

GENERALIZED SEMI-MARKOV PROCESSES: ANTIMATROID STRUCTURE AND SECOND-ORDER PROPERTIES*

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A *generalized semi-Markov scheme* models the structure of a discrete event system, such as a network of queues. By studying combinatorial and geometric representations of schemes we find conditions for second-order properties—convexity/concavity, sub/supermodularity—of their event epochs and event counting processes. A scheme generates a *language* of feasible strings of events. We show that monotonicity of the event epochs is equivalent to this language forming an *antimatroid with repetition*. This connection gives rise to a rich combinatorial structure, and serves as a starting point for other properties. For example, by strengthening the antimatroid condition we give several equivalent characterizations of the convexity of event epochs within a scheme. All of these correspond, in slightly different ways, to making a certain *score space* a lattice, to closing an ordinary antimatroid under intersections. We also establish second-order properties across schemes tied together through a synchronization mechanism. A geometric view based on the score space facilitates verification of these properties in certain queueing systems.

1. Introduction. A *generalized semi-Markov scheme* models the structure of a stochastic event-driven system, such as a network of queues. We study schemes as essentially deterministic mappings of inputs to outputs. The inputs are the intrinsic lifetimes or *clock samples* associated with events, and the outputs are the actual epochs of occurrence of events, or the corresponding counting processes. In a queueing context, typical inputs are interarrival times and service times; typical outputs are arrival and departure epochs. If α denotes an event type, let $\omega_\alpha(n)$, $T_\alpha(n)$, and $D_\alpha(t)$ be, respectively, the n th clock sample for α , the epoch of the n th occurrence of α , and the number of occurrences of α in $[0, t]$. A generalized semi-Markov scheme implicitly defines mappings of $\omega = \{\omega_\alpha(n)\}$ to $T = \{T_\alpha(n)\}$ and $D = \{D_\alpha(t)\}$ with α ranging over the set of events, n over the positive integers, and t over the positive real numbers. We investigate how structural aspects of a scheme determine qualitative properties of these mappings, and vice-versa.

Conditions on the structure of a generalized semi-Markov scheme lead to strong properties of any generalized semi-Markov *process* (GSMP) based on the scheme—properties that do not depend on the particular probability laws that drive the process. In Glasserman and Yao [8] we identify a condition on schemes, called condition (M), which characterizes monotonicity of T and D as functions of ω . This sample-path monotonicity translates immediately to stochastic monotonicity through a standard argument; see Stoyan [23] and Kamae, Krengel, and O'Brien [10]. Condition (M) combines two more primitive properties: *noninterruption* (in the sense of Schassberger [13]), which requires that a clock, once set, continue to run until the associated event occurs; and a *permutability* condition stating that changing the order

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of events does not change the set of active events in the state reached. Precise statements are given in §2.2.

In this paper, we delve deeper into the structure of schemes. We study properties of the possible sequences, or *feasible strings*, of events generated by a scheme, and show that these properties have important implications for GSMPs based on the scheme. We show that condition (M) is equivalent to the requirement that the *language* of feasible strings form an *antimatroid with repetition*, in the sense of Björner [3], and Shor, Björner, and Lovász [22]. To be an antimatroid with repetition, a language must be *locally free* and *permutable*, and these properties correspond precisely to noninterruption and permutability for schemes as described above. The resulting equivalence of (M) and the antimatroid *strong exchange* property is a starting point for new conditions on schemes.

This connection also gives rise to a fruitful geometric interpretation. To each string is associated a vector indexed by events—a *score*—the α component of which records the number of occurrences of α in the string. Properties of a scheme are reflected in its space of feasible scores. Under condition (M), the score space is closed under componentwise maximum. A condition for closure of the score space under componentwise minimum leads to stronger conclusions, as we now explain.

Since the antimatroid condition is equivalent to (M), it is equivalent to monotonicity of T . We obtain *second-order* properties of T —convexity/concavity, sub/supermodularity—by strengthening the antimatroid condition. Increasing convexity of T , for example, is equivalent to a condition (CX)—stronger than (M)—which makes the score space a lattice, because it closes it under componentwise minimum. Among schemes that satisfy (M), convexity of T is also characterized by supermodularity of an analog of the antimatroid *rank* function. Turning this around, we show that a supermodular rank function implies closure of an (ordinary) antimatroid under intersections. On the score space we define a *characteristic function*, χ , which describes the neighborhood structure of the space, and show that (M) holds if and only if χ is increasing, (CX) holds if and only if χ is also supermodular. In addition to being interesting in their own right, these connections lead to simplified proofs of second-order properties, allowing us to avoid the inductive arguments typical of stochastic comparisons.

We also consider parametric families of schemes, motivated by, for example, families of queueing systems that differ only in the number of jobs, servers, or buffer spaces. For a collection of schemes indexed by a set \mathcal{B} , we give conditions for second-order properties of the corresponding counting processes $\{D^b, b \in \mathcal{B}\}$ as functions of the parameter b . These comparisons hinge on a *synchronization* mechanism, a GSMP analog of the method of uniformization in Markov chains: the \mathcal{G}^b 's are subordinated to a “master” scheme in which events run autonomously. In comparisons of different schemes, χ becomes a function of two arguments, the parameter b and a score. Second-order properties of D^b follow from joint second-order properties of the characteristic function. In some queueing examples, we easily obtain explicit expressions for χ , greatly simplifying verification of, for example, concavity across systems.

Second-order properties complement the first-order (monotonicity) results here and in [8], give insight into the behavior of a system, and provide vital qualitative information for optimization of performance. (The field is active; see recent results in Anantharam and Tsoucas [1], Meester and Shanthikumar [12], Shaked and Shanthikumar [16, 17], Shanthikumar and Yao [18, 19, 20, 21].) A statement that, for example, T is convex in ω , interpreted in a queueing context can imply that arrival epochs, departure epochs, and certain sojourn times are convex functions of interarrival times and service times. A statement that D is increasing and concave in buffer

size (for fixed ω) indicates a decreasing marginal increase in throughput from added buffer capacity. In a stochastic setting, these types of results show when a system *preserves* a stochastic convexity property. If the clock times are stochastically increasing and convex in a parameter (as in, e.g., [16] or [21]) then, with appropriate structure, so are the event epochs.

Looking at generalized semi-Markov schemes, rather than specific systems, leads to fairly general results and, more importantly, reveals interesting connections among second-order properties in a diversity of systems. We present illustrative examples along the way, but our principal goal is presenting a unified view of certain structural properties, rather than establishing these properties in individual complex systems. Results for a *specific* system could be established directly, without our methods. But reading proofs of stochastic comparison results one is led to ask what key properties underlie the results and whether intricate construction arguments could not be subsumed by a more general framework. We intend here to present such a framework and to develop some of its implications.

Relations among schemes, languages, scores, and antimatroids are laid out in §2. Theorem 2.3 summarizes the implications of these connections for monotonicity. The following three sections study second-order properties of T as a function of ω ; §3 develops conditions for convexity, §4 for concavity, §5 for submodularity and supermodularity. In §6, we describe two settings in which to compare different schemes, using a *subscheme* relation and using the synchronization mechanism mentioned above. In both cases, we establish monotonicity through properties of schemes and languages. In §7, we provide conditions for second-order properties across synchronized families of schemes. We focus exclusively on schemes with *deterministic routing*, but in the concluding remarks of §8 we point out how our results extend to *state-independent* probabilistic routing (as defined in [8]), and discuss other possible extensions.

A cautionary remark is in order: Antimatroids are sometimes called models of “abstract convexity”. This notion has no direct bearing on (ordinary) convexity of T and D as we mean it.

2. Antimatroid structure. This section introduces basic tools and structures central to the rest of the paper. The initial complexity of the notation is compensated by the eventual simplicity and generality of the results. §2.1 describes schemes and GSMPs in detail. §2.2 introduces the language of feasible strings generated by a scheme, the antimatroid property, and various structural conditions. §2.3 develops a geometric representation of a language of strings, and summarizes connections among the various concepts introduced and monotonicity. Using the geometric interpretation, we obtain explicitly recursions for the event epochs in terms of the operations $+$, \min , and \max .

2.1. Generalized semi-Markov schemes. Our first task is to define generalized semi-Markov schemes and introduce attendant notation. A scheme with *deterministic routing*, fixed initial state (and no *speeds*) is specified by $\mathcal{S} = (\mathbf{S}, \mathbf{A}, \mathcal{E}, \phi, s_0)$ where \mathbf{S} is the state space—the set of physical configurations of a system; $\mathbf{A} = \{\alpha_1, \dots, \alpha_m\}$ is a finite set of event types; $\mathcal{E}(s) \subseteq \mathbf{A}$, for each $s \in \mathbf{S}$, is the *event list* or set of active events in state s ; $\phi(s, \alpha)$ is the state to which the system moves from s upon the occurrence of α , if $\alpha \in \mathcal{E}(s)$; and s_0 is the state in which the system starts. (In a scheme with probabilistic routing, ϕ is replaced by transition probabilities.) An input to the scheme is a sequence $\omega = \{\omega_\alpha(n), \alpha \in \mathbf{A}, n = 1, 2, \dots\}$; $\omega_\alpha(n)$ is the n th *clock sample* for α . The input ω drives the evolution of the system and gives rise to outputs $T = \{T_\alpha(n), \alpha \in \mathbf{A}, n = 1, 2, \dots\}$ and $D = \{D_\alpha(t), \alpha \in \mathbf{A}, t \geq 0\}$, where (as noted in

the introduction) $T_\alpha(n)$ is the epoch of the n th occurrence of α and $D_\alpha(t)$ is the number of occurrences of α in $[0, t]$.

Briefly, the evolution of the system (the generalized semi-Markov process, or GSMP) is as follows. At time $t = 0$, clocks are set for events in $\mathcal{E}(s_0)$: if $\alpha \in \mathcal{E}(s_0)$, the clock for α is initialized to the first sample for α ; i.e., it is set to $\omega_\alpha(1)$. When the first of these runs out—i.e., at time $t = \min\{\omega_\alpha(1) : \alpha \in \mathcal{E}(s_0)\}$ —the corresponding event β , say, occurs, and the system moves to state $s_1 \equiv \phi(s_0, \beta)$. (If two or more clocks run out simultaneously, the order of events is taken to be their order in $\mathbf{A} = \{\alpha_1, \dots, \alpha_m\}$.) Clocks for “old” events (i.e., events in both the new event list $\mathcal{E}(s_1)$ and in $\mathcal{E}(s_0) - \{\beta\}$) continue to run in s_1 . For any newly active events (i.e., events in $\mathcal{E}(s_1)$ which are not in $\mathcal{E}(s_0) - \{\beta\}$) a new clock sample is drawn from the sequence ω . The clock for an event in $(\mathcal{E}(s_0) - \{\beta\}) \setminus \mathcal{E}(s_1)$ is said to be *interrupted* and the associated event becomes inactive. Just after entry to s_1 , the procedure repeats. When events $\beta, \beta^1, \dots, \beta^k$ have occurred, the system is in state $\phi(s_0, \beta\beta^1 \cdots \beta^k) = \phi(\cdots \phi(\phi(s_0, \beta), \beta^1), \dots, \beta^k)$. See Schassberger [13, 14, 15] and Whitt [24] for more detailed descriptions.

A few conventions are in force throughout the rest of this paper. The space Ω of possible inputs is the set of ω 's for which $0 < \omega_\alpha(n) \leq \infty$, all $\alpha \in \mathbf{A}$, all $n = 1, 2, \dots$, and $\sum_{n=1}^\infty \omega_\alpha(n) = \infty$ for all $\alpha \in \mathbf{A}$. Comparisons and operations $(+, \wedge, \vee, \leq)$ on ω , T and D are always taken componentwise. Thus, T increasing in ω means that if $\omega_\alpha(n) \leq \omega'_\alpha(n)$, for all α and n , then $T_\alpha(n)(\omega) \leq T_\alpha(n)(\omega')$ for all α and n . We take $D_\alpha(t) = \sup\{n \geq 0 : T_\alpha(n) \leq t\}$, so D is right-continuous in t . We assume that none of our schemes has extraneous events or states: $\mathbf{A} = \bigcup_{s \in \mathbf{S}} \mathcal{E}(s)$, and for all $s \in \mathbf{S}$, there is a sequence β^0, \dots, β^k through which s is reached from s_0 ; that is, $\beta^0 \in \mathcal{E}(s_0)$, $\beta^i \in \mathcal{E}(\phi(s_0, \beta^0 \cdots \beta^{i-1}))$, $i = 1, \dots, k$, and $s = \phi(s_0, \beta^0 \cdots \beta^k)$. For all $s \in \mathbf{S}$, $\mathcal{E}(s)$ is nonempty.

2.2. *Languages and schemes.* Let us, for a moment, view \mathbf{A} as simply a set of symbols—an *alphabet*. A *string* is a finite (possibly empty) sequence of elements of \mathbf{A} . A *prefix* of a string is any initial portion of the sequence; the empty string, ϵ , is a prefix of every string. For any $\alpha \in \mathbf{A}$ and any string σ , let $N_\alpha(\sigma)$ be the number of occurrences of α in σ . Let $N(\sigma)$ be a vector indexed by \mathbf{A} whose α component is $N_\alpha(\sigma)$. Following Shor, Björner, and Lovász [22], call $N(\sigma)$ the *score* of σ .

A *language* \mathcal{L} over \mathbf{A} is a collection of strings. The language is *left-hereditary* (equivalently, *prefix-closed*) if any prefix of a string in \mathcal{L} is itself in \mathcal{L} . As in [22], call \mathcal{L} *locally free* when it satisfies

(LF) if $\sigma, \sigma\alpha, \sigma\beta \in \mathcal{L}$ and $\alpha \neq \beta$ then $\sigma\alpha\beta \in \mathcal{L}$;
and call \mathcal{L} *permutable* when it satisfies

(P) if $\sigma_1, \sigma_2 \in \mathcal{L}$, $N(\sigma_1) = N(\sigma_2)$, and $\sigma_1\alpha \in \mathcal{L}$ then $\sigma_2\alpha \in \mathcal{L}$.

A generalized semi-Markov scheme \mathcal{S} with starting state s_0 generates a language of *feasible* strings via

$$\mathcal{L} = \{\beta_1 \cdots \beta_n : n \geq 0, \beta_i \in \mathcal{E}(\phi(s_0, \beta_1 \cdots \beta_{i-1}))\}, i = 1, \dots, n\}.$$

The empty string is always in \mathcal{L} , and satisfies $\phi(s, \epsilon) = s$ for all $s \in \mathbf{S}$. The language \mathcal{L} is automatically left-hereditary. Define the *characteristic function* $\chi : \mathcal{L} \rightarrow \mathbf{Z}_+^m$ componentwise by

$$(1) \quad \chi_\alpha(\sigma) = N_\alpha(\sigma) + \mathbf{1}\{\alpha \in \mathcal{E}(\phi(s_0, \sigma))\}.$$

The function χ captures the one-step behavior of the evolution of feasible strings: $\chi_\alpha(\sigma) > N_\alpha(\sigma)$ if and only if σ can be followed by α . Moreover, χ determines \mathcal{L} because $\beta_1 \cdots \beta_n$ is in \mathcal{L} if and only if $N(\beta_1 \cdots \beta_{i-1}) < \chi_{\beta_i}(\beta_1 \cdots \beta_{i-1})$ for all $i = 1, \dots, n$.

Much of our analysis of schemes is based on the languages they generate; structural properties of \mathcal{S} are reflected in \mathcal{L} , and vice-versa. The connection is particularly close in the presence of the following basic condition from Glasserman and Yao [8]:

(M) If $\sigma_1, \sigma_2 \in \mathcal{L}$ and $N(\sigma_2) \geq N(\sigma_1)$ then

$$\mathcal{E}(\phi(s, \sigma_1)) \setminus A_{\sigma_1\sigma_2} \subseteq \mathcal{E}(\phi(s, \sigma_2)),$$

where $A_{\sigma_1\sigma_2} = \{\alpha: N_\alpha(\sigma_2) > N_\alpha(\sigma_1)\}$.

Under this condition, T is an increasing function of ω . A similar but slightly stronger condition is used in Glasserman [6, 7] to ensure a continuity property. The results in this paper follow from other ways of strengthening (M).

We showed in [8] that (M) is equivalent to a combination of two properties (stated together as (M') in [8]); namely, that \mathcal{S} is *noninterruptive*:

$$(2) \quad \forall s \in \mathbf{S}, \quad \alpha, \beta \in \mathcal{E}(s), \quad \alpha \neq \beta \Rightarrow \beta \in \mathcal{E}(\phi(s, \alpha));$$

and that

$$(3) \quad \forall \sigma_1, \sigma_2 \in \mathcal{L}, \quad N(\sigma_1) = N(\sigma_2) \Rightarrow \mathcal{E}(\phi(s_0, \sigma_1)) = \mathcal{E}(\phi(s_0, \sigma_2)).$$

Evidently, we have

LEMMA 2.1. \mathcal{L} is locally free if and only if \mathcal{S} is noninterruptive. \mathcal{L} is permutable if and only if \mathcal{S} satisfies (3).

PROOF. \mathcal{L} is locally free if and only if whenever $\sigma \in \mathcal{L}$ and $\alpha, \beta \in \mathcal{E}(\phi(s_0, \sigma))$, $\alpha \neq \beta$, then $\beta \in \mathcal{E}(\phi(\phi(s_0, \sigma), \alpha))$. Since (by our convention) every state is $\phi(s_0, \sigma)$ for some σ , this is equivalent to noninterruption. \mathcal{L} is permutable if and only if whenever $\sigma_1, \sigma_2 \in \mathcal{L}$ and $N(\sigma_1) = N(\sigma_2)$, for all $\alpha \in \mathbf{A}$, $\sigma_1\alpha \in \mathcal{L} \Leftrightarrow \sigma_2\alpha \in \mathcal{L}$; i.e., if and only if $\alpha \in \mathcal{E}(\phi(s_0, \sigma_1)) \Leftrightarrow \alpha \in \mathcal{E}(\phi(s_0, \sigma_2))$; i.e., if and only if (3) holds. \square

In light of this connection, let us call \mathcal{S} *permutable* if it satisfies (3).

Shor, Björner, and Lovász [22] show that, for an arbitrary left-hereditary language, \mathcal{L} , (LF) and (P) are equivalent to a certain *strong exchange property* of Björner [3]. To state it, let $N(\sigma_1) \vee N(\sigma_2)$ be the componentwise maximum of $N(\sigma_1)$ and $N(\sigma_2)$; for future reference, let \wedge yield the componentwise minimum. The property is

(SE) If $\sigma_1, \sigma_2 \in \mathcal{L}$, then there is a σ such that $\sigma_1\sigma \in \mathcal{L}$ and $N(\sigma_1\sigma) = N(\sigma_1) \vee N(\sigma_2)$.

Two important consequences of (SE) are immediate. It implies that if $N(\sigma_2) \geq N(\sigma_1)$, then σ_1 may be “grown out” to a string $\sigma_1\sigma$ (while maintaining feasibility) whose score is that of σ_2 . It also implies that for any $\sigma_1, \sigma_2 \in \mathcal{L}$ there is a $\sigma_3 \in \mathcal{L}$ satisfying $N(\sigma_3) = N(\sigma_1) \vee N(\sigma_2)$. In this sense, we say that (SE) makes \mathcal{L} *max-closed*. The strong exchange property is equivalent to (M):

PROPOSITION 2.2. *A scheme \mathcal{S} satisfies (M) if and only if the language it generates satisfies (SE).*

PROOF. The equivalence of (LF) plus (P) and (SE) (for left-hereditary languages) is Lemma 1.2 of [22]. The equivalence of (2) plus (3) and (M) is the third part of

Proposition 3.1 of [8]. The equivalence of these pairs of conditions is Lemma 2.1 above. \square

As discussed in [22], (SE) is a strong version of the *greedoid exchange property* of Korte and Lovász [11]. Among left-hereditary languages in which each symbol occurs at most once in any string (i.e., in which $N(\sigma) \leq (1, \dots, 1)$, for all $\sigma \in \mathcal{L}$), (SE) defines a special class of greedoids called *antimatroids*. Without this bound on $N(\sigma)$, the language is an *antimatroid with repetition*. The language generated by a \mathcal{L} satisfying (M) is an antimatroid with repetition—from now on, called simply an antimatroid.

Ordinary antimatroids are often defined as collections of sets, just as matroids are. For reference and comparison we state one set of axioms. (There are many others; Dietrich [5] surveys the subject.) Let E be a finite set and \mathcal{A} a collection of subsets of E called feasible sets. Then (E, \mathcal{A}) is an antimatroid if

- (4) $\emptyset \in \mathcal{A};$
- (5) $X \in \mathcal{A}, X \neq \emptyset \Rightarrow \exists x \in X \text{ s.t. } X - x \in \mathcal{A};$
- (6) $X, Y \in \mathcal{A}, X \not\subseteq Y \Rightarrow \exists x \in X \setminus Y \text{ s.t. } Y \cup x \in \mathcal{A}.$

Clearly, (4) and (5) correspond to left-heredity and (6) to (SE). Property (6) is equivalent to closure of \mathcal{A} under union.

2.3. *The geometry of score space.* Let $\mathbf{Z}_+ = \{0, 1, 2, \dots\}$ and $m = |\mathbf{A}|$. For any scheme \mathcal{S} with language \mathcal{L} define

$$\mathcal{N} = \{ \mathbf{x} \in \mathbf{Z}_+^m : \mathbf{x} = N(\sigma), \text{ for some } \sigma \in \mathcal{L} \};$$

\mathcal{N} is the set of scores of \mathcal{L} , the *score space* of \mathcal{S} . For any $\mathbf{x} \in \mathbf{Z}_+^m$ we use x_α to indicate the α component of \mathbf{x} , and we denote by e_α the unit vector in direction α . Let \mathbf{Z}_+^m be endowed with the componentwise partial order, \leq , and the componentwise minimum and maximum operations, \wedge and \vee . Let $|\mathbf{x}|$ be the l_1 -norm of \mathbf{x} .

Figure 1 depicts part of the score space of a 5-state *birth-death scheme* that starts in state 2: $\mathbf{S} = \{0, 1, 2, 3, 4\}$; $\mathbf{A} = \{\alpha, \beta\}$; $\mathcal{E}(0) = \{\alpha\}$, $\mathcal{E}(s) = \{\alpha, \beta\}$, $s = 1, 2, 3$, and $\mathcal{E}(4) = \{\beta\}$; $\phi(s, \alpha) = s + 1$, if $\alpha \in \mathcal{E}(s)$, and $\phi(s, \beta) = s - 1$ if $\beta \in \mathcal{E}(s)$; $s_0 = 2$. This scheme models a finite-capacity queue in which arrivals are shut off when the

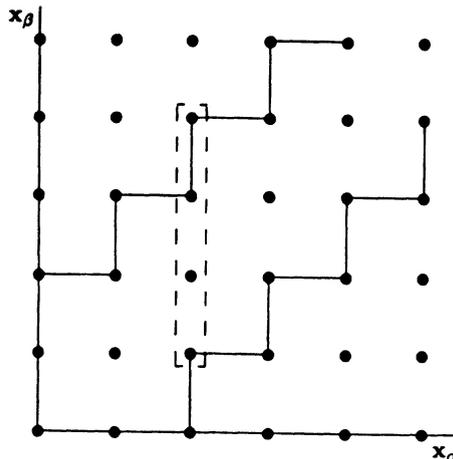


FIGURE 1. Part of the Score Space for a 5-State Birth-Death Scheme. The pattern repeats upward and to the right. The points in the box form $\mathcal{N}_{\alpha,3}$.

buffer is full; that is, a clock for an arrival is only set when there is room in the buffer, so arrivals are never blocked. The system starts with two jobs present. Up to two arrivals can occur without a departure, and up to two departures without an arrival. More generally, a score $(\mathbf{x}_\alpha, \mathbf{x}_\beta)$ is feasible if and only if $-2 \leq \mathbf{x}_\alpha - \mathbf{x}_\beta \leq 2$. These are the points on or between the two “staircases” in the figure.

The birth-death scheme satisfies (M). In any scheme that satisfies (M), \mathcal{L} is completely determined by \mathcal{N} : a string is in \mathcal{L} if and only if all its prefixes have scores in \mathcal{N} .

When $\sigma \in \mathcal{L}$, write $\mathcal{E}(\sigma)$ for $\mathcal{E}(\phi(s_0, \sigma))$. If the scheme is permutable, and $\mathbf{x} \in \mathcal{N}$, let $\mathcal{E}(\mathbf{x})$ be $\mathcal{E}(\sigma)$, where σ is any string in \mathcal{L} for which $N(\sigma) = \mathbf{x}$. For a permutable scheme, the characteristic χ becomes a function on the score space, defined componentwise by

$$(7) \quad \chi_\alpha(\mathbf{x}) = \mathbf{x}_\alpha + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{x})\}.$$

If the scheme is not permutable, $\mathcal{E}(\mathbf{x})$ is not well defined so $\chi(\mathbf{x})$ is not either. When $\chi(\mathbf{x})$ exists, it also satisfies

$$(8) \quad \chi(\mathbf{x}) = \text{l.u.b.}\{\mathbf{y} \in \mathcal{N} : \mathbf{y} \geq \mathbf{x}, |\mathbf{y}| = |\mathbf{x}| + 1\};$$

i.e., $\chi(\mathbf{x})$ is the least upper bound of points in \mathcal{N} that cover \mathbf{x} . For a permutable scheme, χ describes the neighborhood structure of \mathcal{N} . A neighbor $\mathbf{x} + e_\alpha$ of \mathbf{x} is in \mathcal{N} only if $\chi_\alpha(\mathbf{x}) > \mathbf{x}_\alpha$. Call χ (as a function on \mathcal{L}) *increasing* if whenever $N(\sigma_1) \leq N(\sigma_2)$, $\chi(\sigma_1) \leq \chi(\sigma_2)$. For a permutable scheme, χ is increasing if $\mathbf{x}^1 \leq \mathbf{x}^2$ implies $\chi(\mathbf{x}^1) \leq \chi(\mathbf{x}^2)$. We now have

THEOREM 2.3. *The following are equivalent:*

- (i) \mathcal{L} satisfies (M);
- (ii) \mathcal{L} is an antimatroid;
- (iii) χ is increasing;
- (iv) T is increasing in ω .

Any of these implies

- (v) \mathcal{N} is closed under \vee .

PROOF. (i) \Leftrightarrow (ii) is Proposition 2.2. (i) \Rightarrow (iv) is Lemma 3.4 of [8]; Theorem 3.5 of [8] is the reverse implication. Directly from (M) we get $N(\sigma_1) \leq N(\sigma_2)$, $\sigma_1, \sigma_2 \in \mathcal{L} \Rightarrow \chi(\sigma_1) \leq \chi(\sigma_2)$, so (i) \Rightarrow (iii). For the converse, if χ is increasing and $N(\sigma_1) = N(\sigma_2)$, $\sigma_1, \sigma_2 \in \mathcal{L}$, then $\chi(\sigma_1) = \chi(\sigma_2)$; i.e., $\mathbf{1}\{\alpha \in \mathcal{E}(\sigma_1)\} = \mathbf{1}\{\alpha \in \mathcal{E}(\sigma_2)\}$, for all α , which is permutability. For a permutable scheme, (M) is equivalent to noninterruption. If χ is increasing and $\alpha, \beta \in \mathcal{E}(\sigma)$, then $N_\beta(\sigma\alpha) + \mathbf{1}\{\beta \in \mathcal{E}(\sigma\alpha)\} \geq N_\beta(\sigma) + \mathbf{1}\{\beta \in \mathcal{E}(\sigma)\}$, which implies $\beta \in \mathcal{E}(\sigma\alpha)$, which is noninterruption. Thus, (iii) also implies (i). For the last part, if $\mathbf{x}^1, \mathbf{x}^2 \in \mathcal{N}$, there are $\sigma_1, \sigma_2 \in \mathcal{L}$ with $N(\sigma_i) = \mathbf{x}^i$, $i = 1, 2$. If \mathcal{L} is an antimatroid it satisfies (SE), so there is a $\sigma_3 \in \mathcal{L}$ with $N(\sigma_3) = N(\sigma_1) \vee N(\sigma_2)$; i.e., $\mathbf{x}^1 \vee \mathbf{x}^2 \in \mathcal{N}$. \square

REMARK. Of the conditions in Theorem 2.3, the one based on χ is perhaps the most difficult to appreciate at first. But this characterization is, in fact, the one that most easily generalizes (see Theorem 3.1 and the results of §6 and §7), one of the most useful for verification (see especially §7.2), and the one most closely linked with more standard, constructive proofs of monotonicity. Constructive proofs typically use induction and show (often implicitly) that the system in question preserves an ordering as time evolves. Monotonicity of χ is a succinct statement of this principle. It says that if $N(\sigma_1) \leq N(\sigma_2)$ then this ordering will not be violated by allowing the two strings to concatenate feasible events.

The most useful consequence of introducing \mathcal{N} is an explicit representation of the event epochs $\{T_\alpha(n)\}$. Suppose (M) holds and define, for each $\alpha \in \mathbf{A}$ and $n = 1, 2, \dots$,

$$\mathcal{N}_{\alpha,n} = \{\mathbf{x} \in \mathcal{N} : \mathbf{x}_\alpha = n - 1, \alpha \in \mathcal{E}(\mathbf{x})\},$$

which could be empty. A score \mathbf{x} is in $\mathcal{N}_{\alpha,n}$ if it can lead to the n th occurrence of α , in the sense that there is a $\sigma \in \mathcal{L}$ with $N(\sigma) = \mathbf{x}$, $N_\alpha(\sigma) = n - 1$ and $\sigma\alpha \in \mathcal{L}$. For any nonempty $\mathcal{N}_{\alpha,n}$ let $\mathbf{x}^1, \mathbf{x}^2, \dots$ be the minimal elements of $\mathcal{N}_{\alpha,n}$; that is, elements for which $\mathbf{x} \in \mathcal{N}_{\alpha,n}$ implies $\mathbf{x} \geq \mathbf{x}^i$ for some i , and $\mathbf{x} \not\leq \mathbf{x}^i$ for all i . Since \mathbf{A} is finite, there are finitely many such elements. If $\mathcal{N}_{\alpha,n}$ is empty, let “minimal element” refer to the vector (∞, \dots, ∞) . For all $\alpha \in \mathbf{A}$ define $T_\alpha(0) = 0$ and $T_\alpha(\infty) = \infty$.

THEOREM 2.4. *If \mathcal{G} satisfies (M), then T admits the min-max representation*

$$T_\alpha(n) = \omega_\alpha(n) + \min_i \max_{\beta \in \mathbf{A}} \{T_\beta(\mathbf{x}_\beta^i)\},$$

where $\{\mathbf{x}^i\}$ is the set of minimal elements of $\mathcal{N}_{\alpha,n}$.

PROOF. Let $S_\alpha(n)$ be the epoch of the n th setting of a clock for α ; for a noninterruptive scheme, $S_\alpha(n) = T_\alpha(n) - \omega_\alpha(n)$. Recall that $D_\alpha(t)$ is the number of occurrences of α in $[0, t]$. If, for some t , $D(t)$ is strictly dominated by every point in $\mathcal{N}_{\alpha,n}$, then either $D_\alpha(t) < n - 1$, or $D_\alpha(t) = n - 1$ but α is not in the event list at time t . Either way, $S_\alpha(n) > t$. On the other hand, if $D(t)$ dominates some point in $\mathcal{N}_{\alpha,n}$, then either $D_\alpha(t) \geq n$, or $D_\alpha(t) = n - 1$ and α is in the event list at time t . In either case, $S_\alpha(n) \leq t$. Thus,

$$\begin{aligned} S_\alpha(n) &= \inf\{t \geq 0 : \exists \mathbf{x} \in \mathcal{N}_{\alpha,n}, D(t) \geq \mathbf{x}\} \\ &= \inf\{t \geq 0 : D(t) \geq \mathbf{x}^i, \text{ some minimal } \mathbf{x}^i \text{ of } \mathcal{N}_{\alpha,n}\}. \end{aligned}$$

But, for arbitrary \mathbf{x} , $D(t) \geq \mathbf{x}$ if and only if $t \geq \max_\beta \{T_\beta(\mathbf{x}_\beta)\}$, so

$$\begin{aligned} S_\alpha(n) &= \inf\left\{t \geq 0 : t \geq \min_i \max_\beta \{T_\beta(\mathbf{x}_\beta^i)\}\right\} \\ &= \min_i \max_\beta \{T_\beta(\mathbf{x}_\beta^i)\}, \end{aligned}$$

which is what the lemma claims. \square

REMARK. That T is increasing in ω (whenever (M) holds) is immediately clear from the min-max representation in the theorem.

In Figure 1, the scores in the dashed box constitute $\mathcal{N}_{\alpha,3}$, which has just one minimal element, $(2, 1)$. From Theorem 2.4 we conclude that $T_\alpha(3) = \omega_\alpha(3) + \max\{T_\alpha(2), T_\beta(1)\}$. This is intuitively clear: the third arrival cannot occur until the second has and until at least one job has completed service.

We close this section by introducing for score spaces another antimatroid concept. Consider (E, \mathcal{A}) as in (4)–(6). Because \mathcal{A} is closed under union, every $A \subseteq E$ contains a unique maximal element of \mathcal{A} , called the *basis* of A . The cardinality of its basis is the *rank* of A . Now let \mathcal{N} be the score space of a scheme satisfying (M). Since \mathcal{N} is closed under \vee , for each $\mathbf{x} \in \mathbf{Z}_+^m$ the set of elements of \mathcal{N} that \mathbf{x} dominates has a unique maximal element. Call this the basis of \mathbf{x} . The rank of \mathbf{x} , $\rho(\mathbf{x})$, is the l_1 -norm of the basis of \mathbf{x} .

For any $\mathbf{x} \in \mathbf{Z}_+^m$, $\rho(\mathbf{x}) \leq |\mathbf{x}|$, and ρ is obviously increasing. It determines the score space because $\mathcal{N} = \{\mathbf{x} \in \mathbf{Z}_+^m : \rho(\mathbf{x}) = |\mathbf{x}|\}$. In addition, the rank function is *locally supermodular*, meaning that, for $\alpha \neq \beta$,

$$\rho(\mathbf{x}) = \rho(\mathbf{x} + e_\alpha) = \rho(\mathbf{x} + e_\beta) \Rightarrow \rho(\mathbf{x}) = \rho(\mathbf{x} + e_\alpha + e_\beta);$$

but it is not in general supermodular. (Recall that supermodularity means $\rho(\mathbf{x} \vee \mathbf{y}) + \rho(\mathbf{x} \wedge \mathbf{y}) \geq \rho(\mathbf{x}) + \rho(\mathbf{y})$.) Korte and Lovász [11] and Dietrich [5] discuss the corresponding ordinary antimatroid result. Corollary 3.4 below characterizes antimatroids with supermodular rank functions.

3. Convexity of event epochs. In this section we strengthen (M) to obtain conditions under which T is convex in ω , as well as increasing. We give equivalent conditions for convexity in terms of \mathcal{S} , \mathcal{L} and χ , and show their geometric implications for \mathcal{N} . These results are in §3.1. In §3.2, we interpret convexity results in a stochastic setting. §3.3 presents examples illustrating our conditions.

3.1. *Characterizations of convexity.* Our main condition for convexity is

(CX) If $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{L}$, then

$$N(\sigma_3) \geq N(\sigma_1) \wedge N(\sigma_2) \Rightarrow [\mathcal{E}(\sigma_1) \cap \mathcal{E}(\sigma_2)] \setminus A \subseteq \mathcal{E}(\sigma_3),$$

where $A = \{\alpha : N_\alpha(\sigma_3) > N_\alpha(\sigma_1) \wedge N_\alpha(\sigma_2)\}$.

That (CX) implies (M) becomes obvious upon taking $\sigma_1 = \sigma_2$. This condition, like (M), is equivalent to a combination of two more primitive conditions, noninterruption and

$$(9) \quad N(\sigma_3) = N(\sigma_1) \wedge N(\sigma_2) \Rightarrow \mathcal{E}(\sigma_1) \cap \mathcal{E}(\sigma_2) \subseteq \mathcal{E}(\sigma_3).$$

It is evident that (9) is strictly stronger than (3), permutability. Note that (9) does *not* assume that

$$(10) \quad \sigma_1, \sigma_2 \in \mathcal{L} \Rightarrow \exists \sigma_3 \in \mathcal{L}, \quad N(\sigma_3) = N(\sigma_1) \wedge N(\sigma_2).$$

Let us call a language satisfying (10) *min-closed*.

Condition (CX) does for convexity what (M) does for monotonicity:

THEOREM 3.1. *The following are equivalent:*

- (i) \mathcal{S} satisfies (CX);
- (ii) \mathcal{L} is a min-closed antimatroid;
- (iii) χ is increasing and supermodular;
- (iv) T is increasing and convex in ω .

Any of these implies

- (v) \mathcal{N} is a lattice.

Let us restate and prove (i) \Leftrightarrow (ii) as a lemma.

LEMMA 3.2. \mathcal{S} satisfies (CX) if and only if \mathcal{L} satisfies (SE) and is min-closed.

PROOF. First suppose that \mathcal{L} satisfies (SE) and is min-closed. By Proposition 2.2, \mathcal{S} must satisfy (M); in particular, it is noninterruptive. We need to show that it satisfies (9). Suppose, therefore, that $N(\sigma_3) = N(\sigma_1) \wedge N(\sigma_2)$, $\sigma_i \in \mathcal{L}$, $i = 1, 2, 3$. Let α be in $\mathcal{E}(\sigma_1) \cap \mathcal{E}(\sigma_2)$; we need to show that $\alpha \in \mathcal{E}(\sigma_3)$. Min-closure of \mathcal{L} implies there is $\bar{\sigma} \in \mathcal{L}$, $N(\bar{\sigma}) = N(\sigma_1 \alpha) \wedge N(\sigma_2 \alpha)$. Clearly, $N(\bar{\sigma}) > N(\sigma_3)$, so by the

strong exchange property there is a σ such that $\sigma_3\sigma \in \mathcal{L}$ and $N(\sigma_3\sigma) = N(\bar{\sigma})$. But $N(\bar{\sigma})$ is just $N(\sigma_3) + e_\alpha$ so σ must be α . Since $\sigma_3\alpha \in \mathcal{L}$, we conclude that $\alpha \in \mathcal{E}(\sigma_3)$.

Now consider the converse. Given $\sigma_1, \sigma_2 \in \mathcal{L}$, we need to find a σ_3 whose score is the minimum of the scores of σ_1, σ_2 . Consider the set of strings whose scores are dominated by $N(\sigma_1)$ and $N(\sigma_2)$. The corresponding set of scores is closed under \vee (because \mathcal{N} is if (M) holds and (CX) implies (M)). Thus, there is a unique maximal feasible score dominated by both $N(\sigma_1)$ and $N(\sigma_2)$. Take σ_3 to be any element of \mathcal{L} achieving this score. We claim that $N(\sigma_3) = N(\sigma_1) \wedge N(\sigma_2)$.

By (SE) (which follows from (CX) through (M)), there are feasible strings $\bar{\sigma}_1 = \sigma_3\gamma_1 \cdots \gamma_j$ and $\bar{\sigma}_2 = \sigma_3\beta_1 \cdots \beta_l$, with $N(\bar{\sigma}_i) = N(\sigma_i)$, $i = 1, 2$. We need only show that $\gamma_1 \cdots \gamma_j$ and $\beta_1 \cdots \beta_l$ have no common elements. Suppose they do. Suppose they have prefixes $\gamma_1 \cdots \gamma_i\alpha$ and $\beta_1 \cdots \beta_k\alpha$ and that α is the first common element. Applying (CX) to $\sigma_3\gamma_1 \cdots \gamma_i$ and $\sigma_3\beta_1 \cdots \beta_k$ we conclude that $\alpha \in \mathcal{E}(\sigma_3)$. But then $\sigma_3\alpha \in \mathcal{L}$ and $N(\sigma_3\alpha) \leq N(\sigma_i)$, $i = 1, 2$, which contradicts the fact that $N(\sigma_3)$ is maximal. \square

PROOF OF THEOREM 3.1. The lemma shows (i) \Leftrightarrow (ii). We now show (ii) \Rightarrow (v), (i) \Leftrightarrow (iii) and (ii) \Leftrightarrow (iv).

(ii) \Rightarrow (v). We know from Theorem 2.3 that \mathcal{N} is closed under \vee . The assumption that \mathcal{L} is min-closed translates immediately to closure of \mathcal{N} under \wedge .

(i) \Leftrightarrow (iii). χ is supermodular if $\chi(\sigma_1) + \chi(\sigma_4) \geq \chi(\sigma_2) + \chi(\sigma_3)$ whenever $N(\sigma_1) = N(\sigma_2) \wedge N(\sigma_3)$, $N(\sigma_4) = N(\sigma_1) \vee N(\sigma_2)$, and all four strings are feasible. If χ is increasing then (M) holds and we may treat χ as a function on \mathcal{N} . It is supermodular if, whenever $\mathbf{x}, \mathbf{y}, \mathbf{x} \vee \mathbf{y}$ and $\mathbf{x} \wedge \mathbf{y}$ are all in \mathcal{N} , $\chi(\mathbf{x} \vee \mathbf{y}) + \chi(\mathbf{x} \wedge \mathbf{y}) \geq \chi(\mathbf{x}) + \chi(\mathbf{y})$. If (M) holds, \mathcal{L} is noninterruptive, so we only need to show that (9) holds. By definition, for each $\alpha \in \mathbf{A}$,

$$\begin{aligned} (11) \quad & \chi_\alpha(\mathbf{x} \vee \mathbf{y}) + \chi_\alpha(\mathbf{x} \wedge \mathbf{y}) \\ &= (\mathbf{x} \vee \mathbf{y})_\alpha + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{x} \vee \mathbf{y})\} + (\mathbf{x} \wedge \mathbf{y})_\alpha + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{x} \wedge \mathbf{y})\} \\ &= \mathbf{x}_\alpha + \mathbf{y}_\alpha + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{x} \vee \mathbf{y})\} + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{x} \wedge \mathbf{y})\}, \end{aligned}$$

and

$$(12) \quad \chi_\alpha(\mathbf{x}) + \chi_\alpha(\mathbf{y}) = \mathbf{x}_\alpha + \mathbf{y}_\alpha + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{x})\} + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{y})\}.$$

Supermodularity implies (11) is no smaller than (12), so for all α

$$(13) \quad \alpha \in \mathcal{E}(\mathbf{x}) \cup \mathcal{E}(\mathbf{y}) \Rightarrow \alpha \in \mathcal{E}(\mathbf{x} \vee \mathbf{y}) \cup \mathcal{E}(\mathbf{x} \wedge \mathbf{y}),$$

$$(14) \quad \alpha \in \mathcal{E}(\mathbf{x}) \cap \mathcal{E}(\mathbf{y}) \Rightarrow \alpha \in \mathcal{E}(\mathbf{x} \vee \mathbf{y}) \cap \mathcal{E}(\mathbf{x} \wedge \mathbf{y}).$$

The second of these implies that $\mathcal{E}(\mathbf{x}) \cap \mathcal{E}(\mathbf{y}) \subseteq \mathcal{E}(\mathbf{x} \wedge \mathbf{y})$, which restates (9).

For the converse, suppose (CX) holds—in particular, (M) holds. If \mathbf{x} and \mathbf{y} are in \mathcal{N} , then $\mathbf{x} \vee \mathbf{y}$ and $\mathbf{x} \wedge \mathbf{y}$ are too, because \mathcal{N} is now a lattice. That χ is increasing follows from (M), so we only need to establish supermodularity. It is enough to show that (13) and (14) hold for all α , since then (11) is greater than or equal to (12) for all α . First suppose that $\alpha \in \mathcal{E}(\mathbf{x})$, so $\mathbf{x} + e_\alpha \in \mathcal{N}$. If $\mathbf{x}_\alpha \geq \mathbf{y}_\alpha$, then $(\mathbf{x} + e_\alpha) \vee \mathbf{y} = (\mathbf{x} \vee \mathbf{y}) + e_\alpha$ is in \mathcal{N} , which means that $\alpha \in \mathcal{E}(\mathbf{x} \vee \mathbf{y})$. If, on the other hand, $\mathbf{x}_\alpha < \mathbf{y}_\alpha$, then $(\mathbf{x} + e_\alpha) \wedge \mathbf{y} = (\mathbf{x} \wedge \mathbf{y}) + e_\alpha$ is in \mathcal{N} and $\alpha \in \mathcal{E}(\mathbf{x} \wedge \mathbf{y})$. Reversing the roles of \mathbf{x} and \mathbf{y} , we conclude that if α is in either $\mathcal{E}(\mathbf{x})$ or $\mathcal{E}(\mathbf{y})$, it must be in either

$\mathcal{E}(\mathbf{x} \vee \mathbf{y})$ or $\mathcal{E}(\mathbf{x} \wedge \mathbf{y})$, which is (13). Now suppose $\alpha \in \mathcal{E}(\mathbf{x}) \cap \mathcal{E}(\mathbf{y})$. Either $\mathbf{x}_\alpha \geq \mathbf{y}_\alpha$ or $\mathbf{y}_\alpha \geq \mathbf{x}_\alpha$, so $\alpha \in \mathcal{E}(\mathbf{x} \vee \mathbf{y})$. Moreover, (9) states that $\alpha \in \mathcal{E}(\mathbf{x} \wedge \mathbf{y})$; thus, (14) holds.

(ii) \Leftrightarrow (iv). Suppose (ii) holds; we have shown that this makes \mathcal{N} a lattice. If $\mathbf{x}, \mathbf{y} \in \mathcal{N}_{\alpha,n}$ then $\mathbf{x}' = \mathbf{x} \wedge \mathbf{y}$ is in \mathcal{N} . But $\mathbf{x}'_\alpha = \mathbf{x}_\alpha \wedge \mathbf{y}_\alpha = n - 1$, and (9) implies that $\alpha \in \mathcal{E}(\mathbf{x}')$ since $\alpha \in \mathcal{E}(\mathbf{x}) \cap \mathcal{E}(\mathbf{y})$; hence, $\mathbf{x}' \in \mathcal{N}_{\alpha,n}$. In other words, every $\mathcal{N}_{\alpha,n}$ is closed under \wedge . It follows that every $\mathcal{N}_{\alpha,n}$ has a unique minimal element $\mathbf{x}^{\alpha,n}$. It follows from Theorem 2.4 that T admits the *pure-max* representation

$$(15) \quad T_\alpha(n) = \omega_\alpha(n) + \max_\beta \{T_\beta(\mathbf{x}_\beta^{\alpha,n})\}.$$

Since $+$ and \max are increasing convex operations, and $T_\alpha(1) = \omega_\alpha(1)$ for $\alpha \in \mathcal{E}(s_0)$, every $T_\alpha(n)$ is an increasing convex function of ω .

Now suppose T is increasing and convex. Increasing alone implies (M) (Theorem 2.3), so T has the min-max representation of Theorem 2.4. But if T is also convex, this must reduce to a pure-max representation; otherwise, if some $T_\alpha(n)$ involves a min over more than one minimal element, it is possible to choose $\omega^1, \omega^2, \omega^3 \in \Omega$, $\omega^3 = p\omega^1 + (1 - p)\omega^2$, some $0 < p < 1$, for which $T_\alpha(n)(\omega^3) > pT_\alpha(n)(\omega^1) + (1 - p)T_\alpha(n)(\omega^2)$. (Min is concave.) Thus, if T is increasing and convex, every $\mathcal{N}_{\alpha,n}$ must have a unique minimal element. It remains to show that this implies that \mathcal{L} is min-closed.

Let σ_1, σ_2 be arbitrary elements of \mathcal{L} . We need to show there is a string in \mathcal{L} with score $N(\sigma_1) \wedge N(\sigma_2)$. Suppose there is not. There is a maximal feasible score dominated by both $N(\sigma_1)$ and $N(\sigma_2)$. Let $\sigma \in \mathcal{L}$ achieve this maximal score. Suppose for some α , $N_\alpha(\sigma) < N_\alpha(\sigma_1) \wedge N_\alpha(\sigma_2) \equiv n$. Each σ_i , $i = 1, 2$, has a prefix with a score in $\mathcal{N}_{\alpha,n}$. Let σ'_i , $i = 1, 2$, be the shortest such prefixes. Min-closure of $\mathcal{N}_{\alpha,n}$ implies there is a $\sigma'_3 \in \mathcal{N}_{\alpha,n}$ for which $N(\sigma'_3) = N(\sigma'_1) \wedge N(\sigma'_2)$. Since $N(\sigma_i)$ dominates $N(\sigma'_i)$, $i = 1, 2$, $N(\sigma) \geq N(\sigma'_3)$. In particular, $N_\alpha(\sigma) \geq n - 1$, hence $N_\alpha(\sigma) = n - 1$. Since $N_\alpha(\sigma'_3) = N_\alpha(\sigma)$ and $\alpha \in \mathcal{E}(\sigma'_3)$, (M) implies that $\alpha \in \mathcal{E}(\sigma)$. But this contradicts the hypothesis that $N(\sigma)$ is maximal. \square

Among schemes that satisfy (M), convexity of T can be characterized by ρ .

PROPOSITION 3.3. *If \mathcal{L} satisfies (M), T is convex if and only if the rank function ρ is supermodular.*

PROOF. Suppose ρ is supermodular. In light of Theorem 3.1(ii) and Theorem 2.3(v), we only need to show that \mathcal{L} is min-closed; and \mathcal{L} is min-closed if and only if \mathcal{N} is closed under \wedge . Let \mathbf{x} and \mathbf{y} be elements of \mathcal{N} ; then $\mathbf{x} \vee \mathbf{y} \in \mathcal{N}$. Supermodularity implies

$$\begin{aligned} \rho(\mathbf{x} \wedge \mathbf{y}) &\geq \rho(\mathbf{x}) + \rho(\mathbf{y}) - \rho(\mathbf{x} \vee \mathbf{y}) \\ &= |\mathbf{x}| + |\mathbf{y}| - |\mathbf{x} \vee \mathbf{y}| \\ &= |\mathbf{x} \wedge \mathbf{y}|. \end{aligned}$$

Since, also $\rho(\mathbf{x} \wedge \mathbf{y}) \leq |\mathbf{x} \wedge \mathbf{y}|$ (by definition), we have $\rho(\mathbf{x} \wedge \mathbf{y}) = |\mathbf{x} \wedge \mathbf{y}|$, so $\mathbf{x} \wedge \mathbf{y} \in \mathcal{N}$.

Now suppose that T is convex; then \mathcal{N} is a lattice. For $\mathbf{x} \in \mathbf{Z}_+^m$, let $\psi(\mathbf{x})$ be the basis of \mathbf{x} . Since $\rho(\mathbf{x}) = |\psi(\mathbf{x})|$, supermodularity of ψ would imply supermodularity of ρ ($|\cdot|$ is linear). Let \mathbf{x}, \mathbf{y} be any points in \mathbf{Z}_+^m . Since ψ is increasing (by definition), $\psi(\mathbf{x}) \vee \psi(\mathbf{y}) \leq \psi(\mathbf{x} \vee \mathbf{y})$. Supermodularity would follow if we could show that $\psi(\mathbf{x}) \wedge$

$\psi(\mathbf{y}) \leq \psi(\mathbf{x} \wedge \mathbf{y})$. But $\psi(\mathbf{x}) \wedge \psi(\mathbf{y})$ is in \mathcal{N} (\mathcal{N} is a lattice) and is dominated by $\mathbf{x} \wedge \mathbf{y}$ (ψ is increasing) so it is dominated by the basis of $\mathbf{x} \wedge \mathbf{y}$; i.e., $\psi(\mathbf{x}) \wedge \psi(\mathbf{y}) \leq \psi(\mathbf{x} \wedge \mathbf{y})$. \square

The same argument yields the following observation regarding ordinary antimatroids:

COROLLARY 3.4. *An antimatroid in the sense of (4)–(6) is closed under intersection if and only if its rank function is supermodular.*

3.2. Stochastic setting. To interpret Theorem 3.2 stochastically, let Ω be endowed with a σ -algebra, \mathcal{F} , and consider a family of probability measures $\{\mathcal{P}^\theta, \theta \in \Theta\}$ on (Ω, \mathcal{F}) . Take Θ to be a convex subset of Euclidean space; other cases are not fundamentally different. As in Shanthikumar and Yao [19, 21] call $\{\mathcal{P}^\theta\}$ *strongly stochastically convex* if for any $\theta_1, \theta_2, \theta_3 \in \Theta$ with $\theta_3 = p\theta_1 + (1 - p)\theta_2$, some $0 < p < 1$, there exist, on a common probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, random elements X^{θ_i} , $i = 1, 2, 3$, of Ω , each X^{θ_i} having law \mathcal{P}^{θ_i} , for which $X^{\theta_3} \leq pX^{\theta_1} + (1 - p)X^{\theta_2}$, \tilde{P} -a.s. Call $\{\mathcal{P}^\theta\}$ *stochastically increasing* if the X^θ 's can be chosen so that $X^{\theta_1} \leq X^{\theta_2}$, \tilde{P} -a.s., whenever $\theta_1 \leq \theta_2$. Each realization of some $X^\theta = \{X_\alpha^\theta(n)\}$ is a possible input to \mathcal{S} . Let T^θ be the resulting value of T ; $\{T_\alpha^\theta(n)\}$ are the event epochs of the generalized semi-Markov process induced by X^θ . Let f map values of T to the real line, with $f \circ T$ measurable on (Ω, \mathcal{F}) . Let \mathbf{E}_θ denote expectation on (Ω, \mathcal{F}) with respect to \mathcal{P}^θ .

COROLLARY 3.5. *If \mathcal{S} satisfies (CX), and $\{\mathcal{P}^\theta\}$ is strongly stochastically increasing and convex, then $\mathbf{E}_\theta[f(T)]$ is an increasing convex function of θ for all increasing convex f for which the expectations exist.*

PROOF. Pick any θ_i 's, $i = 1, 2, 3$, with $\theta_3 = p\theta_1 + (1 - p)\theta_2$, for some $0 < p < 1$, and construct X^{θ_i} , $i = 1, 2, 3$, as above. $\mathbf{E}_{\theta_3}[f(T)] = \mathbf{E}[f(T^{\theta_3})] = \mathbf{E}[f \circ T(X^{\theta_3})]$, where \mathbf{E} is expectation on $(\tilde{\Omega}, \tilde{F}, \tilde{P})$. Since the composition of increasing convex functions is increasing and convex,

$$f \circ T(X^{\theta_3}) \leq pf \circ T(X^{\theta_1}) + (1 - p)f \circ T(X^{\theta_2}), \quad \tilde{P}\text{-a.s.},$$

and if $\theta_1 \leq \theta_2$ then $f \circ T(X^{\theta_1}) \leq f \circ T(X^{\theta_2})$. Hence, the same inequalities hold for the expectations. \square

Condition (M) by itself implies stochastic monotonicity of T in the strong stochastic ordering (usually denoted \leq_{st}). Convexity of T leads to monotonicity in a different ordering, the *increasing convex* ordering, denoted by \leq_{icx} ; see Stoyan [23]. To simplify the discussion let us take $\{X^\theta, \theta \in \Theta\}$ to be given and defined on a common space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. (Otherwise, adapt the more involved statement in terms of \mathcal{P}^θ given above.) Write $X^{\theta_1} \leq_{icx} X^{\theta_2}$ if $\mathbf{E}[g(X^{\theta_1})] \leq \mathbf{E}[g(X^{\theta_2})]$, for all increasing convex real-valued functions g on Ω for which the expectations exist. (The ordering \leq_{st} is obtained by requiring that this inequality hold for all increasing g .) Call the family $\{X^\theta\}$ increasing in the \leq_{icx} order if $X^{\theta_1} \leq_{icx} X^{\theta_2}$ whenever $\theta_1 \leq \theta_2$. Define analogous concepts for $\{T^\theta\}$ accordingly.

COROLLARY 3.6. *If \mathcal{S} satisfies (CX) and $\{X^\theta\}$ is increasing in the \leq_{icx} order, then $\{T^\theta\}$ is increasing in the \leq_{icx} order too.*

In general, monotonicity in the sense of Corollary 3.6 neither implies nor is implied by the monotonicity obtained by replacing (both instances of) \leq_{icx} with the strong stochastic ordering \leq_{st} . It happens that since (CX) implies (M), it implies monotonicity in both orderings. Since $x \rightarrow x^2$ is increasing and convex, Corollary 3.6 also

allows us to conclude that if all $X^{\theta_1} \leq_{\text{icx}} X^{\theta_2}$ and $\mathbf{E}[T_\alpha^{\theta_1}(n)] = \mathbf{E}[T_\alpha^{\theta_2}(n)]$, then $\mathbf{Var}[T_\alpha^{\theta_1}(n)] \leq \mathbf{Var}[T_\alpha^{\theta_2}(n)]$, all α and n . In this sense, \leq_{icx} is a *variability* ordering.

REMARK. As mentioned in §1, these types of results indicate when a stochastic convexity property is *preserved* by a system. Identifying when, and in what sense, the input $\{X^\theta\}$ is convex in θ is itself an important problem and an area of ongoing research. Various notions of parametric stochastic convexity are developed in [16, 17, 21].

To apply Corollaries 3.5 and 3.6, one must be able to verify the assumed properties of the inputs—either that \mathcal{P}^θ is strongly stochastically increasing and convex or that X^θ is \leq_{icx} -increasing. In practice this is difficult, except when the inputs are i.i.d. sequences.

Stochastic conclusions like Corollary 3.5 can be drawn from all our results on T and D as deterministic functions of ω . Since the argument is essentially the same in every case, we will not repeat it, or even reformulate the results stochastically.

3.3. *Examples.* We now turn to some examples of schemes satisfying (CX).

EXAMPLE 3.7. Consider k single-server queues in tandem, the i th having buffer capacity (including server) b_i and $b_1 = \infty$. Service completion at the i th node is event β_i ; an external arrival is β_0 . Take the blocking to be *manufacturing*—a job entering a service position starts service immediately, but (after completing service) cannot depart until there is room in the next queue. (*Communication* blocking works too.) Take the initial state to be the one in which the system is empty. Shanthikumar and Yao [19] show that the service completion epochs at each node are increasing convex functions of the service and interarrival times. Let us verify that (CX) holds. In [8] we show that this example satisfies (M); hence, we only need to verify that \mathcal{L} (equivalently, \mathcal{N}) is min-closed. The score space is a subset of \mathbf{Z}_+^{k+1} with vectors indexed by $\beta_0, \beta_1, \dots, \beta_k$. A score, \mathbf{x} , is feasible if and only if it satisfies

$$\begin{aligned} \mathbf{x}_{\beta_i} &\leq \mathbf{x}_{\beta_{i-1}} \wedge (\mathbf{x}_{\beta_{i+1}} + b_{i+1} + 1), & i = 1, \dots, k - 1; \\ \mathbf{x}_{\beta_k} &\leq \mathbf{x}_{\beta_{k-1}}. \end{aligned}$$

Obviously, if \mathbf{x} and \mathbf{y} satisfy these inequalities then so does $\mathbf{x} \wedge \mathbf{y}$.

The above arguments extend readily to more general blocking mechanisms. Consider, for instance, so-called “Kanban blocking”: Here server i , upon completion of a job, if not able to release it to the next queue for lack of buffer space there, will continue serving other jobs in its own buffer. It will be blocked only when it has completed all those jobs and the buffer of the next queue remains full. Clearly, the only modification needed is to replace the first inequality above by

$$\mathbf{x}_{\beta_i} \leq \mathbf{x}_{\beta_{i-1}} \wedge (\mathbf{x}_{\beta_{i+1}} + b_i + b_{i+1}), \quad i = 1, \dots, k - 1.$$

Min-closure again follows immediately. (The verification of (M) is also similar to the manufacturing blocking case.) The analysis here also applies readily to other variations and extensions of Kanban blocking, for instance, those considered in Cheng [4].

EXAMPLE 3.8. Consider a *fork-join* system in which an arriving job (event β_0) is split into q subjobs. The i th subjob must pass through nodes $(i, 1), \dots, (i, k_i)$ (with service completion events $\beta_{ij}, j = 1, \dots, k_i$). Node (i, j) has buffer capacity b_{ij} , and $b_{i1} = \infty, i = 1, \dots, q$. After completing all its stages of service, the i th subjob waits in an infinite capacity queue to be joined with its siblings. This system satisfies (CX), and each service completion epoch has the pure-max representation $T_{\beta_{ij}}(n) = \omega_{\beta_{ij}}(n) + (T_{\beta_{ij}}(n - 1) \vee T_{\beta_{ij+1}}(n - b_{ij+1}))$. (Take $b_{ik_i+1} \equiv \infty$ and $T(-\infty) \equiv 0$.) The epochs of

departures of joined jobs do not correspond to any fixed event β_{ij} ; however, because these epochs are just $\max_i\{T_{\beta_{ik}}(n)\}$, $n = 1, 2, \dots$, they, too, are increasing and convex.

EXAMPLE 3.9. A *multi-dimensional birth-death scheme* is a simple model of component failure and repair. A system requires l components; k_i spare components of type i , $i = 1, \dots, l$, are available initially. When a component of type i fails (event α^i), it is replaced with a spare if one is available. One failed component of each type can be repaired at a time (event β^i), and repairs continue until all components are fixed. The corresponding scheme has the structure of an l -dimensional finite birth-death process and always satisfies (M). The feasible scores $\mathbf{x} = (\mathbf{x}_{\alpha^1}, \mathbf{x}_{\beta^1}, \dots, \mathbf{x}_{\alpha^l}, \mathbf{x}_{\beta^l})$ are characterized by $0 \leq \mathbf{x}_{\alpha^i} - \mathbf{x}_{\beta^i} \leq k_i$, $i = 1, \dots, l$. It is easy to see that the set of such scores is closed under \wedge . It follows that the failure and repair epochs are increasing convex functions of the component lifetimes and the repair times.

EXAMPLE 3.10. Any scheme with just two events α, β that satisfies (M) satisfies (CX). A minimal element $(\mathbf{x}_\alpha, \mathbf{x}_\beta)$ of any $\mathcal{N}_{\alpha, n}$ must have the form $(n - 1, \mathbf{x}_\beta)$. Since any two such scores are comparable, every $\mathcal{N}_{\alpha, n}$ has a unique minimal element, and every $T_\alpha(n)$ has a pure-max representation.

4. Concavity of event epochs. The conditions for T to be increasing and concave as a function of ω are not quite as symmetric to the conditions for convexity as one might hope or expect. Nor do the conditions for concavity lend themselves to a unified result along the lines of Theorems 2.3 and 3.1. We explain why concavity is different, point out a class of schemes for which the relation with convexity is more symmetric, then give general conditions for concavity of T .

The obvious “dual” to (9) is the requirement that

$$(16) \quad N(\sigma_3) = N(\sigma_1) \vee N(\sigma_2) \Rightarrow \mathcal{E}(\sigma_3) \subseteq \mathcal{E}(\sigma_1) \cup \mathcal{E}(\sigma_2),$$

whenever $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{L}$. One might expect that (16) and noninterruption are sufficient for concavity, but these conditions are not quite enough, as the following example illustrates.

Consider the portion of a scheme illustrated in Figure 2. If we take $\sigma_1 = \alpha$, $\sigma_2 = \beta$ and $\sigma_3 = \alpha\beta$, these three strings satisfy (16). In particular, $\alpha \in \mathcal{E}(\sigma_3)$ and $\alpha \in \mathcal{E}(\sigma_2)$. However, it is easy to see that $T_\alpha(2) = \omega_\alpha(2) + \max\{\omega_\alpha(1), \omega_\beta(1)\}$, so $T_\alpha(2)$ is convex and strictly nonconcave. Indeed, the portion of the scheme in the figure satisfies (CX). The problem is that while (16) is satisfied, the α in $\mathcal{E}(\beta)$ represents the first occurrence of α , while that in $\mathcal{E}(\alpha\beta)$ represents the second occurrence. In this sense, the α in $\mathcal{E}(\sigma_1) \cup \mathcal{E}(\sigma_2)$ is not quite “the same” as the α in $\mathcal{E}(\sigma_3)$.

To see why this matters, consider again (CX). Condition (9) may be thought of as requiring that if $N(\sigma_3) \geq N(\sigma_1) \wedge N(\sigma_2)$ holds initially, then it is preserved under the evolution of the three strings: any event that can be concatenated to σ_1 and σ_2 can be concatenated to σ_3 . For concavity we would like $N(\sigma_3) \leq N(\sigma_1) \vee N(\sigma_2)$ to be preserved. But in the scheme of Figure 2, even though $N(\sigma_3) = N(\sigma_1) \vee N(\sigma_2)$ and (16) holds, $N(\sigma_3\alpha) > N(\sigma_1) \vee N(\sigma_2\alpha)$. In order that $N(\sigma_3) \leq N(\sigma_1) \vee N(\sigma_2)$ be preserved under the evolution of the strings, we need a stronger condition. We must

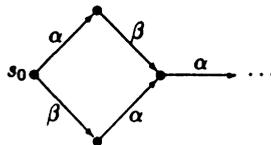


FIGURE 2. A Portion of a Scheme Satisfying (16).

require, roughly, that if $\alpha \in \mathcal{E}(\sigma_3)$ and $N_\alpha(\sigma_3) = N_\alpha(\sigma_1) \vee N_\alpha(\sigma_2)$, then α should be contained in the event list of the σ_i , $i = 1, 2$, with the *greater* α -score. The precise condition is (CV) below.

There is a class of schemes in which this complication does not arise. These are schemes in which each event can occur at most once—schemes whose languages have no repetition. For such schemes, (16) (plus noninterruption) is equivalent to (CV), which implies concavity of T . If each event can only occur once and $\alpha \in \mathcal{E}(\sigma_3)$ and $N_\alpha(\sigma_3) = N_\alpha(\sigma_1) \vee N_\alpha(\sigma_2)$, then $N_\alpha(\sigma_1) = N_\alpha(\sigma_2) = 0$, so it does not matter which $\mathcal{E}(\sigma_i)$, $i = 1, 2$, contains α . If \mathbf{A} is finite and each event can only occur once, then any GSMP based on such a scheme must obviously terminate after finitely many events. Systems of this type arise in reliability models and in stochastic activity networks. Also, Bailey [2], in studying greedy optimization on stochastic networks, points out that the moves of the greedy algorithm can be viewed as the events of certain GSMPs. In the schemes that arise this way, each event occurs at most once.

For more general GSMPs, such as queuing systems, the exclusion of repetition is too strong and (16) is inadequate. For concavity of T we need

(CV) If $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{L}$ and $N(\sigma_3) \leq N(\sigma_1) \vee N(\sigma_2)$, then

$$\mathcal{E}(\sigma_3) \setminus A \subseteq \mathcal{E}(\sigma_1) \cup \mathcal{E}(\sigma_2),$$

where $A = \{\alpha: N_\alpha(\sigma_3) < N_\alpha(\sigma_1) \vee N_\alpha(\sigma_2)\}$, and, for all $\alpha \in \mathbf{A}$,

$$\alpha \in \mathcal{E}(\sigma_3), \quad N_\alpha(\sigma_3) = N_\alpha(\sigma_1) > N_\alpha(\sigma_2) \Rightarrow \alpha \in \mathcal{E}(\sigma_1).$$

Clearly, (CV) is stronger than (M). It combines noninterruption with a condition stronger than permutability; indeed, (CV) implies (16) which implies permutability. The scheme in Figure 2 violates the second part of (CV): $N_\alpha(\alpha) > N_\alpha(\beta)$ but $\alpha \in \mathcal{E}(\alpha\beta)$ and $\alpha \notin \mathcal{E}(\alpha)$.

THEOREM 4.1. *The following implications hold:*

- (i) \mathcal{S} satisfies (CV) $\Leftrightarrow T$ is increasing and concave;
- (ii) \mathcal{S} satisfies (CV) $\Rightarrow \chi$ is increasing and submodular.

PROOF. (i). Suppose (CV) holds. Let $\mathcal{N}_{\alpha,n}$ be nonempty and have minimal elements $\{\mathbf{x}^i\}$. We claim that for each \mathbf{x}^i there is just one event γ_i for which $\mathbf{x}^i - e_{\gamma_i}$ is feasible. For suppose $\mathbf{x}^i - e_\gamma$ and $\mathbf{x}^i - e_\beta$ are feasible, $\beta \neq \gamma$. Since $\alpha \in \mathcal{E}(\mathbf{x}^i)$, (CV) (through (16)) implies that $\alpha \in \mathcal{E}(\mathbf{x}^i - e_\gamma)$ or $\alpha \in \mathcal{E}(\mathbf{x}^i - e_\beta)$. Moreover, $(\mathbf{x}^i - e_\gamma)_\alpha \vee (\mathbf{x}^i - e_\beta)_\alpha = \mathbf{x}^i_\alpha = n - 1$. (CV) further requires that at least one of $(\mathbf{x}^i - e_\gamma)$ and $(\mathbf{x}^i - e_\beta)$ has α -score $n - 1$ and has α in its event list. But then either $(\mathbf{x}^i - e_\gamma)$ or $(\mathbf{x}^i - e_\beta)$ is in $\mathcal{N}_{\alpha,n}$, contradicting minimality of \mathbf{x}^i .

With γ_i so defined, any $\sigma \in \mathcal{L}$ with $N(\sigma) = \mathbf{x}^i$ must have γ_i as its last element; thus, we have $\max_\beta \{T_\beta(\mathbf{x}^i)\} = T_{\gamma_i}(\mathbf{x}^i)$, for all ω . (If $\mathcal{N}_{\alpha,n}$ is empty, choose γ_i arbitrarily from \mathbf{A} .) It follows that T has the *pure-min* representation

$$(17) \quad T_\alpha(n) = \omega_\alpha(n) + \min_i \left\{ T_{\gamma_i}(\mathbf{x}^i) \right\}.$$

Since $+$ and \min are increasing and concave, every $T_\alpha(n)$ is increasing and concave.

Now suppose T is increasing (which implies a min-max representation) and concave (which further implies the max drops out). In other words, for every α and n there are numbers $\{n_\gamma, \gamma \in \mathbf{A}\}$ (possibly infinite) such that $T_\alpha(n) = \omega_\alpha(n) + \min_\gamma \{T_\gamma(n_\gamma)\}$ (for all ω). This can be restated as $S_\alpha(n) = \min_\gamma \{T_\gamma(n_\gamma)\}$, where $S_\alpha(n)$ is the epoch of the n th setting of a clock for α . This representation implies that for any

$\mathbf{x} \in \mathcal{N}$, and any $\gamma \in \mathbf{A}$,

$$(18) \quad \mathbf{x}_\gamma \geq n_\gamma \quad \text{for some } \gamma \text{ if and only if } \mathbf{x} \in \mathcal{N}_{\alpha,n} \text{ or } \mathbf{x}_\alpha \geq n.$$

We will show that if (CV) does not hold, then (18) cannot hold for all α and n .

Suppose (CV) fails; there exist feasible $\sigma_1, \sigma_2, \sigma_3$, $N(\sigma_3) \leq N(\sigma_1) \vee N(\sigma_2)$, and $\alpha \in \mathcal{E}(\sigma_3)$ such that $\alpha \in \mathcal{E}(\sigma_i) \Rightarrow N_\alpha(\sigma_i) < N_\alpha(\sigma_3)$, $i = 1, 2$. Define n by $n - 1 = N_\alpha(\sigma_3)$. We may suppose, without loss of generality, that no prefix σ of σ_3 has $N_\alpha(\sigma) = n - 1$ and $\alpha \in \mathcal{E}(\sigma)$. (Otherwise, replace σ_3 with such a prefix.) Let γ be the last element of σ_3 . Then if $T_\alpha(n)$ has a pure-min representation, n_γ must be $N_\gamma(\sigma_3)$. By relabeling if necessary, let $N_\gamma(\sigma_1) \geq N_\gamma(\sigma_2)$; then $N_\gamma(\sigma_1) = N_\gamma(\sigma_3) = n_\gamma$. However, $N_\alpha(\sigma_1) < n$, and, by hypothesis, $N_\alpha(\sigma_1) = n - 1$ implies $\alpha \in \mathcal{E}(\sigma_1)$, so $N(\sigma_1) \notin \mathcal{N}_{\alpha,n}$, contradicting (18).

(ii). If (CV) holds then χ is increasing (and is a function of scores) because (M) holds. Suppose that \mathbf{x}, \mathbf{y} (hence, $\mathbf{x} \vee \mathbf{y}$) and also $\mathbf{x} \wedge \mathbf{y}$ are in \mathcal{N} . To establish submodularity, we need to show that $\chi(\mathbf{x} \wedge \mathbf{y}) + \chi(\mathbf{x} \vee \mathbf{y}) \leq \chi(\mathbf{x}) + \chi(\mathbf{y})$. We show this componentwise. Since χ is increasing,

$$(\mathbf{x} \wedge \mathbf{y})_\alpha + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{x} \wedge \mathbf{y})\} \leq [\mathbf{x}_\alpha + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{x})\}] \wedge [\mathbf{y}_\alpha + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{y})\}];$$

and (CV) states that

$$(\mathbf{x} \vee \mathbf{y})_\alpha + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{x} \vee \mathbf{y})\} \leq [\mathbf{x}_\alpha + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{x})\}] \vee [\mathbf{y}_\alpha + \mathbf{1}\{\alpha \in \mathcal{E}(\mathbf{y})\}].$$

Adding these two inequalities we get submodularity. \square

It is natural to ask whether the other characterizations in Theorems 2.3 and 3.1 extend to concavity. Unfortunately, the geometric interpretation of (CV) is not as simple as that of (CX). It does, however, lead to a determination of the indices appearing in the pure-min representation (17).

For a permutable scheme, let $\hat{\mathcal{N}}_{\alpha,n} = \{\mathbf{x} \in \mathcal{N} : \exists \mathbf{y} \in \mathcal{N}_{\alpha,n}, \mathbf{x} \geq \mathbf{y}\}$ be the feasible scores that dominate some score in $\mathcal{N}_{\alpha,n}$. Let $\hat{\mathcal{N}}_{\alpha,n}^c = \mathcal{N} \setminus \hat{\mathcal{N}}_{\alpha,n}$. Call an arbitrary subset of \mathcal{N} relatively closed under \wedge (\vee) if for any \mathbf{x}, \mathbf{y} in the subset, $\mathbf{x} \wedge \mathbf{y}$ ($\mathbf{x} \vee \mathbf{y}$) is in the subset whenever it is in \mathcal{N} . Call the subset a relative lattice if it is relatively closed under both \wedge and \vee .

PROPOSITION 4.2. For a permutable scheme,

- (i) if (CX) holds then every $\hat{\mathcal{N}}_{\alpha,n}$ is a lattice;
- (ii) if (CV) holds then every $\hat{\mathcal{N}}_{\alpha,n}^c$ is a relative lattice.

PROOF. (i). If (M) holds then $\hat{\mathcal{N}}_{\alpha,n}$ is closed under \vee . If (CX) holds then $\mathcal{N}_{\alpha,n}$ is a lattice (and has a unique minimal element), so the defining condition for $\hat{\mathcal{N}}_{\alpha,n}$ is preserved under \wedge .

(ii). (M) alone makes $\hat{\mathcal{N}}_{\alpha,n}^c$ relatively closed under \wedge . Clearly, $\mathbf{x} \in \mathcal{N}$ is in $\hat{\mathcal{N}}_{\alpha,n}^c$ if and only if $\chi_\alpha(\mathbf{x}) < n$. If \mathbf{x}, \mathbf{y} are in $\hat{\mathcal{N}}_{\alpha,n}^c$ and $\mathbf{x} \wedge \mathbf{y}$ is in \mathcal{N} then $\chi(\mathbf{x} \wedge \mathbf{y}) \leq \chi(\mathbf{x}) \wedge \chi(\mathbf{y})$ so $\mathbf{x} \wedge \mathbf{y} \in \hat{\mathcal{N}}_{\alpha,n}^c$. Now suppose that (CV) holds and $\mathbf{x}^1, \mathbf{x}^2$ are in $\hat{\mathcal{N}}_{\alpha,n}^c$. Clearly, $(\mathbf{x}^1 \vee \mathbf{x}^2)_\alpha < n$; if $(\mathbf{x}^1 \vee \mathbf{x}^2)_\alpha < n - 1$ then $\chi_\alpha(\mathbf{x}^1 \vee \mathbf{x}^2) < n$ so $\mathbf{x}^1 \vee \mathbf{x}^2 \in \hat{\mathcal{N}}_{\alpha,n}^c$. Suppose $(\mathbf{x}^1 \vee \mathbf{x}^2)_\alpha = n - 1$; then $(\mathbf{x}^1 \vee \mathbf{x}^2) \notin \hat{\mathcal{N}}_{\alpha,n}^c$ only if, for either $i = 1$ or $i = 2$, $\mathbf{x}_\alpha^i = n - 1$ and $\alpha \in \mathcal{E}(\mathbf{x}^i)$; i.e., only if either \mathbf{x}^1 or \mathbf{x}^2 is in $\mathcal{N}_{\alpha,n}$. \square

Strengthening (M) to (CX) therefore closes $\hat{\mathcal{N}}_{\alpha,n}$ under \wedge , and ensures that $\hat{\mathcal{N}}_{\alpha,n}$ has a unique minimal element. This minimal element provides the pure-max representation (15) of $T_\alpha(n)$. Strengthening (M) instead to (CV) closes $\hat{\mathcal{N}}_{\alpha,n}^c$ under \vee ; but $\hat{\mathcal{N}}_{\alpha,n}^c$ need not have a maximal element since it may be unbounded. If, however, we allow infinity as a possible value and let $\mathbf{y}^{\alpha,n} = \sup\{\mathbf{x} : \mathbf{x} \in \hat{\mathcal{N}}_{\alpha,n}^c\}$, the indices $n_\gamma =$

$\mathbf{y}_\gamma^{\alpha,n}$, $\gamma \in \mathbf{A}$, of this “maximal element” of $\hat{\mathcal{N}}_{\alpha,n}^c$ provide the pure-min representation (17), which we may rewrite as $T_\alpha(n) = \omega_\alpha(n) + \min_\gamma \{T_\gamma(\mathbf{y}_\gamma^{\alpha,n})\}$.

5. Submodularity and supermodularity. In general, submodularity and supermodularity of a function neither imply nor are implied by convexity and concavity. It turns out, however, that the conditions for T to be (increasing and) submodular or supermodular are strictly stronger than the conditions for T to be (increasing and) convex or concave.

Since we only consider increasing T , we may restrict attention to schemes that satisfy (M). Consider schemes for which T admits one the following pair of representations:

(i) For every α and n there are subsets A_1, \dots, A_k of \mathbf{A} , $0 \leq k \leq \infty$, and indices $\{n_\beta^i, \beta \in A_i, i = 1, \dots, k\}$, such that

$$(19) \quad T_\alpha(n) = \omega_\alpha(n) + \max_{\beta \in A_1} \{\omega_\beta(n_\beta^1)\} + \dots + \max_{\beta \in A_k} \{\omega_\beta(n_\beta^k)\} \quad \text{for all } \omega.$$

(ii) For every α and n , (i) holds but with

$$(20) \quad T_\alpha(n) = \omega_\alpha(n) + \min_{\beta \in A_1} \{\omega_\beta(n_\beta^1)\} + \dots + \min_{\beta \in A_k} \{\omega_\beta(n_\beta^k)\} \quad \text{for all } \omega.$$

It is possible to characterize what the indices $\{n_\beta^i\}$ in (19) and (20) must be if such a representation holds. These indices are related to the minimal and maximal elements $\mathbf{x}^{\alpha,n}$ and $\mathbf{y}^{\alpha,n}$ introduced in §3 and §4. The connection is complicated and unilluminating and is therefore omitted. Instead, we go directly to

THEOREM 5.1. *The following implications hold:*

- (i) T is increasing and submodular \Leftrightarrow (19) holds.
- (ii) T is increasing and supermodular \Leftrightarrow (20) holds.

PROOF. Since max and min are submodular and supermodular, respectively, and both properties are preserved by addition, (19) and (20) imply submodularity and supermodularity. We need to show that if T is increasing and submodular (supermodular) it has the representation (19) ((20)). T increasing implies the min-max representation; but T cannot be submodular unless the min drops out and cannot be supermodular unless the max does. A pure-max representation of $T_\alpha(n) - \omega_\alpha(n)$ different from (19) is a composition of functions, the last of which has the general form $g(x_1, x_2, x_3, x_4) = \max(x_1 + x_2, x_3 + x_4)$, which is not submodular. A pure-min representation different from (20) is a composition of functions, the last of which has the general form $g(x_1, x_2, x_3, x_4) = \min(x_1 + x_2, x_3 + x_4)$, which is not supermodular. \square

These results indicate that submodularity and supermodularity of T are rather restrictive properties that can only be found in specially structured systems. A typical example is

EXAMPLE 5.2. The representation (19) describes parallel processing, in a generic sense which includes certain kinds of assembly, in addition to parallel computer algorithms. Individual units or machines work independently on tasks which are part of a larger job. Upon completing a task, each machine must wait for the others to complete their tasks in order to exchange information, parts, etc., before beginning the next set of tasks. Thus, overall work on the job is divided into cycles, each machine completing one task in each cycle. At the end of a cycle the machines synchronize.

Let completion of a task on machine i be event β_i . Let A_1 be the set of machines that can start on a task before any other is complete; let A_{j+1} be the set of machines that can initiate their next task only when all the tasks from the j th cycle are complete. If β_i is in A_j , let $n_{\beta_i}^j$ be the number of sets among A_1, \dots, A_j containing β_i ; thus, in the j th cycle, machine i works on its $n_{\beta_i}^j$ th task. If $\beta_i \in A_k$, then the $n_{\beta_i}^k$ th task at machine i is completed at

$$T_{\beta_i}(n_{\beta_i}^k) = \omega_{\beta_i}(n_{\beta_i}^k) + \max_{\alpha \in A_1} \{\omega_{\alpha}(n_{\alpha}^1)\} + \dots + \max_{\alpha \in A_{k-1}} \{\omega_{\alpha}(n_{\alpha}^{k-1})\},$$

which is a special case of (19).

6. Comparing different schemes. Thus far, we have focused exclusively on properties of T as a function of ω . Different results are obtained by changing the function under investigation (from T to D) and by changing the argument of interest (from ω to a scheme parameter). It is easy to see that monotonicity of T and D coincides—one is increasing whenever the other is decreasing. But no similar relation applies for second-order properties; indeed, since each $D(t)$, $t \geq 0$, is piecewise constant in each $\omega_{\alpha}(n)$, it cannot be convex or concave. On the other hand, in comparing different schemes (driven by the same ω) D turns out to be the more convenient output to examine.

We consider two ways in which different schemes can be related. When one scheme is a *subscheme* of another, in a sense to be defined, we can establish monotonicity across schemes, reformulating some results from [8] in the framework of languages and scores. We then introduce a mechanism for *synchronizing* a family of schemes, and establish monotonicity in this setting. In §7, we provide conditions for second-order properties across families of synchronized schemes.

6.1. Subschemes. In [8], we defined a partial order on schemes. One scheme \mathcal{G}^S is a *subscheme* of another \mathcal{G}^B (denoted $\mathcal{G}^S \subseteq \mathcal{G}^B$) if $\mathbf{S}^S \subseteq \mathbf{S}^B$; $\mathbf{A}^S \subseteq \mathbf{A}^B$; $\mathcal{E}^S(s) \subseteq \mathcal{E}^B(s)$, for all $s \in \mathbf{S}^S$; $\phi^S(s, \alpha) = \phi^B(s, \alpha)$, for all $s \in \mathbf{S}^S$ and all $\alpha \in \mathcal{E}^S(s)$; and $s_0^S = s_0^B$. The subscheme relation captures the idea of one system being contained in another, and models, for example, queueing systems that differ only in that one has more buffers, more servers, more jobs, etc. Any string of events feasible in the smaller system is also feasible in the bigger system. In other words, the subscheme relation implies containment of languages:

LEMMA 6.1. *If $\mathcal{G}^S, \mathcal{G}^B$ generate languages $\mathcal{L}^S, \mathcal{L}^B$ then $\mathcal{G}^S \subseteq \mathcal{G}^B$ implies $\mathcal{L}^S \subseteq \mathcal{L}^B$.*

PROOF. A string $\beta^1 \dots \beta^n$ is in \mathcal{L}^S only if $\beta_i \in \mathcal{E}^S(\phi^S(s_0^S, \beta^1 \dots \beta^{i-1}))$, $i = 1, \dots, n$, which holds only if $\beta_i \in \mathcal{E}^B(\phi^B(s_0^B, \beta^1 \dots \beta^{i-1}))$, $i = 1, \dots, n$; that is, only if $\beta^1 \dots \beta^n \in \mathcal{L}^B$. \square

Henceforth, when we refer to a subscheme $\mathcal{G}^S \subseteq \mathcal{G}^B$, we make the simplifying assumption that $\mathbf{A}^S = \mathbf{A}^B$. This is not much of a limitation in practice and it substantially lightens the notation. With this assumption, we may compare inputs and outputs of \mathcal{G}^S and \mathcal{G}^B directly. Without it, we must view $(\omega_{\alpha}, \alpha \in \mathbf{A}^S)$ as a subvector of $(\omega_{\alpha}, \alpha \in \mathbf{A}^B)$, and $(D_{\alpha}, \alpha \in \mathbf{A}^S)$ as a subvector of $(D_{\alpha}, \alpha \in \mathbf{A}^B)$.

We showed in [8, Theorem 4.3] that if \mathcal{G}^B satisfies (M), if \mathcal{G}^S is noninterruptive, and if $\mathcal{G}^S \subseteq \mathcal{G}^B$, then $T^B \leq T^S$ (so $D^B \geq D^S$), for all $\omega \in \Omega$. We can recast this result in a slightly stronger form:

THEOREM 6.2. *If two schemes \mathcal{G}^B and \mathcal{G}^S generate languages \mathcal{L}^B and \mathcal{L}^S , \mathcal{L}^B an antimatroid and \mathcal{L}^S locally free, then $\mathcal{L}^S \subseteq \mathcal{L}^B$ implies $T^B \leq T^S$ and $D^B \geq D^S$.*

In §6.2, we consider pairs of schemes, both satisfying (M). In this setting, the ordering of outputs follows from ordering of characteristic functions:

LEMMA 6.3. *If \mathcal{G}^B and \mathcal{G}^S satisfy (M), then $\mathcal{L}^S \subseteq \mathcal{L}^B$ if and only if $\chi^S \leq \chi^B$ wherever both are defined. Either of these implies $D^B \geq D^S$.*

PROOF. If $\mathcal{L}^S \subseteq \mathcal{L}^B$, the intersection of the domains of χ^S and χ^B is \mathcal{N}^S , the score space of \mathcal{L}^S . If $\mathbf{x} \in \mathcal{N}^S$, then, for every $\alpha \in \mathbf{A}$, $\chi_\alpha^B(\mathbf{x}) - \chi_\alpha^S(\mathbf{x}) = \mathbf{1}\{\alpha \in \mathcal{E}^B(\mathbf{x})\} - \mathbf{1}\{\alpha \in \mathcal{E}^S(\mathbf{x})\}$. We argue that this must be nonnegative: If $\alpha \in \mathcal{E}^S(\mathbf{x})$, then there is a string in \mathcal{L}^S with score $\mathbf{x} + e_\alpha$. The same string is in \mathcal{L}^B . The strong exchange property now requires that $\alpha \in \mathcal{E}^B(\mathbf{x})$.

Suppose $\chi^S \leq \chi^B$ wherever both are defined. Let σ be any string in \mathcal{L}^S , and let $\sigma_i, i = 0, \dots, |N(\sigma)|$, be the length- i prefix of σ . Suppose that some σ_i is in \mathcal{L}^B . Let $\sigma_{i+1} = \sigma_i\beta$. Then $\mathbf{1}\{\beta \in \mathcal{E}^B(\sigma_i)\} = \chi^B(N(\sigma_i)) - N(\sigma_i) \geq \chi^S(N(\sigma_i)) - N(\sigma_i) = \mathbf{1}\{\beta \in \mathcal{E}^S(\sigma_i)\} = 1$; that is, $\beta \in \mathcal{E}^B(\sigma_i)$, so $\sigma_{i+1} \in \mathcal{L}^B$. Since $\sigma_0 = \epsilon$ is in \mathcal{L}^B , we conclude that $\sigma \in \mathcal{L}^B$. The last part of the theorem restates Theorem 6.2. \square

6.2. *Synchronized schemes.* Second-order properties do not carry over from one scheme to multiple schemes the way first-order properties do. Given schemes $\mathcal{G}^1 \subseteq \mathcal{G}^2 \subseteq \mathcal{G}^3$, one might ask for broad conditions under which the output is concave across schemes, in the sense that $D^3 - D^2 \leq D^2 - D^1$. Such conditions, if they exist, have eluded us. It appears that a certain amount of *synchronization* across schemes is needed for second-order properties.

The mechanism we study is a GSMP analog of the method of uniformization in Markov processes. We start with a family $\{\mathcal{G}^b, b \in \mathcal{B}\}$ of schemes with a common event set \mathbf{A} and subordinate them to a “master” scheme \mathcal{G}^* , called the *shuffle*. The shuffle scheme has $\mathbf{S}^* = \{0\}$, $\mathbf{A}^* = \mathbf{A}$, $\mathcal{E}(0) = \mathbf{A}$, $\phi(0, \cdot) \equiv 0$, and $s_0 = 0$. The language \mathcal{L}^* it generates is the “shuffle” of the languages consisting of arbitrary repetitions of individual elements of \mathbf{A} . Every finite sequence of elements of \mathbf{A} is in \mathcal{L}^* . Thus, every $\mathcal{L}^b, b \in \mathcal{B}$ is a subset of \mathcal{L}^* . (Any scheme generating a language containing all the \mathcal{L}^b 's could serve as “master”; the shuffle scheme is convenient because it always applies.)

Given an input ω , the shuffle scheme generates a sequence of events. Denote by a_k^* the type of the k th event, and by τ_k^* its epoch of occurrence. The sequence $\{\tau_k^*\}$ is the superposition of the sequences $\{T_\alpha^*(n), \alpha \in \mathbf{A}\}$. For the shuffle, these are given by $T_\alpha^*(n) = \sum_{i=1}^n \omega_\alpha(i)$. Naturally, the counting process D^* changes values only at the τ_k^* 's. We may focus, therefore, on the sequence of values $\{D^*(\tau_k^*)\}$, denoted simply $\{D^*(k)\}$.

Rather than drive each \mathcal{G}^b separately, let us synchronize them to the shuffle scheme, taking the event epochs of each \mathcal{G}^b to be a subsequence of those of \mathcal{G}^* . For simplicity, assume from the outset that every D^b satisfies (M). Each D^b can only change value at the τ_k^* 's. Denote $D^b(\tau_k^*)$ by $D^b(k)$, let $D^b(0) \equiv 0$, and, for each $\alpha \in \mathbf{A}$, let

$$(21) \quad D_\alpha^b(k + 1) = D_\alpha^b(k) + \mathbf{1}\{a_{k+1}^* = \alpha\} \cdot \mathbf{1}\{\alpha \in \mathcal{E}^b(D^b(k))\}.$$

In this model, each event runs autonomously (in the shuffle scheme) generating a sequence of *potential* event epochs. The actual event epochs are thinned from this sequence by \mathcal{G}^b through its event list. In other words, \mathcal{G}^b accepts the event a_{k+1}^* only if $a_{k+1}^* \in \mathcal{E}^b(D^b(k))$.

REMARK. When the (stochastic) input to the \mathcal{G}^b 's is a vector of independent Poisson processes (one for each event), the probabilistic evolution of each D^b is the same whether it is generated in the ordinary way (as in previous sections) or

subordinated to the shuffle, as in (21). In this case, the mechanism of (21) is just uniformization.

REMARK. There are several reasons for considering the somewhat nonstandard synchronization mechanism described above. Without it, it appears that D^b cannot satisfy any second-order property in b . Further, as remarked above, this mechanism can be used to compare systems driven by Poisson input. In that context, our set-up clarifies what role is played by the Poisson assumption. Second-order properties hold whenever schemes are synchronized as above and our other conditions hold. The Poisson assumption is used to equate the resulting system with “unsynchronized” systems, but is not otherwise needed.

It is also worth noting that (21) is an appropriate representation whenever potential event epochs are autonomously generated. In other words, (21) makes sense as a model in its own right, and not merely as an approximation to some other model. We mention a few examples: If a server takes a vacation whenever its queue empties, and if vacation times are indistinguishable from service times, then the epochs of potential departures are determined *a priori*. As a bus moves through its route, the event epochs—bus and passenger arrivals—are generated autonomously; arriving passengers face a *residual*, not a *new*, interarrival time. In some synchronous communication channels, potential transmission points occur whether or not there is anything to transmit.

Suppose that the set \mathcal{B} of scheme indices is a set of vectors in some \mathbf{R}^k endowed with the componentwise partial order and closed under (componentwise) \wedge and \vee . Properties of $\{\mathcal{L}^b, b \in \mathcal{B}\}$ can be studied through $\chi(\cdot, \cdot)$, where $\chi(b, \cdot)$ is the characteristic function of \mathcal{L}^b . It is easy to see that (21) can be rewritten as

$$(22) \quad D_\alpha^b(k + 1) = [D_\alpha^b(k) + \mathbf{1}\{a_{k+1}^* = \alpha\}] \wedge \chi_\alpha(b, D^b(k))$$

(compare (8)), and as

$$(23) \quad D_\alpha^b(k + 1) = \begin{cases} \chi_\alpha(b, D^b(k)), & a_{k+1}^* = \alpha; \\ D_\alpha^b(k), & a_{k+1}^* \neq \alpha. \end{cases}$$

Say that χ is monotone, convex, etc., if it has the property as a function of both arguments. Monotonicity of χ (and D^b) is closely tied to the following condition on a pair of schemes, $\mathcal{L}^1, \mathcal{L}^2$ with languages $\mathcal{L}^1, \mathcal{L}^2$:

(M²) If $\sigma_i \in \mathcal{L}^i, i = 1, 2$, and $N(\sigma_2) \geq N(\sigma_1)$, then

$$\mathcal{E}^1(\phi^1(s_0^1, \sigma_1)) \setminus A_{\sigma_1\sigma_2} \subseteq \mathcal{E}^2(\phi^2(s_0^2, \sigma_2)),$$

where $A_{\sigma_1\sigma_2} = \{\alpha: N_\alpha(\sigma_2) > N_\alpha(\sigma_1)\}$.

THEOREM 6.4. For a synchronized family of schemes $\{\mathcal{L}^b, b \in \mathcal{B}\}$ the following conditions are equivalent:

- (i) $\mathcal{L}^{b_1}, \mathcal{L}^{b_2}$ satisfy (M²) whenever $b_1 \leq b_2$;
- (ii) χ is increasing;
- (iii) every \mathcal{L}^b satisfies (M) and \mathcal{L}^b is increasing in b .

Any of these implies

- (iv) D^b is increasing in b .

PROOF. (i) \Leftrightarrow (ii). χ is increasing if and only if whenever $b_1 \leq b_2$ and $N(\sigma_1) \leq N(\sigma_2), \sigma_i \in \mathcal{L}^{b_i}$,

$$N_\alpha(\sigma_1) + \mathbf{1}\{\alpha \in \mathcal{E}^{b_1}(\sigma_1)\} \leq N_\alpha(\sigma_2) + \mathbf{1}\{\alpha \in \mathcal{E}^{b_2}(\sigma_2)\} \quad \text{for all } \alpha;$$

i.e., if and only if $\alpha \in \mathcal{E}^{b_1}(\sigma_1)$ implies that either $\alpha \in \mathcal{E}^{b_2}(\sigma_2)$ or $N_\alpha(\sigma_1) < N_\alpha(\sigma_2)$; i.e., if and only if (M^2) holds.

(ii) \Leftrightarrow (iii). Suppose χ is increasing. By taking $b_1 = b_2 = b$ and applying (M^2) to $\mathcal{E}^{b_1}, \mathcal{E}^{b_2}$ we verify that \mathcal{E}^b satisfies (M) . From Lemma 6.3 we conclude that $\mathcal{L}^{b_1} \subseteq \mathcal{L}^{b_2}$ whenever $b_1 \leq b_2$. If instead we assume (iii), Lemma 6.3 implies that χ is increasing.

(ii) \Rightarrow (iv). That D^b is increasing is an immediate consequence of (23). \square

7. Second-order properties across schemes. In this section, we introduce conditions for second-order properties of D^b as a function of the parameter b . We use the synchronization mechanism of §6.2. §7.1 presents structural conditions analogous to those used earlier. In §7.2, we present examples.

7.1. Conditions and results. For a family of schemes $\{\mathcal{E}^b, b \in \mathcal{B}\}$, synchronized as in (21), call $\{D^b, b \in \mathcal{B}\}$ convex (concave) if $2b_3 = b_1 + b_2$ implies $2D^{b_3} \leq (\geq) D^{b_1} + D^{b_2}$, whenever $b_i \in \mathcal{B}, i = 1, 2, 3$. Call $\{D^b, b \in \mathcal{B}\}$ directionally convex (directionally concave) if $b_1 \leq b_2, b_3 \leq b_4, b_1 + b_4 = b_2 + b_3$ implies $D^{b_1} + D^{b_4} \geq (\leq) D^{b_2} + D^{b_3}$. The notion of directional convexity (concavity) was developed in [17] and shown to be equivalent to convexity (concavity) in each component of b plus supermodularity (submodularity).

For these properties we introduce conditions that strengthen (M^2) . The first two are conditions on triples $(\mathcal{E}^1, \mathcal{E}^2, \mathcal{E}^3)$ of schemes, where each \mathcal{E}^i is an arbitrary scheme generating a language \mathcal{L}^i . To reduce repetition in stating the conditions, we assume that σ_i is always an element of \mathcal{L}^i .

(CX-D) If $2N(\sigma_3) \leq N(\sigma_1) + N(\sigma_2)$, then

$$\mathcal{E}^3(\sigma_3) \setminus A_1 \subseteq \mathcal{E}^1(\sigma_1) \cup \mathcal{E}^2(\sigma_2),$$

$$\mathcal{E}^3(\sigma_3) \setminus A_2 \subseteq \mathcal{E}^1(\sigma_1) \cap \mathcal{E}^2(\sigma_2),$$

where $A_1 = \{\alpha: 2N_\alpha(\sigma_3) < N_\alpha(\sigma_1) + N_\alpha(\sigma_2) - 1\}$, $A_2 = \{\alpha: 2N_\alpha(\sigma_3) < N_\alpha(\sigma_1) + N_\alpha(\sigma_2)\}$.

(CV-D) If $2N(\sigma_3) \geq N(\sigma_1) + N(\sigma_2)$, then

$$[\mathcal{E}^1(\sigma_1) \cup \mathcal{E}^2(\sigma_2)] \setminus A_1 \subseteq \mathcal{E}^3(\sigma_3),$$

$$[\mathcal{E}^1(\sigma_1) \cap \mathcal{E}^2(\sigma_2)] \setminus A_2 \subseteq \mathcal{E}^3(\sigma_3),$$

where $A_1 = \{\alpha: 2N_\alpha(\sigma_3) > N_\alpha(\sigma_1) + N_\alpha(\sigma_2)\}$ and $A_2 = \{\alpha: 2N_\alpha(\sigma_3) > N_\alpha(\sigma_1) + N_\alpha(\sigma_2) + 1\}$.

THEOREM 7.1. For a synchronized family of schemes $\{\mathcal{E}^b, b \in \mathcal{B}\}$ the following conditions are equivalent:

- (i) $\mathcal{E}^{b_1}, \mathcal{E}^{b_2}, \mathcal{E}^{b_3}$ satisfy (CX-D) ((CV-D)) whenever $2b_3 = b_1 + b_2$;
- (ii) χ is increasing and convex (concave).

Either of these implies

- (iii) D^b is increasing and convex (concave) in b .

PROOF. χ is increasing and convex if and only if whenever b_1, b_2, b_3 are as in (i) and $2N(\sigma_3) \leq N(\sigma_1) + N(\sigma_2), \sigma_i \in \mathcal{L}^{b_i}, 2\chi(b_3, \sigma_3) \leq \chi(b_1, \sigma_1) + \chi(b_2, \sigma_2)$; i.e., if

and only if, for every α ,

$$2N_\alpha(\sigma_3) + 2 \cdot \mathbf{1}\{\alpha \in \mathcal{E}^{b_3}(\sigma_3)\} \leq N_\alpha(\sigma_1) + \mathbf{1}\{\alpha \in \mathcal{E}^{b_1}(\sigma_1)\} \\ + N_\alpha(\sigma_2) + \mathbf{1}\{\alpha \in \mathcal{E}^{b_2}(\sigma_2)\},$$

which restates (CX-D). Thus (i) \Leftrightarrow (ii). Since the composition of increasing convex functions is increasing and convex, it follows from (23) that D^b is increasing and convex in b . Concavity works the same way. \square

Submodularity and supermodularity are not, in general, preserved under composition; hence, even if χ has one of these properties, D^b may not. We need to look, therefore, at the stronger properties of directional convexity and concavity. We use the following conditions:

(DX) If $N(\sigma_1) \leq N(\sigma_2)$, $N(\sigma_3) \leq N(\sigma_4)$, $N(\sigma_1) + N(\sigma_4) \geq N(\sigma_2) + N(\sigma_3)$, then

$$[\mathcal{E}^2(\sigma_2) \cup \mathcal{E}^3(\sigma_3)] \setminus A^{\cup} \subseteq \mathcal{E}^1(\sigma_1) \cup \mathcal{E}^4(\sigma_4),$$

$$[\mathcal{E}^2(\sigma_2) \cap \mathcal{E}^3(\sigma_3)] \setminus A^{\cap} \subseteq \mathcal{E}^1(\sigma_1) \cap \mathcal{E}^4(\sigma_4),$$

where $A^{\cup} = \{\alpha: N_\alpha(\sigma_2) + N_\alpha(\sigma_3) < N_\alpha(\sigma_1) + N_\alpha(\sigma_4)\}$ and $A^{\cap} = \{\alpha: N_\alpha(\sigma_2) + N_\alpha(\sigma_3) < N_\alpha(\sigma_1) + N_\alpha(\sigma_4) - 1\}$.

(DV) If $N(\sigma_1) \leq N(\sigma_2)$, $N(\sigma_3) \leq N(\sigma_4)$, $N(\sigma_1) + N(\sigma_4) \leq N(\sigma_2) + N(\sigma_3)$, then

$$[\mathcal{E}^1(\sigma_1) \cup \mathcal{E}^4(\sigma_4)] \setminus A^{\cup} \subseteq \mathcal{E}^2(\sigma_2) \cup \mathcal{E}^3(\sigma_3),$$

$$[\mathcal{E}^1(\sigma_1) \cap \mathcal{E}^4(\sigma_4)] \setminus A^{\cap} \subseteq \mathcal{E}^2(\sigma_2) \cap \mathcal{E}^3(\sigma_3),$$

where $A^{\cup} = \{\alpha: N_\alpha(\sigma_1) + N_\alpha(\sigma_4) < N_\alpha(\sigma_2) + N_\alpha(\sigma_3)\}$ and $A^{\cap} = \{\alpha: N_\alpha(\sigma_1) + N_\alpha(\sigma_4) < N_\alpha(\sigma_2) + N_\alpha(\sigma_3) - 1\}$.

THEOREM 7.2. *For a synchronized family of schemes $\{\mathcal{G}^b, b \in \mathcal{B}\}$ the following conditions are equivalent:*

(i) $\mathcal{G}^{b_1}, \mathcal{G}^{b_2}, \mathcal{G}^{b_3}, \mathcal{G}^{b_4}$ satisfy (DX) ((DV)) whenever $b_1 \leq b_2$, $b_3 \leq b_4$ and $b_1 + b_4 = b_2 + b_3$;

(ii) χ is increasing and directionally convex (concave).

Either of these implies

(iii) D^b is increasing and directionally convex (concave) in b .

PROOF. By simply writing out what it means for χ to be increasing and directionally convex (concave) it becomes obvious that (ii) is the same as (i), much as in the proof of Theorem 7.1. The composition of increasing directionally convex (or concave) functions is increasing and directionally convex (or concave), so D^b inherits the corresponding property from χ via (23). \square

COROLLARY 7.3. *If $\mathcal{G}^{b_1}, \mathcal{G}^{b_2}, \mathcal{G}^{b_3}, \mathcal{G}^{b_4}$ satisfy (DX) ((DV)) merely when $b_1 = b_2 \wedge b_3$ and $b_4 = b_2 \vee b_3$, then D^b is merely increasing and supermodular (submodular) in b .*

7.2. Examples. We now discuss some examples of second-order properties across synchronized schemes.

EXAMPLE 7.4. Consider queues in tandem as in Example 3.7; use the notation used there. Anantharam and Tsoucas [1] and Meester and Shanthikumar [12] show

that the counting processes of departures from each node are concave in the buffer sizes when service time at every node is exponential. We can reproduce this result, replacing exponential service times with the slightly more general synchronization mechanism (21). Both [1] and [12] point out that their methods extend to manufacturing blocking but consider only communication blocking in detail, so we consider manufacturing blocking. The parameter is $b = (b_2, \dots, b_k)$, where b_i is the buffer size at node i (we always take $b_1 \equiv \infty$). If the system starts empty, a score \mathbf{x} is feasible if $\mathbf{x}_{\beta_i} \leq \mathbf{x}_{\beta_{i-1}} \wedge (\mathbf{x}_{\beta_{i+1}} + b_{i+1})$, $i = 1, \dots, k$ (taking $b_{k+1} \equiv \infty$, $\mathbf{x}_{\beta_{k+1}} \equiv \infty$). Also, for feasible \mathbf{x} ,

$$\chi_{\beta_i}(b, \mathbf{x}) = \mathbf{x}_{\beta_i} + \mathbf{1}\{\mathbf{x}_{\beta_i} < \mathbf{x}_{\beta_{i-1}} \wedge (\mathbf{x}_{\beta_{i+1}} + b_{i+1})\}, \quad i = 1, \dots, k.$$

We show that χ is concave in feasible pairs (b, \mathbf{x}) . That χ is increasing is automatic.

Let $2b^3 = b^1 + b^2$ and $2\mathbf{x}^3 = \mathbf{x}^1 + \mathbf{x}^2$, each (b^i, \mathbf{x}^i) feasible. Concavity of χ would follow from

$$(24) \quad 2 \cdot \mathbf{1}\{\mathbf{x}_{\beta_i}^3 < \mathbf{x}_{\beta_{i-1}}^3 \wedge (\mathbf{x}_{\beta_{i+1}}^3 + b_{i+1}^3)\} \\ \geq \mathbf{1}\{\mathbf{x}_{\beta_i}^1 < \mathbf{x}_{\beta_{i-1}}^1 \wedge (\mathbf{x}_{\beta_{i+1}}^1 + b_{i+1}^1)\} + \mathbf{1}\{\mathbf{x}_{\beta_i}^2 < \mathbf{x}_{\beta_{i-1}}^2 \wedge (\mathbf{x}_{\beta_{i+1}}^2 + b_{i+1}^2)\}.$$

Suppose $\mathbf{x}_{\beta_i}^1 < \mathbf{x}_{\beta_{i-1}}^1 \wedge (\mathbf{x}_{\beta_{i+1}}^1 + b_{i+1}^1)$. Feasibility requires $\mathbf{x}_{\beta_i}^2 \leq \mathbf{x}_{\beta_{i-1}}^2 \wedge (\mathbf{x}_{\beta_{i+1}}^2 + b_{i+1}^2)$. Adding these inequalities, we get

$$2\mathbf{x}_{\beta_i}^3 = \mathbf{x}_{\beta_i}^1 + \mathbf{x}_{\beta_i}^2 < [\mathbf{x}_{\beta_{i-1}}^1 \wedge (\mathbf{x}_{\beta_{i+1}}^1 + b_{i+1}^1)] + [\mathbf{x}_{\beta_{i-1}}^2 \wedge (\mathbf{x}_{\beta_{i+1}}^2 + b_{i+1}^2)] \\ \leq (\mathbf{x}_{\beta_{i-1}}^1 + \mathbf{x}_{\beta_{i-1}}^2) \wedge (\mathbf{x}_{\beta_{i+1}}^1 + b_{i+1}^1 + \mathbf{x}_{\beta_{i+1}}^2 + b_{i+1}^2) \\ = 2\mathbf{x}_{\beta_{i-1}}^3 \wedge (2\mathbf{x}_{\beta_{i+1}}^3 + 2b_{i+1}^3).$$

In other words, whenever (b, \mathbf{x}) is feasible and the first indicator on the right side of (24) is unity, $\mathbf{x}_{\beta_i}^3 < \mathbf{x}_{\beta_{i-1}}^3 \wedge (\mathbf{x}_{\beta_{i+1}}^3 + b_{i+1}^3)$, so the indicator on the left is unity as well. Reversing the roles of \mathbf{x}^1 and \mathbf{x}^2 does not change the argument, so (24) holds.

A variant of this example allows different classes of jobs with the same serial route but different service requirements. Completion of service by a class c job at node i is event β_i^c . Here, too, χ is increasing and concave.

EXAMPLE 7.5. We can expand on the previous example by supposing that there is just one class of jobs, and that initially there are n_i jobs in queue i . In this case, the condition for feasibility becomes

$$\mathbf{x}_{\beta_i} \leq (\mathbf{x}_{\beta_{i-1}} + n_i) \wedge (\mathbf{x}_{\beta_{i+1}} + b_{i+1} - n_{i+1}),$$

and χ is given by

$$\chi_{\beta_i}(b, \mathbf{x}) = \mathbf{x}_{\beta_i} + \mathbf{1}\{\mathbf{x}_{\beta_i} < (\mathbf{x}_{\beta_{i-1}} + n_i) \wedge (\mathbf{x}_{\beta_{i+1}} + b_{i+1} - n_{i+1})\}.$$

Suppose we insist that $n_i > 0$ only if $b_i = \infty$. Then the $-n_{i+1}$ drops out in both expressions, and χ is increasing and concave in (b, n, \mathbf{x}) . In particular, with this restriction on n , the counting process $D^{b,n}$ is jointly increasing and concave in buffer capacity and initial state.

EXAMPLE 7.6. Consider, next, a closed cyclic network of k infinite capacity queues. Suppose node i starts with n_i jobs. A score is feasible if $\mathbf{x}_{\beta_i} \leq \mathbf{x}_{\beta_{i-1}} + n_i$, $i = 1, \dots, k$ (taking $\beta_0 \equiv \beta_k$, since the network is cyclic). For feasible \mathbf{x} , $\chi_{\beta_i}(\mathbf{x}) = \mathbf{x} + \mathbf{1}\{\mathbf{x}_{\beta_i} < \mathbf{x}_{\beta_{i-1}} + n_i\}$. This χ is increasing and concave on feasible pairs (n, \mathbf{x}) . It turns out to be essentially a special case of the χ in Example 7.5; the score of the additional event (external arrival) in that example is unconstrained because we required $b_1 = \infty$. A consequence of the concavity of χ is that the network throughput is (increasing and) concave in the number of jobs; this reproduces a result of Shanthikumar and Yao [18].

Suppose, instead, we allow some of the queues to be finite, while insisting that at least one be infinite. The feasibility condition becomes $\mathbf{x}_{\beta_i} \leq (\mathbf{x}_{\beta_{i-1}} + n_i) \wedge (\mathbf{x}_{\beta_{i+1}} + b_{i+1} - n_{i+1})$ (taking $\beta_{k+1} \equiv \beta_1$), and χ is modified accordingly. If $n_i > 0$ only when $b_i = \infty$, then χ is concave in (n, b) . (Since the jobs must start *somewhere*, the condition $n_i > 0 \Rightarrow b_i = \infty$ implies that at least one node has sufficient capacity for all jobs. This is the condition used in Shanthikumar and Yao [20] for concavity of D in network population with b fixed.)

The *joint* concavity in Examples 7.5 and 7.6 is new, so we state it as

PROPOSITION 7.7. *For synchronized tandem or cyclic networks of queues (as in Examples 7.5–7.6) with b_i buffer spaces at node i , n_i jobs initially at i , and the condition $n_i > 0 \Rightarrow b_i = \infty$, $D^{b,n}$ is increasing and jointly concave in b and n .*

EXAMPLE 7.8. Finally, consider the simple model of component failure and repair of Example 3.9. The feasibility condition is $0 \leq \mathbf{x}_{\alpha^i} - \mathbf{x}_{\beta^i} \leq k_i$, $i = 1, \dots, l$, where α^i denotes failure of a type- i component, β^i its repair, k_i the (initial) number of spare type- i components, and l the number of component types. Also, $\chi_{\alpha^i}(\mathbf{x}) = \mathbf{x}_{\alpha^i} + \mathbf{1}\{\mathbf{x}_{\alpha^i} - \mathbf{x}_{\beta^i} < k_i\}$ and $\chi_{\beta^i}(\mathbf{x}) = \mathbf{x}_{\beta^i} + \mathbf{1}\{0 < \mathbf{x}_{\alpha^i} - \mathbf{x}_{\beta^i}\}$. The argument of Example 7.4 shows that χ is increasing and concave in the vector $k = (k_i)_{i=1}^l$, so D^k is increasing and concave in k , too.

8. Concluding remarks. We have focused exclusively on generalized semi-Markov schemes with deterministic routing, in which state transitions are completely governed by the mapping ϕ . It is possible to develop analogs of our conditions and results for schemes with *state-independent* routing as defined in Glasserman and Yao [8]. Let $p(s'; s, \alpha)$ be the probability that the system moves to state s' from s upon the occurrence of α , if $\alpha \in \mathcal{E}(s)$. State-independent routing requires that, for each α , all the possible transitions out of different states containing α in their event lists be in one-to-one correspondence, with corresponding transitions having equal probabilities $p(\cdot; \cdot, \alpha)$. This condition allows us (in [8]) to enlarge Ω to include a sequence $\pi = \{\pi_\alpha(n), \alpha \in \mathbf{A}, n = 1, 2, \dots\}$, such that $\pi_\alpha(n)$ determines the state transition upon the n th occurrence of α . We modify ϕ to ϕ_π , a function which uses π to select the next state, given the current state and event. For each π , the system evolves deterministically; putting a measure on the space of π 's causes ϕ_π to mimic the effect of the transition probabilities $p(\cdot; \cdot, \cdot)$.

Given $(\mathbf{S}, \mathbf{A}, \mathcal{E}, s_0)$, our conditions on schemes become conditions on ϕ . As shown in [8], a condition on deterministic schemes extends naturally to a condition on schemes with state-independent routing by requiring that the condition hold for ϕ_π for every π . For example, to extend (CX) we would strengthen (9) to

$$N(\sigma_3) = N(\sigma_1) \wedge N(\sigma_2) \Rightarrow \mathcal{E}(\phi_\pi(s_0, \sigma_1)) \cap \mathcal{E}(\phi_\pi(s_0, \sigma_2)) \subseteq \mathcal{E}(\phi_\pi(s_0, \sigma_3)),$$

for all π , i.e., for all outcomes of the routing decisions. In this setting, the indices appearing in the min-max, pure-max, and pure-min representations of T also depend on π . Our examples based on queues in tandem extend thus to tree-like networks in which departure streams are subject to Bernoulli splitting. Another direction for generalization makes comparison based on clock *speeds*, as in [8]; however, for second-order properties, the extension to speeds is by no means routine.

In studying the connection between a “chip-firing” game on a graph and antimatroids with repetition, Shor, Björner, and Lovász [22] note that the game they consider can be viewed as a Petri net, and that their results should be extensible to that setting. Our conditions and results could be adapted to *stochastic* Petri nets, like those in Haas and Shedler [9]. Indeed, since Petri nets focus directly on events with almost no reference to states, they provide a natural setting for the kind of comparisons we make here and in [8].

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