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Shortfall Risk in Long-Term Hedging with Short-Term Futures Contracts

Paul Glasserman

1 Introduction

Consider a firm with a commitment to deliver a fixed quantity of oil at a specified date T in the future. The commitment exposes the firm to the price of oil at time T . Suppose the firm buys futures contracts for an equal quantity of oil and for settlement at the same date T . In so doing, it has eliminated its exposure to the price of oil at T , but has it entirely eliminated its risk? If the futures contracts are marked-to-market – requiring, in particular, that the firm make payments should the futures price drop – but the forward commitment is not, then in eliminating its price exposure at time T the firm has potentially increased the risk of a cash shortfall before time T because of the funding requirements of the hedge. The possibility of an increased risk is even clearer if the original horizon T is long (say five years) but the futures contracts have a short maturity (say one month). The firm may seek to hedge the long-dated commitment through a sequence of short-term contracts, but this exposes the firm to price risk each time one contract is settled and the next is opened. In particular, should the price of oil decrease, funding the hedge will require infusions of additional cash.¹

The purpose of this chapter is to propose and illustrate a simple measure of the risk of a cash shortfall arising from the funding requirements of a futures hedge. We give particular attention to the probability of a large shortfall anytime *up to* a specified horizon as opposed to merely *at* that horizon. Rough approximations to such probabilities are available through the theory of Gaussian extremes (as in Adler (1990) and Piterbarg (1996)) and the theory of large deviations (as in Dembo and Zeitouni (1998) and Stroock (1984)); we compare the shortfall risk in alternative hedging strategies through these approximations.

Our analysis is motivated in part by the recent debate regarding the widely publicized derivatives losses of Metallgesellschaft Refining and Marketing (MGRM);

¹ See Appendix A for a brief review of futures and forward contracts.

see Benson (1994), Culp and Miller (1995), Edwards and Canter (1995), and Mello and Parsons (1995a) for accounts of this incident, and see Brennan and Crew (1995), Carverhill (1998), Hilliard (1996), Neuberger (1995), and Ross (1995) for related analyses. Briefly, MGRM had entered into long-term contracts to supply oil at fixed prices and was (ostensibly) hedging these commitments with one-month futures contracts. In 1993, as the price of oil dropped and the hedging strategy required increasingly large infusions of cash, MGRM's parent company found it necessary to abandon the strategy, resulting in derivatives losses reported in press accounts to exceed \$1 billion. In theory, as the price of oil dropped the value of the supply contracts increased, but in fact MGRM was forced to unwind its contracts on unfavorable terms.

Because of the complexities of this case and the many aspects that remain undisclosed, we do not attempt a direct application. We focus instead on an admittedly simple model of a central aspect of MGRM's strategy: the use of a *rolling stack* of short-dated futures contract to hedge long-term supply commitments. In this strategy, futures contracts are rolled into the next maturity as they expire, but the number of contracts is decreased over time to reflect the decrease in the remaining commitment in the supply contracts.

A primary objective of such a hedging strategy is to protect the firm from the effects of large price fluctuations. It is therefore reasonable to examine how effectively the rolling stack accomplishes this. In the simple single-factor model we study, the rolling stack eliminates the effect of spot price fluctuations completely – but only at the end of the hedging horizon. Early in the life of the hedge, the use of short-dated contracts increases the risk of a cash shortfall; we quantify this effect.

As a prelude to our analysis, consider the comparison in Figure 1. The solid lines plot the variance of the cash balance resulting from a long-term supply contract with and without hedging, based on a simple model of independent and identically distributed price changes. (The precise assumptions leading to these graphs are reviewed in Section 2.) Not surprisingly, the variance in the unhedged case increases over time. The variance of the hedged cumulative cashflow at the end of the horizon is zero, but (as noted by Mello and Parsons 1995b) early in the life of the contract the hedged variance is larger. This is certainly suggestive of an increased risk, but it is not immediately clear how to make this suggestion precise. At best, the curves give an indication of the relative probabilities of a cash shortfall at each *fixed* time t – what we will call the *spot* risk at time t – with and without hedging. They do not explicitly compare the more relevant probabilities of a cash shortfall any time *up to* time t , which we will call the *running* risk. We will argue that comparing spot risks understates the real shortfall risk resulting from the hedge. Indeed, one of our main conclusions, following from a result on Gaussian extremes, is that the unhedged variance should be compared with the running maximum of the hedged

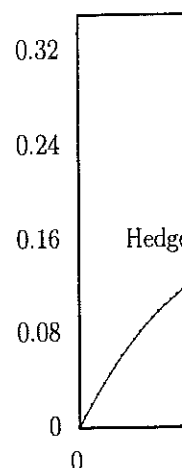


Fig. 1. Variance of unhedged cash balance. The dotted line indicates the variance of the hedged cash balance.

variance, indicated by the solid line, to the greater risk to the hedging strategy.

If the objective of a hedge is to avoid a cash shortfall, then the running maximum of the cash balance is a measure of running risk, which we will discuss in later sections, but we highlight here that the risk of a cash shortfall is not reduced by the hedge, a cash shortfall is not avoided. With no hedging it is most likely to occur under conditions that make the smaller hedge ratio minimize the risk of a cash shortfall at the optimal hedging horizon.

We elaborate these concepts in the next section (not require) mean reversal. We will argue that not hedged or not. We will argue that the impact on both the appropriate hedging strategy and dated futures. For each cash balance, we identify the most likely path of the cash balance such path solves a problem of minimizing large deviations. These "most likely" scenarios occur and not just their probabilities. We are testing" scenarios of the type that we have arrived at through a pathwise analysis.

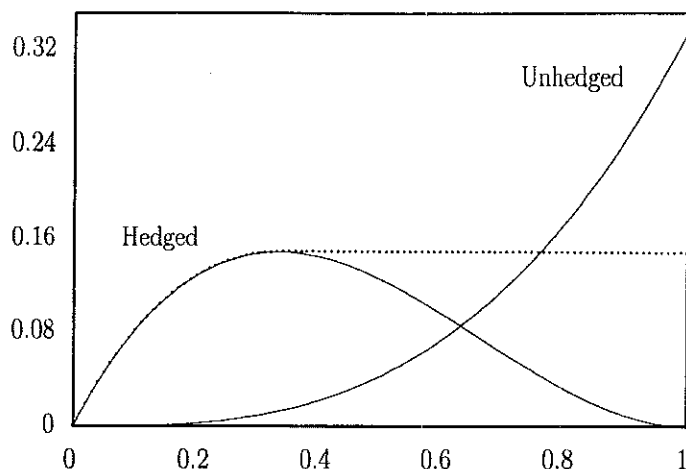


Fig. 1. Variance of unhedged and fully hedged cash balance over the life of the exposure. The dotted line indicates the running maximum of the hedged variance.

variance, indicated by the dotted line in Figure 1. Clearly, the dotted line assigns greater risk to the hedging strategy than does the corresponding solid line.

If the objective of a hedge is (at least in part) to reduce the chance of a cash shortfall, then the running risk is a relevant measure. Based on this premise and a measure of running risk, we make several observations. These will be detailed in later sections, but we highlight a few here. (a) A full rolling-stack hedge increases the risk of a cash shortfall for roughly 3/4 of the hedging horizon. (b) Under a full hedge, a cash shortfall is most likely to occur near 1/3 of the hedging horizon, and with no hedging it is most likely to occur near the end of the horizon. (c) Even under conditions that make the minimum-variance hedge ratio 1, a substantially smaller hedge ratio minimizes the running risk. (d) With a hedge ratio of 1, the optimal hedging horizon is substantially shorter than the full horizon.

We elaborate these conclusions in a model of spot prices that allows (but does not require) mean reversion. So, we have four basic cases: mean reverting or not, hedged or not. We will see that the degree of mean reversion has a major impact on both the appropriate extent and the effectiveness of hedging with short-dated futures. For each case, in addition to comparing risks of a cash shortfall, we identify the *most likely path* to a shortfall, in a sense to be made precise. Each such path solves a problem in the calculus of variations suggested by the theory of large deviations. These "optimal" paths give information about *how* risky events occur and not just their probability of occurrence. They may be thought of as "stress testing" scenarios of the type commonly formulated in practice on an *ad hoc* basis, here arrived at through a precise methodology.

A shortcoming of our analysis is that it rests on a single-factor model of spot and futures prices. As a consequence, we cannot fully model an unexpected shift from backwardation to contango of the type that seems to have precipitated MGRM's crisis. Indeed, as discussed by Benson (1994) and analyzed by Edwards and Canter (1995), the shape of the term structure of commodity prices is central to the rolling stack as a profit-generating strategy, as opposed to merely a hedge. (See Brennan and Crew (1997), Brennan (1991), Garbade (1993), Gibson and Schwartz (1990), Hilliard (1996), and Neuberger (1999) for some relevant multifactor models of commodity prices.) The tools we apply may, however, be extended to multifactor models.

Although we develop just one application here, it seems likely that the methods we use are relevant to other problems in risk management. There is, in particular, a close formal parallel between the model we consider and the exposure over time in an interest rate swap when interest rates follow the Vasicek (1977) model. The approach we follow in identifying price paths leading to shortfalls may be useful in constructing stress testing scenarios in other settings, or as a means of approximating value-at-risk. The evolution of exposures over time also plays a role in setting counterparty credit limits for swaps and other transactions. For background on these ideas, see Frye (1997), Jorion (1997), Picoult (1998), Wakeman (1999), and Wilson (1999).

The rest of this paper is organized as follows. Section 2 introduces the mechanics of the rolling stack and details our model of spot and futures prices, starting from a discrete-time formulation and then making a continuous-time approximation. Section 3 presents a measure of risk; Sections 4 and 5 develop the consequences of this measure with and without mean reversion, respectively. Section 6 presents the most likely paths to a cash shortfall. Section 7 compares our analysis (which is based on the continuous-time model) with simulations in discrete time. Some concluding remarks are collected in Section 8 and some technical issues are deferred to two appendices.

2 A model of exposure and hedging

Our point of departure is a simple model containing the essential features of examples discussed by Culp and Miller (1995) and Mello and Parsons (1995b) in their discussions of MGRM's hedging strategy. Consider a firm that commits to supplying a fixed quantity q of a commodity at a fixed price a at dates $n = 1, \dots, N$. The market price of the commodity at these dates is described by the sequence

$$S_n = c + \sum_{i=1}^n X_i, \quad n = 1, 2, \dots \quad (1)$$

At this point, we do not make the firm's cost equals the market cumulative cashflow to time k is

$$C_k = q \sum_{n=1}^k (a - S_n)$$

Let $F_{n,n+1}$ be the time- n futures price of the commodity maturing at $n+1$, and set h_n for an explicit model of the rolling stack hedging strategy at time n . Each contract bought at $n+1$, so the cumulative cash

$$H_k = q \sum_{n=1}^k (F_{n,n+1} - S_n)$$

$$= q \sum_{n=1}^k (F_{n,n+1} - S_n)$$

Interchanging the order of summation

$$C_k = qk(a - c) - q \sum_{n=1}^k (S_n - c)$$

Combining (3) and (4) and taking the limit as the delivery contract and hedge contract

$$\bar{C}_N = C_N + H_N = qN(a - c) - q \sum_{n=1}^N (S_n - c) + q \sum_{n=1}^N (F_{n,n+1} - S_n)$$

In particular, the hedging strategy is a martingale at N , but – comparing the coefficients of S_N –

In the Mello–Parsons example, the spot and futures are uncorrelated random variables. The expected payoff $qN(a - c)$ is the expected profit. The firm locks this in perfectly.² In the MGRM example, the spot price risk and “play the market” (stochastic or not). Again the firm locks this in at the terminal date N . Under the MGRM strategy, far the hedging strategy deviates from the Mello–Parsons (or isolating the basis) before

² Note, however, that (2)–(4) show that the identity that does not rely on stochasticity

At this point, we do not make any assumptions about the price increments X_i . If the firm's cost equals the market price, then at time n it earns $q(a - S_n)$, and its cumulative cashflow to time k is

$$C_k = q \sum_{n=1}^k (a - S_n) = q \left(k(a - c) - \sum_{n=1}^k \sum_{i=1}^n X_i \right). \quad (2)$$

Let $F_{n,n+1}$ be the time- n futures price for a contract on the underlying commodity maturing at $n + 1$, and set $b_{n,n+1} = F_{n,n+1} - S_n$. We use $b_{n,n+1}$ as a surrogate for an explicit model of the determinants of the cash-futures spread. Consider a *rolling stack* hedging strategy that buys $q(N - n)$ of these short-dated contracts at time n . Each contract bought at time n generates a profit or loss of $S_{n+1} - F_{n,n+1}$ at $n + 1$, so the cumulative cashflow to time k from the hedge is given by

$$\begin{aligned} H_k &= q \sum_{n=1}^k (N - n + 1) [S_n - F_{n-1,n}] \\ &= q \sum_{n=1}^k (N - n + 1) (X_n - b_{n-1,n}). \end{aligned} \quad (3)$$

Interchanging the order of summation in (2) yields

$$C_k = qk(a - c) - q \sum_{n=1}^k (k - n + 1) X_n. \quad (4)$$

Combining (3) and (4) and taking $k = N$, we see that the cash balance from the delivery contract and hedge combined, at the terminal date N , is

$$\bar{C}_N = C_N + H_N = qN(a - c) - q \sum_{n=1}^N (N - n + 1) b_{n-1,n}. \quad (5)$$

In particular, the hedging strategy exactly cancels the price increments X_n at time N , but – comparing the coefficients on X_n in (3) and (4) – only at time N .

In the Mello-Parsons example, the $b_{n-1,n}$ are all zero and the increments X_n are uncorrelated random variables with mean zero and variance σ^2 . As a result, $qN(a - c)$ is the expected profit from the delivery contract, and the rolling stack locks this in perfectly.² In the Culp-Miller example, the firm hedges to eliminate spot price risk and “play the basis”, meaning maintaining exposure to the $b_{n-1,n}$ (stochastic or not). Again the rolling stack accomplishes this perfectly – but only at the terminal date N . Under either interpretation, it is interesting to examine how far the hedging strategy deviates from its objective (be it locking in expected profits or isolating the basis) before the terminal date N .

² Note, however, that (2)–(4) show that this perfect-lock property of the rolling stack is the result of an algebraic identity that does not rely on stochastic assumptions.

Mello and Parsons (1995b) show that under their assumptions about the price increments the variance of the hedged cumulative cashflow is given by

$$\text{Var}[\bar{C}_k] = \text{Var}[C_k + H_k] = q^2 \sigma^2 (N - k)^2 k;$$

in particular, it is zero at $k = N$. The variance of the unhedged position at k is

$$\text{Var}[C_k] = q^2 \sigma^2 \sum_{i=1}^k i^2.$$

Mello and Parsons (1995b) point out that the hedged variance can therefore be greater than the unhedged one for small k . (Figure 1 graphs continuous versions of the two variances with units chosen so that $q = 1$ and $\sigma = 1$.) While this is certainly suggestive of an increased liquidity risk early in the life of the exposure as a result of the hedge, it is at best a comparison of risks at a fixed time k (if the distributions can reasonably be compared through their variances) but not, without further justification, a comparison of risks *up to* time k . We will argue that comparing *spot risks* as measured by variances at fixed times actually understates the *running risk* of a cash shortfall up to a fixed time.

The derivation leading to (5) relied solely on algebraic identities. A second interpretation of the rolling stack that is useful in more general settings is developed in Appendix B. We show there that any hedging strategy generating cumulative cashflows H_k satisfying

$$H_k - E[H_k] = E[C_k] - E_k[C_N] \quad (6)$$

locks in terminal value. (Here, E_k denotes conditional expectation given the price history to time k .) At intermediate dates, the exposure (actual cash balance minus expected) resulting from a hedge satisfying (6) is

$$\bar{C}_k - E[\bar{C}_k] = C_k - E_k[C_N]; \quad (7)$$

see Appendix B for details. Equation (7) sometimes provides a convenient shortcut.

We now give more detailed model assumptions, generalizing the setting considered so far. For simplicity, we take $q = 1$ from now on. We include mean reversion in the price dynamics to allow for more interesting behavior; specifically, we set

$$S_{n+1} = (1 - \alpha)S_n + \alpha c_n + \sigma Z_{n+1}. \quad (8)$$

Here, $0 \leq \alpha < 1$ measures the speed of mean reversion, c_n is the level toward which the price reverts at time n , and the Z_n are uncorrelated with mean 0 and variance 1. (When $\alpha = 0$ there is no mean reversion.) We express the futures price as

$$F_{n,n+1} = E_n[S_{n+1}] + B_{n,n+1}.$$

Notice that $b_{n,n+1} = B_{n,n+1} +$ not by itself entail any assumption deterministic.³ This is a short addressed only through a model Culp and Miller (1995) present small fraction of those in spot validity.⁴

By setting $V_n = E[S_n] - S_n$

$$C_k - E[C_k]$$

with V_n satisfying

$$V_n$$

Simple algebra verifies that

and

$$\sum_{n=1}^k V_n$$

so an application of (6) (or perfect terminal hedge is achieved

one-period futures contracts

³ Assuming $B_{n,n+1}$ deterministic can be of Duffie (1989) or 7.4.2 of Edwards would change the number of contracts in analysis.

⁴ Various notions of basis are common (1993), for example, all give different of the terms "contango" and "backwardation", respectively by Stoll and Whaley (1993), one should

⁵ When $\alpha = 0$, this and all similar effects. In fact, most discussions and assessments to the remaining commitment, which that the size of the position should be most effective in hedging terminal

Notice that $b_{n,n+1} = B_{n,n+1} + E_n[S_{n+1}] - S_n$, so this change in representation does not by itself entail any assumptions. However, we do assume that the $B_{n,n+1}$ are deterministic.³ This is a shortcoming of our analysis, but one that can be suitably addressed only through a model of commodity prices with at least two factors. Culp and Miller (1995) present evidence that fluctuations in the oil basis are a small fraction of those in spot prices, so our approximation is not without some validity.⁴

By setting $V_n = E[S_n] - S_n$ we can express the unhedged exposure as

$$C_k - E[C_k] = \sum_{n=1}^k (E[S_n] - S_n) = \sum_{n=1}^k V_n, \quad (9)$$

with V_n satisfying

$$V_{n+1} = (1 - \alpha)V_n - \sigma Z_{n+1}.$$

Simple algebra verifies that

$$V_n = \sum_{i=1}^n (1 - \alpha)^{n-i} Z_i$$

and

$$\sum_{n=1}^k V_n = -\sigma \sum_{n=1}^k \frac{1 - (1 - \alpha)^{k-n+1}}{\alpha} Z_n,$$

so an application of (6) (or a derivation akin to that leading to (5)) shows that a perfect terminal hedge is achieved by buying

$$h_n^\alpha = \frac{1 - (1 - \alpha)^{N-n}}{\alpha} \quad (10)$$

one-period futures contracts at time n .⁵ The resulting cumulative hedge cashflows

³ Assuming $B_{n,n+1}$ deterministic can be interpreted as assuming a deterministic risk premium; see Section 6.4 of Duffie (1989) or 7.4.2 of Edwards and Ma (1992). Assuming $b_{n,n+1}$ deterministic rather than $B_{n,n+1}$ would change the number of contracts in a perfect terminal hedge but would not significantly affect our analysis.

⁴ Various notions of basis are commonly used: Culp and Miller (1995), Duffie (1989), and Stoll and Whaley (1993), for example, all give different definitions. The ambiguity in terminology is related to that in the use of the terms "contango" and "backwardation". See Appendix A. To equate positive and negative basis with contango and backwardation, respectively, using the latter terms in the sense preferred by Duffie (1989) and by Stoll and Whaley (1993), one should take $B_{n,n+1}$ rather than $b_{n,n+1}$ as the basis.

⁵ When $\alpha = 0$, this and all similar expressions should be interpreted in the limit as $\alpha \downarrow 0$. Thus, $h_n^0 = N - n$. In fact, most discussions and assessments of the rolling stack equate the size of the futures position at time n to the remaining commitment, which corresponds to setting $h_n^\alpha = N - n$ in our setting. Our derivation shows that the size of the position should be adjusted to reflect the speed of mean reversion for the rolling stack to be most effective in hedging terminal value. Ross (1995) makes a related observation.

are

$$\begin{aligned} H_k &= \sum_{n=1}^k h_{n-1}^\alpha [S_n - F_{n-1,n}] \\ &= \sum_{n=1}^k h_{n-1}^\alpha \sigma Z_n - \sum_{n=1}^k h_{n-1}^\alpha B_{n-1,n}. \end{aligned}$$

If we set $\bar{C}_k = C_k + H_k$, then from the expressions above for C_k and H_k or more directly via (7), we find that the resulting exposure is

$$\bar{C}_k - E[\bar{C}_k] = -\frac{(1-\alpha) - (1-\alpha)^{N-k+1}}{\alpha} V_k. \quad (11)$$

Thus, we seek to compare the risks in (9) and (11).

We also consider other hedging strategies. A strategy is defined by $g = (g_1, \dots, g_N)$, where g_i denotes the number of futures contracts to buy at time i . The resulting cumulative hedge cashflows are

$$H_k(g) = \sum_{n=1}^k g_n \sigma Z_n - \sum_{n=1}^k g_n B_{n-1,n},$$

leaving an exposure of

$$(C_k + H_k(g)) - E[C_k + H_k(g)] = \sigma \sum_{n=1}^k \left(g_n - \frac{1 - (1-\alpha)^{k-n}}{\alpha} \right) Z_n. \quad (12)$$

For tractability, we work with continuous-time counterparts of the expressions above. Specifically, we replace (8) with

$$dS_t = -\alpha(S_t - c_t) dt + \sigma dW_t \quad (13)$$

with $\alpha \geq 0$, W a standard Wiener process, and c_t a deterministic function of time representing the level towards which the price reverts at time t .⁶ The firm contracts to deliver the commodity continuously at the rate of 1 unit of the commodity per unit of time throughout the interval $[0, T]$. The contracted price is a_t at time t . The cumulative cashflow process is now

$$C_t = \int_0^t (a_s - S_s) ds$$

with an exposure of

$$C_t - E[C_t] = \int_0^t (E[S_s] - S_s) ds = \int_0^t V_s ds,$$

⁶ The continuous-time and discrete-time speeds of mean reversion α_c and α_d are related via $\alpha_d = 1 - \exp(-\alpha_c)$.

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$$dV_t = \alpha V_t dt - \sigma dW_s, \quad V_0 = 0.$$

The terminal unhedged exposure is

$$\int_0^T V_s ds = -\sigma \int_0^T \int_0^s e^{-\alpha(s-u)} dW_u ds.$$

Interchanging the order of integration and simplifying shows that this equals

$$-\sigma \int_0^T \frac{1}{\alpha} (1 - e^{-\alpha(T-u)}) dW_u.$$

In this continuous-time setting, we do not model futures explicitly, though it is convenient at times to think of contracts with maturities dt (as in Ross (1995)). We return to real maturities in Section 7. By analogy with (12),

$$\sigma \int_0^t g(s) - \frac{1}{\alpha} (1 - e^{-\alpha(t-s)}) dW_s$$

represents the exposure under the strategy of buying $g(s)$ contracts at time s . In particular, a rolling stack of $(1 - \exp[-\alpha(T-s)])/ \alpha$ contracts at time s results in a terminal exposure of zero. We interpret this expression as $(T-s)$ when $\alpha = 0$.

We conclude this section with a remark on *tailing* the hedge – that is, locking in expected *present value*. Discounted at a continuously compounded rate r , the unhedged exposure becomes

$$\int_0^T e^{-rs} V_s ds = -\sigma \int_0^T \left(\frac{e^{-ru} - e^{-(\alpha+r)(T-u)}}{\alpha + r} \right) dW_u.$$

A tailed rolling stack holding

$$\frac{e^{-ru} - e^{-(\alpha+r)(T-u)}}{\alpha + r}$$

futures contracts at time u thus cancels the present value of the unhedged exposure and in so doing locks in the expected present value of the contract. An analogous modification applies in discrete time. Tailing the hedge complicates our analysis without fundamentally affecting it, so for the most part we exclude it from consideration.

3 Spot risk and running risk

For reasons discussed in Section 1, we presume that the firm seeks to hedge expected cashflows from its delivery contract throughout the life of the contract and not just at the terminal date. In particular, we suppose that the firm hedges to try

to prevent the actual cash balance from falling short of the expected cash balance by an amount x , which we take to be large. Write A_t for the actual cash balance at time t under an arbitrary hedging strategy, and say that a *shortfall* occurs when $A_t \leq E[A_t] - x$. Small shortfalls are unlikely to have a significant impact on the firm, so we are primarily interested in large x .

By the *spot risk* at time t we mean

$$P(A_t - E[A_t] < -x),$$

the probability of a shortfall at time t . If, as in our setting, the cash balance is Gaussian, the spot variance $\sigma_t^2 = \text{Var}[A_t]$ measures this risk perfectly. But a more relevant measure is

$$P(\min_{0 \leq s \leq t} (A_s - E[A_s]) < -x), \quad (14)$$

the probability of a shortfall any time up to t , which we call the *running risk* to t .

Calculating the running risk exactly is difficult,⁷ even in our simple model, so we compare risks based on an asymptotic measure that applies for large x . It follows from the Gaussian property of our model that the shortfall probability (hedged or not) can be written as

$$P(\min_{0 \leq s \leq t} (A_s - E[A_s]) < -x) = e^{-\gamma x^2 + o(x^2)}, \quad (15)$$

where

$$\gamma = - \lim_{x \rightarrow \infty} \frac{1}{x^2} \log P(\min_{0 \leq s \leq t} (A_s - E[A_s]) < -x)$$

depends on the hedging strategy and t , and $o(x^2)$ denotes a quantity converging to 0 as $x \rightarrow \infty$, when divided by x^2 . If one hedging strategy has a larger γ than another, it results in smaller probability of a shortfall of magnitude x , for all sufficiently large x . In this sense, a larger γ means less risk.

We use two tools for evaluating γ in particular and the running cashflow risk in general. The first is a remarkable result of Marcus and Shepp (1971)⁸ that, so long as A_t is Gaussian with sample paths that are bounded on bounded intervals (e.g., continuous)

$$\gamma = \frac{1}{2\nu_t^2}, \quad (16)$$

with

$$\nu_t = \sup_{0 \leq s \leq t} \sigma_t.$$

⁷ Adler (1990), p. 5, calls this "an almost impossible problem" for general Gaussian processes and notes that (14) is known for very few examples.

⁸ See Adler (1990) for a more extensive treatment and numerous references to related results.

Thus, the running risk is measured over some interval $[0, t]$, on another, then the shortfall probability is large x . (This is not true with an even better measure of risk, the maximum defining ν_t is attained under conditions are satisfied, then

$P(\min_{0 \leq s \leq t} (A_s - E[A_s]) < -x)$

with Φ denoting the standard normal distribution function (see p. 121, quoting a result of Titchmarsh to this point in Section 7.) The probability that the cash balance falls below level x in $[0, t]$ is well approximated by the probability that the variable lands more than x/ν_t standard deviations below the mean.

Our second tool for studying risk is the "most likely paths" identity, which is not restricted to the Gaussian case. It gives more detailed information about the "most likely paths" identity, which identifies the risks to which different hedging strategies are most exposed. The "most likely paths" identity compares hedged and unhedged cash flows, and identifies the most likely paths to $-x$ found via

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In this section, we specialize to the case of risks in a few hedging strategies. We justify the following results.

- (i) A full hedge has greater risk than a partial hedge ($\sqrt{1/3}/2$) of the life of the exposure.
- (ii) A full hedge has greater risk than a partial hedge ($(4/9)^{1/3}$) of the life of the exposure.
- (iii) The optimal fixed fraction hedge has greater risk than the optimal fixed horizon hedge.
- (iv) The optimal fixed horizon hedge has greater risk than the exposure.

Before explaining how we prove these results, we note that the crossover point in (i) crosses the dotted line in Figure 1 cross. In contrast, the crossover point in (ii) variance crosses the dotted line.

Thus, the running risk is measured by the running maximum standard deviation. If, over some interval $[0, t]$, one hedging strategy has a larger maximum variance than another, then the shortfall probabilities are ordered the same way, for all sufficiently large x . (This is not true without the Gaussian assumption.) In fact, v_t is frequently an even better measure of risk than suggested by (15). If, for example, the supremum defining v_t is attained at a unique point and some additional smoothness conditions are satisfied, then

$$\frac{P(\min_{0 \leq s \leq t} (A_s - E[A_s]) < -x)}{\Phi(-x/v_t)} \rightarrow 1,$$

with Φ denoting the standard normal cumulative distribution. (See Adler (1990), p. 121, quoting a result of Talagrand (1988), and Piterbarg (1996), p. 19; we return to this point in Section 7.) This result states that the probability of a shortfall below level x in $[0, t]$ is well approximated by the probability that a normal random variable lands more than x/v_t standard deviations below its mean.

Our second tool for studying the running risk is the theory of large deviations, which is not restricted to the Gaussian case, and – more importantly in our context – gives more detailed information about when and how a shortfall is likely to occur. The “most likely paths” identified by a large deviations analysis illustrates the types of risks to which different strategies are exposed. In the next three sections, we compare hedged and unhedged positions using $1/\gamma$ as a measure of risk and most likely paths to $-x$ found via large deviations.

4 Without mean reversion

In this section, we specialize to $\alpha = 0$ and compare risks in the unhedged position with risks in a few hedging strategies, including the full hedge that locks in terminal value. We justify the following conclusions:

- (i) A full hedge has greater spot risk than no hedge for approximately 63% ($3(1 - \sqrt{1/3})/2$) of the life of the exposure.
- (ii) A full hedge has greater running risk than no hedge for approximately 76% $((4/9)^{1/3})$ of the life of the exposure.
- (iii) The optimal fixed fraction to hedge for the full horizon is approximately 63%.
- (iv) The optimal fixed horizon for a full hedge is approximately 73% of the life of the exposure.

Before explaining how we arrive at these observations, we make a few remarks. The crossover point in (i) corresponds to the point at which the two solid curves in Figure 1 cross. In contrast, the point identified in (ii) is where the unhedged variance crosses the dotted line. In view of the discussion in Section 3, we arrive

at the rather surprising conclusion that for any $t < 0.76T$, the probability of a cash shortfall of magnitude x at some time in $[0, t]$ is greater for the hedged position than the unhedged position, for large x . To put (iii) in perspective, notice that in our single-factor model of commodity prices, the minimum-variance hedge ratio would be 1. (For discussions of minimum-variance hedging with futures see Chapter 7 of Duffie (1989) or Chapter 6 of Edwards and Ma (1992).) But the minimum-variance criterion considers the risk at a fixed date only; our measure, which reflects risk throughout the life of the exposure, results in a substantially smaller hedge ratio. Finally, (iv) shows that if one does use a hedge ratio of 1 (as in the standard rolling stack), then the hedging horizon should be shortened to minimize risk.

We now proceed with the verification of (i)–(iv), beginning with some preliminary results. If $\alpha = 0$, then $V_t = -\sigma W_t$. Standard calculations give

$$\sigma_t^2 = \text{Var} \left[\int_0^t V_s ds \right] = \frac{\sigma^2}{3} t^3$$

for the variance of the unhedged exposure. Under a full hedge, the exposure at time t is

$$\int_0^t V_s ds - E_t \left[\int_0^T V_s ds \right] = (T - t)V_t.$$

Thus, under a full hedge we have a spot variance of

$$\bar{\sigma}_t^2 = (T - t)^2 \sigma^2 t.$$

As discussed in Section 2, a deterministic hedging strategy is a function g on $[0, T]$, with $g(s)$ interpreted as the number of futures contracts to hold at time s . In the absence of mean reversion, full hedging corresponds to $g(s) = (T - s)$ and no hedging corresponds to $g(s) \equiv 0$. The exposure under any strategy g is (using integration by parts for the first integral)

$$\int_0^t V_s ds + \sigma \int_0^t g(s) dW_s = \sigma \int_0^t [s - t + g(s)] dW_s,$$

which has variance

$$\sigma_t^2(g) = \sigma^2 \int_0^t [s - t + g(s)]^2 ds. \quad (17)$$

We use this repeatedly to compare the risks in different strategies.⁹

For (i) we set $\sigma_t^2 = \bar{\sigma}_t^2$ and solve to get $t = (3T/2)(1 - \sqrt{1/3})$. For (ii), we first note that the spot variance of the full hedge is maximized at $T/3$, where it takes the value $4\sigma^2/27$. The running variance of the full hedge thus remains at this level in

⁹ The problem of minimising (over g) the maximum (over t) of (17) has been given a fascinating solution by Larcher and Leobacher (2000).

the interval $[T/3, T]$. For the equal (the spot variance increases) becomes less risky than the

i.e., at $t = (4/9)^{1/3}T$.

We next consider (iv). By hedging to a horizon τ (and remaining unhedged in contracts at time s , rather than

$g_\tau(s)$

The optimal fixed-horizon hedge is the entire interval $[0, T]$. For using (17). The maximal spot is riskiest) or at T (where that the spot variances at the

$$\sigma^2 \int_0^\tau (T - \tau)^2 ds +$$

respectively. The optimal τ spot variances at these times in principle, be given explicitly (iv). Figure 2 displays the with that for a full hedge – is

We now turn to (iii). Full corresponds to the strategy $g_\pi(s)$ of

σ^2

This is evidently a cubic function

The other possible location is $(1 - \pi)^2 \sigma^2 T$. The optimal Numerically, we find that with $(1/4)^{1/3}$. The resulting

the interval $[T/3, T]$. For the unhedged position, the running and spot variance are equal (the spot variance increases monotonically); hence, the unhedged position becomes less risky than the full hedge when

$$\frac{\sigma^2}{3}t^3 = \frac{4\sigma^2}{27}T^3,$$

i.e., at $t = (4/9)^{1/3}T$.

We next consider (iv). Recall that a full hedge makes the spot risk at T zero. By hedging to a horizon $\tau \leq T$, we mean hedging to make the spot risk at τ zero (and remaining unhedged in $[\tau, T]$). This is achieved by holding $(\tau - s)$ futures contracts at time s , rather than $(T - s)$; i.e., by the strategy

$$g_\tau(s) = \begin{cases} (\tau - s), & 0 \leq s \leq \tau; \\ 0, & s > \tau. \end{cases}$$

The optimal fixed-horizon hedge is the one that minimizes the running risk over the entire interval $[0, T]$. For any τ , we can evaluate the spot variance under g_τ using (17). The maximal spot risk occurs either at $\tau/3$ (where the hedged portion is riskiest) or at T (where the unhedged portion is riskiest). Using (17), we find that the spot variances at these times are $4\sigma^2\tau^3/27$ and

$$\sigma^2 \int_0^\tau (T - \tau)^2 ds + \sigma^2 \int_\tau^T (T - s)^2 ds = \sigma^2 \left(\frac{2}{3}\tau^3 - T\tau^2 + \frac{1}{3}T^3 \right),$$

respectively. The optimal τ – the one that minimizes the running risk – makes the spot variances at these times equal. This is the root of a cubic equation which can, in principle, be given explicitly; numerically, we find $\tau \approx 0.733T$ as indicated in (iv). Figure 2 displays the resulting variance over the life of the exposure along with that for a full hedge – i.e., with a hedging horizon of T .

We now turn to (iii). Fully hedging a fixed fraction π throughout $[0, T]$ corresponds to the strategy $g_\pi(s) = \pi(T - s)$ and therefore results in a spot variance of

$$\sigma^2 \int_0^t (\pi T + (1 - \pi)s - t)^2 ds.$$

This is evidently a cubic function of t ; it achieves a local maximum at

$$t^* = \frac{\pi T(1 + \pi - \sqrt{\pi})}{\pi^2 + \pi + 1}.$$

The other possible location of the maximal variance is T , where the spot variance is $(1 - \pi)^2\sigma^2T$. The optimal π sets the values of the spot variance at t^* and T equal. Numerically, we find that the optimal π is 0.62996, which appears to coincide with $(1/4)^{1/3}$. The resulting variance over time is graphed in Figure 2. Both the

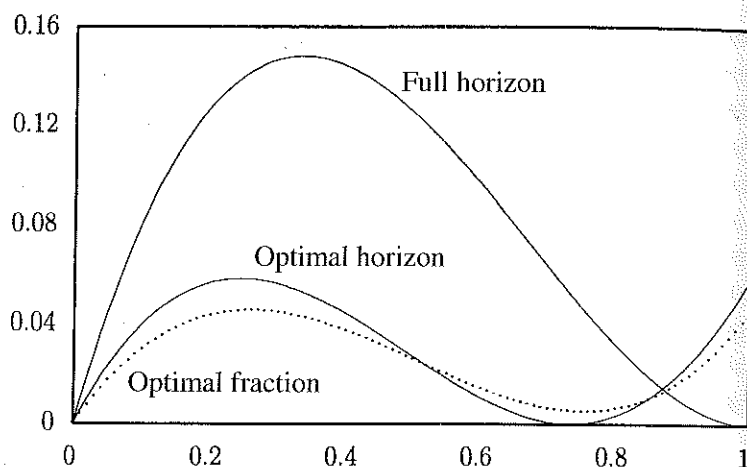


Fig. 2. Comparison of variances under different hedging strategies. The full hedge uses a hedge ratio of 1 for the full horizon T . The optimal fixed-horizon hedge uses a hedge ratio of 1 until time $\tau \approx 0.733T$ and thus balances the risk from the hedge early in the interval with the original risk later in the interval. The optimal fraction hedge uses a hedge ratio of $\pi \approx 0.63$ for the full interval $[0, T]$.

optimal hedge ratio and the optimal fixed horizon result in substantial reduction in the running risk, compared to a full stacked hedge. Hedging the optimal fixed fraction is slightly more effective than hedging fully for the optimal horizon.

We conclude this section with some observations on the impact of tailing the hedge, as described at the end of Section 2. Table 1 shows the location and value of the maximum variance with a full hedge and with no hedging, for various values of the discount rate r . The results indicate little change over a broad range of rates. Indeed, although maximum variances decrease with r (as they should), their ratio remains essentially unchanged.

5 With mean reversion

The possibility of mean reversion introduces more varied behavior in the dynamics of commodity prices and in the hedged and unhedged exposures. If we take $c_t \equiv c$ in (13), then expected future prices satisfy

$$E_t[S_{t+s}] = e^{-\alpha s} S_t + (1 - e^{-\alpha s})c.$$

A graph of expected future prices is thus upward sloping, flat, or downward sloping depending on whether S_t is below, at, or above c , and bears some resemblance to graphs in Figure 3 of Brennan and Crew (1997), Figure 8 of Edwards and Canter

Table 1. The effect of r

Rate	Hedge	
	Location	M
0	0.333	
0.01	0.333	
0.05	0.330	
0.10	0.326	
0.15	0.322	
0.20	0.319	

The columns labeled which the maximal magnitude of the maximum column gives the ratio of unhedged positions.

(1995), and Figure 1 of Neufeld

at various points in time. The presence of mean reversion in commodity prices are mean reversion hedge: unusually large price movements over time. To lock in expected returns with a greater speed α of mean reversion.

For the most part, our observations are about the effect of α . In thinking about what happens as $1/\alpha$ as the expected time for mean reversion. (Data in Bessembinder et al. (1995) time measured in years.) In our conclusions in terms of measuring time in multiples of $1/\alpha$, the results are more complicated than those for a random walk. As a consequence, our results are exact and numerical results,

- (i') The spot risk of the full hedge is the rate of mean reversion.
- (ii') Unless αT is greater than 2.06 , the full hedge is more effective than no hedge for most of the time. The off is $\alpha T \approx 2.06$.
- (iii') The optimal fixed fraction hedge is 75%.

Table 1. *The effect of tailing the hedge using a range of discount rates.*

Rate	Hedged		Unhedged		Ratio
	Location	Maximum	Location	Maximum	
0	0.333	0.148	1	0.333	44.4%
0.01	0.333	0.146	1	0.329	44.4%
0.05	0.330	0.139	1	0.313	44.3%
0.10	0.326	0.130	1	0.294	44.1%
0.15	0.322	0.121	1	0.277	43.9%
0.20	0.319	0.114	1	0.260	43.7%

The columns labeled "Location" and "Maximum" give the time at which the maximal variance is attained (as a fraction of T) and the magnitude of the maximal variance (as a fraction of $\sigma^2 T^3$). The last column gives the ratio of the maximal variances of the hedged and unhedged positions.

(1995), and Figure 1 of Neuberger (1999) showing the term structure of oil prices at various points in time.

The presence of mean reversion has important implications for hedging. If commodity prices are mean reverting, an exposure to them has a type of built-in hedge: unusually large price movements in the short term will be naturally offset over time. To lock in expected terminal profits, less hedging should be required with a greater speed α of mean reversion.

For the most part, our observations in this section depend on the magnitude of α . In thinking about what values of α are plausible, it is convenient to view $1/\alpha$ as the expected time for prices to revert about two-thirds of the way to their mean. (Data in Bessembinder *et al.* (1995) suggests $\alpha \approx 0.77$ for oil prices, with time measured in years.) In particular, α depends on the unit of time, so we state our conclusions in terms of the dimensionless quantity αT . This is equivalent to measuring time in multiples of the horizon T . The expressions we obtain for $\alpha > 0$ are more complicated than those we obtained for $\alpha = 0$ in the previous section; as a consequence, our results are somewhat less explicit. Through a combination of exact and numerical results, we make the following observations:

- (i') The spot risk of the fully hedged position is maximized at $T/3$, regardless of the rate of mean reversion.
- (ii') Unless αT is greater than about 2.375, a full hedge has greater running risk than no hedge for most of the life of the exposure. For the spot risk, the cut off is $\alpha T \approx 2.06$.
- (iii') The optimal fixed fraction to hedge for the full horizon is approximately 63–75%.

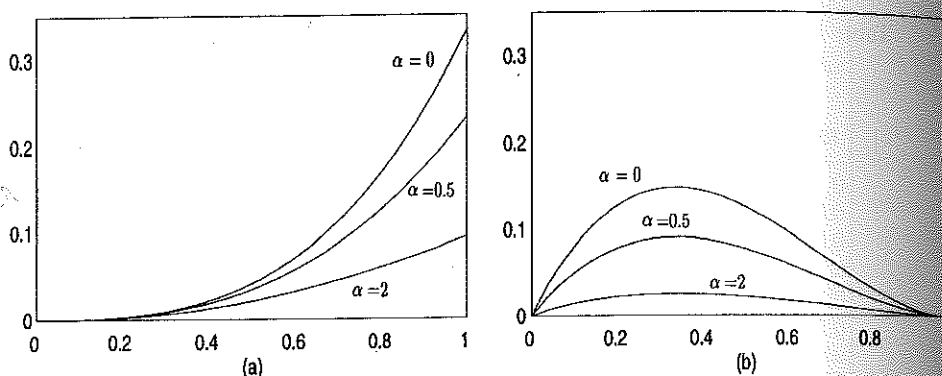


Fig. 3. Variance of (a) unhedged and (b) hedged cash balance over time for three values of the mean-reversion speed α .

(iv') The optimal fixed horizon for a full hedge is approximately 72–78% of the life of the exposure.

A useful result for the case $\alpha > 0$ is

$$\text{Cov}[V_s, V_t] = E[V_s V_t] = \frac{1}{2\alpha} e^{-\alpha(t+s)} (e^{2\alpha s} - 1), \quad s < t;$$

see, e.g., p. 358 of Karatzas and Shreve (1991). From this we can calculate the spot risk of the unhedged exposure to be

$$\begin{aligned} \sigma_t^2 &= \text{Var} \left[\int_0^t V_s ds \right] = 2 \int_0^t \int_0^s E[V_u V_s] du ds \\ &= \frac{\sigma^2}{\alpha^3} \left[\alpha t + 2(e^{-\alpha t} - 1) - \frac{1}{2}(e^{-2\alpha t} - 1) \right]. \end{aligned} \quad (18)$$

The fully hedged position has an exposure of (see (7))

$$\int_0^t V_s ds - E_t \left[\int_0^T V_s ds \right] = -\frac{1}{\alpha} V_t (1 - e^{-\alpha(T-t)}) \quad (19)$$

and a spot risk of

$$\bar{\sigma}_t^2 = \text{Var} \left[\frac{1}{\alpha} V_t (1 - e^{-\alpha(T-t)}) \right] = \frac{\sigma^2}{2\alpha^3} (1 - e^{-\alpha(T-t)})^2 (1 - e^{-2\alpha t}).$$

Some tedious but straightforward calculus shows that $\bar{\sigma}_t^2$ is maximized at $T/3$, as indicated in (i'); in particular, the location of the maximum is independent of α . For the unhedged position, σ_t^2 is, of course, always maximized at T . Figure 3 illustrates the dependence on α . With larger α there is less risk and the full hedge

Table 2. Crossover po

Rever	ra	α
0	0	0
0.1	0.1	0.1
0.2	0.2	0.2
0.3	0.3	0.3
0.4	0.4	0.4
0.5	0.5	0.5
0.6	0.6	0.6
0.7	0.7	0.7
0.8	0.8	0.8
0.9	0.9	0.9
1.0	1.0	1.0

is more effective in reducing hedge resulting from mean re

To justify (ii'), we located $\max_{s \leq t} \bar{\sigma}_s^2$, respectively. These of α values. The crossover p horizon until αT exceeds 2.0 larger values of α , σ_t^2 crosses two crossover points in Table

For an arbitrary hedging st

$$\sigma_t^2(g) = \sigma^2$$

which reduces to the express horizon strategy g_τ given by

$$g_\tau(s) = \begin{cases} \frac{1}{\alpha} \\ 0, \end{cases}$$

makes the spot variance 0 a either $\tau/3$ or T ; the spot vari

and

$$\frac{\sigma^2}{\alpha^3} \left[-\frac{1}{2}(e^{-\alpha t} \right.$$

Table 2. Crossover points as a fraction of the life T of the exposure.

Reversion rate αT	Spot risk crossover	Running risk crossover
0	0.63	0.76
0.10	0.63	0.75
0.5	0.60	0.71
1	0.57	0.65
2	0.50	0.53
5	0.31	0.31
10	0.16	0.16
100	0.02	0.02

is more effective in reducing what risk there is. Both properties reflect the natural hedge resulting from mean reversion.

To justify (ii'), we located the points $t > 0$ at which $\sigma_t^2 = \bar{\sigma}_t^2$ and $\max_{s \leq t} \sigma_s^2 = \max_{s \leq t} \bar{\sigma}_s^2$, respectively. These crossover points are displayed in Table 2 for a range of α values. The crossover points occur more than halfway through the life of the horizon until αT exceeds 2.06 for the spot risk and 2.375 for the running risk. For larger values of α , σ_t^2 crosses $\bar{\sigma}_t^2$ before $T/3$; because $\bar{\sigma}_t^2$ increases in $[0, T/3)$, the two crossover points in Table 2 are the same for larger α .

For an arbitrary hedging strategy g , the spot variance is

$$\sigma_t^2(g) = \sigma^2 \int_0^t \left[g(s) - \frac{1}{\alpha} (1 - e^{-\alpha(t-s)}) \right]^2 ds \quad (20)$$

which reduces to the expression in (17) as $\alpha \downarrow 0$. For each $\tau \in [0, T]$, the partial-horizon strategy g_τ given by

$$g_\tau(s) = \begin{cases} \frac{1}{\alpha} (1 - \exp(-\alpha(\tau - s))), & 0 \leq s \leq \tau; \\ 0, & \tau < s \leq T \end{cases}$$

makes the spot variance 0 at τ . The maximum spot variance under g_τ occurs at either $\tau/3$ or T ; the spot variances at these points are

$$\frac{\sigma^2}{2\alpha^3} \left(1 - e^{-\frac{2\alpha\tau}{3}} \right)^3 \quad (21)$$

and

$$\frac{\sigma^2}{\alpha^3} \left[-\frac{1}{2} (e^{-\alpha\tau} - e^{-\alpha T})^2 + e^{-\alpha(T-\tau)} - 1 + \alpha(T-\tau) \right], \quad (22)$$

path. This tendency to follow the most likely path becomes most pronounced as the event becomes rare, which corresponds to x becoming large in our setting. These statements are made precise by the theory of large deviations; see Dembo and Zeitouni (1998) and Stroock (1984) for background. This is a highly technical topic, so we will keep our discussion informal and proceed as directly as possible to the calculation of most likely paths.

We noted in Section 3 that the limit

$$\lim_{x \rightarrow \infty} \frac{1}{x^2} \log P(\mathcal{A}_x) = -\gamma$$

gives the exponential rate of decrease of $P(\mathcal{A}_x)$ in x^2 . The most likely path $\phi^* \in \mathcal{A}_x$ has the following property: if we define a strip around ϕ^* of width ϵ , then the probability that the Wiener process stays within this strip throughout $[0, T]$ decays at an exponential rate nearly equal to that of $P(\mathcal{A}_x)$, the difference vanishing as $\epsilon \downarrow 0$. Moreover, the probability that the Wiener process leaves this strip *conditional* on \mathcal{A}_x occurring vanishes exponentially as x increases. Thus, given that \mathcal{A}_x occurs, with high probability it occurs by the Wiener process staying close to the most likely path.

Finding the most likely path is a problem in the calculus of variations. For any absolutely continuous function ϕ on $[0, T]$, denote by $\dot{\phi}$ its derivative with respect to time. The most likely path in \mathcal{A}_x solves

$$\text{minimize}_{\phi \in \mathcal{A}_x} \frac{1}{2} \int_0^T [\dot{\phi}(t)]^2 dt. \quad (23)$$

This is known as Schilder's Theorem; see Dembo and Zeitouni (1998) or Stroock (1984) (especially pp. 66–7 for the mean-reverting case). Membership in \mathcal{A}_x defines a constraint on ϕ . Still with $\alpha = 0$, for the unhedged exposure

$$\mathcal{A}_x = \{\phi : \sigma \int_0^t \phi(s) ds > x, \text{ for some } t \in [0, T]\},$$

since this defines a cash shortfall in this setting. (In this and all subsequent cases, the requirement $\phi(0) = 0$ is implicit.) In the fully hedged case, a shortfall occurs when $(T - t)V_t > x$, so (recalling that $V_t = -\sigma W_t$)

$$\mathcal{A}_x = \{\phi : \sigma \phi(t) < -x/(T - t) \text{ for some } t \in [0, T]\}.$$

The solutions to (23) in these two cases are displayed in Figure 4a, b; the derivations are given in Appendix C. In each case, if ϕ^* is the minimizing path, then

$$\gamma = \frac{1}{2} \int_0^T [\dot{\phi}^*(t)]^2 dt,$$

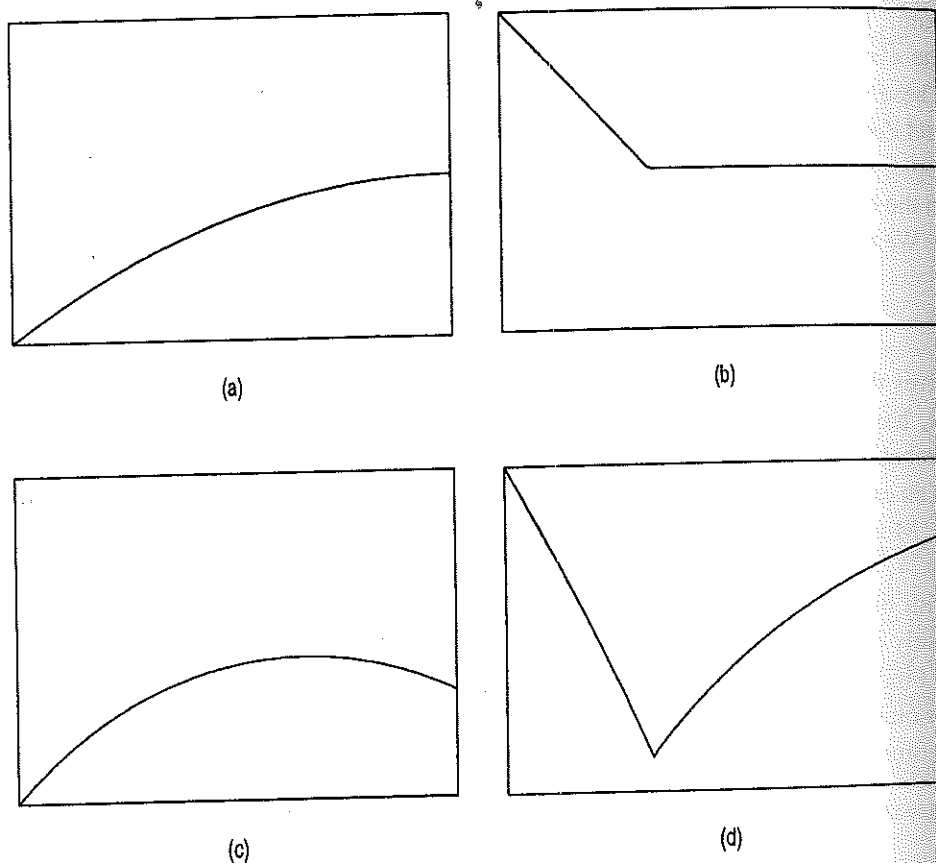


Fig. 4. Most likely paths of $S_t - E[S_t]$ to a cash shortfall. (a) and (b) are with $\alpha = 0$, (c) and (d) with $\alpha = 2$. (a) and (c) are for unhedged exposures, (b) and (d) are for fully hedged exposures.

with γ as defined in (15). In other words, the exponential rate of decrease of the shortfall probability is the also the "cost" of the minimum-cost path to a shortfall.

We now consider the case $\alpha > 0$. In light of the relation

$$V_t = -\sigma \int_0^t e^{-\alpha(t-s)} dW_s,$$

any event defined in terms of V can be expressed through conditions on W . More specifically, to each path ψ of V there corresponds a path ϕ of W via

$$\psi(t) = -\sigma \int_0^t e^{-\alpha(t-s)} \dot{\phi}(s) ds;$$

i.e.,

$$\dot{\psi}(t) =$$

and therefore

$$\dot{\phi}(t) =$$

If we now let \mathcal{A}_x be the set of ψ than x , then substituting (24) in

$$\text{minimize}_{\psi \in \mathcal{A}_x}$$

to determine the most likely path

$$\mathcal{A}_x = \left\{ \psi : \int_0^t \psi(s) ds \leq x \right\}$$

whereas in the hedged case it is

$$\mathcal{A}_x = \{ \psi : \sigma \psi(t) < -\alpha x \}$$

In each of these problems, x for arbitrary x is just x times the solution for $x = 1$. The volatility therefore be set to 1 as well. We the problems above:

- $\alpha = 0$, unhedged:

- $\alpha = 0$, hedged:

$$\phi(t) =$$

- $\alpha > 0$, unhedged:

where $a = \alpha / (3 - 2\alpha T)$ and $c = -(a + b)$.

- $\alpha > 0$, hedged:

$$\psi(t) =$$

with $c_1 = \alpha \exp(-\alpha T/3)$

i.e.,

$$\dot{\psi}(t) = -\alpha\psi(t) - \sigma\dot{\phi}(t)$$

and therefore

$$\dot{\phi}(t) = -\frac{1}{\sigma}[\dot{\psi}(t) + \alpha\psi(t)]^2. \quad (24)$$

If we now let \mathcal{A}_x be the set of ψ paths resulting in a shortfall of magnitude greater than x , then substituting (24) in (23) we arrive at the objective

$$\text{minimize}_{\psi \in \mathcal{A}_x} \frac{1}{2\sigma^2} \int_0^T [\dot{\psi}(t) + \alpha\psi(t)]^2 dt \quad (25)$$

to determine the most likely path. In the unhedged case the constraint is

$$\mathcal{A}_x = \left\{ \psi : \int_0^t \psi(s) ds > x, \text{ for some } t \in [0, T] \right\},$$

whereas in the hedged case it is (see (19))

$$\mathcal{A}_x = \{ \psi : \sigma\psi(t) < -\alpha x / (1 - \exp[-\alpha(T-t)]) \text{ for some } t \in [0, T] \}.$$

In each of these problems, x merely serves to scale the solution: the solution for arbitrary x is just x times the solution for $x = 1$; hence, it suffices to give the solution for $x = 1$. The volatility parameter σ is also a scale parameter and may therefore be set to 1 as well. With these simplifications, we present the solutions to the problems above:

- $\alpha = 0$, unhedged:

$$\phi(t) = \frac{3}{T^2}t - \frac{3}{2T^2}t^2;$$

- $\alpha = 0$, hedged:

$$\phi(t) = \begin{cases} -(9/2T^2)t, & 0 \leq t \leq T/3; \\ -3/2T, & T/3 < t \leq T. \end{cases}$$

- $\alpha > 0$, unhedged:

$$\psi(t) = ae^{\alpha t} + be^{-\alpha t} + c,$$

where $a = \alpha / ((3 - 2\alpha T) \exp(\alpha T) + \exp(-\alpha T) - 4)$, $b = (2 \exp(\alpha T) - 1)a$ and $c = -(a + b)$.

- $\alpha > 0$, hedged:

$$\psi(t) = \begin{cases} -2c_1 \sinh(t) & 0 \leq t \leq T/3; \\ -c_2 e^{-\alpha t} & T/3 < t \leq T, \end{cases}$$

with $c_1 = \alpha \exp(-\alpha T/3) (1 - \exp(-2\alpha T/3))^{-2}$ and $c_2 = (\exp(2\alpha T/3) - 1)c_1$.

These paths are graphed in Figure 4(a)–(d), the last two with $\alpha = 2$. The graphs are all on the same (dimensionless) scale, but with the origin in the upper-left corner of (b) and (d) and the lower-left corner of (a) and (c). In each case, the curve shows the most likely path by which the commodity price S_t deviates from the expected price $E[S_t]$ in generating a cash shortfall. Appropriately, in the unhedged cases (a) and (c) the shortfall results from an unexpected price increase and in the hedged cases (b) and (d) it results from an unexpected decrease: the rolling stack creates a large long position in the commodity early in the life of the exposure. In (a), the price increases throughout the life of the exposure, leveling off at the end, where the optimal path has derivative zero. With mean reversion, (c) shows that the most likely scenario has the price deviation reaching a maximum before T ; the curvature of the path increases with α . The graphs in (b) and (d) show the rather different risks to which the firm is most exposed under a full hedge. In both cases, there is a sharp drop in price until $T/3$ where the shortfall occurs. In (b), the price then stays flat, whereas in (d) it reverts towards its mean. Indeed, after $T/3$, the paths in (b) and (d) are unconstrained by the corresponding event \mathcal{A}_x , so the paths follow their mean behavior; the most likely paths are interesting only up to $T/3$ in these cases. Figure (d) is reminiscent of the sharp drop followed by a gradual recovery in the price of oil around the time of MGRM's crisis.

7 Assessing the approximations

The analysis in Sections 3–6 relied on two approximations to the model initially developed in Section 2: we replaced the discrete-time model with a continuous-time one, and we replaced the exact (unknown) risk of a cash shortfall with the running maximum variance, which is valid when the magnitude x of the shortfall is large. In this section, we examine the validity of these approximations.

We begin with a closer look at approximations based on (15) and the surrounding discussion, still in continuous time. It follows from Theorem D.3 of Piterbarg (1996) that for the unhedged exposure

$$C_t - E[C_t] = \int_0^t V_s ds,$$

the shortfall probability satisfies

$$\lim_{x \rightarrow \infty} \frac{P(\min_{0 \leq s \leq t} \{C_s - E[C_s]\} < -x)}{\Phi(-x/\nu_t)} = 1, \quad (26)$$

for each $t \in (0, T]$, indicating that the running maximum standard deviation ν_t is an even better measure of the running risk than suggested by (15) and (16), in the

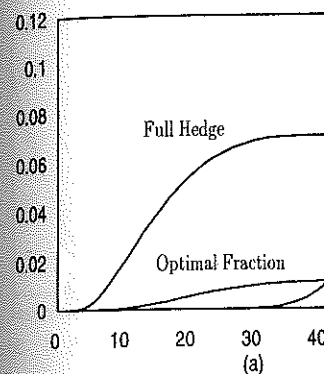


Fig. 5. Cumulative probability (a), $\alpha = 0$ and the horizon is 60

unhedged case.¹⁰ In the hedged

$\bar{C}_t -$

Theorem D.4 of Piterbarg (1

$$\Phi(-x/\nu_t) \leq P(\min_{0 \leq s \leq t} \bar{C}_s$$

but not the analog of (26). T underestimate the risk of the

To assess the reliability of the variance, we conducted simulations directly for the discrete-time model. A number of experiments with 1000 trials show estimated cumulative probabilities for a full hedge, and the optimal hedge (a) are based on 60 periods (one-month contracts) and $\alpha = 0.1$ magnitude of the shortfall was 10%. The overall appearance of the running maximum variance suggest that Figure 1 even under the comments following (27). The and other simulation results

¹⁰ Piterbarg formulates his result in terms of the interval over which the maximum is attained at the boundary.

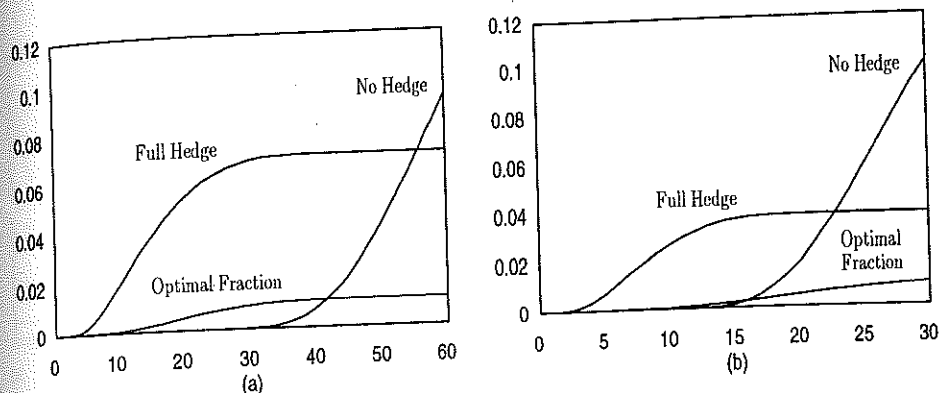


Fig. 5. Cumulative probability over time of a cash shortfall, estimated by simulation. In (a), $\alpha = 0$ and the horizon is 60 periods; in (b), $\alpha T = 2$ and the horizon is 30 periods.

unhedged case.¹⁰ In the hedged case, with an exposure of

$$\bar{C}_t - E[\bar{C}_t] = -\frac{1}{\alpha} V_t (1 - e^{-\alpha(T-t)}),$$

Theorem D.4 of Piterbarg (1996) gives

$$\Phi(-x/v_t) \leq P(\min_{0 \leq s \leq t} \{\bar{C}_s - E[\bar{C}_s]\} < -x) \leq \text{constant} \cdot x \Phi(-x/v_t), \quad (27)$$

but not the analog of (26). This suggests that the running maximum variance may underestimate the risk of the hedge, relative to no hedge, when x is not too large.

To assess the reliability of risk comparisons based on the running maximum variance, we conducted simulation experiments to estimate shortfall probabilities directly for the discrete-time model. The graphs in Figure 5 are indicative of a large number of experiments with different parameter values. The curves in the graphs show estimated cumulative probabilities of a shortfall over time with no hedge, a full hedge, and the optimal hedge ratio from Sections 4 and 5. The graphs in (a) are based on 60 periods (intended to suggest a five-year exposure hedged with one-month contracts) and $\alpha = 0$, those in (b) use 30 periods and $\alpha T = 2$. The magnitude of the shortfall was chosen to get a cumulative probability of roughly 10%. The overall appearance of the graphs is strikingly similar to the comparison of the running maximum variances in Figure 1. Indeed, the simulation results suggest that Figure 1 even understates the risk of a full hedge, consistent with the comments following (27). The general pattern we have observed based on these and other simulation results is that the riskiness of the full hedge (relative to no

¹⁰ Piterbarg formulates his result in the case that the point of maximal variance is in the interior of the time interval over which the maximum is computed, but then notes that the result extends to the case in which the maximum is attained at the boundary, as in our setting.

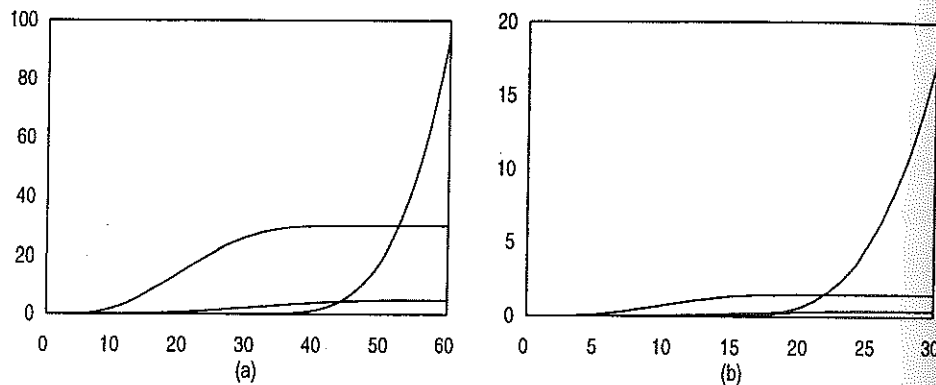


Fig. 6. Cumulative expected cash shortfall with no hedge, a full hedge, and the optimal-fraction hedge. (a) and (b) are based on the same parameters as in Figure 5. As before, the curves are ordered with the optimal-fraction hedge having smallest cumulative risk, the full hedge in the middle, and no hedge having the largest cumulative risk.

hedge) decreases with the magnitude of the shortfall and with the speed of mean reversion.

Figure 5 also indicates that substantial risk reduction can be achieved by using the optimal fixed-fraction hedge rather than a hedge ratio of 1. It should be possible to get further risk reduction for any number of periods N by solving numerically for the strategy (g_1, \dots, g_N) that minimizes the maximum variance over the hedging horizon. This is an easily solved optimization problem; we have found that the resulting strategy is surprisingly erratic and does not appear to lend itself to simple specification. Of course, even this strategy is at best the optimal deterministic strategy; in practice, a firm is likely to adjust its hedge in light of new price information.

The shortfall probability is open to criticism as a measure of risk because it treats all shortfalls of magnitude greater than x equally. A simple alternative weights shortfalls in proportion to the amount by which their magnitudes exceed x . Let ϵ_n denote the exposure at the end of period n , hedged or not. By the expected cumulative shortfall to time k we mean

$$\sum_{n=1}^k E[\max(0, -x - \epsilon_n)].$$

Artzner *et al.* (1996) have developed an axiomatic approach to risk measures in which the only "coherent" measures of risk are generalizations of this expression with $x = 0$.

Figure 6 shows cumulative expected shortfalls estimated through simulation with a full hedge, no hedge, and the optimal fixed-fraction hedge. The parameters

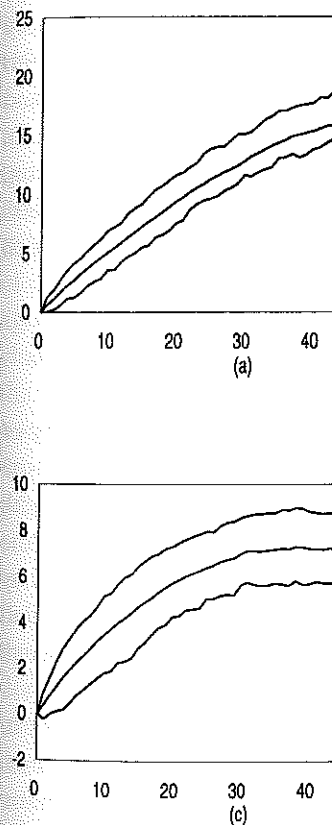


Fig. 7. Simulated paths on which the average over all simulated paths is shown. The middle path shows the interquartile range.

are exactly as in Figure 5. The results are similar to that in Figure 1. The motivation in Section 3 focuses on the results suggest that the running of risk.

We next turn to the most likely based on continuous time and are relevant to the original model, with and without mean we simulated roughly 20 000. The magnitudes of the required the probability of a shortfall the conditional law of the ex

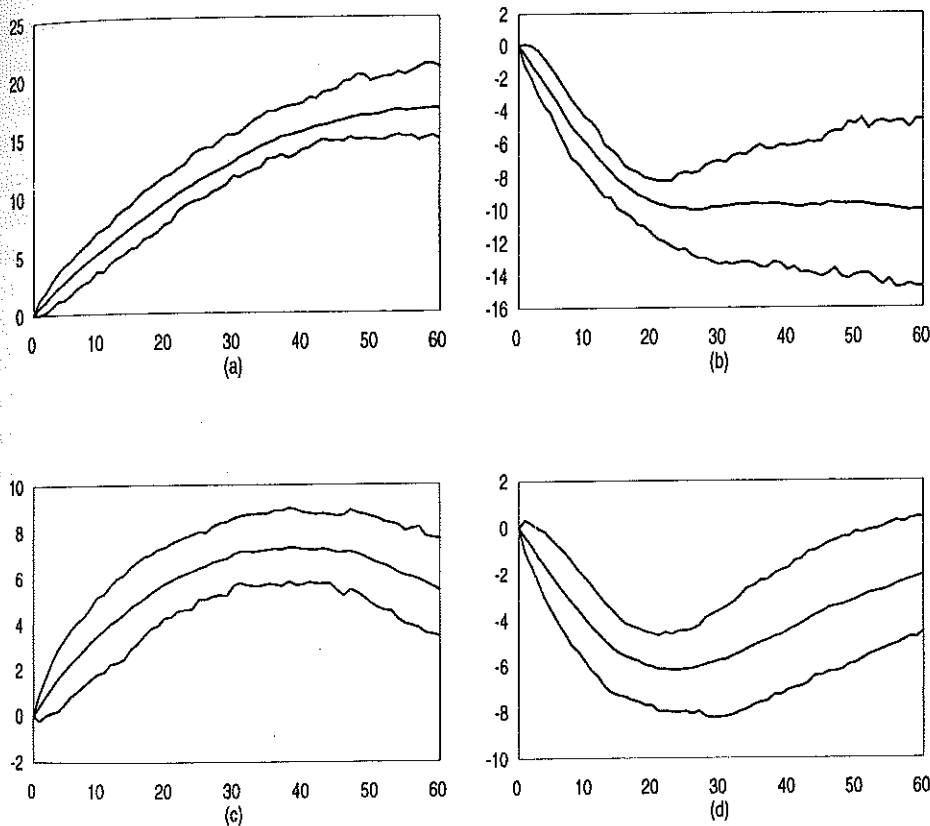


Fig. 7. Simulated paths on which a shortfall occurs. In each case, the center path is the average over all simulated paths on which a shortfall occurs, and the band around the center path shows the interquartile range. (a) and (b) are for $\alpha = 0$, (c) and (d) for $\alpha = 2$.

are exactly as in Figure 5. Again, the overall behavior of the risks is strikingly similar to that in Figure 1. The similarity is even more notable given that the motivation in Section 3 focused exclusively on the shortfall probability. These results suggest that the running maximum variance is a reasonably robust measure of risk.

We next turn to the most likely paths found in Section 6. That analysis was also based on continuous time and large x . To determine whether the paths found there are relevant to the original setting, we again simulated the original discrete-time model, with and without mean reversion, with and without hedging. For each case, we simulated roughly 20 000 paths, and saved those on which a shortfall occurred. The magnitudes of the required shortfalls were varied for different cases to keep the probability of a shortfall in the range of 2–5%. The saved paths approximate the conditional law of the exposure process given a shortfall. In Figure 7 we

have graphed the mean and the 25th and 75th percentiles (computed separately for each time period) of the paths. These show good qualitative agreement with the theoretical paths in Figure 4. As explained in Section 6, the paths in (b) and (d) are constrained only up until a shortfall occurs (near one-third of the horizon), so only this portion of the path is interesting. After the first third of the horizon, the spread in (b) reflects the ordinary \sqrt{n} diffusion associated with a random walk. Indeed, the contrast in (b) before and after the first third shows the extent to which the occurrence of a shortfall alters the usual evolution of the path.

8 Concluding remarks

We have proposed a measure of liquidity risk that approximates the probability of a cash shortfall any time in the life of an exposure, and used it to compare the risks in various strategies for a firm hedging long-term commodity contracts with short-dated futures. The implications of our analysis include an assessment of the cashflow risks produced by a seemingly perfect terminal hedge of the type used by Metallgesellschaft. We have also identified the particular price patterns to which a hedged or unhedged firm is most exposed, and examined the impact of mean reversion in the spot price.

Although we focused on a rather specific context, our analysis is relevant to other settings in which the variance of a position may fail to be monotone over time. Swaps, for example, typically have this property, and, like the fully hedged position in our context, have zero terminal variance. Indeed, our basic setup applies to the cumulative payments on a floating-for-fixed interest rate swap with the floating rate described by the Vasicek (1977) model. Hedging strategies based on discrete rebalancing can also be expected to have nonmonotone variance. The current and growing emphasis – in the finance industry, among regulators, and even in corporate finance – on measuring value-at-risk over multiple horizons suggests broader potential application for the perspective developed here.

Acknowledgements.

I thank Frank Edwards for discussions that motivated this work and Suresh Sundaresan for helpful discussions and detailed comments. For additional comments and helpful discussions I thank Sid Browne, John Parsons, Larry Shepp, and Tim Zajic.

Appendix A: Futures and forwards

This section gives a brief summary of some concepts and terminology pertinent to futures and forward contracts. More thorough treatments of these topics are given

in, for example, Duffie (1993).

A forward contract is an agreement to buy or sell a fixed price and date in the future for a specified quantity of, e.g., a commodity at a fixed delivery price. The forward contract has a value of zero. If a forward contract is entered into at the time of the agreement as the delivery date approaches, the value of the agreement with no exchange of cash changes whereas the value of the forward contract – which is the difference between the long party and negative value of the short party – drops, the value of the forward contract becomes negative for the long party.

A futures contract is similar to a forward contract, but the delivery date is in the future; the value of the futures contract is zero. The parties involved, future transactions, and the implications of the strategies that use them. The precise quantity, grade, and quality of the commodity involved. In contrast, futures contracts are marked-to-market and yet meet the needs of a large number of maturities and the precise underlying asset.

The most important difference between futures and forward contracts is that futures contracts are marked-to-market. For a forward contract, no payments are made until the delivery date. For a futures contract, no payments are made until the delivery date. These transactions are marked-to-market, which can be somewhat confusing. This illustrates the key point.

Consider a futures contract. At the time of the agreement, the value of the futures contract is zero. As the delivery date approaches, the value of the futures contract changes. If the price of the commodity rises, the value of the futures contract becomes positive. If the price of the commodity falls, the value of the futures contract becomes negative.

rather than at some point

in, for example, Duffie (1989), Edwards and Ma (1992), and Stoll and Whaley (1993).

A forward contract is an agreement between two parties to make a transaction at a fixed price and date in the future. The long party commits to buying a specified quantity of, e.g., a commodity or financial asset from the short party at a specified delivery price. The forward price is the delivery price that makes the value of the contract zero. If a forward contract specifies the current forward price at the time of the agreement as the delivery price (the typical case), then the parties enter the agreement with no exchange of payments. At later dates, the forward price may change whereas the contractual delivery price will not. If the forward price rises, the forward contract – worth zero at inception – will take on positive value for the long party and negative value for the short party. Conversely, if the forward price drops, the value of the forward contract becomes positive for the short party and negative for the long party.

A futures contract is similarly a commitment to execute a sale at a specified price and date in the future; the futures price is the delivery price that makes entry into a futures contract costless. Whereas forward contracts are arranged directly between the parties involved, futures contracts are traded through exchanges. This distinction has many implications for the design of the contracts and hence for hedging strategies that use them. Forward contracts can be highly customized, specifying the precise quantity, grade, delivery date and delivery location that suits the parties involved. In contrast, futures contracts must be standardized for exchange trading and yet meet the needs of many market participants; they thus admit a relatively small number of maturities, fixed quantities, flexibility in the timing of delivery and the precise underlying grade or asset to be delivered.

The most important distinction for the purposes of this article is that futures contracts are marked-to-market and forward contracts are not. With a forward contract, no payments are made at the inception of a contract and no payments are made subsequently until the contract matures, at which time the two parties execute the agreed-upon transaction. A party entering into a futures contract neither makes nor receives a payment upon entry, but on each subsequent day the exchange will credit the party for any profits and charge the party for any losses on its position. These transactions are made through a margin account, the precise mechanics of which can be somewhat involved. A simple example should nevertheless serve to illustrate the key point.

Consider a futures or forward contract maturing in three days and suppose the current futures or forward price is 100. Suppose that over the next three days the futures or forward price fluctuates to 98, 101, and then 103. At the end of the third day, the contract matures and thus reduces to a commitment to buy immediately rather than at some point in the future. Accordingly, 103 must be the spot price

(the price for immediate purchase) at the end of the third day. Consider the case of a forward contract: the contract specifies a delivery price of 100 though the spot price is 103, so the long party can buy at 100 and then sell at 103 for a profit of 3 at the end of the third day. In the case of a futures contract, at the end of the first day the exchange would require a payment of 2 from the long party, reflecting the drop in the futures price to 98. At the end of the next day, the exchange would credit the long party 3, reflecting the increase to 101, and on the next day the exchange would make a further payment of 2. The long party could close its position without taking physical delivery of the underlying, earning a profit of $-2+3+2=3$. Thus, in this example, the final profit resulting from the two contracts is the same, but the futures contract entails intermediate cashflows whereas the forward contract does not. It is precisely this distinction that gives rise to the possibility of a cash shortfall in offsetting a short forward position with a long futures position. It should be noted that this distinction in the timing of cashflows also leads to the conclusion that futures prices and forward prices will not generally be equal (as they are in the example) if interest rates are correlated with the underlying asset, though we will not address that issue here.

We briefly consider the relation between futures prices and the price of the underlying asset or commodity. Fix a date T and let F_t denote the time- t futures price for a contract maturing at T . Let S_t denote the price of the underlying at time t . Under simplifying assumptions (including costless transactions and unlimited short-selling) the futures and spot price are related via $F_t = S_t e^{c(T-t)}$, where c is the *cost of carry*. The cost of carry could be positive or negative and reflects both costs and benefits associated with holding the underlying, such as financing and storage costs and any dividends paid by the underlying. In a world with a deterministic cost of carry, changes in the futures price are perfectly correlated with changes in the spot price, so the risk in one can be eliminated through trading in the other.

The term *basis* refers broadly to differences between futures and spot prices. The relevant spot price may not be precisely the one underlying the futures contracts. For example, hedging an exposure to the price of jet fuel with futures contracts on heating oil is said to entail *basis risk* due to imperfect correlation between the futures price of heating oil and the spot price of jet fuel. The simplest definitions of basis take it to be $S_t - F_t$ or $F_t - S_t$ (consistent with $b_{n,n+1}$ in Section 2), but other definitions are used as well. Duffie (1989), for example, defines the basis to be $F_T - S_T$ even at time $t < T$. This difference would generally be nonzero (but unknown) if, e.g., S_t is the price of jet fuel and F_t is the futures price for heating oil.

A related ambiguity concerns the terms *backwardation* and *contango*. Broadly speaking, these describe conditions in which futures prices are, respectively, lower

than or higher than spot price. See Section 4.3 of Duffie (1989), and $E_t[S_T] > F_t$ and $E_t[S_T] < F_t$ through the older condition. To observe whether in fact a market is in backwardation, one must observe this definition, the oil market is in backwardation than con-

Appendix B: The

In this appendix, we argue that perfect terminal hedging p

Consider, again, the set of prices $b_{n-1,n}$ are all conditional expectation of price history to time k :

$$\begin{aligned} E_k[C_N] &= E_k\left[\sum_{n=k}^N b_{n,n+1}\right] \\ &= \sum_{n=k}^N E_k[b_{n,n+1}] \\ &= \sum_{n=k}^N N \\ &= N \end{aligned}$$

Comparing the last two terms of the rolling stack hedge

More generally (i.e., drop N whenever we can find a h

we get (using (29) with k

$$\bar{C}_N = C_N$$

than or higher than spot prices. According to the interesting discussion in Section 4.3 of Duffie (1989), modern usage associates these terms with the conditions $E_t[S_T] > F_t$ and $E_t[S_T] < F_t$ respectively. An advantage of defining these terms through the older conditions $S_t > F_t$ and $S_t < F_t$ is that it becomes possible to observe whether in fact a futures market is in backwardation or contango. With this definition, the oil market and many other commodity markets are more often in backwardation than contango.

Appendix B: The rolling stack and conditional expectations

In this appendix, we argue that (6) and (7) are the key properties underlying the perfect terminal hedging property of the rolling stack.

Consider, again, the setting leading to (3) and (4). Suppose the X_n have mean zero and the $b_{n-1,n}$ are all zero, as in the Mello-Parsons setting, and compute the conditional expectation of the terminal value of the unhedged position, given the price history to time k :

$$\begin{aligned} E_k[C_N] &= E_k \left[\sum_{n=1}^N (a - S_n) \right] \\ &= \sum_{n=1}^k (a - S_n) + (a - S_k)(N - k) \\ &= N(c - a) + \sum_{i=1}^k (k - i + 1)X_i + (N - k) \sum_{i=1}^k X_i \\ &= N(c - a) + \sum_{i=1}^k (N - i + 1)X_i. \end{aligned}$$

Comparing the last two terms with (4) and (3) (at $k = N$) we conclude that under the rolling stack hedge

$$E_k[C_N] = E[C_k] + H_k. \quad (28)$$

More generally (i.e., dropping the assumption that $E[X_n] = 0$ and $b_{n,n+1} = 0$), whenever we can find a hedging strategy with cumulative cashflows H_k satisfying

$$H_k - E[H_k] = E[C_k] - E_k[C_N], \quad (29)$$

we get (using (29) with $k = N$ for the third equality)

$$\begin{aligned} \bar{C}_N = C_N + H_N &= E_N[C_N] + H_N \\ &= E[C_N] + E[H_N] = E[\bar{C}_N], \end{aligned}$$

showing that the hedged cash balance \bar{C}_N is riskless at the terminal date N . Equation (28) is a special case of (29) with $E[H_k] = 0$ because we took all $b_{n,n+1}$ to be zero. At intermediate dates, the exposure (actual cash balance minus expected) resulting from a hedge satisfying (29) is

$$\begin{aligned}\bar{C}_k - E[\bar{C}_k] &= C_k + H_k - E[C_k] - E[H_k] \\ &= C_k - E[C_k] + E[C_k] - E_k[C_N] \\ &= C_k - E_k[C_N],\end{aligned}$$

as claimed in (7). Thus, under any hedging strategy satisfying (29), the resulting exposure at intermediate times is given directly by (7). The same argument applies if the discrete time index is replaced with a continuous one. We used this shortcut in (10), (11) and (19).

Appendix C: Derivation of optimal paths

The derivations of the optimal paths use standard techniques from the calculus of variations, especially Sections 2.12 and 3.14 from Gelfand and Fomin (1963) for the unhedged and hedged cases, respectively. We detail the cases with $\alpha > 0$; the calculations for $\alpha = 0$ are similar but slightly simpler.

When there is no hedge, it is easy to see that we can replace the inequality constraint defining \mathcal{A}_x with an equality, since the integral of the optimal path will not be any larger than required by the constraint. We thus need to find an extremal for

$$\frac{1}{2} \int_0^T [\dot{\psi}(t) + \alpha \psi(t)]^2 + \lambda[\psi(t) - x] dt,$$

with λ a Lagrange multiplier. As already noted, we may take $x = 1$ since x merely scales the path. The Euler equations give

$$\alpha^2 \psi - \ddot{\psi} = \text{constant}, \quad \psi(0) = 0 \quad (30)$$

$$\dot{\psi}(T) + \alpha \psi(T) = 0 \quad (31)$$

$$\int_0^T \psi = 1. \quad (32)$$

From (30) we obtain the general solution

$$\psi(t) = ae^{\alpha t} + be^{-\alpha t} - (a + b).$$

From (31) we get $b = (2 \exp(\alpha T) - 1)a$, and by eliminating b we can solve for a using (32).

Finding the optimal path
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has already been met. A
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Finding the optimal path in the hedged case is a free-endpoint problem because we do not know in advance the time τ at which

$$\psi(\tau) = h(\tau) \equiv -\frac{\alpha}{1 - \exp(T - \tau)}; \quad (33)$$

i.e., the time at which the shortfall occurs. The Euler equations give

$$\alpha^2 \psi - \ddot{\psi} = 0, \quad \psi(0) = 0$$

with the general solution $\psi(t) = 2c_1 \sinh(t)$. To find c_1 and τ we use (33) and the transversality condition

$$\frac{\alpha}{2} \psi(\tau) + \dot{h}(\tau) - \frac{1}{2} \dot{\psi}(\tau) = 0.$$

Some algebra shows that c_1 is as given in Section 6 and $\tau = T/3$. On $(\tau, T]$, the minimum-cost path should contribute no cost at all since the constraint for \mathcal{A}_x has already been met. A zero cost path must have $\dot{\psi} + \alpha\psi = 0$; i.e., $\psi(t) = \psi(\tau) \exp(-\alpha(t - \tau))$, so that $c_2 = \psi(\tau) \exp(\alpha\tau)$.

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Numerical Compa Me

David Heat

At present there is much the hedging of derivatives instance in the presence of This chapter provides cor methodologies, namely loc

We first describe the the Some comparative studies the asymptotics of these c addition, the density funct costs are estimated as we frequency. Numerical res Stein–Stein stochastic vol

To produce accurate an equation and simulation te terest. Some explicit soluti hedging are also describe difficult to implement th volatility models. In par cases difficult to identifi optimal hedge, due to its g combination of partial de character of an optimal co

The importance of this volatility models some c hedging approaches. We c have implications for the