

Biased Beliefs, Asset Prices, and Investment: A Structural Approach

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ABSTRACT

We structurally estimate a model in which agents' information processing biases can cause predictability in firms' asset returns and investment inefficiencies. We generalize the neoclassical investment model by allowing for two biases—overconfidence and overextrapolation of trends—that distort agents' expectations of firm productivity. Our model's predictions closely match empirical data on asset pricing and firm behavior. The estimated bias parameters are well identified and exhibit plausible magnitudes. Alternative models without either bias or with efficient investment fail to match observed return predictability and firm behavior. These results suggest that biases affect firm behavior, which in turn affects return anomalies.

This is the pre-peer-reviewed version of the article accepted for publication in the *Journal of Finance*.

* Aydoğan Altı is from the University of Texas at Austin. Paul Tetlock is from Columbia University. A prior version of this manuscript was entitled "How Important Is Mispricing?" We appreciate helpful suggestions from the associate editor, an anonymous referee, Nick Barberis, Jonathan Berk, Alon Brav, John Campbell, Murray Carlson, Josh Coval, Wayne Ferson, Cam Harvey, Martin Lettau, Erica Li, Sheridan Titman, Toni Whited, Jeff Wurgler, and seminar participants at the Michigan SFS, the NBER Behavioral, and the WFA conferences, as well as Carnegie Mellon, Columbia, Harvard, Stanford, UC Berkeley, UNC Chapel Hill, and UT Austin. We thank UT Austin and Columbia for research support. Please send all correspondence to aydogan.alti@mcombs.utexas.edu and paul.tetlock@columbia.edu.

There is disagreement over how much predictability in firms' asset returns comes from mispricing rather than risk premiums. On one side of this dispute, several researchers evaluate the merits of risk-based explanations using *structural modeling and estimation*.¹ On the other side, although numerous models of mispricing have been proposed, there have been no attempts to structurally estimate behavioral biases or link return anomalies to firm behavior.² Thus, we lack a clear answer to basic questions of model fit, parameter identification, parameter plausibility, and the economics of mispricing. For example, which models of biases best fit the key features of the data? Do the data allow one to distinguish among various behavioral biases? Are the implied magnitudes of biases plausible? Is there feedback between firm behavior and investors' biases and thus return anomalies?

To address such questions, we propose and estimate a model of mispricing in which agents' information processing biases can cause differences in firms' asset returns and investment inefficiencies. We generalize the neoclassical investment model by allowing for two biases—overconfidence and overextrapolation of trends—that distort agents' expectations of firm productivity. In the model, firm managers make investment decisions and investors make pricing decisions based on their information about firm-specific productivity. Their choices, which are affected by biases, generate endogenous relationships among firm investment, profitability, valuation, and asset returns. We estimate the model parameters by matching key features of firms' asset return and accounting data. Notably, we identify the magnitude of information processing biases from return anomalies—*i.e.*, cross-sectional variation in firms'

¹ Studies that estimate or calibrate risk-based models include Berk, Green, and Naik (1999), Gomes, Kogan, and Zhang (2003), Bansal, Dittmar, and Lundblad (2005), Gomes, Yaron, and Zhang (2006), Lettau and Wachter (2007), and Liu, Whited, and Zhang (2009).

² Examples of theories in which errors in investors' expectations lead to return predictability include Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subrahmanyam (1998), and Hong and Stein (1999). Models in which investors' preferences lead to return predictability include Benartzi and Thaler (1995), Barberis and Huang (2001), Barberis, Huang, and Santos (2001), and Barberis and Huang (2008).

asset returns that remains unexplained after adjusting for risk premiums. To our knowledge, we are the first to use asset prices to structurally estimate behavioral biases.

This structural approach confers four benefits, which are our primary contribution. First, we can rigorously test whether models of biases fit the data. Second, we can separately identify behavioral biases, such as overconfidence and overextrapolation. Third, we can evaluate whether the estimated magnitudes of these biases are plausible. Fourth, by conducting formal model comparisons, we can test whether behavioral biases affect real outcomes, such as firm investment, and whether firm behavior affects return anomalies.

Our model is designed to serve as a natural benchmark for the structural estimation of behavioral biases. As such, it features *homogeneous* investors who value firm assets and managers who choose firm investment to maximize the firm's current asset price. Firm productivity varies over time, but adjusting the capital stock is costly. Thus, if a firm's expected productivity increases, its manager dynamically responds by gradually increasing the firm's capital stock, generating persistence in the firm's growth opportunities and valuation ratios.

Agents cannot observe firm productivity, so they must estimate it using two pieces of information: realized profits and a soft information signal. Current profits are a noisy indicator of the firm's productivity. The soft information signal summarizes intangible information about productivity that complements the information from profits.

Agents in the model may suffer from two biases in estimating firm productivity: overconfidence and overextrapolation. Overconfident agents believe the precision of the soft signal to be higher than it actually is, much like agents in the models of Odean (1998), Daniel, Hirshleifer, and Subrahmanyam (1998), and Scheinkman and Xiong (2003). Agents who overextrapolate believe the persistence of firm productivity to be higher than it actually is, much

like agents in the model of Barberis, Shleifer, and Vishny (1998). Previous research proposes that these two biases could explain why value stocks earn higher returns than growth stocks.³

We estimate the structural model and evaluate its fit using the simulated method of moments (SMM). Simulated model moments closely replicate the relevant features of the empirical data. Most notably, the model's predictions closely match return anomalies, including the average asset returns of firms ranked by their value (book-to-market or B/M), investment rates, and profitability.⁴ The model also accurately predicts firm investment, profitability, and Tobin's q . Based on the overidentifying restrictions on the model's parameters, we cannot reject the hypothesis that the model is correct at even the 10% level.

Simulations of our benchmark model demonstrate that the two behavioral biases are well identified by their predictions of return anomalies. As in other models, both biases contribute in a similar direction to the observed return predictability arising from firms' B/M ratios and investment rates. Thus, these return anomalies help us identify the combined magnitude of the two biases. However, we are able to distinguish the two biases using their opposing predictions of the profitability anomaly. In estimating firm productivity, overconfident agents place too much weight on the soft information signal at the expense of the profit signal, so they underprice profitable firms. Extrapolative agents, on the other hand, think profitable firms will remain profitable longer than in reality, so they overprice such firms. As a result, when agents are overconfident (overextrapolate), profitability positively (negatively) predicts asset returns.

The estimated parameter values shed light on the plausibility of the information processing biases needed to explain return anomalies. Comparing the biases implied by return

³ Further evidence on investor overconfidence appears in Odean (1999) and Barber and Odean (2001). Empirical evidence consistent with overextrapolation appears in Lakonishok, Shleifer, and Vishny (1994), La Porta (1996), La Porta et al. (1997), and Benartzi (2001). For the sake of parsimony, we do not model other biases, such as conservatism or limited attention, though these may prove fruitful in future work.

⁴ See Fama and French (2008) for a review of these return anomalies.

anomalies to biases estimated using direct measures of expectations from surveys and professional forecasts, we find that implied overconfidence is large whereas implied overextrapolation is small. We gauge overconfidence in relation to survey estimates of business school students, money managers, and corporate executives and overextrapolation in relation to stock analysts' earnings forecast errors.⁵ Measuring overconfidence by the miscalibration of agents' declared confidence intervals, the overconfidence bias implied by return anomalies is one-third larger than that of survey participants. In contrast, the overextrapolation bias implied by return anomalies is only half the overextrapolation bias exhibited by stock analysts.

We estimate and simulate alternative versions of the model to evaluate whether modeling overconfidence, overextrapolation, and inefficient investment is necessary to match the data. Each of these elements appears to be important in our model: we strongly reject the null hypotheses that investors do not exhibit each bias and that managers invest efficiently. Alternative versions of the model without biases fail to match not only return anomalies, but also firm behavior. Furthermore, alternative versions in which rational managers choose investment efficiently—that is, to maximize long-run value, rather than the firm's current asset price—fail to fit the return anomalies. In the context of our framework, these two results imply that modeling biases is necessary to understand firm behavior and that modeling firm behavior is necessary to understand return anomalies.

Simulations of the model reveal how the two belief biases relate to observable firm characteristics. Although firm characteristics such as Tobin's q endogenously reflect the level of agents' belief biases, they are extremely noisy proxies for belief biases because much of their variation comes from rational variation in firms' expected productivity. As a result, asset

⁵ The survey evidence on overconfidence comes from studies by Alpert and Raiffa (1969), Russo and Schoemaker (1992), and Ben-David, Graham, and Harvey (2010).

mispricing can be much larger than what return anomalies based on observable proxies for belief biases would suggest. Even though both biases are latent, the overextrapolation bias is much easier to detect under our parameter estimates. Overextrapolative belief errors are highly correlated with observable variables, particularly a firm's past cash flow, while errors based on overconfidence are not. The reason is that overconfidence is based on the soft signal, which we estimate to be far less informative about productivity than a firm's cash flow. As a result, return predictability generated by overextrapolation is much easier for econometricians to identify and for investors to exploit using observable variables, such as q and cash flows.

This study's quantitative structural approach differs from the reduced-form approach adopted in earlier studies that test the validity of their models' qualitative predictions. Examples of this approach include Baker, Stein, and Wurgler (2003), Gilchrist, Himmelberg, and Huberman (2005), and Polk and Sapienza (2009). These papers focus on how mispricing affects rational managers' investment decisions in the presence of financing constraints or incentives to cater to investors. Our models address the benchmark case in which financing is frictionless, though we do consider alternative models of manager behavior.

Our study is unique among structural estimation studies in that we analyze the impact of information processing biases on asset prices and firm behavior. Liu, Whited, and Zhang (2009) estimate an investment-based asset pricing model using moment conditions based on return anomalies such as size and value, but they do not allow for biases or use moments based on firm behavior. Hennessy and Whited (2007) estimate a model of optimal firm behavior, but they do not allow for biases or use asset pricing moments.

Like our study, Chirinko and Schaller (2001) and Panageas (2005) model firms' asset prices and investment in the presence of mispricing. Both studies use aggregate time-series data

and do not use data on the cross-section of firms to estimate parameters, as we do.⁶ Panageas (2005) models mispricing using heterogeneous investors and short-sales constraints. In contrast, our model features investors with homogeneous beliefs that cause both pricing and investment inefficiencies, even without constraints. Thus, our insights and analyses are quite different.

The organization of the paper is as follows. Section I develops our dynamic model of firm investment and asset prices. Appendix A provides technical details of our solution method. Section II explains the empirical identification strategy and methodology. Appendix B describes the empirical and estimation methods in greater detail. Section III presents the estimation results and evaluates the model's fit to the data. Section IV considers various alternative model specifications, including specifications in which investors exhibit only one or none of the two behavioral biases and a specification in which managers maximize the firm's long-term rational value. Section V explores implications of the benchmark model for the detection and correction of mispricing. Section VI concludes by discussing implications of our results for future research.

I. A Model of Investment and Asset Prices with Biased Information Processing

A. Modeling Framework

This section presents the model that we analyze empirically in later sections. The model builds on the standard dynamic neoclassical framework by allowing agents to exhibit information processing biases. There are two groups of agents in the economy: investors who price assets, and managers who choose firm investment. All agents have homogeneous information. For much of the analysis, we assume that managers maximize the firm's current asset price. In this case, a representative investor/manager effectively determines both asset

⁶ A subsequent working paper by Warusawitharana and Whited (2012) examines firm behavior in the presence of mispricing using cross-sectional data. They employ a reduced-form model of mispricing, rather than modeling information processing biases, and they do not use standard return anomalies to identify mispricing parameters.

prices and firms' investment policies. This representative agent case is a natural benchmark for a structural model that attempts to infer agents' beliefs from asset prices.⁷

Here we describe the model for an individual firm; later in our empirical analysis we consider a cross-section of such firms with differing productivity and capital. To facilitate identification, we assume that model parameters are constant across firms and over time. Although firms are ex ante identical, they differ widely in size and growth rates after incurring a series of persistent productivity shocks, as explained below. Because we do not model tax or financing frictions, capital structure is irrelevant. Without loss of generality, one can assume that investors finance firms using only equity.

We denote the continuous passage of time by t . The firm pays cash flows $d\pi_t$ given by

$$d\pi_t = dh_t m_t^{1-\alpha} K_t^\alpha. \quad (1)$$

The firm's capital stock is K_t , and $\alpha \in (0,1]$ is a returns-to-scale parameter. The capital stock evolves according to

$$dK_t = I_t dt - \delta K_t dt, \quad (2)$$

where I_t denotes the firm's instantaneous investment rate and δ is the depreciation rate. To allow for variation in growth opportunities, we adopt the standard assumption that adjusting the capital stock is costly. When the firm invests at a rate I_t , it pays the cost of the newly installed capital and a quadratic adjustment cost of

$$\Psi(I_t, K_t) = \frac{\phi}{2} \left(\frac{I_t}{K_t} - \delta \right)^2 K_t. \quad (3)$$

In equation (3), the parameter ϕ determines the cost of adjusting the capital stock.

The variable m_t in equation (1) denotes the observable economy-wide component of

⁷ Introducing risk-averse arbitrageurs in the model is unlikely to change the qualitative patterns in firms' asset prices because prices in such models depend on a weighted average of biased and rational investors' expectations.

productivity, which follows a geometric Brownian motion with growth g and volatility σ_m :

$$\frac{dm_t}{m_t} = gdt + \sigma_m d\omega_t^m, \quad (4a)$$

where $d\omega_t^m$ denotes a Wiener process representing innovations in m_t .⁸ We allow investors to be risk averse by assuming that economy-wide productivity m_t is a priced risk factor. The parameter μ determines the price of risk, which is $(\mu - r)/\sigma_m > 0$. Defining a Wiener process $d\hat{\omega}_t^m$ under the risk-neutral measure, the aggregate productivity process can be expressed as

$$\frac{dm_t}{m_t} = [g - (\mu - r)]dt + \sigma_m d\hat{\omega}_t^m. \quad (4b)$$

We define the firm-specific component of the cash flow process dh_t in equation (1) as

$$dh_t \equiv f_t dt + \sigma_h d\omega_t^h. \quad (5)$$

The f_t term in equation (5) is firm-specific *productivity*, which reverts to its long-term mean of \bar{f} according to the law of motion

$$df_t = -\lambda(f_t - \bar{f})dt + \sigma_f d\omega_t^f. \quad (6)$$

The parameter λ measures the extent of mean reversion in productivity. The $\sigma_h d\omega_t^h$ term in equation (5) represents noise in cash flows that is uncorrelated with the firm's true productivity.

Agents in the economy do not observe productivity f_t directly. Instead, they estimate it using two available pieces of information. First, agents observe the firm-specific component of the cash flow process dh_t , which is informative about f_t as shown in equation (5). Second, agents observe a *soft information signal* s_t that is correlated with innovations in f_t . The signal s_t summarizes all productivity-related information, such as subjective interpretations of news events, other than cash flow realizations. We assume that the soft signal s_t evolves according to

⁸ We include the m process so that the firm's asset price includes a component representing economy-wide growth.

$$ds_t = \eta d\omega_t^f + \sqrt{1-\eta^2} d\omega_t^s. \quad (7)$$

The $d\omega_t^s$ term represents noise in the signal. Equation (7) ensures that the variance of the signal is equal to one for any value of signal informativeness, $\eta \in [0,1]$.⁹ We define signal precision as

$$\theta \equiv \frac{\eta}{\eta + \sqrt{1-\eta^2}}. \quad (8)$$

A value of one (zero) for θ represents a fully revealing (uninformative) signal.

In summary, the model consists of four jointly independent sources of randomness that follow Brownian motions: $d\omega_t^m$ (aggregate productivity innovation), $d\omega_t^f$ (firm productivity innovation), $d\omega_t^h$ (cash flow noise), and $d\omega_t^s$ (signal noise).

B. Information Processing

We model two information processing biases, *overconfidence* and *overextrapolation*. As in prior studies, we assume that overconfident agents think their information is more accurate than it actually is. Formally, agents believe the soft signal informativeness to be $\eta_B > \eta$ and hence the signal precision to be $\theta_B > \theta$. The difference $\theta_B - \theta$ thus measures how overconfident the agents are. Extrapolative agents believe firm productivity to be more persistent than it actually is. Formally, these agents believe the productivity mean reversion parameter in equation (6) to be $\lambda_B < \lambda$. The difference $\lambda - \lambda_B$ measures agents' overextrapolation of productivity.¹⁰

As agents observe cash flow and signal realizations, they revise their beliefs about productivity f_t using Bayes' rule, given their possibly incorrect beliefs (λ_B and η_B) about the productivity and signal processes. We denote the conditional estimate of f_t given information

⁹ The modeling of the soft information signal follows Scheinkman and Xiong (2003).

¹⁰ In estimating the model, we allow for θ_B and λ_B to be higher or lower than θ and λ , respectively. As we show below, however, the empirical estimates indicate overconfidence ($\theta_B > \theta$) and overextrapolation ($\lambda_B < \lambda$).

available at time t by \hat{f}_t , which follows the law of motion

$$d\hat{f}_t = -\lambda_B (\hat{f}_t - \bar{f}) dt + \sigma_f \eta_B \underbrace{ds_t}_{\equiv d\bar{\omega}_t} + \frac{\gamma_B}{\sigma_h} \underbrace{\frac{dh_t - \hat{f}_t dt}{\sigma_h}}_{\equiv d\bar{\omega}_t}, \quad (9)$$

where $d\bar{\omega}_t \equiv ds_t$, and $d\bar{\omega}_t \equiv (dh_t - \hat{f}_t dt) / \sigma_h$ are Brownian motions under biased beliefs, and γ_B is the steady-state variance of the estimation error $\hat{f}_t - f_t$. The error variance γ_B is the solution to

$$\underbrace{\frac{\sigma_f^2}{2\lambda_B}}_{\text{variance of } f} = \frac{1}{2\lambda_B} \underbrace{\left[\sigma_f^2 \eta_B^2 + \frac{\gamma_B^2}{\sigma_h^2} \right]}_{\text{variance of } \hat{f}} + \underbrace{\frac{\gamma_B}{\sigma_h}}_{\text{variance of estimation error}}, \quad (10)$$

which gives

$$\gamma_B = \sigma_h \left(-\lambda_B \sigma_h + \sqrt{\lambda_B^2 \sigma_h^2 + (1 - \eta_B^2) \sigma_f^2} \right). \quad (11)$$

The learning process for a rational agent satisfies equations (9) through (11), except that we replace η_B and λ_B with their rational counterparts, η and λ .¹¹

To evaluate agents' overall reaction to new information, we compute the variance in biased beliefs caused by the arrival of new information in the last two terms in equation (9):¹²

$$\text{Var} \left[\sigma_f \eta_B d\bar{\omega}_t + \frac{\gamma_B}{\sigma_h} d\bar{\omega}_t \right] = \sigma_f^2 \eta_B^2 + \frac{\gamma_B^2}{\sigma_h^2} = \sigma_f^2 - 2\lambda_B \gamma_B. \quad (12)$$

Equations (9), (11), and (12) reveal how biased agents respond to information, which has implications for return anomalies. Both the overconfidence ($\eta_B > \eta$) and the overextrapolation

¹¹ We assume that initial productivity f_0 is normally distributed with mean \bar{f} and variance $\sigma_f^2 / (2\lambda)$. There is also an initial signal $s_0 = f_0 + \varepsilon_0$, where ε_0 is normally distributed with mean zero and variance $v_0 = \gamma \sigma_f^2 / (\sigma_f^2 - 2\gamma\lambda)$. To ensure the model starts at the steady state, we choose the initial signal variance such that the biased agents' perceived initial estimation error variance matches their perceived steady-state estimation error variance γ_B .

¹² The overall reaction to new information of biased agents in equation (12) is the same under rational or biased beliefs. From the perspective of a rational observer, biased agents' beliefs follow equation (9), except that the drift term depends on the bias in agents' productivity estimates because $d\bar{\omega}_t$ is not a Brownian motion.

($\lambda_B < \lambda$) bias increase the variance of beliefs in response to new information.¹³ Overconfident agents overreact because they perceive information from the current soft signal to be more accurate than it is. Their overreaction is driven solely by the soft signal, as the second and third terms in equation (9) show. Overextrapolative agents overreact mainly because they place too much weight on past soft and profit signals, thinking productivity is more persistent than it is. The overreaction arising from either bias leads to return predictability from firms' valuation ratios and investment rates. High (low) perceptions of a firm's productivity cause its valuation ratio and investment rate to be high (low), but such firms later experience low (high) asset returns as investors are negatively (positively) surprised by realized productivity.

Despite this similarity in reactions to overall information, the two biases have opposing impacts on agents' responses to information from profit signals. While overextrapolative agents overreact to profit signals ($\gamma_B > \gamma$), overconfident agents *underreact* to profit signals ($\gamma_B < \gamma$), as shown by equations (9) and (11). Overextrapolative agents overweight the profit signal because it is based on the level of firm productivity, which these agents expect to persist. In contrast, overconfidence reduces agents' posterior variance of firm productivity, implying they underweight all signals except the soft signal.¹⁴ The differing reactions to profit signals lead to opposing empirical predictions: with overconfidence (overextrapolation), profitability positively (negatively) forecasts returns. A second difference is that only the overextrapolation bias causes productivity information to have an excessive impact on agents' estimates of future (not just

¹³ To see the overconfidence result, note that $\eta_B > \eta$ implies $\gamma_B < \gamma$ in equation (11) and thus $\sigma_f^2 - 2\lambda_B\gamma_B > \sigma_f^2 - 2\lambda_B\gamma$ in equation (12). For the overextrapolation result, note that $\lambda_B < \lambda$ implies $\gamma_B > \gamma$ and $\sigma_f^2\eta_B^2 + \gamma_B^2/\sigma_h^2 > \sigma_f^2\eta^2 + \gamma^2/\sigma_h^2$.

¹⁴ This contrasts with Daniel, Hirshleifer, and Subrahmanyam's (1998) model, in which overconfidence does not cause underreaction to other signals. The reason is that overconfident agents in their model perceive the soft signal's variance to be lower than it is, whereas agents in our model correctly recognize that the soft signal's variance is 1.0. We adopt this constant signal variance specification, as in Scheinkman and Xiong (2003), for two reasons. First, agents can estimate variance with infinite precision in continuous-time models. Second, agents can objectively measure the variance of many real-world soft signals, such as the number of clicks on a firm's web site.

current) firm productivity. For this reason, the overextrapolation bias has an especially large impact on asset prices and return predictability.

C. The Firm's Optimization Problem

In the biased representative agent model, a manager who maximizes the firm's current asset price given investors' information selects the firm's investment policy (I) according to

$$V(K_t, \hat{f}_t, m_t) = \max_I E_t^Q \left(\int_{u=t}^{\infty} e^{-r(u-t)} \left[\hat{f}_u m_u^{1-\alpha} K_u^\alpha - I_u - \Psi(I_u, K_u) \right] du \right), \quad (13)$$

subject to the evolution of capital, estimated productivity, and economy-wide productivity,

$$\begin{aligned} dK_u &= I_u du - \delta K_u du, \\ d\hat{f}_u &= -\lambda_B (\hat{f}_u - \bar{f}) du + \sigma_f \eta_B d\bar{\omega}_u + \frac{\gamma_B}{\sigma_h} d\bar{\omega}_u^2, \text{ and} \\ \frac{dm_u}{m_u} &= [g - (\mu - r)] du + \sigma_m d\hat{\omega}_u^m, \end{aligned} \quad (14)$$

respectively. In the maximization above, E^Q denotes an expectation under the risk-neutral measure. The firm's asset price is the expected discounted value of future *net* cash flows—that is, cash flows minus investment and adjustment costs—according to agents' beliefs and information. The Hamilton-Jacobi-Bellman equation for this maximization is

$$rV(K_t, \hat{f}_t, m_t) = \max_{I_t} \left\{ \begin{aligned} &\hat{f}_t m_t^{1-\alpha} K_t^\alpha - I_t - \Psi(I_t, K_t) + (I_t - \delta K_t) V_K - \lambda_B (\hat{f}_t - \bar{f}) V_f \\ &+ [g - (\mu - r)] m_t V_m + \frac{1}{2} \left(\sigma_f^2 \eta_B^2 + \frac{\gamma_B^2}{\sigma_h^2} \right) V_{ff} + \frac{1}{2} \sigma_m^2 m_t^2 V_{mm} \end{aligned} \right\}. \quad (15)$$

Substituting adjustment costs from equation (3) and maximizing with respect to I yields

$$\frac{I_t}{K_t} = \delta + \frac{V_K - 1}{\varphi}. \quad (16)$$

Equation (16) is the standard neoclassical investment policy: the firm invests more than enough to replace depreciated capital if and only if its *marginal* q , V_K , exceeds one. The rate of

adjustment of the capital stock is inversely related to adjustment costs ϕ . Substituting equation (16) into equation (15) and suppressing the t subscripts to simplify notation, we obtain a partial differential equation that characterizes the firm's asset price V :

$$\begin{aligned}
rV = & \hat{f}m^{1-\alpha}K^\alpha - \delta K + \frac{1}{2\phi}(V_K - 1)^2 K - \lambda_B(\hat{f} - \bar{f})V_f \\
& + [g - (\mu - r)]m_t V_m + \frac{1}{2}\left(\sigma_f^2 \eta_B^2 + \frac{\gamma_B^2}{\sigma_h^2}\right)V_{ff} + \frac{1}{2}\sigma_m^2 m_t^2 V_{mm}.
\end{aligned} \tag{17}$$

We solve for the firm's asset price V in equation (17) and the resulting investment policy I in equation (16) using the numerical methods described in Appendix A.

In two later analyses, we characterize firm value when the agent pricing the firm (“investor”) holds different beliefs from those of the agent (“manager”) determining the firm's investment policy. First, we analyze a model in which the manager makes investment decisions that maximize rational long-term firm value, and investors value the firm's cash flows using biased beliefs. Second, we analyze a model in which a manager with biased beliefs chooses investment, and investors value the firm's cash flows using rational beliefs. In both cases, we determine the firm's investment policy by solving equations (16) and (17), where the belief parameters η and λ are those of the manager. We then use a modified version of equation (15) to compute the value of the firm from the perspective of an investor who takes the manager's investment policy as given. The investor also realizes that there is an additional state variable representing the difference between the manager's and investor's productivity estimates. We then solve for firm value using the same numerical methodology described in Appendix A.¹⁵

¹⁵ The partial differential equation for firm value in these cases is available from the authors upon request.

II. Estimation and Identification of the Model

We determine three model parameters using direct estimation and the remaining ten using SMM. First, we directly estimate g , r , and δ using economy-wide capital stock growth, the real risk-free rate of interest, and the average of firms' depreciation rates, as explained in Appendix B. We also normalize the long-term mean of productivity \bar{f} to be one without loss of generality.¹⁶ Next, we apply our SMM procedure to determine the values of the remaining ten model parameters based on 15 additional moments—ten based on firms' real characteristics, two based on the economy, and three based on firms' asset returns. We discuss these moments at a general level below and refer the reader to Appendix B for detailed definitions of the relevant empirical quantities. Although each of the technology and information processing parameters affects all 15 of these moments, some moments convey far more information about certain parameters than other moments. We estimate each empirical moment using the time-series average of cross-sectional estimates, following Fama and MacBeth (1973). In all moments using firms' asset returns, we use abnormal returns computed relative to a single-factor market model, as described in Appendix B.

Here we provide intuition for identifying each parameter, starting with the two economy-wide parameters. We use the value-weighted average of firms' asset returns to identify the market price of risk (μ). Because capital structure does not affect firms' asset values in our model, our empirical analysis uses firms' total asset returns and total assets, not just the equity portions. We measure asset returns using unlevered stock returns as defined in Appendix B. We

¹⁶ This normalization is inconsequential because we analyze moments, such as q , profitability rates, and investment rates, that are normalized by firm size in an economy in which all firms have reached the steady state.

use the volatility of the growth of aggregate firm assets to identify the volatility of the economy-wide component of productivity (σ_m).¹⁷

We infer firms' technology parameters from moments based on firms' operating and investment characteristics. We use the averages of firms' Tobin's q (the ratio of market assets to book assets or M/B) and profitability (return on assets) to identify return-to-scale (α) in production. As α decreases, firms' average profitability (i.e., cash flows in the model) and Tobin's q increase. We infer the adjustment cost parameter (ϕ) from the coefficients in a regression of investment on q and profitability. High adjustment costs cause firm investment to respond less to growth opportunities. The volatility of firms' abnormal returns, profitability, investment, and q jointly provide information about the volatility of productivity and cash flows (σ_f and σ_h). We use the coefficient in a regression of firm abnormal returns on profitability to identify the precision of the soft signal (θ_B). A high coefficient indicates that agents rely heavily on realized profits to learn about productivity, implying that they perceive the precision of the soft signal to be low. The persistence in firm profitability helps us infer the true rate of mean reversion in firm productivity (λ).

We use three asset return anomalies to identify agents' information processing biases. Recall that the difference between λ and λ_B measures the overextrapolation of trends in productivity, while the difference between θ_B and θ captures the magnitude of overconfidence in the precision of the soft productivity signal. An increase in either of these biases produces an increase in the value and investment anomalies, as discussed in Section I.B.

We measure the value (investment) anomaly using the difference between the one-year average returns of firms in the top and bottom deciles ranked by the inverse of Tobin's q

¹⁷ As explained in Appendix B, we use a monotonic transformation of volatility for ease of computation.

(investment) in the previous year. Sorting on inverse q (i.e., B/M) is the appropriate way to measure the value anomaly because we model total firm assets, not just equity. When forming portfolios based on B/M, investment, and other accounting variables, we use data from each firm's most recently ended fiscal year, allowing for a three-month reporting lag. We rebalance all portfolios monthly. Each anomaly moment is based on abnormal portfolio returns that adjust for the return premium attributable to market risk, as described in Appendix B. Adjusting for market risk is the appropriate treatment in our model, which has a single priced risk factor.

To distinguish which of the two biases produces the observed value and investment anomalies, we exploit their conflicting predictions about return predictability coming from firm profitability, as discussed in Section I.B. Recall that overconfidence in the soft signal leads to underreaction to profit signals and causes a positive profitability anomaly, whereas overextrapolation of productivity amplifies reactions to profit signals and causes a negative profitability anomaly. Thus, the extent to which firms with high current profitability exhibit high future returns reveals the relative importance of the overextrapolation and overconfidence biases. We compute the profitability anomaly using the difference between the averages of one-year returns for firms in the top and bottom deciles of profitability.

In Appendix B, we describe the estimation and simulation methodology. We compute the 15 empirical moments using standard procedures, except that the return anomaly moments are measured using firms' asset returns—rather than equity returns—to conform to the model. Appendix B also explains how we use SMM to estimate the model parameters. In brief, the estimated parameters are those that minimize the distance between the 15 moments predicted by the model and the 15 empirical moments, where distance is measured in units based on the moments' empirical covariance matrix. To compute the model's predicted moments, we generate

simulated data for a cross-section of 1,000 firms, using 20 samples of 40 years each. For each sample firm, we draw initial values of true productivity and agents' productivity estimate from their respective steady-state distributions.¹⁸ Appendix B contains further details.

III. Estimation Results

A. Evaluating the Model's Fit to the Data

This section analyzes the biased representative agent model's ability to match the empirical moments described in Section II. The first two columns in Panel A of Table I show the estimates and standard errors of the 15 empirical moments. The average Tobin's q of $1.580 > 1$ suggests that firms have valuable growth opportunities. The cross-sectional standard deviation of q , 0.937 , is high as a fraction of average q mainly because more than 5% of firms have q values exceeding 4.375 . The persistence of profitability is 0.790 , suggesting that firm productivity is highly persistent. The investment sensitivities to q and profitability are 0.067 and 0.660 , which are within the respective ranges of values reported in Kaplan and Zingales (1997).

[Insert Table I here.]

The three anomaly rows in Panel A of Table I report the differences in asset returns between the top and bottom decile portfolios sorted on B/M, investment, and profitability. The annualized return differences from sorts on B/M and investment are 6.38% and -6.71% , respectively, one year after portfolio formation. The significantly positive value (B/M) anomaly is consistent with the well-known value effect in equity returns found in Fama and French

¹⁸ For convenience, we assume that productivity shocks are independent across firms. Because many firms in this economy exhibit predictable and independent asset returns, a rational arbitrageur could obtain an extremely high Sharpe ratio. There are no such rational agents in our model. Moreover, introducing productivity shocks that are correlated across firms but not priced would eliminate the possibility of nearly risk-free arbitrage. This is unlikely to affect our analysis because we estimate the model using average anomaly returns, not Sharpe ratios.

(1992). The negative investment anomaly is consistent with the investment and asset growth effects found in Titman, Wei, and Xie (2004) and Cooper, Gulen, and Schill (2008), respectively. The profitability anomaly of 1.39% per year is positive but statistically insignificant. Nevertheless, this anomaly is informative about the relative importance of the two information processing biases, as discussed in Section II. Some anomaly estimates appear slightly smaller than estimates in the literature because we measure anomalies using firms' asset returns, not just their equity returns.

The third column in Table I presents the results from SMM estimation of the model. The last two rows summarize how the model fits the data, using χ^2 statistics with degrees of freedom equal to the number of moments (15) minus the number of unrestricted parameters (ten or fewer). The $\chi^2(5)$ value of 7.80 for the unrestricted model indicates that the 15 simulated model moments match the empirical moments quite well. The p -value of 0.167 implies that we cannot reject the hypothesis that the five overidentifying restrictions are valid at even the 10% level.

Figure 1 graphically summarizes the fit of the model to the 15 empirical moments. The height of each vertical bar represents the t -statistic of the model's prediction error. The t -statistic is the difference between the model's prediction and the empirical moment, divided by the empirical standard error adjusted for the number of simulated samples. The ten leftmost bars in Figure 1 represent the prediction errors for technology moments, the next two bars are the economy-wide moments, and the three rightmost bars represent the errors for return anomalies. The model matches each of the 15 moments within 1.25 standard errors, which is quite close. None of the model's 15 prediction errors is statistically significant at even the 10% level.

[Insert Figure 1 here.]

We evaluate the economic magnitudes of the model's prediction errors by measuring each prediction error as a percentage of the empirical moment. The percentage model errors for seven of the ten technology moments are small at less than 6%. The model's most notable errors are its high predictions of investment sensitivity to q (+36% error) and its low predictions of mean cash flow and the standard deviation of investment (errors of -33% and -37%, respectively). One possible reason for the investment volatility error is that the model does not include fixed adjustment costs, which could produce lumpy and hence more volatile firm investment. Even so, this model error is not statistically significant.

The model is also able to match key properties of the aggregate economy. It predicts an inflation-adjusted asset market return of 4.60%, which is close to the empirical return of 5.31%. The predicted volatility of aggregate investment at 2.07% also closely matches the empirical volatility of 2.48%.

Importantly, the model matches the three empirical return anomalies quite well in both statistical and economic terms. The three model prediction errors for the profitability, value, and investment anomalies are all less than one empirical standard error. The model fits the profitability and investment anomalies almost perfectly, though it predicts a value anomaly that is too small by 1.55% per year. Thus, the model predicts an investment anomaly that is slightly too large relative to the value anomaly, though this effect is not statistically significant. The model fits the data well overall. Next we analyze whether the model parameters are properly identified and exhibit plausible magnitudes.

B. Parameter Estimates and Their Plausibility

Panel B of Table I shows the estimates of model parameters that produce the best fit to the data, along with parameters' standard errors. Reassuringly, the standard errors of all ten parameters are small relative to the parameter values, indicating that there is enough statistical power to identify each parameter. The estimates of the technology parameters are reasonable given the empirical data. The returns to scale (α) estimate of 0.730 is significantly below one, indicating diminishing returns to scale.¹⁹ The adjustment cost parameter (ϕ) of 2.44 is smaller than the estimates of 4.81 to 6.55 in Gomes, Yaron, and Zhang (2006), who use aggregated investment data, which tend to be less volatile. Our cost estimate implies that growth of 10% in the capital stock reduces its value by 0.60%. The estimate of mean reversion in productivity (λ) of 0.158 is low because profitability is quite persistent. Panel A of Table I shows that the model matches this feature of the data well. Because other model parameters such as σ_f , σ_h , σ_m , and μ are latent, there are no natural benchmarks for their values. We note, however, that our model and empirical moments provide enough information to identify these parameters.

Next we evaluate the information processing parameter estimates in Panel B. The perceived precision of the soft information signal (θ_B) is high at 0.852, but significantly less than 1.0. This high precision in the soft signal enables the model to match the weak response of asset returns to profitability. The estimated true precision of the soft signal (θ) is 0.496, which is significantly lower than the perceived precision of 0.852, implying that agents overreact to the soft information signal. The difference between the true and perceived signal precisions (-0.356) helps the model match the return predictability from B/M and investment.

¹⁹ The returns to scale estimate of 0.730 is slightly higher than recent estimates, such as those in Hennessy and Whited (2007), mainly because our model allows firm productivity to grow at a positive rate of g .

Another way for the model to reproduce the return predictability from B/M and investment is to set the perceived mean reversion in productivity (λ_B) lower than the true mean reversion (λ). Indeed, the estimate $\lambda - \lambda_B = 0.047$ does satisfy this overextrapolation condition. The overextrapolation bias appears statistically insignificant based on the local standard errors in Panel B, though a comparison of the χ^2 statistics in Panel A shows that this conclusion is misleading. Although the economic magnitude of the overextrapolation bias seems small, it is large enough to offset most of the impact of the overconfidence bias on the profitability anomaly, which we illustrate later using simulations.

Now we gauge the combined economic magnitude and plausibility of the two information processing biases. Panel B of Table I reports the perceived standard deviations of the errors in estimating productivity for a rational agent (0.302) and a biased agent (0.132), along with the standard deviation of true productivity of 0.913. On average, the biased agent perceives an estimation error about half as large as the rational agent's error, suggesting the two biases are large. However, the difference in the biased and rational agents' perceived errors of $0.169 = 0.302 - 0.132$ is only 19% of the magnitude of the volatility in true productivity (0.913). This comparison suggests such biases may be difficult to detect and correct.

We now compare the bias parameters implied by return anomalies to biases based on direct measures of expectations from surveys and professional forecasts. For our direct measures of overconfidence, we use estimates from general settings, including financial settings, where agents are quite uncertain about outcomes. The direct measures of overconfidence are based on surveys of Harvard Business School (HBS) students in Alpert and Raiffa (1969), money managers in Russo and Schoemaker (1992), and executives in Ben-David, Graham, and Harvey (2010). These studies solicit confidence intervals from HBS students regarding general

knowledge, from money managers regarding industry-relevant knowledge, and from executives regarding their projections of one-year returns on the S&P 500. Table II summarizes survey participants' stated confidence intervals along with the actual accuracy of these confidence intervals. In all three studies, participants' confidence intervals were far narrower than their empirical accuracy warranted, indicating substantial overconfidence.

[Insert Table II here.]

Table II compares the direct measures of overconfidence to the degree of overconfidence implied by our model estimates of the true and perceived precisions of the signal of firm productivity (θ and θ_B , respectively). To compute the model counterpart to the survey responses, we evaluate agents' productivity confidence intervals, which are based solely on the soft signal. Our estimates of θ and θ_B imply that agents' 80% confidence interval for productivity is 0.438 units wide. Based on the actual standard deviation of the productivity error of 0.767 units, one would expect a confidence interval that is 0.438 units wide to be correct only 22% of the time. Using similar logic, agents' 90% confidence intervals for productivity would be correct just 29% of the time and their 98% confidence intervals would be right only 40% of the time. Only the difference between money managers' overconfidence and model agents' overconfidence is statistically significant at the 5% level. In summary, the estimates of overconfidence implied by return anomalies are somewhat higher than survey measures, though not implausibly high.

While overconfidence is quite general, the overextrapolation bias inherently depends on the properties of the stochastic process that agents are forecasting. To gauge overextrapolation in empirical data, we focus on stock analysts whose goal is to forecast a process that mimics firm productivity, namely, firm earnings. We measure analysts' overextrapolation using the difference in the log of regression coefficients of actual earnings on forecasted earnings at the one-year and

two-year horizons. To model this difference in coefficients, we assume that firm earnings follow a continuous time AR(1) process just like productivity in our model. Clearly, the difference should be zero under rational expectations because both one-year and two-year forecasts should be unbiased predictors of actual earnings. Under biased expectations, however, the difference in the log of the one-year and two-year earnings forecast coefficients provides an estimate of $\lambda - \lambda_B$, where λ is the true rate and λ_B is the perceived rate of mean reversion in firm earnings.

To estimate this difference, we use analysts' one- and two-year earnings forecasts in the Institutional Broker Estimate System (IBES) from 1985 through 2006. As with our other moments, we winsorize forecasts and actual earnings at the 5% level to reduce the influence of outliers.²⁰ The yearly averages of the estimated cross-sectional coefficients are 0.995 and 0.904 (standard errors of 0.016 and 0.029, respectively) for the regressions of actual on forecasted earnings at the one- and two-year horizons, respectively. Based on the yearly average of the difference in the log coefficients, our estimate of a plausible overextrapolation bias is given by $\lambda - \lambda_B = 10.5\%$ (standard error = 3.0%). This analyst bias is double the overextrapolation bias of $\lambda - \lambda_B = 4.7\%$ (standard error = 4.7%) implied by return anomalies.²¹ The magnitudes suggest that a plausibly small amount of overextrapolation can help explain return anomalies.

Modeling priced risk generally affects the magnitudes of bias estimates because risk and biases jointly determine the extent of return predictability. We compare the three long-short anomaly portfolios' raw returns to their risk-adjusted returns (i.e., alphas) to evaluate the impact

²⁰ For one-year forecasts, we use consensus forecasts occurring eight months prior to the end of the fiscal year being predicted, which allows time for firms to report their prior year's earnings. For two-year forecasts, we use consensus forecasts occurring 20 months prior to the end of the fiscal year being predicted. DeBondt and Thaler (1990) use similar definitions of analyst forecasts when trying to estimate overreaction biases. We thank Zhi Da for his assistance in providing the cleaned and merged forecast data as used in Da and Warachka (2011).

²¹ The statistical significance of the difference in biases depends on the assumed correlation in sampling errors between the biases of model agents and stock analysts. At a correlation of 0.8 or higher, the difference in biases is significant at the 5% level.

of risk. The risk adjustment accounts for the portfolio's beta on the common productivity shock and the price of risk. In our model, growth firms are actually riskier (i.e., more exposed to aggregate productivity shocks) than value firms because growth options have implicit leverage.²² Mainly for this reason, the model predicts the raw returns of the value and investment anomalies (2.98% and -6.11%) to be closer to zero than their alphas (4.83% and -6.69%, respectively) shown in Table I, Panel A. Consistent with our model of priced risk, the raw returns of the empirical value and investment anomalies are also closer to zero (2.97% and -5.72%) than their alphas (6.38% and -6.71%). These comparisons indicate that risk, as we model it and empirically measure it, generates little return predictability. Of course, alternative models and measures of risk could produce different conclusions.

IV. Estimation and Analysis of Alternative Models

This section considers two types of alternatives to our benchmark model, which features investors who exhibit two behavioral biases and managers who invest to maximize the firm's current asset price. First, we consider alternatives with one bias or no bias to distinguish the individual impacts of the overconfidence and overextrapolation biases on the model's predictions. Second, we evaluate an economy in which managers choose firm investment to maximize long-term rational firm value—but biased investors still determine asset prices—to analyze how return anomalies depend on managerial behavior.

²² This feature of our model is quite general as argued in Grinblatt and Titman (2002).

A. Distinguishing the Impact of Overconfidence and Overextrapolation

Here we analyze the separate impacts of the overconfidence and overextrapolation biases on the model's predictions using new estimations and simulations. The estimations indicate whether each model fits the data. Simulations based on the benchmark parameter estimates help us build intuition for the model's comparative statics with respect to each bias.

First, we evaluate the fit of alternative models in which we restrict certain biases to be zero but allow all other model parameters to vary. We consider an extrapolation-only estimation in which overconfidence is restricted to be zero ($\theta_B = \theta$), an overconfidence-only estimation in which overextrapolation is restricted to be zero ($\lambda_B = \lambda$), and a no-bias estimation in which both biases are restricted to be zero ($\theta_B = \theta$ and $\lambda_B = \lambda$). In contrast to the simulations, the estimations do not place any restrictions on the true values of θ and λ and the other parameters. By allowing the data to determine the model's best possible fit, the estimations permit formal model comparisons and tests of whether each bias is necessary to fit the data. The last three columns in Table I show the model predictions from estimations of the alternative models, alongside the empirical moments and predictions from the benchmark model.

The high χ^2 statistics based on the tests of the models' overidentifying restrictions indicate that we can reject the null hypothesis that each of the three restricted models is valid at even the 1% level. Because the benchmark model nests these three alternatives, we can compute likelihood ratio statistics based on the differences in χ^2 values to test whether the bias restrictions are valid. The no-bias restrictions are consistently rejected at the 1% level in favor of the benchmark model with both biases, implying that *both* biases are needed to fit the data. The rejection of the no overextrapolation hypothesis in Panel A contrasts with the failure to reject based on the standard error of the parameter estimate of $\lambda - \lambda_B$ in Panel B. The test based on the

differences in χ^2 values is more powerful and accurate because it exploits the global shape of the SMM criterion function rather than the local properties used in the standard error computation.

Of the three alternative models, the overconfidence-only model has the lowest χ^2 value at 22.51, implying it fits the data best. Based on the difference in χ^2 values of 29.54, the no-bias model has a far worse fit to the data ($\chi^2(7) = 52.05$) than the overconfidence-only model ($\chi^2(6) = 22.51$). This suggests that overconfidence is worth modeling, regardless of whether one models the overextrapolation bias. However, the small difference of 0.95 in χ^2 values between the no-bias ($\chi^2(7) = 52.05$) and extrapolation-only ($\chi^2(6) = 51.10$) models shows that the overextrapolation bias only improves the model's fit when it is modeled with overconfidence.

One reason for the rejections of the alternative models is their poor fit to the three return anomalies. The overidentifying restriction that the three anomaly moments are equal to their approximately zero values in the no-bias model is rejected at the 5% level ($\chi^2(3) = 11.29$). The anomalies are not exactly zero in the no-bias model partly because of sampling errors and partly because of small approximation errors in solving the model. The empirical challenge for the overconfidence-only model is that the value and investment anomalies are much bigger than the profitability anomaly. The estimation sets the overconfidence parameter so that the predicted value and investment anomalies are less than half of their empirical values, while the predicted profitability anomaly is more than double its empirical value. The extrapolation-only model would actually generate an incorrect sign for the profitability anomaly if agents overextrapolate. This partly explains why the estimated bias implies slight underextrapolation, though this bias is close to zero.

Importantly, the two behavioral biases enable the model to fit several of the real moments, not just the return anomalies. Although many predictions of the two-bias and no-bias

models are quite close, the two-bias model makes more accurate predictions for 11 of the 12 real moments. Based on just these non-anomaly moments, one can reject the no-bias model at the 1% level ($\chi^2(12) = 31.30$). Furthermore, the no-bias and the extrapolation-only model make the counterfactual prediction that stock prices do not respond to earnings announcements because the soft signal perfectly reveals firm productivity ($\theta = 1.000$). This result demonstrates that modeling both biases is necessary for the model to fit the real moments. It also suggests that modeling firm behavior is helpful for identifying the biases.

To provide further intuition, using the parameter estimates from the benchmark model (i.e., the first column in Table I.B) as a starting point, we simulate the impact of three parameter changes. We eliminate agents' overconfidence by setting perceived signal precision equal to true signal precision ($\theta_B = \theta = 0.496$), while retaining all other parameter values, including the overextrapolation parameters ($\lambda_B = 0.112$ and $\lambda = 0.158$). Next we eliminate agents' overextrapolation by setting $\lambda_B = \lambda = 0.158$ while retaining all other parameter values, including the overconfidence parameters ($\theta = 0.496$ and $\theta_B = 0.852$). Finally, we consider a completely rational economy in which neither overconfidence nor overextrapolation biases are present ($\theta_B = \theta = 0.496$ and $\lambda_B = \lambda = 0.158$). We simulate economies with these three sets of parameters to isolate the impact of the two individual biases, holding other parameters constant.

Table III reports the moments from the simulation in which agents overextrapolate, the simulation in which agents are overconfident, and the fully rational economy with neither bias. For comparison, the last two columns in Table III redisplay the simulated model moments with both biases and the empirical moments. The bottom rows in Table III show that overextrapolation and overconfidence biases have opposing impacts on the profitability anomaly. Profitability negatively predicts returns in the extrapolative investor simulation and positively

predicts returns in the overconfident investor simulation. An overconfident investor ignores public profit signals in favor of his soft information signal. In contrast, an extrapolative investor erroneously thinks that current estimates of productivity, which depend on recent profit signals, will be representative of the firms' future productivity. In the benchmark model in Table I, these two effects roughly offset, producing a small net positive profitability anomaly. In contrast, both overconfidence and overextrapolation biases contribute to the value and investment anomalies.

[Insert Table III here.]

The model with extrapolative agents predicts that the investment anomaly will be larger than the value anomaly, whereas the model with overconfident agents does not. This difference arises because firm investment is more highly correlated with errors in investors' expectations when they exhibit the overextrapolation bias. Recall that extrapolative (overconfident) agents place excessive (insufficient) weight on cash flow in forming productivity expectations, so cash flow is a proxy for agents' irrational optimism (pessimism). In general, firm investment and cash flow are positively correlated because of managers' rational responses to informative cash flow signals. Thus, investment is more highly positively correlated with irrational optimism in the extrapolative model, where cash flow is an optimism proxy, implying that firms with high investment have lower returns in the extrapolative model.

Table III shows that many real moments depend critically on which of the two biases agents exhibit. Overconfidence causes agents to respond more to soft signals and less to profit signals, while overextrapolation causes them to overreact to both signals. This is why Tobin's q is so much more variable in the model with overextrapolation. It also explains why stock returns

and firm investment are more sensitive to profits in the overextrapolation model.²³ Because the biases make opposing predictions for these real moments, these moments help us identify the relative magnitudes of the biases in the benchmark estimation.

This logic also helps to explain the failure of the no-bias model estimated in Table I to match key real moments. Table III shows that eliminating the biases, especially overconfidence, increases the sensitivity of returns to profitability well above its empirical value and reduces the volatility of q below its empirical value. To compensate for these failings, the no-bias estimation increases the informativeness of the soft signal until it becomes perfectly revealing. But the resulting increased volatility of q makes investment sensitivity to Tobin's q too low. To offset the increased volatility of q , the estimation reduces the volatility of aggregate productivity. However, this causes the model's prediction of aggregate investment volatility to be too low. In summary, the no-bias model struggles to match the volatility of returns and investment because it links both of these variables to true productivity, rather than perceived productivity.

B. Analysis of the Model in which Managers Maximize Long-Term Rational Firm Value

The models analyzed so far feature managers who maximize firms' asset prices, and thus choose investment according to investors' information and beliefs. In this subsection, we consider a model with managers who maximize long-term rational firm value. The manager in this model efficiently selects investment to maximize true firm value, rather than investors' perception of firm value.²⁴ Because the manager uses the true rate of mean reversion in

²³ The sensitivity of firm investment to q in the overconfidence model is much higher than in the overextrapolation model mainly because the volatility of q is two times higher in the overextrapolation model.

²⁴ In this alternative model, a rational manager could exploit market mispricing through timing equity and debt issuance. Such market-timing behavior would have no impact on investment policy in our model because investment and financing decisions are separable when capital structure does not affect firm value. If our model included a capital structure trade-off, then market-timing considerations would affect investment policy as in Stein (1996).

productivity and true signal precision to forecast productivity, her forecast differs from investors' forecasts. Biased investors know that the manager is not pursuing their ideal policy and think that this reduces firm value.²⁵ Here we estimate this alternative model to test whether it fits the data and develop intuition for the impact of managerial behavior.

The rightmost column in Table I shows that the efficient manager model fails to fit the data. Based on the $\chi^2(5)$ value of 20.09, we reject this alternative model's overidentifying restrictions at even the 1% level. We explicitly compare the fit of the efficient manager and benchmark models using Singleton's (1985) test statistic for competing nonnested models. Comparing the null hypothesis of the benchmark model against the efficient manager alternative, we cannot reject the benchmark model at the 10% level ($\chi^2(1) = 0.92$; p -value 0.339). However, comparing the efficient manager null hypothesis against the benchmark alternative, we reject the efficient manager model at the 1% level ($\chi^2(1) = 21.77$; p -value < 0.001).

A key reason for the benchmark model's superior fit is that the efficient manager model predicts that the investment anomaly is too small relative to the value anomaly. Intuitively, the value anomaly arises when investors' productivity expectations are too high for growth firms and too low for value firms. When they are run efficiently, value firms have higher investment rates than investors think is ideal. This makes investors even more pessimistic about value firms, thereby exacerbating the value anomaly. That is, the realized cash flows of efficiently managed value firms positively surprise biased investors even more than the realized cash flows of inefficiently managed value firms. Biased investors are, however, more negatively surprised by the cash flows of efficiently managed growth firms. They mistakenly think that efficiently

²⁵ We represent the differing beliefs of investors and managers using a third state variable in the firm value function in equation (17). We solve for the value function using the numerical method in Appendix A.

managed growth firms do not currently invest enough and thus have valuable growth opportunities that will soon be exploited as managers observe high signals of productivity.

In comparison, the investment anomaly arises when managers tend to invest in situations in which investors are overly optimistic. If managers maximize firms' current asset prices, which are set by biased investors, they invest more whenever investors' irrational optimism increases, *ceteris paribus*—so this managerial behavior unambiguously generates an investment anomaly. If managers invest efficiently, however, high investment may not coincide with irrational optimism. Both investment and optimism depend positively on the soft productivity signal, but they can be negatively correlated because overconfident investors may underreact to the cash flow signal. These competing effects explain why the model with efficient investment produces an investment anomaly that is smaller than the empirical anomaly and can have the wrong sign.

V. The Detection and Correction of Mispricing

In this section, within the context of the benchmark model, we explore how one can detect mispricing using observable proxies for behavioral biases. Empirical asset pricing research uses the predictability in average *realized* returns as a proxy for mispricing because measuring *ex ante* rationally expected risk-adjusted returns (i.e., alphas) and firms' fundamental values is impossible in practice. In our model, however, we can measure firms' *true* alphas and values from the perspective of a rational agent with investors' information.²⁶

A firm's (true) alpha is the rational expectation of the firm's raw return minus the appropriate discount rate—that is, the sum of the risk-free rate and a firm's exposure to aggregate risk times the price of risk. Alpha measures the rate of convergence to true firm value.

²⁶ We estimate firms' alphas from the perspective of a rational agent by computing firms' average returns, including both price changes and dividends, based on 2,500 simulated paths of the economy.

True firm value is the rationally expected present value of the firm's cash flows discounted at the appropriate rate. True firm value is a solution to a partial differential equation analogous to equation (17), except that it includes an extra state variable representing the rational estimate of productivity. We solve this equation using the method described in Appendix A. We define *mispricing* as the firm's market value divided by its true value, so that a ratio above one implies that price exceeds value. This definition is featured in Black (1986), who states "we might define an efficient market as one in which price is within a factor of 2 of value" (p. 533). In his view, efficiently priced firms exhibit mispricing values between 0.5 and 2.0.

Panel A in Table IV summarizes true one-year alphas and mispricing in the benchmark model. The statistics in Panel A reveal that the cross-sectional standard deviations of alphas and mispricing are high in the model (5.23% and 16.5%) relative to the empirical anomaly estimates, implying significant pricing inefficiencies. In the model, the average value-weighted true alpha of -0.81% is lower than the equal-weighted alpha of 0.61%, implying that larger firms have lower alphas, as explained below.²⁷

[Insert Table IV here.]

Panel A shows that the equal-weighted mean mispricing (i.e., price-to-value) ratio is 1.104 in the model, meaning that the average firm is significantly overpriced. This occurs because investors overestimate the value of firms' growth options. Both information processing biases cause agents to think their estimates of future productivity are more accurate than they actually are. Consequently, they expect firms to make better-informed investment decisions than they actually will, which leads agents to overprice firms' assets.

²⁷ Value-weighted true alphas computed across all firms need not be zero because agents can systematically over- or underestimate the future returns of the average firm in the economy.

Panel B in Table IV shows the distribution of one-year alphas for portfolios sorted on firm characteristics. The first column in Panel B shows that simulations of the benchmark model produce large variation in alphas: the top-to-bottom decile difference in firms' alphas is 18.41% (11.01% – (–7.40%)). The distribution of alphas is positively skewed, much like empirical data on firms' realized alphas.

The key result shown in Panel B is that the *majority* of the differences in portfolios' true alphas remain unexplained by observable variables, such as B/M, investment, and profitability. The top-to-bottom decile spreads in alphas are 4.38% for B/M, 6.65% for investment, and 1.71% for profitability, all of which are small compared to the spread of 18.41% in firms' true alphas. By this criterion, investment is the characteristic that is most able to explain differences in alphas. This implies that investment is the best proxy—though it is quite noisy—for the rate at which agents correct their expectations of firms' cash flows.

Panel C on mispricing shows that the model predicts that market prices often deviate widely from firm values. The average firms in the bottom and top mispricing deciles are mispriced by $0.821 - 1 = -17.9\%$ and $1.404 - 1 = 40.4\%$, which one could interpret as a sign of material market inefficiency. The top-to-bottom decile spread in mispricing is 58.3% ($1.404 - 0.821$) in the model, even though the model predicts a value anomaly that is slightly too small. Interestingly, Panel C shows that B/M and investment do a good job in explaining mispricing, implying they are reasonable proxies for errors in agents' expectations of firms' cash flows. Comparing the explanatory power of B/M in Panels B and C, one can deduce that B/M captures a long-term component of mispricing, most of which is not eliminated in the next year.

Table V presents cross-sectional regressions of true alphas on five observable firm characteristics using simulated data. The five characteristics are B/M, investment, the log of

market capitalization (size), profitability, and last year's asset returns. All five firm characteristics are standardized to have a mean of zero and unit standard deviation. The bottom row in Table V reports the R^2 statistics from these regressions.

[Insert Table V here.]

The low R^2 statistics reveal that the ability of each of the individual firm characteristics to explain true alphas is quite low. The R^2 s from the five univariate regressions in Table V range from 0.2% to 16.6%. The main reason for the low R^2 s is that the econometrician cannot directly measure the soft signal to which investors overreact. This is why using data from the simulated economy is necessary to measure the true extent of mispricing.

In the simulated model, the return predictability coefficients on firm size in the univariate regression (4) and the multivariate regression (8) are highly negative. Although the estimation does not include moments based on the empirical size anomaly, the model predicts a negative size premium. This is consistent with the empirical findings in Banz (1981) and Fama and French (1992). In the model, firms are ex ante identical, so they become large only after a series of positive productivity signal realizations. Investors tend to be too optimistic about large firms because they overweight positive soft signals and overextrapolate positive productivity trends. Both mechanisms account for the negative size premium.

The coefficient on past one-year returns in predicting one-year alphas is approximately zero, showing that the model does not predict positive return momentum of the sort identified empirically in Jegadeesh and Titman (1993). This is interesting because, in the model, profitability positively predicts returns as overconfident agents underreact to profitability signals. This effect alone would produce return momentum because profitability is positively associated with returns. There is, however, an offsetting effect that comes from the positive correlation

between the soft information signal and returns. Agents overreact to the soft signal, which could cause negative autocorrelation in returns. The two countervailing effects roughly offset at the empirical parameter estimates in this specification. Naturally, including moments that measure return momentum in the estimations could lead to model parameters that produce positive momentum. A satisfactory explanation of momentum, however, could require introducing a new parameter to capture limited investor attention or conservatism.

To disentangle the impacts of the two biases, Table VI analyzes the cross-section of true alphas in both one-bias simulations for portfolios consisting of firms sorted into deciles using firm characteristics, such as B/M and profitability. This table is analogous to Panel B in Table IV, which shows the results from the benchmark model. As before, the analysis focuses on top-to-bottom decile spreads in alphas.

[Insert Table VI here.]

The most striking fact in Table VI is that the simulation with overextrapolation produces a spread in true alphas of only 6.72%, and two different observable variables, profitability and investment, capture nearly all of this spread (5.80% and 4.93%, respectively). The corrections in extrapolative agents' beliefs occur mainly through observable profit signals because profits are quite informative about latent productivity. In contrast, the simulation with just overconfident agents produces a much larger spread in alphas (14.91%), and none of the observable variables such as B/M can reproduce a spread in returns of more than 5.59%. The reason is that corrections in overconfident agents' beliefs are driven by the soft productivity signal, which is not directly observable and is less strongly linked to true productivity. Because overconfidence causes most of the differences in firms' alphas, the model predicts that the cross-section of alphas cannot be

explained by “tangible” information, such as past profits, consistent with empirical findings in Daniel and Titman (2006).

VI. Concluding Discussion

We structurally estimate a model in which information processing biases cause mispricing and distort firms’ investment decisions. Our benchmark model fits asset return and investment data far better than alternative models without any one of three key ingredients: the overconfidence bias, the overextrapolation bias, and inefficient investment. The key overconfidence and overextrapolation parameters are well identified by the data and line up reasonably well with direct measures of agents’ expectations. The two behavioral biases help the model match investment data as well as anomalies, and inefficient managerial behavior helps it match asset return data as well as firm behavior. These findings illustrate the importance of modeling and estimating the joint behavior of investors and managers.

One need not interpret the parameter estimates from our model as constant over time. Investors and managers may use empirical data on asset returns to learn about their mistaken parameter beliefs, as suggested by Brav and Heaton (2002). If so, asset return predictability patterns in the next 40 years could be markedly different from those in the past 40 years. As the economy’s latent structural parameters change, behavioral biases may dissipate or be replaced by new biases. Furthermore, rational agents may learn about such changes only gradually, generating return patterns that are predictable only in hindsight. Under these alternative interpretations, our model still provides a useful ex post description of the underlying mispricing and investment inefficiencies that occurred over the past 40 years. This description of the past may or may not apply to the future.

Our model is intended to serve as a benchmark for future work that could further develop the behavioral side or the risk side of our model. On the behavioral side, one could model rational arbitrageurs, whose actions presumably affect return anomalies—mitigating some, while perhaps exacerbating others. One could also consider alternative information structures in which managers and investors have access to different, possibly complementary, sources of information. Furthermore, allowing for additional biases, such as limited investor attention, could enable the model to make richer predictions of both anomalies and firm behavior.

The risk side of our model features only one source of priced risk. Although we reject the model with one risk and no biases in favor of the version with biases, we do not test alternative risk-based explanations of return anomalies. Future models could include additional sources of risk-based return predictability that could reduce the explanatory power of biases. Regardless of how sophisticated such models become, our empirical approach would still apply: to test behavioral against risk-based mechanisms, one must evaluate both sets of predictions against the same data.

Appendix A. Numerical Solution of the Model

Here we solve for the firm's value function V . The partial differential equation in equation (17) characterizes firm value as a function of three state variables: capital stock K_t , productivity estimate \hat{f}_t , and state of the economy m_t . Now we introduce a variable transformation that reduces the number of state variables in equation (17) to two. We define

(A1)

so that equation (17) simplifies to

$$\begin{aligned}
 (\mu - g)J = & \hat{f}X^\alpha - \delta X + \frac{1}{2\phi}(J_X - 1)^2 X - (\mu - r - g)XJ_X \\
 & + \frac{1}{2}\sigma_m^2 X^2 J_{XX} - \lambda_B(\hat{f} - \bar{f})J_f + \frac{1}{2}\left(\sigma_f^2 \eta_B^2 + \frac{\gamma_B^2}{\sigma_h^2}\right)J_{ff},
 \end{aligned}
 \tag{A2}$$

where

$$J(X, \hat{f}) = m^{-1}V(K, \hat{f}, m).
 \tag{A3}$$

Next, one can solve for the function J in equation (A2) and use equation (A3) to obtain the value function V . By differentiating both sides of equation (A3) with respect to K , one obtains

(A4)

Equation (A4) states that marginal q , V_K , equals the derivative of J with respect to X .

We use numerical approximations to solve for functions J and J_X . The approximations apply over a mesh grid of (X, \hat{f}) points over intervals $X \in [X_L, X_H]$ and $\hat{f} \in [f_L, f_H]$, which we choose such that the realized state variables X and \hat{f} attain values outside the bounds with less than 1% probability.

We start with a set of initial guesses of J_X over the grid. Next, we approximate the value function J using a sum of products of Chebyshev polynomials²⁸:

$$J(X, \hat{f}) = \sum_{m=0}^M \sum_{n=0}^N a_{m,n} T_m \left(2 \frac{X - X_L}{X_H - X_L} - 1 \right) T_n \left(2 \frac{\hat{f} - f_L}{f_H - f_L} - 1 \right). \quad (\text{A5})$$

In equation (A5), M and N are integers that specify approximation degrees, $\{a_{m,n}\}$ is a set of unknown polynomial coefficients, and $\{T_m\}$ and $\{T_n\}$ are Chebyshev polynomials.²⁹ To solve for the polynomial coefficients $\{a_{m,n}\}$, we write equation (A2) in terms of the Chebyshev approximation and its partial derivatives. Because the nonlinear terms in equation (A2) only involve J_X , for which we have initial guesses, substituting these guesses in the approximation equations results in a linear system of equations in terms of $\{a_{m,n}\}$. By choosing the numbers of distinct X and \hat{f} values on the grid to be M and N , respectively, we obtain an exactly identified linear system of equations, which we then solve to compute $\{a_{m,n}\}$. Using the resulting polynomial approximation of the value function J , we next compute its partial derivative J_X . Finally, we compute an error criterion function as the difference between the initial guesses and partial derivative values of J_X normalized by ϕ , for each grid point. We iterate this procedure to minimize the sum of squared values of the error criterion using a nonlinear least square routine.

²⁸ We provide only a brief overview of this procedure—see Judd (1998) for details. Chebyshev polynomials are a family of orthogonal functions that constitute a basis for the space of continuous functions.

²⁹ We use approximation degrees $M = 80$ and $N = 10$. In the efficient manager model, the approximation degree for the third state variable (biased minus rational productivity estimate) discussed in footnote 25 is ten.

Appendix B. Estimation and Simulation Methodology

We determine three model parameters using direct estimation and the remaining ten using SMM. First, we directly estimate parameters g , r , and δ using economy-wide dividend growth, the real risk-free rate of interest, and the average firm depreciation rate, as explained below. Next, we apply our SMM procedure to determine the values of the remaining ten model parameters based on 15 additional moments: ten based on firms' real characteristics, two based on the economy, and three based on firms' asset returns. We estimate each empirical moment using the time series average of cross-sectional estimates following Fama and MacBeth (1973).

Our goal in constructing an empirical sample is to provide an appropriate testing ground for the model in Section I. The basis for the sample is public U.S. firms from 1968 through 2006. We restrict the sample to firms that have common stocks trading on the NYSE, NASDAQ, or Amex in CRSP data and those with comprehensive accounting information in Compustat data.

Because the model considers a firm in its steady state, all sample firms must have at least 60 consecutive months of nonmissing stock prices before data collection. The accounting data needed to identify the key model parameters includes market equity, book equity, book debt, cash, property, plant, and equipment (PP&E), profits, investment, sales, and total assets for each of the last two years. The sample excludes firms with negative values of market equity, book debt, PP&E, sales, and noncash assets. The sample also excludes public utilities, regulated firms, and financial firms—those with Standard Industry Classification codes in the ranges 4900-4999, 9000-9999, or 6000-6999—because our model of the capital stock may not apply to these firms. We also exclude firms with stock prices less than \$1 and those without trading volume in the most recent month because these firms' stock returns may contain significant measurement error.

Computing the empirical moments requires two types of raw inputs: accounting and asset market variables. Each firm's accounting variables include book equity, book debt, total assets, Tobin's q , investment, profits, and depreciation. We define book equity as in Davis, Fama, and French (2000), book debt as in Kaplan and Zingales (1997), and total assets using the noncash asset definition in Cooper, Gulen, and Schill (2008). We do not include cash in firm assets because firm assets in the model do not include any cash. Thus, we also subtract cash from firm value when computing q , which is $(\text{market equity} + \text{book debt} - \text{cash}) / (\text{noncash assets})$. Market equity is shares outstanding times the stock price at the end of the firm's fiscal year. Investment is the annual growth rate of assets. Profitability is after-tax operating income before depreciation and before research and development expenses, divided by assets. That is, we treat research and development as an investment, not an expense. Depreciation is the annual change in the difference between gross capital and PP&E, divided by assets.

We compute monthly asset returns for each firm as the weighted average of equity and debt returns, using the firm's capital structure weights. We use standard CRSP data to measure the equity return. A lack of firm-specific bond return data presents a challenge for measuring debt returns. For the years 1973 through 2006, we use the monthly return on the Lehman Brothers investment grade bond index as a proxy for firms' debt returns. For the years 1968 to 1972, when the Lehman index is unavailable, we use the equal-weighted monthly return on U.S. Treasuries with maturities between five and 10 years from the CRSP U.S. Monthly Treasury Database. From 1973 to 2006, a regression of the investment grade returns on the Treasury index return yields a coefficient of 1.00 with a standard error of 0.03 and an R^2 of 83%. The correlation of 0.91 and regression coefficient near 1.0 suggest that the Treasury index return is a reasonable proxy for investment grade bond returns in most economic climates.

We use abnormal asset returns in all moments involving returns, except the market-wide return moment. Each firm's abnormal return is its raw return minus its required return, which is the risk-free rate plus the firm's equity market beta multiplied by the excess equity market return. We use risk-free rate and excess equity market return data from Kenneth French's website.³⁰ We compute each firm's (rolling) beta by regressing its excess return on the excess equity market return using the past five years of monthly return data. We compute portfolio abnormal returns in the anomaly moments below using the alpha (i.e., regression intercept) from a full-sample time-series regression of the portfolio's excess return on the equity market's excess return.

For our three directly estimated moments, we set the risk-free rate (r) equal to the average yield on five-year nominal Treasury bills deflated by the Consumer Price Index (CPI). This real interest rate is 2.68% for our sample. We obtain the Treasury yield from St. Louis Federal Reserve Economic Data and the CPI from Center for Research on Securities Prices (CRSP). To estimate the economy-wide growth of economy-wide productivity (g), we set the expected growth of dividends ($g - \sigma_m^2/2$) equal to the value-weighted average across all sample-eligible firms of the growth in firm assets (1.87%). The depreciation rate of firm assets (δ) is the equal-weighted average of depreciation across all sample-eligible firms (2.96%).

As described in the text, the two economy-wide moments are the value-weighted average of firms' asset market returns and the volatility of the growth of firm assets, where both are computed using all sample-eligible firms. We actually use the mean squared growth of firm assets to infer the volatility of growth via a monotonic transformation because the moments' covariance matrix is easier to compute using mean squared growth. However, we report the point estimate and standard error for volatility for ease of interpretation.

³⁰ See http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html (retrieved in October 2008).

For the three return anomaly moments, such as the one-year value anomaly, we use the monthly difference between the equal-weighted raw returns of the firms in the top and bottom deciles of the accounting variable. In forming portfolios, we allow for a reporting lag of three months for accounting variables. We compute the averages and standard deviations of variables such as Tobin's q , profitability, investment, and asset returns using monthly cross-sectional estimates. We winsorize q , profitability, and investment at the 5% level to reduce the influence of outliers. We winsorize returns at the 5% level only for computing return volatility, not for the other return moments. For the standard deviation of investment only, we use the two-year average of firm investment as the basis for the computation. The motivation is to smooth possibly lumpy firm-specific investment rates caused by factors beyond our model, including fixed costs and delays in the implementation of investment decisions.

The point estimate for each empirical moment is the time-series average of the cross-sectional moment. Let d be the 15-by-1 vector consisting of all 15 point estimates, and let W be the covariance matrix of d . We estimate the W matrix in a parsimonious manner that accounts for time-series and cross-firm correlations within and across moments. Specifically, we model the yearly time series of each annual moment as a univariate first-order autoregressive (AR(1)) process, except that we allow for correlations in the residuals across moments. First, we estimate the AR(1) coefficients, intercepts, and their covariance matrix. The covariance matrix is based on the formula in Newey and West (1987) with the number of lags selected based on the formula in Newey and West (1994). Next, we compute the gradient matrix measuring the sensitivity of each moment to the AR(1) coefficients and intercepts. Lastly, we use the AR(1) model point estimates, the covariance matrix of these estimates, and the gradient matrix above in the delta method to obtain the covariance matrix of the moments (W).

The goal of the SMM estimation is to select model parameters that produce simulated model moments that are as close as possible to the empirical moments above. We compute the distance between the model and empirical moments (d) using an SMM criterion function that assumes a quadratic form: $(m - d)^T W^{-1} (m - d)$, where m is the 15-by-1 vector of simulated model moments. We select model parameters to minimize this SMM criterion function using a nonlinear least squares (NLS) routine. The NLS algorithm converges to the same parameter vector regardless of the initial guess, suggesting that we are not mistaking a local minimum for a global one. We use the inverted empirical covariance matrix (W^{-1}) to weight the model's prediction errors of the 15 moments ($m - d$). This is the efficient method for selecting model parameters under the conditions described in Cochrane (2001).

Section II summarizes the empirical computation of d and W . The first step in computing the vector of simulated model moments (m) is to obtain numerical approximations of the firm value and policy functions, using methods described in Appendix A. Next, we apply these numerical approximations over discrete time intervals of 40 periods per year to generate simulated sample paths of model firms. We generate simulated data for a cross section of 1,000 firms, using 20 samples of 50 years each. For each sample firm, we draw initial values of true productivity and agents' productivity estimate from their respective steady-state distributions. We assume that the firm-specific component of productivity is independent across firms. To allow the capital stock to reach its steady-state distribution, we discard the first ten years of data in each sample. We use the remaining 40 years of data in each sample to compute the model counterparts of empirical data moments and the resulting SMM criterion function. The 40 years of simulated data correspond closely to the 39 years of empirical data that we attempt to match.

REFERENCES

- Alpert, Marc, and Howard Raiffa, 1969, A progress report on the training of probability assessors, Working paper, Harvard University.
- Baker, Malcolm, Jeremy Stein, and Jeffrey Wurgler, 2003, When does the market matter? Stock prices and the investment of equity-dependent firms, *Quarterly Journal of Economics* 118, 969–1005.
- Bansal, Ravi, Robert F. Dittmar, and Christian T. Lundblad, 2005, Consumption, dividends, and the cross section of equity returns, *Journal of Finance* 60, 1639–1672.
- Banz, Rolf W., 1981, The relationship between return and market value of common stocks, *Journal of Financial Economics* 9, 3–18.
- Barber, Brad, and Terrance Odean, 2001, Boys will be boys: Gender, overconfidence, and common stock investment, *Quarterly Journal of Economics* 116, 261–292.
- Barberis, Nicholas, and Ming Huang, 2001, Mental accounting, loss aversion, and individual stock returns, *Journal of Finance* 56, 1247–1292.
- Barberis, Nicholas, and Ming Huang, 2008, Stocks as lotteries: The implications of probability weighting for security prices, *American Economic Review* 98, 2066–2100.
- Barberis, Nicholas, Ming Huang, and Tano Santos, 2001, Prospect theory and asset prices, *Quarterly Journal of Economics* 116, 1–53.
- Barberis, Nicholas, Andrei Shleifer, and Robert Vishny, 1998, A model of investor sentiment, *Journal of Financial Economics* 49, 307–343.
- Ben-David, Itzhak, John R. Graham, and Campbell R. Harvey, 2010, Managerial miscalibration, NBER Working Paper No. 16215.
- Benartzi, Shlomo, 2001, Excessive extrapolation and the allocation of 401(k) accounts to company stock, *Journal of Finance* 56, 1747–1764.
- Benartzi, Shlomo, and Richard H. Thaler, 1995, Myopic loss aversion and the equity premium puzzle, *Quarterly Journal of Economics* 110, 73–92.
- Berk, Jonathan B., Richard C. Green, and Vasant Naik, 1999, Optimal investment, growth options, and security returns, *Journal of Finance* 54, 1553–1607.
- Black, Fischer, 1986, Noise, *Journal of Finance* 41, 529–543.
- Brav, Alon, and J.B. Heaton, 2002, Competing theories of financial anomalies, *Review of Financial Studies* 15, 575–606.

- Chirinko, Robert S., and Huntley Schaller, 2001, Business fixed investment and “bubbles”: The Japanese case, *American Economic Review* 91, 663–680.
- Cochrane, John H., 2001, *Asset Pricing* (Princeton University Press).
- Cooper, Michael J., Huseyin Gulen, and Michael J. Schill, 2008, Asset growth and the cross-section of stock returns, *Journal of Finance* 63, 1609–1651.
- Da, Zhi, and Mitch Warachka, 2011, The disparity between long-term and short-term forecasted earnings growth, *Journal of Financial Economics* 100, 424–442.
- Daniel, Kent, David Hirshleifer, and Avanidhar Subrahmanyam, 1998, Investor psychology and security market under- and overreactions, *Journal of Finance* 53, 1839–1886.
- Daniel, Kent, and Sheridan Titman, 2006, Market reactions to tangible and intangible information, *Journal of Finance* 56, 1605–1643.
- Davis, James L., Eugene F. Fama, and Kenneth R. French, 2000, Characteristics, covariances, and average returns: 1929 to 1997, *Journal of Finance* 55, 389–406.
- DeBondt, Werner F. M., and Richard H. Thaler, 1990, Do security analysts overreact? *American Economic Review* 80, 52–57.
- Fama, Eugene F., and Kenneth R. French, 1992, The cross-section of expected stock returns, *Journal of Finance* 47, 427–465.
- Fama, Eugene F., and Kenneth R. French, 2008, Dissecting anomalies, *Journal of Finance* 63, 1653–1678.
- Fama, Eugene F., and James D. MacBeth, 1973, Risk, return, and equilibrium: Empirical tests, *Journal of Political Economy* 81, 607–636.
- Gilchrist, Simon, Charles P. Himmelberg, and Gur Huberman, 2005, Do stock price bubbles influence corporate investment? *Journal of Monetary Economics* 52, 805–827.
- Gomes, João, Leonid Kogan, and Lu Zhang, 2003, Equilibrium cross section of returns, *Journal of Political Economy* 111, 693–732.
- Gomes, João, Amir Yaron, and Lu Zhang, 2006, Asset pricing implications of firms’ financing constraints, *Review of Financial Studies* 19, 1321–1356.
- Grinblatt, Mark, and Sheridan Titman, 2002, *Financial Markets and Corporate Strategy* (McGraw-Hill).
- Hennessy, Christopher A., and Toni M. Whited, 2007, How costly is external financing? *Journal of Finance* 62, 1705–1745.

- Hong, Harrison, and Jeremy C. Stein, 1999, A unified theory of underreaction, momentum trading, and overreaction in asset markets, *Journal of Finance* 54, 2143–2184.
- Jegadeesh, Narasimhan, and Sheridan Titman, 1993, Returns to buying winners and selling losers: Implications for stock market efficiency, *Journal of Finance* 48, 65–91.
- Judd, Kenneth L., 1998, *Numerical Methods in Economics* (The MIT Press).
- Kaplan, Steven N., and Luigi Zingales, 1997, Do investment-cash flow sensitivities provide useful measures of financing constraints? *Quarterly Journal of Economics* 112, 169–215.
- La Porta, Rafael, 1996, Expectations and the cross-section of expected returns, *Journal of Finance* 51, 1715–1742.
- La Porta, Rafael, Josef Lakonishok, Andrei Shleifer, and Robert W. Vishny, 1997, Good news for value stocks: Further evidence on market efficiency, *Journal of Finance* 52, 859–874.
- Lakonishok, Josef, Andrei Shleifer, and Robert Vishny, 1994, Contrarian investment, extrapolation, and risk, *Journal of Finance* 49, 1541–1578.
- Lettau, Martin, and Jessica A. Wachter, 2007, Why is long-horizon equity less risky? A duration-based explanation of the value premium, *Journal of Finance* 117, 1105–1139.
- Liu, Laura Xiaolei, Toni M. Whited, and Lu Zhang, 2009, Investment-based expected stock returns, *Journal of Political Economy* 117, 1105–1139.
- Newey, Whitney K., and Kenneth D. West, 1987, A simple positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix estimator, *Econometrica* 55, 703–708.
- Newey, Whitney K., and Kenneth D. West, 1994, Automatic lag selection in covariance matrix estimation, *Review of Economic Studies* 61, 631–653.
- Odean, Terrance, 1998, Volume, volatility, and profit when all traders are above average, *Journal of Finance* 53, 1887–1934.
- Odean, Terrance, 1999, Do investors trade too much? *American Economic Review* 89, 1279–1298.
- Panageas, Stavros, 2005, The neoclassical theory of investment in speculative markets, Working paper, University of Chicago.
- Polk, Christopher, and Paola Sapienza, 2009, The stock market and corporate investment: A test of catering theory, *Review of Financial Studies* 22, 187–217.

- Russo, J. Edward, and Paul J. H. Schoemaker, 1992, Managing overconfidence, *Sloan Management Review* 33, 7–17.
- Scheinkman, Jose A., and Wei Xiong, 2003, Overconfidence and speculative bubbles, *Journal of Political Economy* 111, 1183–1219.
- Singleton, Kenneth J., 1985, Testing specifications of economic agents' intertemporal optimum problems in the presence of alternative models, *Journal of Econometrics* 30, 391–413.
- Stein, Jeremy C., 1996, Rational capital budgeting in an irrational world, *Journal of Business* 69, 429–455.
- Titman, Sheridan, K.C. John Wei, and Feixue Xie, 2004, Capital investment and stock returns, *Journal of Financial and Quantitative Analysis* 39, 677–700.
- Warusawitharana, Missaka, and Toni M. Whited, 2012, Equity market misvaluation, financing, and investment, Working paper, University of Rochester.

Table I
Estimation Results

The table summarizes the results of the estimation of the biased representative agent model. Panel A reports empirical estimates, empirical standard errors, and simulated model values of the 15 moments used in the SMM estimations, along with the χ^2 statistics and the corresponding p -values of the estimations. For each of five estimations, Panel B reports the point estimates of the ten model parameters and their standard errors in parentheses. It also reports the standard deviations of the rational and biased agents' perceived errors in estimating productivity and the standard deviation of productivity. We compute these quantities using the parameter estimates.

Panel A: Empirical and simulated moments							
Moments	Empirical Estimate (Standard Error)		Both Biases	No Biases	Only Over- extrapolation	Only Over- confidence	Rational Manager
Mean Tobin's Q	1.580	(0.369)	1.587	1.293	1.297	1.384	1.713
Mean Profitability	0.132	(0.033)	0.089	0.067	0.067	0.070	0.100
Std. Dev. of Tobin's Q	0.937	(0.299)	0.957	0.773	0.790	0.778	0.817
Std. Dev. of Abnormal Return	0.211	(0.025)	0.207	0.180	0.179	0.175	0.200
Std. Dev. of Profitability	0.094	(0.021)	0.089	0.073	0.074	0.075	0.078
Std. Dev. of Investment	0.150	(0.057)	0.095	0.083	0.083	0.089	0.074
Persistence of Profitability	0.790	(0.017)	0.791	0.787	0.789	0.789	0.785
Abnormal Return Sensitivity to Profitability	1.757	(0.148)	1.867	1.866	1.843	1.760	1.778
Investment Sensitivity to Tobin's Q	0.067	(0.021)	0.091	0.117	0.114	0.107	0.080
Investment Sensitivity to Profitability	0.660	(0.054)	0.647	0.740	0.734	0.680	0.690
Volatility of Aggregate Investment	2.48%	(0.40%)	2.07%	1.41%	1.41%	1.41%	1.32%
Mean Value-Weighted Market Return	5.31%	(2.03%)	4.60%	4.13%	4.21%	4.06%	5.43%
Value Anomaly	6.38%	(2.35%)	4.83%	0.29%	0.13%	2.42%	7.50%
Investment Anomaly	-6.71%	(1.39%)	-6.69%	0.10%	0.45%	-2.35%	-2.85%
Profitability Anomaly	1.39%	(1.43%)	1.55%	0.05%	0.50%	3.76%	-1.07%
χ^2 statistic			7.80	52.05	51.10	22.51	20.09
χ^2 p-value			(0.17)	(0.00)	(0.00)	(0.00)	(0.00)

Table I (continued)

Panel B: Parameter estimates and standard errors										
Parameters	Both Biases		No Biases		Only Over-extrapolation		Only Over-confidence		Rational Manager	
<i>Returns-to-Scale α</i>	0.730	(0.085)	0.865	(0.040)	0.866	(0.071)	0.842	(0.045)	0.703	(0.056)
<i>Adjustment Cost φ</i>	2.435	(0.830)	3.000	(0.514)	2.997	(0.758)	2.563	(0.379)	2.151	(0.282)
<i>Idiosyncratic Productivity Shock σ_f</i>	0.514	(0.056)	0.525	(0.086)	0.519	(0.151)	0.503	(0.138)	0.390	(0.072)
<i>Noise in Cash Flow σ_h</i>	0.282	(0.086)	0.373	(0.041)	0.376	(0.095)	0.359	(0.070)	0.239	(0.048)
<i>True Mean Reversion λ</i>	0.158	(0.049)	0.134	(0.024)	0.129	(0.041)	0.130	(0.033)	0.137	(0.019)
<i>Biased Mean Reversion λ_B</i>	0.112	(0.007)	-	-	0.133	(0.051)	-	-	0.100	(0.037)
<i>True Signal Precision θ</i>	0.496	(0.075)	1.000	(0.520)	1.000	(1.227)	0.551	(0.055)	0.458	(0.265)
<i>Biased Signal Precision θ_B</i>	0.852	(0.112)	-	-	-	-	0.876	(0.061)	0.808	(0.223)
<i>Systematic Productivity Shock σ_m</i>	0.160	(0.041)	0.181	(0.039)	0.179	(0.053)	0.147	(0.073)	0.097	(0.021)
<i>Price of Risk μ</i>	0.112	(0.025)	0.107	(0.035)	0.107	(0.058)	0.105	(0.055)	0.110	(0.022)
Bias in Mean Reversion $\lambda - \lambda_B$	0.047	(0.047)	-	-	-0.005	(0.026)	-	-	0.037	(0.045)
Bias in Signal Precision $\theta - \theta_B$	-0.356	(0.067)	-	-	-	-	-0.324	(0.043)	-0.350	(0.066)
Std. Dev. of estimation error, rational	0.302		0.000		0.000		0.314		0.253	
Std. Dev. of estimation error, biased	0.132		-		0.000		0.117		0.129	
Std. Dev. of productivity	0.913		1.015		1.024		0.986		0.743	

Table II
Comparison of Survey and Model Estimates of Overconfidence

This table compares the bias parameters implied by return anomalies to biases estimated using direct measures of expectations in surveys. We consider direct estimates of overconfidence based on surveys of Harvard Business School (HBS) students in Alpert and Raiffa (1969), money managers in Russo and Schoemaker (1992), and executives in Ben-David, Graham, and Harvey (2010). The columns report which questions these survey participants were asked, their confidence intervals in percentage terms, and their empirical accuracy in percentage terms. We evaluate model agents' overconfidence based on the confidence and accuracy in their forecasts of firm productivity. The last two columns show the difference between the accuracy of model agents and survey participants and the t -statistic of the difference. Standard errors are reported in parentheses.

Survey Participants	Type of Question Asked	Confidence Interval	Participant Accuracy	Model Accuracy	Difference in Accuracy	t -statistic of Difference
Executives	1-yr S&P 500 return	80	33 (16)	22 (7)	11 (15)	0.62
Money managers	Industry knowledge	90	50 (2)	29 (8)	21 (9)	2.47
HBS students	General knowledge	98	54 (2)	40 (11)	14 (11)	1.29

Table III
Simulated Moments with Different Biases in the Biased Representative Agent Model

The table reports simulated values of the 15 estimation moments under different bias specifications. The first four columns correspond to cases with the overextrapolation bias only, overconfidence bias only, no biases, and both biases, respectively. We use the estimated parameters from the benchmark model with both biases in Panel B of Table I as the starting point for each simulation and change only the bias parameters. The last column replicates the empirical moment values from Panel A of Table I.

Bias in the model	Overextrapolation	Overconfidence	No Biases	Both Biases	Empirical
Mean Tobin's q	1.551	1.462	1.483	1.587	1.580
Mean Profitability	0.088	0.091	0.091	0.089	0.132
Std. Dev. Of Tobin's q	0.762	0.360	0.640	0.957	0.937
Std. Dev. of Abnormal Return	0.209	0.167	0.170	0.207	0.211
Std. Dev. of Profitability	0.087	0.087	0.088	0.089	0.094
Std. Dev. of Investment	0.089	0.070	0.073	0.095	0.150
Persistence of Profitability	0.788	0.797	0.793	0.791	0.790
Abnormal Return Sensitivity to Profitability	3.007	1.617	2.512	1.867	1.757
Investment Sensitivity to Tobin's q	0.113	0.179	0.117	0.091	0.067
Investment Sensitivity to Profitability	0.704	0.298	0.564	0.647	0.660
Volatility of Aggregate Capital Stock Growth	2.09%	2.02%	2.04%	2.07%	2.48%
Mean Value-Weighted Market Return	4.83%	5.12%	5.27%	4.60%	5.31%
Value Anomaly	3.62%	2.58%	0.04%	4.83%	6.38%
Investment Anomaly	-4.94%	-2.48%	0.37%	-6.69%	-6.71%
Profitability Anomaly	-5.89%	5.29%	0.21%	1.55%	1.39%

Table IV
Alphas and Mispricing in the Benchmark Model

The table reports summary statistics and the cross-sectional properties of firms' one-year alphas and mispricing generated by the estimated model. Alpha is the rational expectation of the firm's raw return minus the appropriate discount rate—that is, the sum of the risk-free rate and a firm's exposure to aggregate risk times the price of risk—over a one-year horizon. Mispricing is the ratio of the firm's stock price to the rationally expected present value of its future cash flow stream. Panel A reports the summary statistics of one-year alphas and mispricing. Panel B reports the equal-weighted average of alphas for firms sorted into deciles based on the alpha variable itself, B/M, investment, and profitability. Investment is the lagged value of annual capital expenditures normalized by capital stock, and profitability is the lagged value of annual profits normalized by capital stock. Panel C replicates the analysis in Panel B for mispricing.

Panel A: Summary statistics for expected returns and mispricing				
Summary statistic	Value-weighted mean	Equal-weighted mean	Median	Standard deviation
Alpha	−0.81%	0.61%	0.09%	5.23%
Mispricing	1.144	1.104	1.100	0.165

Panel B: Cross-section of alphas				
Sorting variable	Alpha	B/M	Investment	Profitability
Lowest decile	−7.40%	−0.02%	5.35%	0.01%
2	−4.41%	−0.70%	2.48%	0.30%
3	−2.89%	−0.56%	1.37%	0.25%
4	−1.65%	−0.32%	0.67%	0.33%
5	−0.49%	−0.11%	0.21%	0.40%
6	0.68%	0.16%	−0.17%	0.53%
7	1.97%	0.48%	−0.57%	0.66%
8	3.53%	0.97%	−0.87%	0.78%
9	5.69%	1.81%	−1.12%	1.07%
Highest decile	11.01%	4.36%	−1.30%	1.72%

Table IV (continued)

Panel C: Cross-section of mispricing				
Sorting variable	Mispricing	B/M	Investment	Profitability
Lowest decile	0.821	1.240	0.925	1.089
2	0.938	1.196	1.004	1.082
3	1.000	1.167	1.044	1.090
4	1.041	1.142	1.074	1.095
5	1.081	1.121	1.097	1.100
6	1.120	1.100	1.119	1.104
7	1.161	1.078	1.144	1.108
8	1.207	1.052	1.169	1.115
9	1.269	1.013	1.199	1.120
Highest decile	1.404	0.927	1.262	1.135

Table V
Cross-Sectional Regressions of True Alphas on Firm Characteristics

The table reports cross-sectional regressions of one-year true alpha and mispricing on observable firm characteristics using simulated data from the estimated model. Alpha is the rational expectation of the firm's raw return minus the appropriate discount rate—that is, the sum of the risk-free rate and a firm's exposure to aggregate risk times the price of risk—over a one-year horizon. The firm characteristics include B/M, investment, profitability, size (the log of firm's market value), and past one-year raw return. Reported coefficients and R^2 values are time-series averages of cross-sectional regression estimates. All right-hand side variables are standardized in each cross-sectional regression.

Regressions	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
B/M	0.014					0.005	0.002	0.015
Investment		-0.016				-0.013	-0.046	-0.042
Profitability			0.005				0.041	0.065
Size				-0.021				-0.040
Past one-year return					-0.001			0.001
R^2	7.6%	10.0%	1.0%	16.6%	0.2%	10.7%	36.8%	76.1%

Table VI
Alphas with Different Biases in the Biased Representative Agent Model

The table shows the cross-sectional properties of alphas and mispricing in one-bias model specifications in which either the overextrapolation (extrapolat.) bias or the overconfidence (overconf.) bias applies. We use the estimated parameters from the benchmark model with both biases in Panel B of Table I as the starting point for each simulation and change only each bias parameter. Alpha is the rational expectation of the firm's raw return minus the appropriate discount rate—that is, the sum of the risk-free rate and a firm's exposure to aggregate risk times the price of risk—over a one-year horizon. The columns report the equal-weighted average of alphas for firms sorted into deciles based on the alpha variable itself, B/M, investment, and profitability. Investment is the lagged value of annual capital expenditures normalized by capital stock, and profitability is the lagged value of annual profits normalized by capital stock.

Sorting variable	Alpha		B/M		Investment		Profitability	
Bias in the model	Extrapolat.	Overconf.	Extrapolat.	Overconf.	Extrapolat.	Overconf.	Extrapolat.	Overconf.
Lowest decile	-2.22%	-6.88%	-0.47%	-0.80%	3.74%	1.75%	4.10%	-3.08%
2	-1.35%	-3.90%	-0.68%	-0.56%	1.71%	0.98%	1.99%	-1.36%
3	-0.93%	-2.50%	-0.49%	-0.32%	0.97%	0.58%	1.16%	-0.72%
4	-0.56%	-1.39%	-0.32%	-0.21%	0.51%	0.32%	0.61%	-0.21%
5	-0.18%	-0.40%	-0.12%	-0.00%	0.15%	0.16%	0.19%	0.22%
6	0.23%	0.59%	0.12%	0.16%	-0.15%	-0.06%	-0.17%	0.57%
7	0.71%	1.62%	0.39%	0.39%	-0.42%	-0.23%	-0.50%	0.93%
8	1.31%	2.81%	0.76%	0.67%	-0.69%	-0.36%	-0.82%	1.29%
9	2.17%	4.41%	1.36%	1.10%	-0.96%	-0.52%	-1.19%	1.69%
Highest decile	4.50%	8.03%	3.11%	1.96%	-1.19%	-0.76%	-1.70%	2.51%

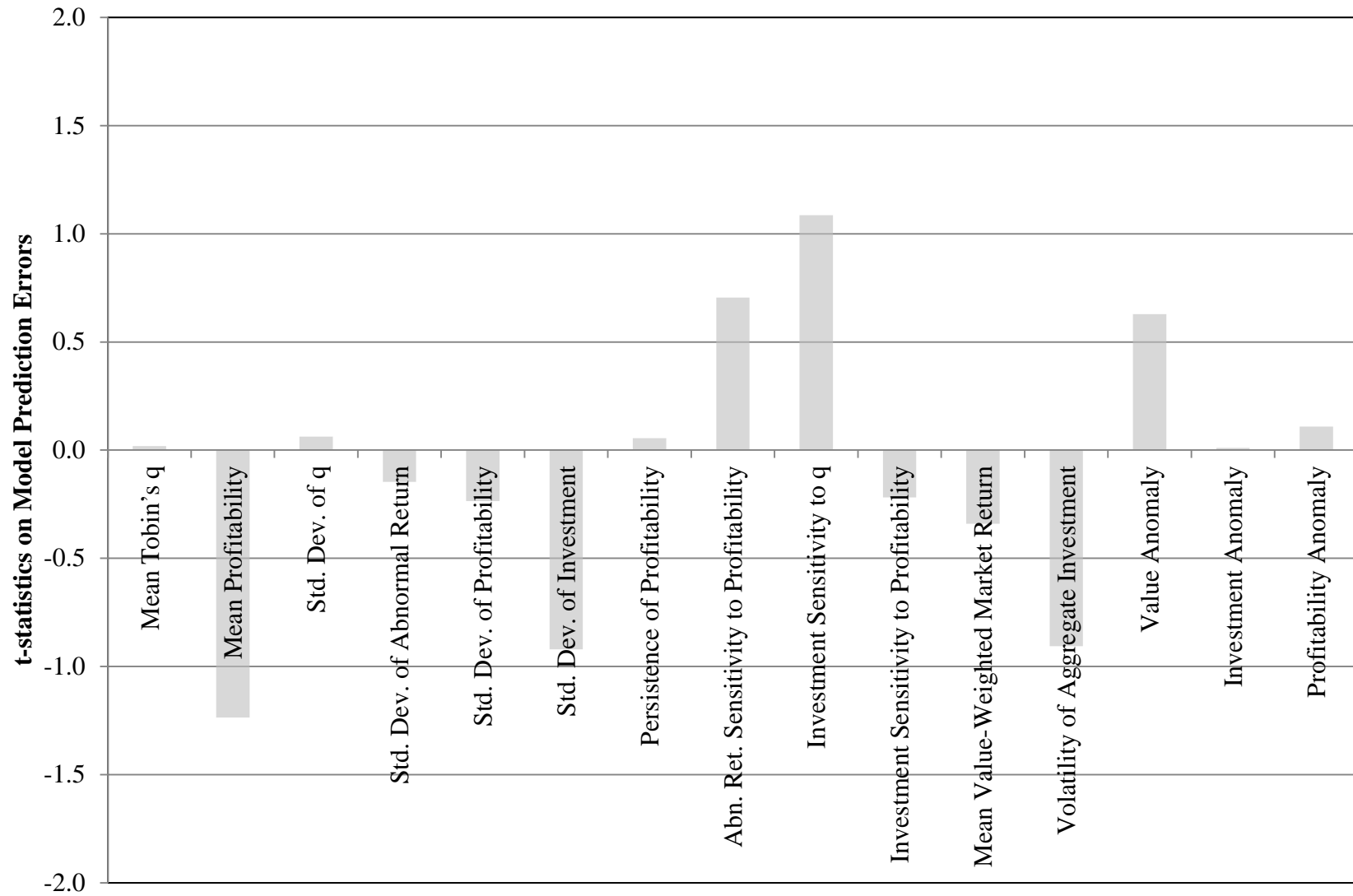


Figure 1. Model prediction errors. The figure shows the statistical significance of the difference between the 15 simulated model moments and the 15 empirical moments used in the estimation of the model: $t\text{-statistic} = (\text{simulated} - \text{empirical moment}) / (\text{empirical moment standard error} * (S + 1) / S)$, where S is the number of simulations, which is 20.