Based on Theorem 1, a design procedure of the decentralized robust control scheme is given as follows.

Step 1) Verify Assumptions 1 and 2.

Step 2) For a given degree of exponential convergence α, choose numbers α_i, ρ_i and β_i, and solve the matrix Riccati equations (3.1).

Step 3) According to (3.4) and (3.5), choose the numbers γ_i and θ_i, and check the condition (3.6) with different sets of the positive numbers ω_{ij}.

Step 4) If in Steps 2 and 3 a set of positive numbers $\alpha_i, \rho_i, \beta_i, \gamma_i, \theta_i$ and ω_{ij} is found such that (3.6) is satisfied, go to next step. Otherwise, repeat Steps 2 and 3 with different set of numbers $\alpha_i, \rho_i, \beta_i, \gamma_i, \theta_i$ and ω_{ij}. If no set of the numbers $\alpha_i, \rho_i, \beta_i, \gamma_i, \theta_i$ and ω_{ij} can be found such that (3.6) is satisfied, this decentralized control scheme is not applicable to the system, and this system is probably not decentralized stabilizable.

Step 5) Use (3.8) to estimate the final bound r of the closed-loop system and construct the decentralized feedback controller (3.3).

It should be noted that in the computing of r the largest numbers ε_i and δ_i which satisfy (3.4) and (3.6) should be used to estimate the smallest final bound of the closed-loop system. Also, the design procedure given above may be repeated with the numbers $\alpha_i, \rho_i, \beta_i, \gamma_i, \theta_i$ and ω_{ij} adjusted to obtain a better control law and a better estimation of the final bound.

IV. Conclusion

In this note, a robust decentralized control scheme for uncertain time-varying interconnected systems is proposed. General uncertainties are considered which do not satisfy the matching conditions and may appear in the subsystems and in the interconnections between the subsystems as well. The proposed control scheme is simple in the structure and yet it guarantees the controlled systems to have a prescribed degree of exponential convergence and a predictable final bound, when the uncertainties satisfy a certain bound condition.

References

[1] D. D. Siljak, *Large-Scale Systems: Stability and Structure.* Amsterdam: North-Holland, 1978.

[2] M. Jamshidi, *Large-Scale Systems: Modeling and Control.* Amsterdam: North-Holland, 1983.

[3] M. K. Sundareshan, "Exponential stabilization of large-scale systems: Decentralized and multilevel schemes," *IEEE Trans. Syst., Man, Cybernetics*, vol. SMC-7, pp. 478–483, 1977.

[4] M. Ikeda, O. Umefuji, and S. Kodama, "Stabilization of large-scale linear systems," *Syst., Computers Contr.*, vol. 7, pp. 34–41, 1976.

[5] M. Ikeda and D. D. Siljak, "Decentralized stabilization of linear time-varying systems," *IEEE Trans. Automat. Contr.*, vol. AC-25, no. 1, pp. 106–107, 1980.

[6] O. Huseyin, M. E. Sezer, and D. D. Siljak, "Robust decentralized control using output feedback," *IEE Proc.* vol. 129, Pt. D, no. 6, pp. 310–314, 1982.

[7] M. Ikeda and D. D. Siljak, "On optimality and robustness of LQ regulators for nonlinear and interconnected systems," in *Proc. IFAC Workshop on Model Error Concepts and Compensation*, 1985, pp. 77–82.

[8] Y. H. Chen, "Decentralized robust control for large-scale uncertain systems: a design based on the bound of uncertainty," *J. Dynamic Systems, Measurement, Contr.*, vol. 114, pp. 1–9, 1992.

[9] L. Shi and S. K. Singh, "Decentralized control for interconnected uncertain systems: Extensions to higher-order uncertainties," *Int. J. Contr.*, vol. 57, pp. 1453–1468, 1993.

[10] Y. H. Chen, G. Leitmann, and X. Zhong Kai, "Robust control design for interconnected systems with time-varying uncertainties," *Int. J. Contr.*, vol. 54, no. 5, pp. 1119–1142, 1991.

[11] S. Gutman, "Uncertain dynamical systems—Lyapunov min-max approach," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 437–443, 1979.

[12] M. J. Corless and G. Leitmann, "Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 1139–1144, 1981.

[13] M. L. Ni and H. X. Wu, "A Riccati equation approach to the design of linear robust controllers," *Automatica*, vol. 29, pp. 1603–1605, 1993.

[14] H. Wu and K. Mizukami, "Exponential stability of a class of nonlinear dynamical systems with uncertainties," *Syst. Contr. Lett.*, vol. 21, pp. 307–313, 1993.

[15] B. R. Barmish and G. Leitmann, "On ultimate boundedness control of uncertain systems in the absence of matching assumptions," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 153–158, 1982.

[16] P. P. Khargonekar, I. R. Petersen, and K. Zhou, "Robust stabilization of uncertain linear systems: quadratic stabilizability and H^∞ control theory," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 356–361, 1990.

[17] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods.* Englewood Cliffs, NJ: Prentice-Hall, 1989.

[18] M. Ikeda, H. Maeda and S. Kodama, "Stabilization of linear systems," *SIAM J. Contr.*, vol. 10, pp. 716–729, 1972.

[19] S. H. Wang and E. J. Davison, "On the stabilization of decentralized control systems," *IEEE Trans. Automat. Contr.*, vol. AC-18, no. 5, pp. 473–478, 1973.

[20] M. E. Sezer and O. Huseyin, "Comments on decentralized state feedback stabilization," *IEEE Trans. Automat. Contr.*, vol. AC-26, no. 2, pp. 547–548, 1981.

[21] Z. Gong and M. Aldeen, "Stabilization under decentralized information structure," in *Proc. Amer. Contr. Conf.*, 1991, pp. 922–927.

Hedging-Point Production Control with Multiple Failure Modes

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***Abstract*—We consider the control of a production facility subject to multiple failure modes. Motivated by work of Akella and Kumar [1] and Bielecki and Kumar [5] on single-failure-mode models, we study hedging-point policies, in which production is controlled to its maximum rate whenever inventory is below a critical level and set to zero whenever inventory is above that level. The maximum production rate varies with the state of the machine. Assuming that the machine state is governed by a semi-Markov process, we evaluate average and discounted inventory costs for any hedging point, thus providing a simple mechanism for identifying optimal hedging points. Our most explicit results require that intervals in which demand exceeds production are exponentially distributed. We drop the exponential assumption at the expense of obtaining asymptotics rather than exact results.**

I. Introduction

We consider the control of a production facility subject to various types of failures. The facility, or machine, can be in any of n states, labeled $1, \cdots, n$. In state i, the machine has maximum production rate $\bar{r}_i$; the production rate can be controlled to any level not exceeding the maximum rate in the current state. The durations of visits to each state are stochastic and mutually independent; those

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to state i have distribution F_i. Upon the completion of a holding time in state i, the machine state becomes j with probability R_{ij} independent of everything else. Demands arrive at a constant rate d. Production in excess of demand accumulates as inventory; when the demand rate exceeds the production rate, the inventory level decreases. Negative inventories reflect unfilled demands. Broadly, the objective is to minimize costs associated with (positive) inventory and unmet demands.

Motivated by work of Akella and Kumar [1], Bielecki and Kumar [5], and Kimemia and Gershwin [10], we analyze the performance of a simple class of control rules based on a critical number z, sometimes called a hedging point: when the inventory level is below z, produce at the maximum possible rate; when the inventory level is at z, produce at the maximum possible rate or at the demand rate, whichever is smaller; when the inventory level is above z, do not produce. Our main result (Theorem 1, below) characterizes average and discounted costs associated with this class of policies under relatively mild assumptions.

To state the result, we need to introduce some further notation. Let $r_i = d - \tilde{r}_i$ be the net rate of decrease of inventory in state i, i.e., the demand rate minus the maximum production rate in that state. To rule out trivial cases, we always assume that there is at least one strictly positive r_i. We assume throughout that the matrix R of machine-state transitions is irreducible; this implies the existence of limiting probabilities $(\pi_1, \cdots, \pi_n)$ for the machine states. For each $\gamma \geq 0$, define an $n \times n$ matrix-valued function $\Phi^\gamma(\cdot)$ with entries

$$\Phi_{ij}^\gamma(\theta) = R_{ij} \int_0^\infty \exp[(r_i\theta - \gamma)x]\, dF_i(x). \quad (1)$$

Each $\Phi^\gamma(\theta)$ has positive entries; when these are finite, the Perron–Frobenius Theorem (see, e.g., [15]) asserts that $\Phi^\gamma(\theta)$ has a real eigenvalue $\rho_\gamma(\theta)$ equal to its spectral radius. Denote by λ_γ the strictly positive solution (whenever it exists) to

$$\rho_\gamma(\lambda_\gamma) = 1, \qquad \rho_\gamma'(\lambda_\gamma) < \infty. \quad (2)$$

Suppose that cost accrues at rate $f(x)$ whenever the inventory level is x, with f mapping $(-\infty, \infty)$ to $[0, \infty)$. Denote by X_t^z the inventory level at time t under policy parameter z, and suppose $X_0^z = z$. Let

$$V(z) = \lim_{t\to\infty} t^{-1} \int_0^t f(X_t^z)\, dt$$

be the average cost under this policy, and let

$$V_i^\gamma(z) = \mathrm{E}_i \int_0^\infty e^{-\gamma t} f(X_t^z)\, dt$$

be the corresponding expected γ-discounted cost starting in machine-state i. Part of the content of our main result is that $V(z)$ is well defined.

Call state i a deficit state if $r_i > 0$. We characterize V and V_γ under the assumption that holding times in deficit states are exponentially distributed. Later, we drop the exponential requirement at the expense of obtaining asymptotics and bounds, rather than exact results.

Theorem 1: Suppose that F_i is an exponential distribution for all i with $r_i > 0$.

i) If $\sum_i \pi_i r_i < 0$ and if $\lambda_0 > 0$ solves (2) at $\gamma = 0$, then there is a $C_0 \in (0, 1)$ such that

$$V(z) = (1 - C_0) f(z) + C_0 \int_0^\infty f(z - t) \lambda_0 e^{-\lambda_0 t}\, dt. \quad (3)$$

ii) For any $\gamma > 0$, if $\lambda_\gamma > 0$ solves (2), then there are constants $C_{\gamma,i}$, $i = 1, \cdots, n$, such that

$$\gamma V_i^\gamma(z) = (1 - C_{\gamma,i}) f(z) + C_{\gamma,i} \int_0^\infty f(z - t) \lambda_\gamma e^{-\lambda_\gamma t}\, dt. \quad (4)$$

In proving this result, we give expressions for the constants C_0 and $C_{\gamma,i}$; these are most easily evaluated when there is a unique deficit state. We also argue that the existence of $\lambda_\gamma > 0$ in (2) should be considered typical.

A consequence of Theorem 1 is that optimal hedging points can be identified by minimizing the expressions in (3) and (4). In a case of particular interest, costs for inventory and unfilled demand are linear; i.e., there are constants $c_+, c_- \geq 0$ for which

$$f(x) = c_+ \max\{x, 0\} + c_- \max\{-x, 0\}. \quad (5)$$

Write $\log_+ x$ for $\max\{0, \log x\}$. Using (5) in (3) and (4) and minimizing over z gives the following corollary.

Corollary 1: Suppose the assumptions of Theorem 1 hold and that f has the form in (5), then

i) under the average-cost criterion, the optimal hedging point is

$$z^* = \frac{1}{\lambda_0} \log_+ \left(\frac{C_0(c_+ + c_-)}{c_+} \right) \quad (6)$$

ii) under the γ-discounted criterion, the optimal hedging point from initial machine state i is

$$z_{\gamma,i}^* = \frac{1}{\lambda_\gamma} \log_+ \left(\frac{C_{\gamma,i}(c_+ + c_-)}{c_+} \right). \quad (7)$$

Akella and Kumar [1] and Bielecki and Kumar [5] consider a two-state version of this model with exponentially distributed holding times and the linear cost function in (5). In that setting, they prove that the class of policies considered here includes the optimal policy under discounted and average cost criteria, respectively. They also present counterparts of (6) and (7). We do not prove (or even suggest) that these policies remain optimal in our more general setting; however, their simplicity and relative tractability makes them appealing.

The key to our analysis is a link between the inventory process and a time-reversed, continuous-time random walk in a semi-Markov environment. Through this connection, we are able to draw on results for random walks, in particular the work of Asmussen [4]. For two-state models, there is an equivalence with queues, as pointed out by Hu and Xiang [8]; see Chen and Yao [6] and Kella and Whitt [9] for related observations. None of these related references considers discounted quantities. The matrix $\Phi_0(\theta)$ and its leading eigenvalue $\rho_0(\theta)$ play an important, related role in the large deviations theory for Markov additive processes, as developed by Ney and Nummelin [14].

We prove Theorem 1 in Sections II and III. Section IV presents asymptotically optimal hedging points when holding times in deficit states have general distributions.

II. Average-Cost Analysis

Let us take the hedging point z to be fixed and write $X_t \equiv X_t^z$ for the inventory level at time t, with $X_0 = z$. Let $\tilde{J}_t$ denote the state of the machine at time t. Our analysis of the inventory level is simplified if we work with the process $Y_t = z - X_t$, $t \geq 0$, recording the deficit in the inventory level. The process Y has the advantage that its law does not depend on z: during intervals in which $\tilde{J}_t = i$, Y_t increases at rate r_i, unless $Y_t = 0$ and $r_i \leq 0$, in which case Y_t remains at zero. This description reveals that Y is the image under reflection at the origin of a free process evolving according to the same rule but without the constraint at zero. More precisely, let $\tilde{S}_0 = 0$ and let $\tilde{S}_t$ increase at rate r_i whenever $\tilde{J}_t = i$. Then

$$Y_t = \sup_{0 \leq u \leq t} (\tilde{S}_t - \tilde{S}_u). \quad (8)$$

(This type of representation is standard in queueing theory.) From this formulation, we will evaluate $V(z)$ by evaluating the expectation of $f(z - Y_t)$ with respect to the stationary distribution of Y_t.

Now let (S_t, J_t) have the law of the time-reversal of $(\tilde{S}_t, \tilde{J}_t)$, defined as follows. The process J is a semi-Markov process on $\{1, \cdots, n\}$, with the same holding-time distributions $F_1, \cdots, F_n$ as $\tilde{J}$, and with embedded transition probabilites $R^*_{ij} = \nu_j R_{ji}/\nu_i$, where $(\nu_1, \cdots, \nu_n)$ are the stationary probabilities associated with the matrix R; i.e., $\nu R = \nu$. (See, e.g., [3, Section II.5] for background on time-reversal.) In particular, then J has the same stationary distribution as $\tilde{J}$, given by $\pi_i = \nu_i m_i/(\sum_j \nu_j m_j)$, $i = 1, \cdots, n$, with m_i the mean of F_i [7, p. 342]. The process S_t starts at zero and increases at rate r_i throughout intervals in which $J_t = i$. We take J to be right continuous.

If we give $\tilde{J}_0$ the stationary distribution π, then, for each $t > 0$, we obtain a stationary version of $\{J_u, 0 \le u \le t\}$ by setting $J_u = \tilde{J}_{t-u}, 0 \le u \le t$. Similarly, we can couple S to $\tilde{S}$ by setting $S_u = \tilde{S}_t - \tilde{S}_{t-u}$, $0 \le u \le t$. With this construction, from (8) we get

$$Y_t = \sup_{0 \le u \le t} (\tilde{S}_t - \tilde{S}_u) = \sup_{0 \le u \le t} (\tilde{S}_t - \tilde{S}_{t-u}) = \sup_{0 \le u \le t} S_u.$$

Therefore, for any $x > 0$ and any $i, j \in \{0, 1, \cdots, n\}$, the events $\{Y_t \le x, \tilde{J}_0 = i, \tilde{J}_t = j\}$ and $\{\sup_{0 \le u \le t} S_u \le x, J_0 = j, J_t = i\}$ coincide, and so have the same probability. Defining $M_t = \sup_{0 \le u \le t} S_u$ and using the stationarity of J and $\tilde{J}$, we conclude that

$$\pi_i P_i(Y_t \le x, \tilde{J}_t = j) = \pi_j P_j(M_t \le x, J_t = i) \quad (9)$$

where the subscripts on P indicate the initial machine state ($\tilde{J}_0$ on the left, J_0 on the right). By summing over i, we get

$$\sum_i \pi_i P_i(Y_t \le x, \tilde{J}_t = j) = \pi_j P_j(M_t \le x). \quad (10)$$

Next, we consider the limit as t increases. Because M_t is almost surely increasing in t, the limit $M \triangleq \lim_{t\to\infty} M_t$ exists with probability one; moreover, under the drift condition $\sum_i \pi_i r_i < 0$ in Theorem 1-i), M is almost surely finite [2, p. 309]. Also, since $(Y_t, \tilde{J}_t)$ is evidently regenerative, it has a limiting distribution not depending on the initial state; let $(Y_\infty, \tilde{J}_\infty)$ have that limiting distribution. Then from (10) we get

$$\begin{aligned} P(Y_\infty \le x, \tilde{J}_\infty = j) &= \lim_{t\to\infty} \sum_i \pi_i P_i(Y_t \le x, \tilde{J}_t = j) \\ &= \lim_{t\to\infty} \pi_j P_j(M_t \le x) \\ &= \pi_j P_j(M \le x). \end{aligned} \quad (11)$$

Summing over j, we have proved the following (compare [3, Proposition 2.2]).

Lemma 1: Under the assumptions in Theorem 1–i), $P(Y_\infty \le x) = \sum_j \pi_j P_j(M \le x)$. Consequently, $V(z) = \mathrm{E}[f(z - Y_\infty)] = \sum_j \pi_j \mathrm{E}_j[f(z - M)]$.

In light of Lemma 1, to prove Theorem 1–i) it suffices to evaluate the distributions $P_j(M \le \cdot)$, $j = 1, \cdots, n$ and to show that the expression in the lemma coincides with (3). The properties we need can in principle be obtained from the analytical results of [2]. Instead, we use a change-of-measure argument based on similar techniques in [4], [13], [17]. We work with the discrete-time process $\{(S_{\tau_n}, J_{\tau_n}), n = 0, 1, \cdots\}$, where τ_n is the epoch of the nth jump of J.

For fixed $\gamma \ge 0$, let λ_γ be as in (2). We need two preliminary results. They are easily verified by induction and similar to results in [4] so we omit their proofs.

Lemma 2: The quantity $\mathrm{E}_j[\exp(\lambda_\gamma S_{\tau_n} - \gamma\tau_n); J_{\tau_n} = i]$ is the ji-entry of $(\Phi^\gamma(\lambda_\gamma))^n$.

By the Perron–Frobenius theorem, $\Phi^\gamma(\lambda_\gamma)$ has a strictly positive right-eigenvector h (depending on γ) associated with the maximal eigenvalue $\rho_\gamma(\lambda_\gamma) \equiv 1$; i.e., $\Phi^\gamma(\lambda_\gamma)h = h$. We have the following lemma.

Lemma 3: $\mathrm{E}_j[\exp(\lambda_\gamma S_{\tau_n} - \gamma\tau_n)h(J_{\tau_n})/h(j)] = 1$, for all j and n.

A consequence of Lemma 3 is that, with $J_0 = j$, $\exp(\lambda_\gamma S_{\tau_n} - \gamma\tau_n)h(J_{\tau_n})/h(j)$ defines a change of measure. More explicitly, with

$$\bar{F}_i(x) = \frac{Rh(i)}{h(i)} \int_0^x e^{(\lambda_\gamma r_i - \gamma)u} F_i(du) \quad (12)$$

and $Rh(j) = \sum_j R_{ij}h(j)$, $\bar{F}_i$ is a probability distribution function. Let $\bar{P}$ and $\bar{\mathrm{E}}$ denote probability and expectation, respectively, when the holding times in state i have distribution $\bar{F}_i$, $i = 1, \cdots, n$. These are related to the original process through a version of Wald's identity.

Lemma 4: For any $\gamma \ge 0$, suppose there is a $\lambda_\gamma > 0$ solving $\rho_\gamma(\lambda_\gamma) = 1$. If N is a stopping time for the process $\{(S_{\tau_n}, J_{\tau_n}), n \ge 0\}$ and if the event A is measurable with respect to $\{(S_{\tau_n}, J_{\tau_n}), 0 \le n \le N\}$, then $P_i(A; N < \infty) = \bar{\mathrm{E}}_i[\exp\{-(\lambda_\gamma S_{\tau_N} - \gamma\tau_N)\}h(i)/h(J_{\tau_N}); N < \infty]$.

Similar results are proved in [4,] [13], and [17]. The proof is standard, so we omit it.

Under the new measure, S has positive drift.

Lemma 5: When the holding times have distributions $\bar{F}_i$, $i = 1, \cdots, n$, $\lim_{t\to\infty} t^{-1}S_t > 0$ (and is independent of the initial state).

Proof: By Lemma 5.3 of [14], $\lim_{t\to\infty} t^{-1}S_t = \rho'_\gamma(\lambda_\gamma)$. From [15, Theorem 3.7], we know that $\rho_\gamma(\cdot)$ is convex. We now separate the cases $\gamma = 0$ and $\gamma > 0$. For the former, we find that $\Phi^0(0) = R$, a stochastic matrix, so $\rho_0(0) = 1$. Moreover, $\rho'_0(0)$ is the drift of S under the original measure and is therefore negative, by hypothesis. Thus, the convexity of ρ_0 implies that if $\lambda_0 > 0$ exists, then ρ_0 is increasing at λ_0. In the case $\gamma > 0$, we see that $\Phi^\gamma(0)$ is strictly substochastic and therefore $\rho_\gamma(0) < 1$. Again, convexity implies that ρ_γ must be increasing at λ_γ. □

Remark: The convexity of the functions $\rho_\gamma(\cdot)$, $\gamma \ge 0$, indicates that the existence of λ_γ solving (2) is typical, at least if the deficit-state holding-time distributions have exponential tails. Indeed, the only alternative then to $\rho_\gamma(\lambda_\gamma) = 1$ is the existence of a $\theta > 0$ for which $\rho_\gamma(\theta-) < 1$ and $\rho_\gamma(\theta+) = \infty$. Such cases must be considered exceptional; see [2] for a treatment of this case. The second part of (2) makes the new drift finite.

We can now evaluate the distribution of M from any initial state. Let $N_x = \inf\{n \ge 0 : S_{\tau_n} > x\}$ be the index of the first jump epoch at which S exceeds x. (In a simplifying abuse of notation, we write S_{N_x} and J_{N_x} for $S_{\tau_{N_x}}$ and $J_{\tau_{N_x}}$, and we write J_{N_x-} for the state of the machine when S first exceeds x.) Under the original (negative-drift) measure, N_x may be infinite, but under the new (positive-drift) measure, N_x is almost surely finite for all $x \ge 0$. By Lemma 4

$$\begin{aligned} P_j(M > x) &\equiv P_j(N_x < \infty) \\ &= \bar{\mathrm{E}}_j[\exp(-\lambda_0 S_{N_x})h(j)/h(J_{N_x})] \\ &= e^{-\lambda_0 x}h(j)\bar{\mathrm{E}}_j[\exp(-\lambda_0[S_{N_x} - x])/h(J_{N_x})]. \end{aligned}$$

To evaluate the expectation on the right, we condition on J_{N_x-}, which is necessarily a deficit state. Given $J_{N_x-} = k$, the overshoot $S_{N_x} - x$ and the next state J_{N_x} are independent. Moreover, F_k exponential implies $\bar{F}_k$ exponential, so the overshoot is exponentially distributed with mean $r_k\bar{m}_k$, where

$$\bar{m}_k = \frac{Rh(k)}{h(k)} \int_0^\infty x e^{\lambda_0 x} F_k(dx) = \frac{m_k}{1 - \lambda_0 r_k m_k} \quad (13)$$

is the mean of $\bar{F}_k$. With $J_0 = k$, S_{τ_1} also has the exponential distribution $\bar{F}_k$. Thus, we have shown that

$$\begin{aligned} &P_j(M > x) \\ &= h(j)\sum_k \bar{P}_j(J_{N_x-} = k)\bar{\mathrm{E}}_k[\exp(-\lambda_0 S_{\tau_1})]\bar{\mathrm{E}}_k[1/h(J_{\tau_1})]e^{-\lambda_0 x} \\ &= h(j)\sum_k \bar{P}_j(J_{N_x-} = k)\frac{1}{1 + \lambda_0 r_k \bar{m}_k} Rh^{-1}(k)e^{-\lambda_0 x}. \end{aligned} \quad (14)$$

From (13) we find that $1/(1+\lambda_0 r_k \bar{m}_k) = 1 - \lambda_0 r_k m_k$. So, setting

$$C_0 = \sum_j \pi_j h(j) \sum_k \bar{P}_j(J_{N_x-} = k)(1 - \lambda_0 r_k m_k) R h^{-1}(k)$$

we find from (11) and (14) that $P(Y_\infty > x) = C_0 e^{-\lambda_0 x}$, for all $x \geq 0$. Since $V(z) = \mathrm{E}[f(z - Y_\infty)]$, this completes the proof of part (i) of Theorem 1.

The only unknown terms in our expression for C_0 are the probabilities $\bar{P}_j(J_{N_x-} = k)$. If there is just one deficit state—call it state 1—then

$$C_0 = \sum_j \pi_j h(j)(1 - \lambda_0 r_1 m_1) R h^{-1}(1). \tag{15}$$

Each of the terms appearing in this expression is easily computed through matrix calculations if the number of machine states is not too large—not more than 30, say. Choosing h so that $\pi h = 1$ eliminates the first factor on the right in (15).

III. Discounted-Cost Analysis

We now turn to the evaluation of $V_i^\gamma(z), \gamma > 0$. As in the previous section, we take z to be fixed and suppress it as an argument. Let L be an exponentially distributed random variable with mean $1/\gamma$, independent of everything else. Then

$$\gamma V_i^\gamma = \gamma \mathrm{E}_i \int_0^\infty e^{-\gamma t} f(X_t)\, dt = \mathrm{E}_i[f(X_L)].$$

Thus, to evaluate V_i^γ, it suffices to find the distribution of the inventory level at the random time L. As in the previous section, we work with the process $Y_t = z - X_t$ rather than the inventory level itself, and now seek to evaluate

$$\gamma V_i^\gamma = \mathrm{E}_i[f(z - Y_L)]. \tag{16}$$

From (9) we get

$$\begin{aligned} P_i(Y_L > x) &= \frac{1}{\pi_i} \sum_j \pi_j P_j(M_L > x, J_L = i) \\ &= \frac{1}{\pi_i} \sum_j \pi_j P_j(T_x < L, J_L = i) \end{aligned} \tag{17}$$

where $T_x = \inf\{t \geq 0 : S_t \geq x\}$ is the first time S reaches level x. Letting $L' = L - T_x$, we get

$$\begin{aligned} P_j(T_x < L, J_L = i) &= P_j(T_x < L) P_j(J_L = i \mid T_x < L) \\ &= \mathrm{E}_j[e^{-\gamma T_x}] P_j(J_{T_x + L'} = i \mid T_x < L). \end{aligned}$$

Given $\{T_x < L\}$, L' is exponentially distributed with mean $1/\gamma$. Thus, we get

$$\begin{aligned} P_j(T_x < L, J_L = i) = \mathrm{E}_j[e^{-\gamma T_x}] &\sum_k P_j(J_{T_x} = k \mid T_x < L) \\ &\times P_k(J_L = i). \end{aligned}$$

We conclude from (17) that

$$\begin{aligned} P_i(Y_L > x) = &\frac{1}{\pi_i} \sum_j \pi_j \mathrm{E}_j[e^{-\gamma T_x}] \\ &\times \sum_k P_j(J_{T_x} = k \mid T_x < L) P_k(J_L = i). \end{aligned} \tag{18}$$

We evaluate $\mathrm{E}_j[e^{-\gamma T_x}]$ by adapting a change-of-measure argument due to Glynn (personal communication) and Kollman [11]. Let $\gamma > 0$ be fixed and let h denote a strictly positive Perron-Frobenius right-eigenvector for $\Phi^\gamma(\lambda_\gamma)$, so that $\Phi^\gamma(\lambda_\gamma)h = h$. Via Lemma 4, define new holding-time distributions as specified by (12). Throughout this section $\bar{P}$ and $\bar{\mathrm{E}}$ refer to probability and expectation based on these distributions. Let $N_x = \inf\{n \geq 0 : S_{\tau_n} > x\}$, as before.

Lemma 6: $\bar{\mathrm{E}}_j[e^{-\gamma T_x}] = b_j e^{-\lambda_\gamma x}$, where

$$b_j = h(j) \sum_k \bar{P}_j(J_{N_x-} = k)(1 + (\gamma - r_k \lambda_\gamma) m_k) R h^{-1}(k).$$

Proof: Because $T_x \leq \tau_{N_x}$, a.s., and because the evolution of S_t is deterministic between jumps, T_x is measurable with respect to $\{(S_{\tau_n}, J_{\tau_n}),\ 0 \leq n \leq \tau_{N_x}\}$; so, we may evaluate the expectation of $\exp(-\gamma T_x)$ by applying a measure transformation to $\{(S_{\tau_n}, J_{\tau_n}), 0 \leq n \leq N_x\}$ as follows

$$\begin{aligned} \mathrm{E}_j[e^{-\gamma T_x}] &= \bar{\mathrm{E}}_j[e^{-\gamma T_x} e^{\gamma \tau_{N_x} - \lambda_\gamma S_{N_x}} h(j)/h(J_{N_x})] \\ &= e^{-\lambda_\gamma x} h(j) \bar{\mathrm{E}}_j[e^{\gamma(\tau_{N_x} - T_x)} e^{-\lambda_\gamma(S_{N_x} - x)}/h(J_{N_x})] \\ &= e^{-\lambda_\gamma x} h(j) \bar{\mathrm{E}}_j[e^{(\gamma - \lambda_\gamma r_{N_x-})(\tau_{N_x} - T_x)}/h(J_{N_x})] \end{aligned} \tag{19}$$

where we have written r_{N_x-} for $r_{J_{N_x-}}$ (the prevailing net rate when S first crosses x) and used the fact that $(S_{N_x} - x) = r_{N_x-}(\tau_{N_x} - T_x)$, a.s. Now we condition on J_{N_x-}, which is necessarily a deficit state. The corresponding holding-time distribution is therefore exponential; and given J_{N_x-}, the next state J_{N_x} is independent of $\tau_{N_x} - T_x$. Thus

$$\begin{aligned} &\mathrm{E}_j[e^{-\gamma T_x}] \\ &\quad = e^{-\lambda_\gamma x} h(j) \sum_k \bar{P}_j(J_{N_x-} = k) \bar{\mathrm{E}}_k[e^{(\gamma - r_k \lambda_\gamma)\tau_1}] \\ &\qquad \cdot \bar{\mathrm{E}}_k[1/h(J_{\tau_1})] \\ &\quad = e^{-\lambda_\gamma x} h(j) \sum_k \bar{P}_j(J_{N_x-} = k) \\ &\qquad \times \frac{R h^{-1}(k)}{1 - (\gamma - r_k \lambda_\gamma)\bar{m}_k}, \\ &\quad = e^{-\lambda_\gamma x} h(j) \sum_k \bar{P}_j(J_{N_x-} = k) \\ &\qquad \times (1 + (\gamma - r_k \lambda_\gamma) m_k) R h^{-1}(k), \\ &\quad = b_j e^{-\lambda_\gamma x} \end{aligned}$$

where we have used

$$\begin{aligned} \bar{m}_k &= \frac{Rh(k)}{h(k)} \int_0^\infty x e^{(-\gamma + r_k \lambda_\gamma)x} F_k(dx) \\ &= \frac{m_k}{1 + (\gamma - \lambda_\gamma r_k) m_k}. \end{aligned}$$

□

If we set

$$C_{\gamma,i} = \frac{1}{\pi_i} \sum_j \pi_j b_j \sum_k P_j(J_{T_x} = k \mid T_x < L) P_k(J_L = i) \tag{20}$$

then from (18) and Lemma 6, we find that $P_i(Y_L > x) = C_{\gamma,i} e^{-\lambda_\gamma x}$. Using this in (16) concludes the proof of part ii) of Theorem 1.

As in the average-cost setting, a more explicit expression for $C_{\gamma,i}$ is available if we assume that only state 1 is a deficit state. With this assumption, $J_{N_x-} = 1$, a.s., and the sums over k in (18) and (20) collapse, resulting in

$$\begin{aligned} C_{\gamma,i} = \frac{1}{\pi_i} &\left(\sum_j \pi_j h(j) \right) (1 + (\gamma - r_1 \lambda_\gamma) m_1) \\ &\times R h^{-1}(1) P_1(J_L = i). \end{aligned} \tag{21}$$

To conclude the evaluation of this constant, we need to find $g_i(j) \triangleq P_j(J_L = i)$ at $j = 1$. For $\gamma > 0$, let D_γ be the diagonal matrix with entries

$$D_\gamma(i,i) = 1 - \sum_j \Phi_{ij}^\gamma(0).$$

Let e_i be the n-dimensional vector $(0, \cdots, 0, 1, 0, \cdots, 0)'$ with ith component equal to one.

Lemma 7: $g_i = (I - \Phi^\gamma(0))^{-1} D_\gamma e_i$.

Proof: Notice that

$$g_i(j) \equiv P_j(J_L = i) = \gamma \mathrm{E}_j\left[\int_0^\infty e^{-\gamma t} \mathbf{1}_{\{J_t = i\}}\, dt\right].$$

The result now follows by combining Proposition 2.20 and (5.14) of Chapter 10 of Çinlar [7], noting that the Q_γ defined in his (2.10) coincides with our $\Phi^\gamma(0)$. □

Remarks:

i) As $\gamma \downarrow 0$, $P_j(J_L = i) \to \pi_i$ for all i, j, and we expect that $\lambda_\gamma \to \lambda_0$. Thus, $C_{\gamma,i} \to C_0$ for all i, and (3) and (4) are consistent as γ decreases to zero.

ii) Under the discounted cost criterion, the optimal hedging point depends on the initial state; denote by $z_{\gamma,i}$ the optimal hedging point starting in machine-state i. An alternative policy controls the system to hedging point $z_{\gamma,i}$ whenever the machine state is i. Such a policy is guaranteed to result in lower cost than following a single $z_{\gamma,i}$. The optimal policy of Akella and Kumar [1] in a two-state model is precisely of this form; however, since in their model the hedging point for the breakdown state is always greater than that for the functional state, the former plays no role, and the policy is indistinguishable from one with a single hedging point. See Kimemia and Gershwin [10], Malhamé [12], and Sharifnia [16] for more on state-dependent hedging points.

IV. Asymptotics for General Deficit Intervals

We now drop the requirement that the holding-time distributions for deficit states be exponential, at the expense of obtaining asymptotics rather than exact results. For the asymptotics, we restrict attention to the linear cost structure in (5) and consider optimal hedging points as $c_- \to \infty$. The case of large penalties for unfilled demands is of practical as well as theoretical interest.

For the following, recall that a distribution is nonlattice if it is not concentrated on any set of the form $\{\cdots, -2\delta, -\delta, 0, \delta, 2\delta, \cdots\}$.

Theorem 2: Suppose that F_i is nonlattice whenever $r_i > 0$.

i) If $\sum_i \pi_i r_i < 0$ and if $\lambda_0 > 0$ solves (2) at $\gamma = 0$, then there is a constant $C_0 \in (0, 1)$ such that, as $c_- \to \infty$

$$z^* = \frac{1}{\lambda_0} \log_+ \left(\frac{C_0(c_+ + c_-)}{c_+}\right) + o(1). \tag{22}$$

ii) Suppose there is a unique deficit state. For any $\gamma > 0$, if $\lambda_\gamma > 0$ solves (2), then there are constants $C_{\gamma,i}$, $i = 1, \cdots, n$, such that, as $c_- \to \infty$

$$z^*_{\gamma,i} = \frac{1}{\lambda_\gamma} \log_+ \left(\frac{C_{\gamma,i}(c_+ + c_-)}{c_+}\right) + o(1). \tag{23}$$

Proof: The argument used for Theorem 1 still applies, except that $S_{N_x} - x$ and $\tau_{N_x} - T_x$ no longer have exponential distributions. In the average-cost setting the argument leading to (15) now shows that

$$P(Y_\infty > x) = \sum_j \pi_j h(j) \bar{\mathrm{E}}_j[e^{-\lambda_0(S_{N_x} - x)}/h(J_{N_x})] e^{-\lambda_0 x}. \tag{24}$$

Set $B_x = S_{N_x} - x$ and $\Gamma_x = J_{N_x}$ and consider the process $\{(B_x, \Gamma_x), x \geq 0\}$, with x playing the role of time parameter. This process is regenerative; the regeneration points are those x for which $S_{N_x} = x$ and $J_{N_x} = j$ for some fixed j. It follows (under our nonlattice condition) that this process has a limiting distribution not depending on the initial state; let $(B_\infty, \Gamma_\infty)$ have that distribution. Then, by the very definition of convergence in distribution, the limit

$$\lim_{x \to \infty} \bar{\mathrm{E}}_j[e^{-\lambda_0(S_{N_x} - x)}/h(J_{N_x})] = \bar{\mathrm{E}}[e^{-\lambda_0 B_\infty}/h(\Gamma_\infty)]$$

exists. We have thus shown that with

$$C_0 = \sum_j \pi_j h(j) \bar{\mathrm{E}}[e^{-\lambda_0 B_\infty}/h(\Gamma_\infty)]$$

we have

$$P(Y_\infty > x) \sim C_0 e^{-\lambda_0 x} \tag{25}$$

the symbol $\sim$ indicating that the ratio of the two expressions converges to one as $x \to \infty$; see also [2]. So long as the distribution of Y_∞ is continuous on $(0, \infty)$, straightforward minimization of $V(z) = c_+ \mathrm{E}[(z - Y_\infty)^+] + c_- \mathrm{E}[(Y_\infty - z)^+]$ shows that the optimal hedging point satisfies

$$P(Y_\infty > z^*) = \frac{c_+}{c_+ + c_-}. \tag{26}$$

The required continuity follows from the nonlattice assumption. But then (25) implies that

$$\frac{1}{C_0} e^{\lambda_0 z_*} \left(\frac{c_+}{c_+ + c_-}\right) \to 1$$

which implies (22). The proof of (23) is similar, but starts from the representation

$$P_i(Y_L > x) = \frac{1}{\pi_i} \sum_j \pi_j h(j) \bar{\mathrm{E}}_j[e^{(\gamma - \lambda_\gamma r_{N_x})(\tau_{N_x} - T_x)}/h(J_{N_x})] \times P_j(J_L = i \mid T_x < L) e^{-\lambda_\gamma x}.$$

When state 1 is the only deficit state, this reduces to

$$P_i(Y_L > x) = \frac{1}{\pi_i} \sum_j \pi_j h(j) R h^{-1}(1) \bar{\mathrm{E}}_j[e^{(\gamma - \lambda_\gamma r_1)(\tau_{N_x} - T_x)}] \times P_1(J_L = i) e^{-\lambda_\gamma x} \tag{27}$$

much as in Section III. We claim that $\gamma - \lambda_\gamma r_1 \leq 0$. Otherwise, all row sums of $\Phi^\gamma(\lambda_\gamma)$ would be strictly less than one, and since the maximum row sum is an upper bound on the spectral radius, we would arrive at the contradiction $\rho_\gamma(\lambda_\gamma) < 1$. By essentially the same regenerative argument used above for C_0, we may define

$$C_{\gamma,i} = \frac{1}{\pi_i} \sum_j \pi_j h(j) R h^{-1}(1) \times P_1(J_L = i) \lim_{x \to \infty} \bar{\mathrm{E}}_j[e^{(\gamma - \lambda_\gamma r_1)(\tau_{N_x} - T_x)}]$$

and conclude that $P_i(Y_L > x) \sim C_{\gamma,i} e^{-\lambda_\gamma x}$. The rest of the proof is the same as part i). □

It is possible that part ii) of this result extends to multiple deficit states. Such an extension would require convergence of $P_j(J_L = i \mid T_x < L)$ as $x \to \infty$, and a guarantee that $\gamma - \lambda_\gamma r_i \leq 0$ whenever $r_i > 0$.

V. Concluding Remarks

i) For simplicity, we have assumed throughout that the time spent in a machine state is independent of the next state visited. One can easily imagine settings in which it would be desirable to relax this requirement. For example, a quick repair might restore the machine to operation sooner than a thorough repair, but the resulting machine states might differ in their susceptibility to further failure. To incorporate such phenomena, one could define holding-time distributions F_{ij} corresponding to current-state i and next-state j. The matrix $\Phi^\gamma(\theta)$ is defined just as in (1), but with F_i replaced by F_{ij}. The analysis goes through as before, except that in expressions like (14)

$$\frac{1}{1 + \lambda_0 r_k \bar{m}_k} \sum_i R_{ki} h^{-1}(i)$$

becomes

$$\sum_i R_{ki} h^{-1}(i)/(1+\lambda_0 r_k \bar{m}_{ki})$$

with $\bar{m}_{ki}$ the mean of F_{ki}.

ii) It also seems possible to carry out an extension in which the hedging point changes with the machine state, with one important modification. Let Z_t be the hedging point at time t, a function of $\tilde{J}_t$. It is possible that upon a change in the machine state the inventory level X_t exceeds the new hedging point Z_t. Under the usual operation of the system, X_t would then decrease linearly at the rate of demand until it reaches Z_t. Suppose we modify the system so that X_t is instantaneously reduced to Z_t whenever a change in machine state results in $X_t > Z_t$; physically, this corresponds to discarding excess inventory rather than waiting to sell it. With this modification, the process $Y_t = Z_t - X_t$ becomes the image under the reflection mapping of the free process $Z_t - \hat{S}_t$, and its distribution can be analyzed through time reversal.

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References

[1] R. Akella and P. R. Kumar, "Optimal control of production rate in a failure prone manufacturing system," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 116–126, 1986.

[2] K. Arndt, "Asymptotic properties of the distribution of the supremum of a random walk on a markov chain," *Th. Probab. Appl.*, vol. 25, pp. 309–324, 1980.

[3] S. Asmussen, *Applied Probability and Queues.* Chichester: Wiley, 1987.

[4] ——, "Risk theory in a markovian environment," *Scand. Actuarial J.*, pp. 69–100, 1989.

[5] T. Bielecki and P. R. Kumar, "Optimality of zero-inventory policies for unreliable manufacturing systems," *Oper. Res.*, vol. 36, pp. 532–541, 1988.

[6] H. Chen and D. D. Yao, "A fluid model for systems with random disruptions," *Oper. Res.* vol. 40, pp. S239–S247, 1992.

[7] E. Çinlar, *Introduction to Stochastic Processes.* Englewood Cliffs, NJ: Prentice-Hall, 1975.

[8] J. Q. Hu and D. Xiang, "The queueing equivalence to a manufacturing system with failures," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 499–502, 1993.

[9] O. Kella and W. Whitt, "A storage model with a two-state random environment," *Oper. Res.*, vol. 40, pp. S257–S262, 1992.

[10] J. Kimemia and S. B. Gershwin, "An algorithm for the computer control of a flexible manufacturing system," *IIE Trans.*, vol. 15, pp. 353–362, 1983.

[11] C. Kollman, "Deep-penetration calculations for scattering neutrons by importance sampling," Statistics Dept., Stanford Univ., Stanford, CA, working paper, 1993.

[12] R. P. Malhamé, "Ergodicity of hedging control policies in single-part multiple-state manufacturing systems," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 34–343, 1993.

[13] H. D. Miller, "A generalization of wald's identity with applications to random walks," *Ann. Math. Statist.*, vol. 32, pp. 549–560, 1961.

[14] P. Ney and E. Nummelin, "Markov additive processes. I. eigenvalues and limit theorems. II. large deviations," *Ann. Probab.*, 15, pp. 561–609, 1986.

[15] E. Seneta, *Non-Negative Matrices.* New York: Wiley, 1973.

[16] A. Sharifnia, "Production control of a manufacturing system with multiple machine states," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 620–625, 1988.

[17] M. C. K. Tweedie, "Generalizations of wald's fundamental identity in sequential analysis to markov chains," *Proc. Cambridge Philos. Soc.*, 56, pp. 205–214, 1960.

A Globally Optimal Minimax Solution for Spectral Overbounding and Factorization

Robert E. Scheid and David S. Bayard

***Abstract*—In this paper, an algorithm is introduced to find a minimum phase transfer function of specified order whose magnitude "tightly" overbounds a specified real-valued nonparametric function of frequency. This method has direct application to transforming nonparametric uncertainty bounds (available from system identification experiments and/or plant modeling) into parametric representations required for modern robust control design software (i.e., a minimum-phase transfer function multiplied by a norm-bounded perturbation).**

I. Introduction

Assume that a discrete-time plant $P(z^{-1})$ is estimated as $\hat{P}(z^{-1})$, and let L denote the uncertainty in the estimate. For example, three common characterizations of plant uncertainty are L_A-additive uncertainty, L_I-input multiplicative uncertainty, and L_O-output multiplicative uncertainty, where [17, p. 224]

$$\begin{aligned} L_A &= P - \hat{P} \\ L_I &= \hat{P}^{-1}(P - \hat{P}) \\ L_O &= (P - \hat{P})\hat{P}^{-1}. \end{aligned} \tag{1}$$

Note that multiplicative representations require a square plant. Let L denote any one of the above three quantities. Suppose, a nonparametric overbound $\ell(\omega)$ on L is known such that

$$\ell(\omega) > \bar{\sigma}(L(e^{-j\omega T})) \quad \text{for all } \omega \in [0, \pi/T] \tag{2}$$

where T is the sampling period and $\bar{\sigma}(L)$ is the maximum singular value of L. Various methods are available to find $\ell(\omega)$ from raw data (cf., [3], [11], [13], [16]). $\ell(\omega)$, however, is a nonparametric function of frequency and cannot be used directly in modern robust control software packages such as the Matlab Robust Control Toolbox [7] and μ synthesis software [2]. Instead, the uncertainty must be represented as a minimum phase transfer function matrix $\mathcal{W}(z^{-1})$ of a specified order such that

$$L(e^{-j\omega T}) = \Delta \mathcal{W}(e^{-j\omega T}) \tag{3}$$

where Δ is norm-bounded, i.e.,

$$\|\Delta\|_\infty < 1.$$

The choice of $\mathcal{W}$ in (3) can be structured or unstructured. For present purposes, the simplest choice is to use a scalar matrix representation

$$\mathcal{W} = W \cdot I \tag{4}$$

where W is a single-input single-output rational function.

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