{|J|}\}$ of the indices $z_{j_1} \geqslant z_{j_2} \geqslant \ldots \geqslant z_{j_{|J|}}$ and $j_{\mathrm{FIRST}(l)} = l$ and $j_{\mathrm{LAST}(i)} = i$. If $\mathrm{FIRST}(l) \geqslant \mathrm{LAST}(i)$, all $z'(k) \leqslant z(k) \leqslant v(k)$, and z' is feasible. Otherwise $z'(k) = z(k)$ for $k \geqslant \mathrm{LAST}(i)$ and $z'(k) > z(k)$ if and only if $\mathrm{FIRST}(l) \leqslant k \leqslant \mathrm{LAST}(i) - 1$.

Thus assume $z \in Z$, $z_1 \geqslant z_2 \geqslant \ldots \geqslant z_{|J|}$. In view of Lemma 3, we have $\{l \in J: z + e^l - e^i \in Z\} = \{l^*(i), \ldots, |J|\}$ for some $1 \leqslant l^*(i) \leqslant |J| + 1$. ($l^*(i) = |J| + 1$ implies the index set is empty.) The values $\{l^*(i), i \in J\}$ can be determined by the following procedure:

Procedure (Determination of $l^*(i)$, $i \in J$):

1. $i := |J|$; *while* $z_i = 0$ *do begin* $l^*(i) = |J| + 1$; $i := i - 1$ *end*;
2. $k := i$;
   repeat
3. *if* $z(k) < v(k)$ *then* $k := k - 1$ *else*
   begin
   $j := k$; $k := \mathrm{FIRST}(k)$
   while $i > k$ *do begin* $l^*(i) := j$; $i := i - 1$ *end*;
   until $k = 0$.

This procedure requires $O(|J|)$ steps. (Note that the values $\mathrm{FIRST}(k)$ are needed only for tight indices; these may be computed in the course of the procedure and need not be stored.)

4. A Bidding Model for Oil and Gas Ventures

In 1960 the Federal Government ruled that every citizen (as well as partnership, association and corporation) should have an equal right to share the revenues from oil and gas deposits found on federally owned lands. Therefore, the Federal Government holds simultaneous drawings every other month that enable the public to acquire leases on a large number of land parcels. Each person (partnership, association, and so forth) can submit only one lease application per parcel, with every filer having an equal chance of acquiring the rights. A fee of approximately $75 per filing is paid to the Bureau of Land Management. A substantial number of parcels have a direct market value which is at least 5–10 times the public's total investment in filing fees. In addition, overriding royalties often amount to a multiple of the direct market value, all fees are tax deductible, and income is taxed as capital gain. In spite of this potential for an unusually high return on investment, very few citizens file for leases. Most people are unaware of the drawings, the filing procedures are complicated and time consuming, and the general public lacks expert information on desirable parcels.

An industry of professional filing services has arisen to assist investors in selecting parcels as well as in the actual filing procedure. The very best among these services gather geological surveys, experienced lease-broker reports, and statistical analyses of past drawings in order to select the best leases. Their clients pay a fixed service fee and authorize the service to file a given number of applications in their name. Prior to each drawing, the filing service faces the problem of determining the parcels on which to apply for each of its clients.

The problem can be formally stated as follows: let I be the client pool and assume client $i \in I$ has paid for a_i applications to be filed in his name. Let J denote the set of relevant parcels and for each $j \in J$ let V_j and F_j denote the estimates of the market value and number of outside filers (exclusive of company clients), as obtained by geological surveys, real estate broker reports, statistical analyses, and the like. In order to maintain and possibly expand its future business, the filing service is interested in optimizing several aggregate performance measures for the entire client pool, in particular

(i) the expected total market value of the parcels won by the client pool, and
(ii) the expected number of winners.

Let

$$x_{ij} = \begin{cases} 1 & \text{if client } i \text{ files for parcel } j; \\ & \quad i \in I, \quad j \in J. \\ 0 & \text{otherwise} \end{cases}$$

z_j = number of clients filing for parcel j, $\quad j \in J$.

Table I
Characteristics of the Land Parcels

Characteristic	Parcel No.									
	1	2	3	4	5	6	7	8	9	10
Value V_j	8	16	18	8	20	4	16	20	10	12
No. of outside filers F_j	14	17	10	7	13	6	12	13	10	10

Table II
Trade-offs between Expected Returns and Expected Number of Winners

IT	E(RET)	E(WIN)	MULT λ	1	2	3	4	5	6	7	8	9	10
1	37.31	2.374	0.0000	0	4	7	2	7	0	6	7	3	4
2	37.30	2.413	0.1194	0	4	7	3	7	0	5	7	3	4
3	37.23	2.515	0.4267	0	3	7	3	7	1	5	7	3	4
4	36.99	2.586	0.7682	0	3	6	3	7	2	5	7	3	4
5	36.84	2.618	0.8230	1	3	6	3	6	2	5	7	3	4
6	36.80	2.624	0.8538	1	4	6	3	6	2	5	6	3	4
7	36.67	2.647	0.8568	1	3	6	4	6	2	5	6	3	4
8	36.49	2.667	0.8992	1	3	6	3	6	3	5	6	3	4
9	36.24	2.693	0.9074	1	3	6	4	5	3	5	6	3	4
10	36.03	2.710	0.9256	1	3	6	4	5	3	5	5	4	4
11	35.79	2.724	0.9439	2	3	6	4	5	3	4	5	4	4
12	35.75	2.726	0.9474	2	3	5	4	5	3	5	5	4	4
13	35.31	2.749	0.9512	2	3	5	4	5	4	4	5	4	4
14	35.01	2.757	0.9723	2	2	5	5	5	4	4	5	4	4
15	34.58	2.766	0.9799	3	2	5	5	4	4	4	5	4	4
16	33.94	2.778	0.9813	3	2	5	5	4	5	4	4	4	4
17	33.56	2.778	1.0000	3	2	4	5	4	5	4	4	5	4
18	33.18	2.778	1.0000	3	2	4	5	3	5	4	4	5	5

Note that objective (i) is maximized by solving (P) with

$$r_j(z_j) = V_j z_j/(z_j + F_j), \quad j \in J;$$

$$u_{ij} = 1; \quad i \in I, \quad j \in J.$$

Likewise, objective (ii) is maximized by solving (P) with $r_j(\cdot)$ replaced by $q_j(z_j) = z_j/(z_j + F_j)$. Observe that both $q_j(\cdot)$ and $r_j(\cdot)$ are concave and nondecreasing. Alternatively, the parameters V_j and F_j, $j \in J$, may be treated as random variables. Let $\phi_j(\cdot)$ denote the cdf of F_j, $j \in J$ and solve (P) with $r_j(\cdot)$ replaced by

$$\bar{r}_j(\cdot) = EV_j \int_0^\infty z_j/(z_j + F_j)\, d\phi_j(F_j), \quad j \in J,$$

and $q_j(\cdot)$ replaced by $\bar{q}_j(\cdot) = \bar{r}_j(\cdot)/(EV_j)$. (Note $\bar{q}_j(\cdot)$ and $r_j(\cdot)$ are concave and nondecreasing as well.)

We now illustrate the use of the MSA procedure with the help of a numerical example. In actual problem instances solved for a particular filing service, we found that solutions based on expert judgment were often significantly below the efficient frontier.

Example. Let $|I| = 8$ with $a_1 = a_2 = 8$; $a_3 = a_4 = 6$; $a_5 = a_6 = 4$ and $a_7 = a_8 = 2$. Table I shows all input parameters. All V_i for $i \in I$ were chosen to be $2{*}Un\{1, 2, \ldots, 10\}$ and $F_i = \frac{1}{2}V_i + Un\{1, 2, \ldots, 10\}$ for $i \in I$ ($Un\{1, 2, \ldots, 10\}$ represents a uniformly drawn integer between 1 and 10.) Table II exhibits how the optimal solution varies as the parameter λ increases from 0 to 1. Eighteen different solutions arise. The table also shows the corresponding values of the two objective functions, $\sum_j r_j(\cdot)$ (expected return) and $\sum_j q_j(\cdot)$ (expected number of winners).

Acknowledgment

We gratefully acknowledge the suggestions of an anonymous Associate Editor which led to an improved exposition of this manuscript.

References

BROWN, J. 1979a. The Sharing Problem. *Opns. Res.* **27**, 324–340.

BROWN, J. 1979b. The Knapsack Sharing Problem. *Opns. Res.* **27**, 341–350.

BRUCKER, P. 1982. Network Flows in Trees and Knapsack Problems with Nested Constraints. In *Proceedings of the 8th Conference on Graph Theo-*

retic Concepts in Computer Science, pp. 25–35, H. Schneider and H. Gottler, (eds.). Hanser, München.

CUNNINGHAM, W. 1981. *Testing Membership in Matroid Polyhedra.* Report 81207—OR, Universität Bonn, Bonn.

DANSKIN, F. 1967. *The Theory of Max-Min.* Springer Verlag, Berlin.

EINBU, J. 1977. On Shih's Incremental Method in Resource Allocations. *Opns. Res. Quart.* **27,** 759–763.

EINBU, J. 1978. Optimal Allocations of Continuous Resources to Several Activities with a Concave Return Function—Some Theoretical Results. *Math. Opns. Res.* **3,** 82–88.

EINBU, J. 1983. Allocation of Resources to Depots. Norwegian Computing Center Working Paper.

EINBU, J. 1984. A Finite Method for the Solution of a Multi-Resource Allocation Problem with Concave Return Functions. Math. Opns. Res. **9,** 232–244.

FEDERGRUEN, A., AND H. GROENEVELT. 1986. The Greedy Procedure for Resource Allocation Problems: Necessary and Sufficient Conditions for Optimality. *Opns. Res.* (to appear).

FEDERGRUEN, A., AND P. ZIPKIN. 1983. Solution Techniques for Some Allocation Problems. *Math. Progr.* **25,** 13–24.

FORD, L., AND D. FULKERSON. 1962. *Flows in Networks.* Princeton University Press, Princeton, N.J.

FOX, B. 1966. Discrete Optimization via Marginal Analysis. *Mgmt. Sci.* **4,** 136–153.

FREDRICKSON, G., AND D. JOHNSON. 1982. The Complexity of Selection and Ranking in $X + Y$ and Matrices with Sorted Columns. *J. Comput. Syst. Sci.* 197–209.

FUJISHIGE, S. 1980. Lexicographically Optimal Base of a Polymatroid with Respect to a Weight Vector. *Math. Opns. Res.* **5,** 186–196.

GALIL, Z., AND N. MEGIDDO. 1979. A Fast Selection Algorithm and the Problem of Optimum Distribution of Efforts. *J. ACM* **26,** 58–64.

GALPERIN, A., AND Z. WACKSMAN. 1979. More on Maximization of Separable Concave Functions Subject to Nested Constraints (unpublished paper).

GROENEVELT, H. 1984. Resource Allocation Problems with Decreasing Marginal Returns to Scale. Ph.D. Dissertation, Columbia University, New York.

GROENEVELT, H. 1985. Maximizing a Concave Separable Function over a Polymatroid Feasible Region. University of Rochester, Graduate School of Management, Working Paper, Rochester, N.Y.

GROSS, O. 1956. A Class of Discrete-Type Minimization Problems. Working Paper RM-1644-PR, The Rand Corporation, Santa Monica, Calif.

GRÖTSCHEL, M., L. LOVASZ AND A. SCHRYVER. 1981. The Ellipsoid Method and Its Consequences in Combinatorial Optimization. *Combinatorica* **1,** 169–197.

GUSFIELD, D., C. MARTEL AND D. FERNANDEZ-BACA. 1985. Fast Algorithms for Bipartite Network Flow (unpublished manuscript).

HARTLEY, R. 1976. On an Algorithm Proposed by Shih. *Opns. Res. Quart.* **27,** 765–767.

HASSIN, R. 1978. On Network Flows. Ph.D. Dissertation, Yale University.

HU, T. C. 1966. Minimum Cost Flows in Convex-Cost Networks. *Naval Res. Logist. Quart.* **13,** 1–9.

IBARAKI, T. 1980. Resource Allocation with a Convex Objective Function and its Generalizations. Working Paper, Faculty of Engineering, Kyoto University, Kyoto, Japan.

ICHIMORI, T., M. ISHII AND T. NISHIDA. 1982. Optimal Sharing. *Math. Program.* **23,** 341–348.

KAO, E. 1976. On Incremental Analysis in Resource Allocations. *Opnl. Res. Quart.* **27,** 759–763.

KATOH, N., T. IBARAKI AND H. MINE. 1979. A Polynomial Time Algorithm for the Resource Allocation Problem with a Convex Objective Function. *Opnl. Res. Quart.* **30,** 449–455.

KENNINGTON, J., AND R. HELGASON. 1980. *Algorithms for Network Programming.* John Wiley & Sons, New York.

LANGLEY, R., KENNINGTON AND C. SHETTY. 1974. Efficient Computational Devices for the Capacitated Transportation Problem. *Naval Res. Logist. Quart.* **21,** 637–647.

LAWLER, E., AND C. MARTEL. 1982. Computing Maximal Polymatroidal Network Flows. *Math. Opns. Res.* **7,** 334–348.

LUSS, H., AND S. GUPTA. 1975. Allocation of Effort Resources among Competing Activities. *Opns. Res.* **23,** 360–366.

MEGIDDO, N. 1974. Optimal Flows in Networks with Multiple Sources and Sinks. *Math. Progr.* **7,** 97–107.

MJELDE, K. 1975. The Optimality of an Incremental Solution of a Problem Related to Distribution of Effort. *Opnl. Res. Quart.* **26,** 765–767.

MJELDE, K. 1976. Evaluation and Incremental Determination of Almost Optimal Allocations of Resources. *Opnl. Res. Quart.* **27,** 581–588.

MJELDE, K. 1983. Resource Allocation with Tree Constraints. *Opns. Res.* **31,** 881–891.

PROLL, L. 1976. Marginal Analysis Revisited. *Opnl. Res. Quart.* **27,** 765–767.

SHIH, W. 1974. A New Application of Incremental Analysis in Resource Allocation. *Opnl. Res. Quart.* **25,** 587–597.

TAMIR, A. 1980. Efficient Algorithm for a Selection Problem with Nested Constraints and its Application to a Production-Sales Planning Model. *SIAM J. Contr. Opt.* **18,** 282–287.

TOPKIS, D. 1983. Adjacency on Polymatroids. Working Paper, Bell Laboratories, Holmdel, N.J.

VEINOTT, A. 1964. Production Planning with Convex Costs: A Parametric Study. *Mgmt. Sci.* **3,** 441–460.

WELSH, D. 1975. *Matroid Theory.* Academic Press, New York.

ZIPKIN, P. 1980. Simple Ranking Methods for Allocation of Resources. *Mgmt. Sci.* **26,** 34–43.