Nontraded Asset Valuation with Portfolio Constraints: A Binomial Approach

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We provide a simple binomial framework to value American-style derivatives subject to trading restrictions. The optimal investment of liquid wealth is solved simultaneously with the early exercise decision of the nontraded derivative. No-short-sales constraints on the underlying asset manifest themselves in the form of an implicit dividend yield in the risk-neutralized process for the underlying asset. One consequence is that American call options may be optimally exercised prior to maturity even when the underlying asset pays no dividends. Applications to executive stock options (ESO) are presented: it is shown that the value of an ESO could be substantially lower than that computed using the Black–Scholes model. We also analyze nontraded payoffs based on a price that is imperfectly correlated with the price of a traded asset.

The economics of asset pricing when one or more of the assets in the opportunity set are either subject to trading restrictions or entirely nontraded is a matter of great interest. Viewed from a practical perspective, we have several important examples of such assets that are subject to trading restrictions. Pensions, which represent perhaps the most significant of assets held by individual households, are subject to trading restrictions. It is typically the case that assets in pensions are not available for immediate consumption. Borrowing against pension assets is subject to significant direct and indirect costs by way of taxes and early withdrawal penalties. Human capital is another example. Housing investment is also illiquid and subject to significant transaction costs. Together, pensions, human capital, and housing constitute a substantial part of a typical household's assets. The significance of such nontraded assets for risk premia has already been noted by Bewley (1982). There are other circumstances where lack of unrestricted trading plays an important role. Executive compensation plans usually take the

form of options that are not allowed to be traded on the open market. They are subject to restrictions on how often and when they may be exercised. In addition, executives who own such options are not permitted to short the underlying shares of the company. In a similar vein, long-dated forward contracts are frequently entered into by counterparties who are fully aware that a liquid secondary market for the contract does not exist. Furthermore, the underlying commodity often is yet to be harvested or cannot be sold short. It seems reasonable then to think of such forward contracts as essentially nontraded assets. These examples stress the role of nontraded derivatives on an underlying asset on which there may be trading restrictions.

The purpose of this article is to provide a constructive framework to value derivative assets that are subject to trading restrictions. This framework relies on simple dynamic programming techniques and can be viewed as a counterpart to the martingale methods in Cvitanic and Karatzas (1992). Our approach, however, delivers significant new insights. In the context of a simple binomial model, we characterize the pricing and the optimal exercise strategies associated with derivative assets that are nontraded. The approach is illustrated with an executive stock option (ESO) example, although it is general and can be applied to any other contexts where trading restrictions are important. In particular, it could be utilized to explain endogenous convenience yields in long-term forward contracts which have a very thin market and hence may be viewed to a first approximation as nontraded assets (NTA). In some instances, such long-term forward contracts are written on assets which may not be shorted easily. Examples are long-dated forwards on crude oil or on commodities that are yet to be harvested. Our contribution pertaining to ESOs draws and builds significantly upon the work of Huddard (1994), Kulatilaka and Marcus (1994), and Carpenter (1998). We briefly review these articles to motivate our own work and place it in the proper perspective.

Huddard (1994) and Kulatilaka and Marcus (1994) consider expected utility maximizing models, which is in the spirit of our own work. But both articles assume that the nonoption wealth (liquid wealth) is invested in the risk-free asset. As Carpenter (1998) notes, this assumption places an artificial constraint on portfolio choice before and after the exercise of the option which may in turn distort the optimal exercise decision. Carpenter (1998) develops two models. The first is an extension of Jennergren and Naslund (1993). In this model she considers an *exogenous* stopping state in which the executive must either optimally forfeit or exercise the option. This setting is well suited to examine issues pertaining to vesting restrictions. In her second model, which is much closer to our own work, she studies an expected utility maximizing model in which the executive is offered an *exogenous* reward for leaving the firm at each instant. This induces the executive to optimally select the exercise (or continuation) policy. Carpenter concludes that the first model, which is much simpler to implement, does as well as the

more elaborate expected utility maximizing model in terms of predicting the actual exercise times and payoffs. Like Huddard (1994) and Kulatilaka and Marcus (1994), Carpenter (1998) assumes an exogenous investment policy for the nonoption wealth: the executive invests in the Merton (1969, 1971) portfolio. She notes an important complication in making this assumption: "Investing non-option wealth in the Merton portfolio is more appealing although not fully optimal in the presence of the option. Full optimality would allow the executive to choose investment and exercise strategies simultaneously. This scenario is intractable because the nonnegativity constraint on the stock holdings would become binding along some stock price paths, but not along other paths. Under these conditions, the optimal portfolio value would be a path-dependent function of the stock price, and backward recursion would be impossible." This is in fact one of the thrusts of our article. We model the simultaneous investment and exercise decision problem. This problem is path dependent as Carpenter (1998) correctly notes. However, an expansion of the state space enables us to formulate the problem as a purely backward problem that can be solved using a dynamic programming algorithm. As we show in the article, the optimal investment policy differs from the Merton policy. Thus our article provides a broad framework which is both constructive and easy to implement numerically.

Section 1 focuses on European-style nontraded assets. We analyze the private valuation of such an asset and the hedging policy when there is a no-short-sales constraint on the underlying asset. One insight arising out of this analysis is that trading restrictions manifest themselves in the form of an implicit dividend yield in the risk-neutralized underlying asset price process. This implicit dividend yield will lead to qualitatively different predictions for the exercise policies of American options on the underlying asset. In this context the private value of the asset is given by the certainty equivalent of its payoff. We show that this certainty equivalent is bounded above by the unconstrained asset value. We also provide a simple computational algorithm and a numerical example which illustrates the algorithm for nontraded European options. The solution of the constrained portfolio problem can be formulated in terms of a backward equation which involves the liquid wealth of the manager and his certainty-equivalent valuation. Due to the trading restrictions, a simple closed form solution such as Black and Scholes cannot be obtained. But this is precisely where our binomial framework lends itself superbly to the computation of the solution of the model.

In Section 2 we examine the private valuation and the early exercise policy associated with American-style nontraded derivatives. We first display simple examples involving call options on a non-dividend-paying stock in which the policy of holding the option to maturity is dominated by early exercise. These examples demonstrate that early exercise (prior to maturity) of an ESO may be optimal even when the underlying asset does not pay

dividends. This result runs counter to the conventional wisdom and seems to contradict a well-known proposition on the suboptimality of early exercise of such claims [see Merton (1973)]. In this context, exercising a call option has two consequences. On the one hand, it reduces welfare since the holder effectively gives up any potential appreciation in the expectation of the discounted call option payoff. On the other hand, early exercise provides an indirect benefit since it alleviates the no-short-sales constraint faced by the investor in the underlying market. Early exercise eliminates the need to hedge the NTA and increases liquid wealth; both of these effects increase the optimal demand for the stock and reduce the occurrence of a binding constraint. In instances in which the constraint is sufficiently binding when the NTA is held to maturity, the benefits of early exercise (relaxing the constraint) may dominate the costs (the loss of gains from appreciation of the discounted payoff) and this leads to the optimality of early exercise.[1] These results enable us to rationalize a well-known empirical regularity: the fact that executives tend to exercise their compensation options prior to maturity, and at times that do not seem to conform to the predictions of conventional options pricing theory. The arguments above show that such an early exercise policy may well be rational even in the absence of an exogenous reward for leaving the firm. The remainder of Section 2 characterizes the optimal exercise policy. Section 3 presents numerical applications of the model to ESO.

In Section 4 we extend our basic model to consider cases in which the nontraded payoff depends on a price S^2 that is imperfectly correlated with the price S^1 of the asset in which the investor can invest. Our analysis is based on a trinomial model. In this context we extend the dynamic programming approach of earlier sections and provide numerical results on the effects of correlation. We show that the private value of a nontraded call option may exceed the unconstrained value when correlation is negative or sufficiently low: in such a situation the nontraded option has diversification benefits that may offset the negative impact of the no-short-sales constraint on the traded asset. When correlation increases toward 1 the nontraded call option on asset 2 becomes a substitute for a nontraded call option on asset 1: the private values of the two contracts converge. For American-style call options on non-dividend-paying assets, early exercise may take place even when the two asset prices S^1 and S^2 are imperfectly correlated.

Appendix A presents background results on the dynamic programming approach to the problem. Proofs are collected in Appendix B. Appendix C details a recursive procedure to construct certainty-equivalent values. Appendix D solves the constrained portfolio problem with two underlying assets in the context of a trinomial model.

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[1] Arnason and Jagannathan (1994) point out that early exercise could be optimal even when the stock does not pay dividends in the presence of a reload feature.

1. European-Style Contingent Claims

In this section we consider a portfolio problem cast in a binomial lattice that can be solved using dynamic programming methods. A backward numerical procedure based on the dynamic programming algorithm is developed and implemented in the context of a simple numerical example.

1.1 The model

Our setting parallels the one in Cox, Ross, and Rubinstein (1979). We assume that the underlying asset price follows a binomial "process" with constant parameters u and d, and probability p.

$$
S_0 \quad
\begin{array}{ll}
^{\text{p}} \nearrow & S_0 u \\[4pt]
{}_{1-\text{p}} \searrow & S_0 d
\end{array}
$$
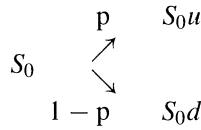
The initial asset value is S_0 and the tree has N steps. There is also a riskfree asset bearing a constant return r.[2] We assume that $u > r > d$.

In this complete market setting the risk-neutral probability is $q = (r - d)/(u - d)$ and the implied state price density (SPD), ξ_n, satisfies

$$
\xi_{n,n+1} \equiv \frac{\xi_{n+1}}{\xi_n} = \frac{1}{r}
\begin{cases}
q/p & \text{w.p. } p \\
(1-q)/(1-p) & \text{w.p. } 1-p
\end{cases}
\tag{1}
$$

subject to the initial condition $\xi_0 = 1$.

Suppose that an investor holds an NTA with payoff Y_N at the terminal date, where the cash flow depends on the asset price and takes the form $Y_N \equiv g(S_N)$ for some function $g(\cdot): \mathbb{R} \to \mathbb{R}$. Assume that the investor has strictly concave, nondecreasing utility function $u(\cdot)$ such that $\lim_{x \to 0} u'(x) = \infty$ and $\lim_{x \to \infty} u'(x) = 0$. Let X denote his liquid wealth (X_n is liquid wealth at date n) and π be the proportion of wealth invested in the risky asset. Suppose that the investor cannot short sell the underlying asset. He then faces the constrained dynamic problem

$$
\max_{\pi} Eu(X_N + Y_N) \quad s.t.
\tag{2}
$$

$$
\begin{cases}
X^u_{n+1} = X_n[r + \pi_n(u - r)] \\
X^d_{n+1} = X_n[r + \pi_n(d - r)]
\end{cases} ; \quad X_0 = x,
\tag{3}
$$

$$
X_n \pi_n \geq 0, \quad \text{for all } n = 0, \ldots, N-1
\tag{4}
$$

$$
X_N + Y_N \geq 0.
\tag{5}
$$

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[2] It is straightforward to extend the analysis to (stochastic) path-dependent coefficients (u, d, r). Likewise path-dependent payoffs can be easily accommodated in our framework.

1.2 A dynamic programming formulation

Let $J(X_n, n)$ be the value function for this constrained portfolio problem.[3] It satisfies, for $n = 0, \ldots, N - 1$,

$$J(X_n, n) = \max_{\pi \geq 0} EJ\left(\widetilde{X}_{n+1}, n + 1\right) \tag{6}$$

where

$$\widetilde{X}_{n+1} = X_n \left[r + \pi_n \left(\widetilde{r} - r\right)\right]. \tag{7}$$

Here the random variable $\widetilde{r}$ is the return on the stock (with values u or d) and $\widetilde{X}_{n+1}$ is liquid wealth at $n + 1$. The wealth process [Equation (7)] satisfies the initial condition $X_0 = x$. This recursion is subject to the boundary condition $J(X_N, N) = u(X_N + Y_N)$.

The Kuhn–Tucker conditions for this constrained problem are standard and are presented in Appendix A. Let $J'(X_{n+1}^u, n + 1)$ denote the marginal value of wealth, y_n the Lagrange multiplier for the budget constraint at date n, q_n^δ the adjusted risk-neutral probability, and $\xi_{n,n+1}^\delta$ the corresponding SPD which satisfies Equation (1) substituting q_n^δ for q. Finally, let $I(\cdot, n + 1)$ be the inverse of the marginal value of wealth at $n + 1$.

Our first theorem presents an equivalence relation between the constrained economy and an artificial unconstrained economy constructed by changing the drift of the risk-neutralized process.

Theorem 1. *Let $\{(X_{n+1}^*, y_n^*, q_n^*, \xi_n^*): n = 0, \ldots, N - 1\}$ denote the solution, described in Appendix A, to the constrained optimization problem subject to the initial condition $X_0 = x$. The constrained portfolio problem is equivalent to an unconstrained portfolio problem in an artificial economy in which the risk neutral measure is $\{q_n^*: n = 0, \ldots, N - 1\}$. In this unconstrained problem the stock price lives on a binomial lattice with parameters $u_n^* = u + \delta_n^*$ and $d_n^* = d + \delta_n^*$ where $\delta_n^* = (q - q_n^*)(u - d)$ and $\xi_{n,n+1}^*$ is the corresponding state price density.[4] The wealth process and the optimal portfolio are, for $n = 0, \ldots, N - 1$,*

$$X_{n+1}^* = I(y_n^* \xi_{n,n+1}^*, n + 1)$$

$$\pi_n^* = \frac{r}{q_n^*(1 - q_n^*)(u - d)} \left(\frac{G_{n+1}^{*u}}{G_n^*} p - q_n^*\right),$$

where $G_n^* \equiv \xi_n^* X_n^*$ and $G_{n+1}^{*u} = (\xi_{n+1}^* X_{n+1}^*)^u$.

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[3] The value function is a function of the stock price as well. For ease of exposition we adopt the simpler notation $J(X, n)$ with two arguments. This notation places emphasis on the fact that liquid wealth X is controlled by the investor through his portfolio choice. Other arguments of the value function are exogenous state variables (or time) that have a parametric effect on the optimal solution.

[4] Equivalence results of this type are known to hold in economies with portfolio constraints [see Cvitanic and Karatzas (1992)].

The equivalence between the two optimization problems implies that ξ_n^* is the pricing kernel for the constrained individual and $q_n^* = (r - d_n^*)/(u_n^* - d_n^*)$ his adjusted risk-neutral measure in the constrained market. The pair (ξ_n^*, q_n^*) encodes the private valuation of the constrained investor taking the environment as given. It reflects the no-short-sales constraint as well as the other exogenous parameters of the model, in particular the fact that he is endowed with a nontraded asset paying off at date N.

Note that the stock price takes the value

$$S_N = S_0 u^{N-k} d^k$$

at N if there are $N - k$ steps up and k steps down, for $k = 0, \ldots, N$. Using the definitions of u_n^* and d_n^* above and the fact that $u_n^* - \delta_n^*$ and $d_n^* - \delta_n^*$ are constant, we can also write

$$S_N = S_0 u^{N-k} d^k = S_0 \prod_{n \in \mathbf{N} - \mathbf{N}_k} (u_n^* - \delta_n^*) \prod_{n \in \mathbf{N}_k} (d_n^* - \delta_n^*)$$

for $k = 0, \ldots, N$, where $\mathbf{N} = \{0, \ldots, N - 1\}$ and $\mathbf{N}_k$ is the subset of k elements of $\mathbf{N}$ corresponding to the relevant down movements in the stock price. Hence the stock price in the constrained market can be viewed as paying an implicit dividend equal to δ_n^* at date $n + 1$. This interpretation also emerges if we use the definition of q_n^* to derive the stock price formula

$$S_n = \frac{1}{r} \left[q_n^* S_{n+1}^u + (1 - q_n^*) S_{n+1}^d \right] + \frac{1}{r} S_n \delta_n^*.$$

This formula shows that the stock price S_n is the discounted value of $S_n \delta_n^*$ augmented by the expected value of the discounted price at $n + 1$ where discounting is at the risk free rate and the expectation is taken under q^*. By analogy with the standard representation result, we can then interpret the stock as a dividend-paying asset with dividend yield δ_n^* under the adjusted risk-neutral measure q^*.[5]

We therefore reach an important conclusion: in the presence of a no-short-sales constraint, a derivative asset on a non-dividend-paying stock is equivalent to a derivative written on a dividend-paying stock under the adjusted risk-neutral measure. Theorem 1 then suggests that the (private)

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[5] The interpretation of the stock as a dividend-paying asset under the adjusted risk-neutral measure q^* is not meant to suggest that the properties of complete market models will hold in this economy. In fact there are significant differences. For example the adjusted risk-neutral measure q^* is not independent of the dividend yield δ^*, and this is a consequence of the no-short-sales constraint. Furthermore, q^* is affected by changes in exogenous variables such as the risk aversion of the investor and the properties of his nontraded payoff (see Sections 3 and 4). Our interpretation also assumes that the dividend yield applies to the initial stock price at date n (i.e., the implicit dividend payment at $n + 1$ is $S_n \delta_n^*$) and this differs from the standard binomial model with proportional dividend yield. Note that the solution of the constrained portfolio problem and our results concerning the rationality (optimality) of early exercise are independent of the interpretation given to the process δ^*.

valuation of a nontraded derivative may be lower than an otherwise identical derivative which is freely traded. This insight has far-reaching implications for the exercise policies associated with ESO, which we explore later. Additional intuition about the costliness of a constraint is provided by the certainty equivalent of the nontraded asset from the perspective of an investor endowed with the nontraded payoff and facing this trading restriction. What is the certain compensation required to induce this investor to give up his claim to the future cash flows associated with the NTA? We examine this issue next.

1.3 Certainty equivalent and unconstrained valuation

In the absence of any constraint the (complete market) value of the European contingent claim with payoff Y_N is $V_0 = (\frac{1}{r})^N E^*[Y_N] = E[\xi_N Y_N]$. In the presence of the no-short-sales constraint the value of the claim is the certainty equivalent $\widehat{Y}_0$ of the payoff Y_N [see Pratt (1964)]. By definition

$$\widehat{Y}_0 = \widehat{J}^{-1}(J(X_0, 0), 0) - X_0,$$

where $\widehat{J}(X_0 + \widehat{Y}_0)$ represents the value function for the constrained problem without cash flow Y_N but starting from initial wealth $X_0 + \widehat{Y}_0$, and $\widehat{J}^{-1}(\cdot, 0)$ is the inverse of this function.

With unrestricted investment the financial market described above is complete. The market value of an asset is then unambiguous: it represents the amount of initial wealth that is required to synthesize the terminal cash flow Y_N. The certainty equivalent, on the other hand, represents the compensation required by an individual for giving up his right to the terminal cash flow Y_N. Clearly these two notions coincide when the market is complete.

What is the relationship between the unconstrained value and the certainty equivalent of the NTA in our constrained problem? Our next result shows that the two notions relate in a simple manner.

Proposition 2. *Consider the constrained investment problem with an NTA paying a terminal cash flow Y_N. Suppose (a) that the payoff Y_N is an increasing function of the stock price and (b) that the short-sales constraint never binds for the pure portfolio problem with initial wealth $X_0 + \widehat{Y}_0$. The following properties hold.*

(i) If the short-sales constraint never binds in the constrained problem, the certainty equivalent and the complete market value are equal: $\widehat{Y}_0 = V_0$.

(ii) Suppose that the short-sales constraint binds with positive probability. Then the certainty equivalent is bounded above by the complete market value of the asset: $V_0 \geq \widehat{Y}_0$.

An investor who is effectively unconstrained in the constrained economy is in fact in a complete market situation. Equality between the two notions follows.

When the constraint binds at certain nodes of the tree the value function decreases (since the set of feasible policies is effectively restricted). The certainty equivalent then unambiguously decreases when condition (b) holds: $V_0 \geq \widehat{Y}_0$.

A numerical illustration of the results of Proposition 2 is given in Section 1.7. Before presenting this example we provide further insights about the solution of the constrained portfolio problem.

1.4 A certainty-equivalent formulation

Further light can be shed on the optimal portfolio policy by defining a certainty-equivalent payoff $\widehat{Y}_n$ for each date n and using it to reformulate the dynamic portfolio problem. Indeed, by definition of the certainty equivalent, the value function at every node equals the value function of a pure portfolio problem without NTA but starting from an adjusted (certainty equivalent) wealth level. It follows that we can write the objective function at date n entirely in terms of the value function of the certainty-equivalent problem at date $n + 1$. This procedure leads to a recursive construction of the certainty-equivalent payoff $\widehat{Y}_n$ which is detailed in Appendix C. We summarize the construction next.

Let $\widehat{J}(X_{n+1} + \widehat{Y}_{n+1}, n + 1)$ denote the value function at date $n + 1$ of the pure portfolio problem without NTA but starting from the adjusted wealth level $X_{n+1} + \widehat{Y}_{n+1}$. By definition the date $n + 1$ certainty equivalent $\widehat{Y}_{n+1}$ solves

$$J(X_{n+1}, n + 1) = \widehat{J}\left(X_{n+1} + \widehat{Y}_{n+1}, n + 1\right).$$

The constrained portfolio problem at time $n + 1$ can then be written as

$$J(X_n, n) = \max_{\pi_n \geq 0} E_n \widehat{J}(\widetilde{X}_{n+1} + \widehat{Y}_{n+1}, n + 1) \quad \text{s.t.} \quad X_n = E_n[\xi^{\delta}_{n,n+1} \widetilde{X}_{n+1}].$$

Let $(X^*_{n+1}, \widehat{Y}^*_{n+1}, q^*_n)$ denote the solution. The value function, certainty equivalent, and optimal portfolio at date n are

$$\begin{cases} J(X_n, n) = E_n \widehat{J}(\widetilde{X}^*_{n+1} + \widehat{Y}^*_{n+1}, n + 1) \\ \widehat{Y}_n(X_n) = \widehat{J}^{-1}(J(X_n, n), n) - X_n \\ \pi^*_n = \frac{r}{q^*_n(1-q^*_n)(u-d)} \left(\frac{G^{*u}_{n+1}}{G^*_n} p - q^*_n\right) \end{cases} \tag{8}$$

where $G^*_n \equiv \xi^*_n X^*_n$ and $G^{*u}_{n+1} = (\xi^*_{n+1} X^*_{n+1})^u$ and where $\widehat{J}^{-1}$ is the inverse of the date n value function $\widehat{J}(\cdot, n)$ of the pure portfolio problem with initial wealth $X_n + \widehat{Y}_n$. The second equation in Equation (8) provides the recursive relation between the certainty equivalents at dates n and $n + 1$.

In the next sections we specialize the model to power utility functions. In this context we present a numerical recipe for solving the problem and examine the behavior of the certainty equivalent.

1.5 Power utility function (CRRA)

Consider the utility function $u(X) = \frac{1}{1-R} X^{1-R}$, where $R > 0$ is the constant relative risk-aversion coefficient. Let $K_n(X_n) = \partial \widehat{Y}_n(X_n)/\partial X_n$ represent the derivative of the CE and let $\widehat{g}_{n+1,N} = E_{n+1}[\widehat{\xi}_{n+1,N}^{1-1/R}]$, where $\widehat{\xi}_{n+1,N}$ is the adjusted state price density for the pure portfolio problem over $\{n + 1, \ldots, N\}$ with initial wealth $\widehat{X}_{n+1} = X_{n+1} + \widehat{Y}_{n+1}$ and subject to a no-short-sales constraint. Now define the function $F_n(a, b) = E_n[\xi_{n,n+1}^{1-1/R}(1 + K_{n+1}(a))^b \widehat{g}_{n+1,N}]$. Appendix C shows that the certainty equivalent and its derivative satisfy the system of recursive equations

$$\widehat{Y}_n(X_n) = \begin{cases} (X_n + W_n) \frac{(F_n(X_{n+1}, 1/R - 1))^{1/(1-R)}}{F_n(X_{n+1}, 1/R)} (\widehat{g}_{n,N})^{-\frac{R}{1-R}} - X_n \\ \quad \text{if } X_{n+1}^u > r X_n \\[2mm] \left(E_n \left[(r X_n + \widehat{Y}_{n+1}(r X_n))^{1-R} \widehat{g}_{n+1,N}^R \right] \right)^{1/(1-R)} (\widehat{g}_{n,N})^{-\frac{R}{1-R}} - X_n \\ \quad \text{if } X_{n+1}^u = r X_n \end{cases}$$

$$1 + K_n = r E_n \left[(X_{n+1} + \widehat{Y}_{n+1})^{-R}(1 + K_{n+1}) \widehat{g}_{n+1,N}^R \right] (X_n + \widehat{Y}_n)^R \widehat{g}_{n,N}^{-R}$$

where $W_n = \frac{1}{r}[q \widehat{Y}_{n+1}^u + (1 - q) \widehat{Y}_{n+1}^d]$. These recursions are subject to the boundary conditions $\widehat{Y}_N = Y_N$ and $K_N = 0$.

1.6 Numerical evaluation of the certainty equivalent

The numerical scheme that we employ implements the dynamic programming equations described above. The procedure is a backward algorithm structured as follows:[6]

    1. Select a grid for wealth: $X(j)$, $j = 1, \ldots, N_x$.
    2. Set $\widehat{Y}_N = Y_N$, $K_N = 0$.
    3. At date $N - 1$: for each node and for $j = 1, \ldots, N_x$,
        (i) fix $X_{N-1} = X(j)$ and solve for $(X_N^u(j), X_N^d(j))$,
        (ii) compute $\widehat{Y}_{N-1}(j)$ and $K_{N-1}(j)$.
    4. At date n: for each node and for $j = 1, \ldots, N_x$,
        (i) fix $X_n = X(j)$ and solve for $(X_{n+1}^u(j), X_{n+1}^d(j))$,
        (ii) compute $\widehat{Y}_n(j)$, $K_n(j)$.
    5. Proceed recursively until $n = 0$.

---

[6] An alternative computational procedure can be developed based on a forward-backward binomial algorithm (FBB). Such a scheme involves the recursive computation of the CE based on estimated state prices (backward binomial procedure) combined with a reestimation of state prices (forward procedure involving the liquid wealth process and the optimal portfolio). Applying this FBB algorithm repeatedly eventually leads to a fixed point (in the space of processes) which represents the solution of our constrained problem. In numerical experiments the FBB algorithm has produced CE values that are identical to those obtained via the dynamic programming procedure in this article.

Several approaches are available for computing the derivative K_n of the certainty equivalent. Direct computation based on the recursive equation for K_n can be performed in parallel with the computation of $\widehat{Y}_n$. An alternative estimate is based on the finite difference $(\widehat{Y}_n(j) - \widehat{Y}_n(j-1))/(X(j) - X(j-1))$. Both approaches are easily implemented and produce similar results for sufficiently fine grids for wealth.

1.7 A numerical example

We illustrate the results in this section by considering a simple numerical example involving a nontraded call option with strike k. The binomial model is calibrated in the standard manner: $u = \exp(\sigma\sqrt{h})$, $d = 1/u$, and $p = \frac{1}{2}(1 + (\mu/\sigma)\sqrt{h})$, where $h = T/N$. The example's parameters are $\mu = .08$, $\sigma = .3$, $r = .05$, $R = 2$, $X_0 = 40$, $k = 80$, $T = 1$, and $N = 8$.

Figure 1 illustrates the relationship between the certainty equivalent $\widehat{Y}_0$ and the unconstrained call value C when the initial stock price S_0 ranges from 0 to 300. For low values of the underlying stock price the option is out of the money and both its private value and the unconstrained value are near 0 (the ratio is 1). As the stock price increases the option payoff increases at every node at the maturity date. As the owner of the nontraded option attempts to hedge the contract he will hit the no-short-sales constraint on the underlying asset and this will reduce his private valuation. In fact, his private valuation declines as a fraction of the unconstrained value (see Figure 1) for moderate values of the underlying stock price. For larger values of the stock price the magnitude of the difference between unconstrained valuation and private valuation increases to an upper bound: the ratio of the two values eventually converges to 1.

This numerical example also vividly illustrates the fact that the private value of the nontraded asset can be at a substantial discount to the unconstrained value: in the example the discount is nearly 17% for an at-the-money option.

The analysis above shows that the nontraded derivative with the short-sales constraint is equivalent to an unconstrained derivative on a dividend-paying stock. This property, whose consequences are illustrated in the numerical example, also foreshadows the result that early exercise may be optimal if we allow for an early exercise feature. Furthermore, by taking this argument to the limit, it is easily seen that the private value of a nontraded European option on a non-dividend-paying stock is equal to the certainty-equivalent value of a traded European option on a stock with the same drift and volatility coefficients but which pays a continuous nonnegative dividend flow. This suggests that the Black–Scholes formula will in fact overestimate the private value of a nontraded option; in our discrete time setting, the binomial model of Cox, Ross, and Rubinstein (1979) is an upper bound for the private value of the executive stock option. The first-order conditions of Equation (C.1) also show that the implicit dividend
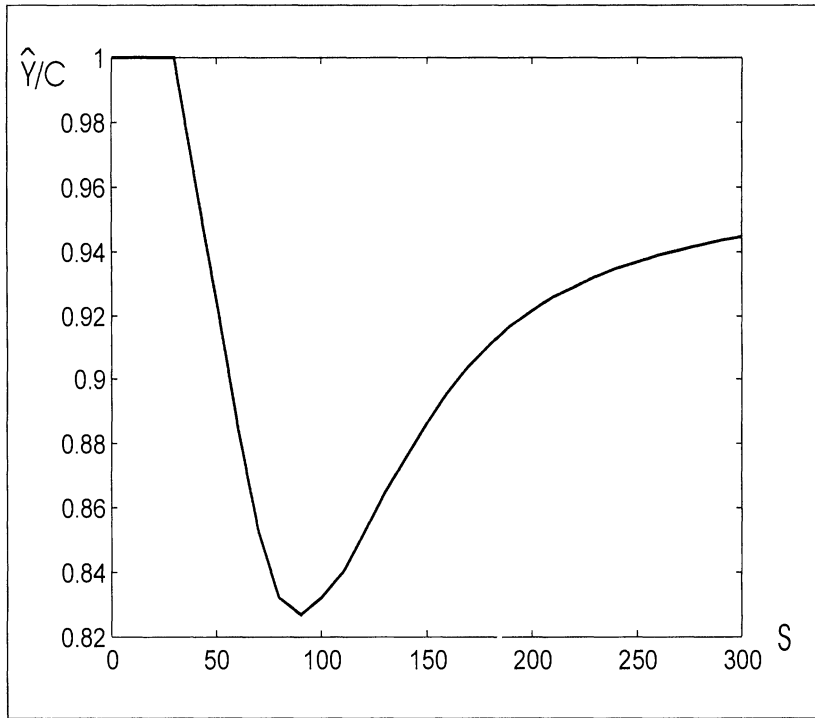
Figure 1
Certainty-equivalent relative to unconstrained value: moneyness effect
Parameter values: $\mu = .08$, $\sigma = .3$, $r = .05$, $R = 2$, $X_0 = 40$, $k = 80$, $T = 1$, $N = 8$. S_0 ranges from 0 to 300.

yield tends to be positive precisely when the nontraded asset owner would like to go short but cannot due to the short-sales constraint. The dividend yield $\delta_n^* = (q - q_n^*)(u - d)$ becomes zero when he is unconstrained. For logarithmic utility this dividend yield can be characterized in greater detail. In this case the dividend yield is (a) a decreasing function of liquid wealth (this clearly illustrates that the lack of diversification is a major source of loss in the private value of an executive stock option) and (b) a decreasing function of the excess return on the stock.

2. American-Style Contingent Claims

We now turn to the case of American contingent claims. We first demonstrate that early exercise of claims such as call options may be an optimal policy even when the underlying asset does not pay dividends. We then characterize the optimal exercise time.

2.1 The optimality of early exercise

In the case of complete markets it is well known that it is never optimal to exercise a call option when the underlying asset does not pay dividends. More generally, it is suboptimal to exercise any claim whose discounted payoff is a strict submartingale under the risk-neutral measure q (i.e., when $r^{-n}E^*Y_n > Y_0$). We now consider the exercise decision when the holder of the NTA is subject to a no-short-sales constraint in the underlying asset. Contrary to conventional wisdom our first result establishes the optimality of early exercise.

Proposition 3. *Early exercise of a contingent claim whose discounted payoff is a submartingale under the risk-neutral measure q may be optimal.*

This proposition states that waiting until maturity to exercise such a claim is a suboptimal policy under certain conditions. In order to prove this proposition we need only exhibit examples that display the property. Our first example below sets the stage: it shows that it is always optimal (in the context of the example) to exercise early any claim whose discounted payoff is a martingale. The second and third examples are numerical examples involving an ESO which demonstrate that a submartingale discounted payoff may also be optimally exercised prior to maturity.

Example 1. Consider an investor with logarithmic utility. Suppose first that the discounted payoff of the claim is a supermartingale (i.e., $r^{-n}E^*Y_n \leq Y_0$) and that $p = q$ (i.e., $E\tilde{r} - r = 0$). In this case the unconstrained optimal portfolio is a pure hedging portfolio equal to

$$X_n\pi_n = -\frac{V_{n+1}^u - V_{n+1}^d}{u - d}$$

for all n (V_{n+1} is the unconstrained value of the claim at $n + 1$). For claims that are positively correlated with the underlying stock price, this portfolio demand is negative. The constrained optimum is then $\pi_n = 0$. The policy of exercising the claim at maturity leads to a random terminal wealth equal to $X_0r^N + Y_N$. Immediate exercise on the other hand leads to the certain amount of terminal wealth (since the optimal unconstrained and constrained portfolios are null) $(X_0 + Y_0)r^N$. Let $J(X_0)$ (resp. $\widehat{J}(X_0 + Y_0)$) denote the value functions if exercise takes place at maturity (resp. immediately). The value functions are related by

$$\begin{aligned}
J(X_0) &= E\log(X_0r^N + Y_N) \\
&< \log(X_0r^N + EY_N) \\
&\leq \log((X_0 + Y_0)r^N) = \widehat{J}(X_0 + Y_0),
\end{aligned}$$

where the first inequality follows from Jensen's inequality and the second from the supermartingale property of the discounted payoff function. In particular, if the discounted payoff is a p-martingale, there is no incentive to wait until maturity. Thus in this example immediate exercise strictly dominates the policy of never exercising prior to maturity.

In this first example there are two distinct effects at play. The first is the effect of the constraint. By preventing a complete hedge of the non-traded position the no-short-sales constraint prevents terminal consumption smoothing. The individual is forced to bear unwanted variability in his terminal payoff and this reduces his expected utility. This effect provides incentives to exercise early. The second effect is the supermartingale behavior of the discounted payoff which also provides incentives for early exercise. Combining both effects results in the suboptimality of waiting until maturity to exercise.

Under the conditions of the example above ($p = q$), a call option on a non-dividend-paying stock is a p-submartingale $r^{-n} E^*(S_n - k)^+ \geq (S_0 - k)^+$. This submartingale behavior works in the opposite direction of the smoothing/constraint effect and may mitigate the negative effect of the constraint on welfare. However, as we show in the next numerical example, this effect may be too weak to fully offset the negative impact of the constraint.

Example 2. Consider an ESO with the following parameters $\sigma = .2$, $r = .1$, $p = .61767$, $u(X) = \log(X)$, $X_0 = 0$, $k = 80$, $S_0 = 100$, $T = 1$, $N = 3$. In this case the value of waiting to maturity is $J(X_0) = -\infty$ while immediate exercise leads to $\widehat{J}(X_0 + Y_0) = 3.0957$. If the individual waits until maturity to exercise the portfolio constraint binds at all nodes and terminal wealth includes highly undesirable outcomes with null payoff $X_0 r^N + Y_N = 0$. Immediate exercise on the other hand ensures strictly positive terminal wealth in all cases. In this example the submartingale property of the discounted option payoff mitigates the effect of the constraint but not sufficiently to offset the suboptimality of waiting to maturity.

Our last example shows that the suboptimality of waiting to maturity may also hold when $p > q$ (i.e., $E\widetilde{r} - r > 0$) and $X_0 > 0$, provided risk aversion is sufficiently large.

Example 3. Consider an ESO with the following parameters: $\sigma = .2$, $r = .1$, $p = .62$, $R = 4$, $X_0 = 10$, $k = 80$, $S_0 = 100$, $T = 1$, $N = 3$. In this case the value functions are, respectively, $J(X_0) = -0.0000265$ and $\widehat{J}(X_0 + Y_0) = -0.0000091$ Again, waiting to maturity is dominated by immediate exercise.

In all examples above there is tension between two conflicting effects. On the one hand, waiting until maturity to exercise enables the holder of

the NTA to capture the benefits associated with the appreciation of the discounted payoff (a submartingale is a positive sum game: $r^{-n}E^*Y_n \geq Y_0$). On the other hand, the portfolio constraint prevents a complete hedge of the claim (i.e., prevents terminal consumption smoothing) and this reduces welfare. Whenever the constraint is binding, early exercise has the important added benefit of alleviating the portfolio constraint. When the welfare losses resulting from the inability to smooth consumption are sufficiently important early exercise becomes optimal.

2.2 A dynamic programming formulation

We consider an American-style contingent claim with payoff $Y = \{Y_n: n = 0, \ldots, N\}$, where Y_n is a function of the stock price. If exercised at date n the payoff is Y_n.[7] Let i_n denote an indicator variable equal to 1 if early exercise did not take place at or before $n - 1$ and equal to 0 if it did. Let $J(X_n, i_n, n)$ be the value function for the portfolio problem with this American-style NTA. It satisfies, for $n = 0, \ldots, N - 1$,

$$\begin{cases} J(X_n, 1, n) = \max \left\{ \max_{\pi \geq 0} E_n J(\widetilde{X}^a_{n+1}, 0, n + 1), \right. \\ \left. \qquad \max_{\pi \geq 0} E_n J(\widetilde{X}^b_{n+1}, 1, n + 1) \right\} \\ J(X_n, 0, n) = \max_{\pi \geq 0} E_n J(\widetilde{X}^b_{n+1}, 0, n + 1). \end{cases} \tag{9}$$

where

$$\begin{cases} \widetilde{X}^a_{n+1} = (X_n + Y_n)[r + \pi_n(\widetilde{r} - r)] \\ \widetilde{X}^b_{n+1} = X_n[r + \pi_n(\widetilde{r} - r)]. \end{cases} \tag{10}$$

Here $\widetilde{X}^a_{n+1}$ (resp. $\widetilde{X}^b_{n+1}$) is liquid wealth at $n + 1$ in the event of exercise (resp. continuation) at n; the wealth process [Equation (10)] is subject to the initial condition $X_0 = x$. The random variable $\widetilde{r}$ is the return on the stock (with values u or d). These recursions are subject to the boundary conditions $J(X_N, 1, N) = u(X_N + Y_N)$ and $J(X_N, 0, N) = u(X_N)$. The first component inside the bracket on the right-hand side of Equation (9) represents the immediate exercise value function, the second is the continuation value function.

Clearly, immediate exercise is optimal at n if and only if the exercise value function exceeds the continuation value function, that is, if and only if $J(X_n, 1, n) = \max_{\pi \geq 0} E_n J(\widetilde{X}^a_{n+1}, 0, n + 1)$. Thus the optimal exercise time is

$$n^* = \inf \left\{ n \geq 0: \; J(X_n, 1, n) = \max_{\pi \geq 0} E_n J \left(\widetilde{X}^a_{n+1}, 0, n + 1 \right) \right\}$$

---

[7] Without loss of generality we assume that $Y_n \geq 0$ for all $n = 0, \ldots, N$. Otherwise replace Y_n by Y_n^+.

or $n^* = N$ if no such time exists in $\{0, \ldots, N - 1\}$, that is, n^* is the first time at which immediate exercise dominates continuation.

For any value taken by $J(X_n, 1, n)$ we can define the certainty equivalent $\widehat{Y}_n$ as the solution to

$$J(X_n, 1, n) = J(X_n + \widehat{Y}_n, 0, n),$$

that is,

$$\widehat{Y}_n = J^{-1}(J(X_n, 1, n), 0, n) - X_n,$$

where $J^{-1}(\cdot, 0, n)$ represents the inverse of $J(\cdot, 0, n)$ relative to the first argument. An alternative characterization of the optimal exercise time is then

$$n^* = \inf\left\{n \geq 0: \widehat{Y}_n \leq Y_n\right\}$$

or $n^* = N$ if no such time exists in $\{0, \ldots, N - 1\}$, that is, n^* is the first time at which the CE is bounded above by the exercise payoff of the claim.

2.3 Solving the dynamic program

The first step in the determination of the exercise policy is the resolution of the portfolio problem in the event that exercise takes place (i.e., the identification of the exercise value function $J(X_n, 0, n)$). This problem was in fact resolved in the context of the previous section.

Suppose that immediate exercise takes place at date n. Then the exercise value function is

$$J(X_n + Y_n, 0, n) = \max_{\pi \geq 0} E_n J\left(\widetilde{X}_{n+1}^a, 0, n + 1\right) = \widehat{J}(X_n + Y_n, n),$$

where $\widehat{J}(X_n, n)$ is the solution defined in Theorem 7, Appendix C, evaluated at initial wealth $X_n + Y_n$. Note that the function $\widehat{J}(\cdot, n): (0, \infty) \rightarrow (0, \infty)$ is strictly increasing since the inverse marginal utility function $I(\cdot)$ is strictly decreasing. Thus $J(\cdot, 0, n)$ is strictly increasing in the first argument and the certainty equivalent $\widehat{Y}_n$ is uniquely defined.

To complete the description of the exercise decision we still need to identify the continuation value function $J^c(X_n, n) \equiv \max_{\pi \geq 0} E_n J(\widetilde{X}_{n+1}^b, 1, n + 1)$. This function can be determined recursively since for $n = 0, \ldots, N - 1$,

$$J^c(X_n, n) = \max_{\pi \geq 0} E_n \left\{ J^c(\widetilde{X}_{n+1}^b, n + 1) 1_{\{\widehat{Y}_{n+1} > Y_{n+1}\}} \right.$$

$$\left. + J(\widetilde{X}_{n+1}^b + Y_{n+1}, 0, n + 1) 1_{\{\widehat{Y}_{n+1} \leq Y_{n+1}\}} \right\} \quad (11)$$

$$\text{s.t.} \quad J^c(X_N, N) = J(X_N + Y_n, 0, N) = u(X_N + Y_N) \quad (12)$$

where X_{n+1}^b satisfies Equation (10). This dynamic programming problem can be solved recursively using the methodology developed earlier, since it

consists of a sequence of static one-period problems. Let $I(\cdot, n+1)$ denote the inverse of the date $n+1$ marginal value function

$$J^{c\prime}(X_{n+1}^{b}, n+1)1_{\{\widehat{Y}_{n+1} > Y_{n+1}\}} + J_{n+1}^{\prime}(X_{n+1}^{b} + Y_{n+1}, 0, n+1)1_{\{\widehat{Y}_{n+1} \leq Y_{n+1}\}}$$

with respect to X_{n+1}^{b}. With this definition the first-order conditions at date n are also given by Equation (A.2) in Appendix A; denote this new system (A.2a). Solving this system for (y_n, q_n^{δ}) resolves the constrained portfolio-exercise decision problem. Indeed, the solution identifies the optimal stopping time n^* as the first time at which the certainty equivalent falls below the exercise payoff. At the exercise time n^* the liquid wealth of the investor increases to $X_{n^*} + Y_{n^*}$, which equals the present value of terminal consumption. Thus

$$X_{n^*} = E_{n^*}\left[\widehat{\xi}_{n^*,N} I(\widehat{y}_{n^*}\widehat{\xi}_{n^*,N})\right] - Y_{n^*},$$

where $\widehat{\xi}_n$ is the state price density process postexercise. Prior to exercise, liquid wealth satisfies $X_n = E_n[\xi_{n,n^*}^c X_{n^*}]$ by construction. Since the state price densities must coincide at exercise $(\widehat{\xi}_{n^*} = \xi_{n^*}^c)$ we can also write

$$X_n = E_n\left[\xi_{n,n^*}^c \left[E_{n^*}[\widehat{\xi}_{n^*,N} I(\widehat{y}_{n^*}\widehat{\xi}_{n^*,N})] - Y_{n^*}\right]\right]$$

$$= (\xi_n^c)^{-1} E_n\left[\widehat{\xi}_N I(\widehat{y}_{n^*}\widehat{\xi}_{n^*,N}) - \xi_{n^*}^c Y_{n^*}\right].$$

Summarizing,

Theorem 4. *Consider the joint portfolio and exercise decision problem with initial wealth $X_0 = x$ and subject to a no-short-sales constraint. Let $\{(X_{n+1}^*, \widehat{Y}_n^*, y_n^*, q_n^*, \xi_n^*): n = 0, \ldots, N-1\}$ denote the solution of the system of backward equations (A.2a) subject to the initial condition $X_0 = x$. The optimal exercise time is*

$$n^* = \inf\left\{n \geq 0 : \widehat{Y}_n^* \leq Y_n\right\}; \text{ or}$$

$$n^* = N \text{ if no such time exists in } \{0, \ldots, N-1\}.$$

At times prior to exercise, $n < n^*$, the optimal wealth and portfolio are

$$X_{n+1}^* = I(y_n^*\xi_{n,n+1}^*, n+1)$$

$$\pi_n^* = \frac{r}{q_n^*(1-q_n^*)(u-d)}\left(\frac{G_{n+1}^{*u}}{G_n^*}p - q_n^*\right),$$

where $G_{n+1}^{*u} = (\xi_{n+1}^* X_{n+1}^*)^u$ and $G_n^* = \xi_n^* X_n^*$. For $n \geq n^*$,

$$\widehat{X}_n = E_n\left[\widehat{\xi}_{n,N} I(\widehat{y}_{n^*}\widehat{\xi}_{n^*,N})\right]$$

$$\widehat{\pi}_n = \frac{r}{\widehat{q}_n(1 - \widehat{q}_n)(u - d)}\left(\frac{\widehat{G}_{n+1}}{\widehat{G}_n}p - \widehat{q}_n\right),$$

where $\widehat{G}_n = E_n[\widehat{\xi}_N I(\widehat{y}_{n^*}\widehat{\xi}_{n^*,N})]$ and $\widehat{q}_n$ satisfies $q - \widehat{q}_n \geq 0, \widehat{\pi}_n \geq 0$ and $(q - \widehat{q}_n)\widehat{\pi}_n = 0$.

Theorem 4 shows quite clearly that the structure of the solution changes after exercise has taken place. This reflects the irreversible nature of the exercise decision which changes the structure of Arrow–Debreu prices. These Arrow–Debreu prices are different from those used to compute the continuation value.

2.4 The certainty-equivalent formulation

The certainty-equivalent formulation of Section 1.4 can be easily adapted to the case of American-style contingent claims. In order to embed the possibility of early exercise in this formulation it suffices to replace the certainty equivalent which appears in the dynamic programming algorithm with the maximum of the exercise payoff and the continuation certainty equivalent $\widehat{Y}_{n+1}^c(X_{n+1})$, that is, $\widehat{Y}_{n+1}(X_{n+1}) \equiv \max\{\widehat{Y}_{n+1}^c(X_{n+1}), Y_{n+1}\} \equiv \widehat{Y}_{n+1}^c(X_{n+1}) \vee Y_{n+1}$.

The continuation value at date n now satisfies

$$J^c(X_n, n) = \max_{\pi_n \geq 0} E_n \widehat{J}(\widetilde{X}_{n+1}^b + (Y_{n+1} \vee \widehat{Y}_{n+1}^c), n + 1)$$

$$\text{s.t. } X_n = E_n[\xi_{n,n+1}^\delta \widetilde{X}_{n+1}^b]$$

and the first-order conditions are given by

$$\begin{cases} \widehat{J}'(X_{n+1}^b + (Y_{n+1} \vee \widehat{Y}_{n+1}^c), n + 1)(1 + K_{n+1}(X_{n+1}^b)) = y_n \xi_{n,n+1}^\delta \\ X_n = E_n[\xi_{n,n+1}^\delta \widetilde{X}_{n+1}^b]; \quad y_n > 0 \\ (X_{n+1}^b)^u - X_n r \geq 0, q - q_n^\delta \geq 0, \\ \qquad \text{and } (q - q_n^\delta)[(X_{n+1}^b)^u - X_n r] = 0 \end{cases}$$

(13)

where

$$K_{n+1}(X_{n+1}^b) = \frac{\partial\left(Y_{n+1} \vee \widehat{Y}_{n+1}^c(X_{n+1}^b)\right)}{\partial X_{n+1}^b} = \frac{\partial \widehat{Y}_{n+1}(X_{n+1}^b)}{\partial X_{n+1}^b}1_{\{Y_{n+1} < \widehat{Y}_{n+1}\}} \quad (14)$$

is the derivative of the certainty equivalent in the event of optimal continuation at $n+1$. The system [Equation (C.1)] with $\widehat{Y}_{n+1}(X_{n+1}) = \widehat{Y}_{n+1}^c(X_{n+1}) \vee Y_{n+1}$ substituting for the CE then characterizes the optimal policy; denote this new system Equation (C.1.a). Solving for (y_n, q_n^δ) gives the solution at date n assuming that the NTA is held one more period. Let $(X_{n+1}^*, \widehat{Y}_{n+1}^*, y_n^*, q_n^*)$ denote the solution. The date n continuation value

function $J^c(X_n, n) = E_n \widehat{J}(X^*_{n+1} + \widehat{Y}^*_{n+1}, n+1)$ then leads to the date n continuation certainty-equivalent payoff

$$\widehat{Y}^c_n(X_n) = \widehat{J}^{-1}(J^c(X_n, n), n) - X_n,$$

where $\widehat{J}^{-1}$ is the inverse of the value function $\widehat{J}(\cdot, n)$ of the pure portfolio problem with initial wealth $X_n + \widehat{Y}_n$. Immediate exercise at date n is optimal if and only if

$$\widehat{Y}^c_n(X_n) \leq Y_n$$

and the date n CE is $\widehat{Y}_n(X_n) = \widehat{Y}^c_n(X_n) \vee Y_n$. Summarizing,

Theorem 5. *Let* $\{(X^*_{n+1}, \widehat{Y}^*_{n+1}, y^*_n, q^*_n): n = 0, \ldots, N-1\}$ *denote the so-lution of the system of backward equations (C.1a) subject to the initial condition $X_0 = x$. The optimal exercise time n^* is*

$$n^* = \inf\{n \geq 0: \widehat{Y}^*_n \leq Y_n\}$$

or $n^* = N$ if no such time exists in $\{0, \ldots, N-1\}$. The optimal portfolio is, for $n < n^*$

$$\pi^*_n = \frac{r}{q^*_n(1-q^*_n)(u-d)}\left(\frac{G^{*u}_{n+1}}{G^*_n}p - q^*_n\right),$$

where $G^*_n \equiv \xi^*_n X^*_n$ and $G^{*u}_{n+1} = (\xi^*_{n+1} X^*_{n+1})^u$ and for $n \geq n^*$

$$\widehat{\pi}_n = \frac{r}{\widehat{q}_n(1-\widehat{q}_n)(u-d)}\left(\frac{\widehat{G}_{n+1}}{\widehat{G}_n}p - \widehat{q}_n\right),$$

where $\widehat{G}_n = E_n[\widehat{\xi}_N I(\widehat{y}_{n^*}\widehat{\xi}_{n^*,N})]$ and $\widehat{q}_n$ satisfies $q - \widehat{q}_n \geq 0, \widehat{\pi}_n \geq 0$ and $(q - \widehat{q}_n)\widehat{\pi}_n = 0$.

Remark 1. (i) The certainty equivalent $\widehat{Y}^*_n$ represents the private value that the investor attaches to the full liquidation of the asset at date n. It represents the cash compensation that yields indifference between ownership of the asset and liquidation. The CE private valuation captures the fact that the asset is not divisible.

(ii) The notion of certainty equivalent has been introduced by Pratt (1964) in the context of static problems under uncertainty. An important difference in our model is the endogenous timing of collection of the random payoff. The CE payoff $\widehat{Y}^*_n$ captures both the intrinsic randomness of the payoff as well as the randomness of the optimal exercise time.

2.5 American-style claims with logarithmic utility
In the case of logarithmic utility the solution of the portfolio problem with initial wealth $\widehat{X}_n = X_n + \widehat{Y}_n$ is given in Corollary 8 with $R = 1$. In the

particular case $E\widetilde{r} - r > 0$ (i.e., $p > q$) the portfolio policy is strictly positive and the adjusted risk-neutral measure is the unconstrained measure $\widehat{q}_m = q$; also $\widehat{\xi}_{n,N} = \xi_{n,N}$. We shall maintain this assumption in the derivations below. The constrained portfolio problem with NTA at date n can now be written

$$\max_{\pi_n \geq 0} E_n \log(X_{n+1} + \widehat{Y}_{n+1}) - E_n \log(\xi_{n+1,N}) \quad \text{s.t.} \quad X_n = E_n[\xi^{\delta}_{n,n+1} X_{n+1}],$$

where $\widehat{Y}_{n+1} = Y_{n+1} \vee \widehat{Y}^c_{n+1}$. Specializing the first-order conditions of Equation (C.3) to the log case and solving gives $\widehat{Y}_N(X_N) = Y_N$ and $K_N(X_N) = 0$ at date N and, at an arbitrary date n,

$$X^u_{n+1} = (X_n + W_n) \frac{1 + K^u_{n+1}(X^u_{n+1})}{1 + E_n K_{n+1}(X_{n+1})} (\xi^{\delta u}_{n,n+1})^{-1} - \widehat{Y}^u_{n+1}(X^u_{n+1})$$

$$W_n = \frac{1}{r} \left[q^{\delta}_n \widehat{W}^u_{n+1} + (1 - q^{\delta}_n) \widehat{W}^d_{n+1} \right]; \quad \widehat{W}_{n+1} = \widehat{Y}_{n+1}(X_{n+1})$$

and

$$K_{n+1}(X_{n+1}) = \frac{\partial \widehat{W}_{n+1}}{\partial X_{n+1}} = \frac{\partial \widehat{Y}_{n+1}}{\partial X_{n+1}} 1_{\{Y_{n+1} < \widehat{Y}_{n+1}\}}.$$

The value function and the certainty equivalent are

$$J^c(X_n, n) = \begin{cases} \log(X_n + W_n) + E_n \log\left(\frac{1 + K^u_{n+1}(X^u_{n+1})}{1 + E_n K_{n+1}(X_{n+1})} \right) - E_n \log(\xi_{n,N}) \\ \quad \text{if } X^u_{n+1} > r X_n \\ E_n \log(r X_n + \widehat{W}_{n+1}(r X_n)) - E_n \log(\xi_{n+1,N}) \\ \quad \text{if } X^u_{n+1} = r X_n \end{cases}$$

$$\widehat{Y}^c_n(X_n) = \begin{cases} (X_n + W_n) \exp\left(E_n \log(\frac{1 + K_{n+1}(X_{n+1})}{1 + E_n K_{n+1}(X_{n+1})}) \right) - X_n \\ \quad \text{if } X^u_{n+1} > r X_n \\ \exp(E_n \log(r X_n + \widehat{W}_{n+1}(r X_n)) - E_n \log(\xi_{n+1,n})) - X_n \\ \quad \text{if } X^u_{n+1} = r X_n \end{cases}$$

$$1 + K_n = r E_n \left[(X_{n+1} + \widehat{Y}_{n+1})^{-1} (1 + K_{n+1}) \widehat{g}_{n+1,N} \right] (X_n + \widehat{Y}_n) \widehat{g}^{-1}_{n,N}$$

$$\widehat{Y}_n(X_n) = Y_n \vee \widehat{Y}^c_n(X_n).$$

Immediate exercise at date n is optimal if and only if $\widehat{Y}_n(X_n) \leq Y_n$. Summarizing,

Corollary 6. *Suppose that* $E\widetilde{r} - r > 0$. *Let* $\{(X^*_{n+1}, \widehat{Y}^*_{n+1}, y^*_n, q^*_n): n = 0, \ldots, N - 1\}$ *denote the solution of the system of backward equations*

above subject to the initial condition $X_0 = x$. The optimal exercise time n^* is

$$n^* = \inf \left\{ n \geq 0 \colon \widehat{Y}_n^* \leq Y_n \right\}$$

or $n^* = N$ if no such time exists in $\{0, \ldots, N - 1\}$. The optimal portfolio is, for $n < n^*$,

$$\pi_n^* = \frac{r}{q_n^*(1 - q_n^*)(u - d)} \left(\frac{G_{n+1}^{*u}}{G_n^*} p - q_n^* \right),$$

where $G_n^* \equiv \xi_n^* X_n^*$ and $G_{n+1}^{*u} = (\xi_{n+1}^* X_{n+1}^*)^u$, and for $n \geq n^*$,

$$\pi_n^* = \frac{r}{q(1 - q)(u - d)} (p - q).$$

## 3. Application: Executive Stock Options

In this section we provide numerical results illustrating the behavior and properties of ESOs. The computations are performed using the backward numerical scheme described in Section 1.

Executive stock options are typical examples of NTAs involving trading restrictions in the underlying asset market. These restrictions imply substantial differences with standard option contracts.

The liquidity of the manager's wealth plays an important role for the value of the ESO. Figure 2 documents the difference between the European ESO value ($\widehat{Y}^e$) and the American ESO value ($\widehat{Y}^a$) as a function of liquid wealth. Values are reported as a fraction of the option value in an unrestricted market (C). Note that the early exercise premium decreases as liquid wealth increases. For a fixed immediate exercise payoff, the incidence of a binding constraint decreases when liquidity increases and this reduces the gains from early exercise. However, both the European and the American ESO values are at a substantial discount to the unconstrained value (the European ESO value may be worth less than 10% of the unconstrained value when the investor experiences severe liquidity shortage).

The ESO is a concave function of liquid wealth when the early exercise premium is sufficiently small (the European ESO is always concave): the marginal impact decreases as X_0 increases. As liquid wealth tends to infinity the ESO value converges to the value of a standard call option if the probability of a binding constraint tends to zero. If the probability of a binding constraint converges to a positive limit the ESO value remains at a discount to a standard call even for large values of X_0.

Unlike conventional option prices, the ESO value depends on the risk aversion of the owner. As risk aversion increases the ESO holder invests more conservatively in the risky asset and this leads to an increased probability of a binding constraint. The ESO value then decreases. As illustrated

855
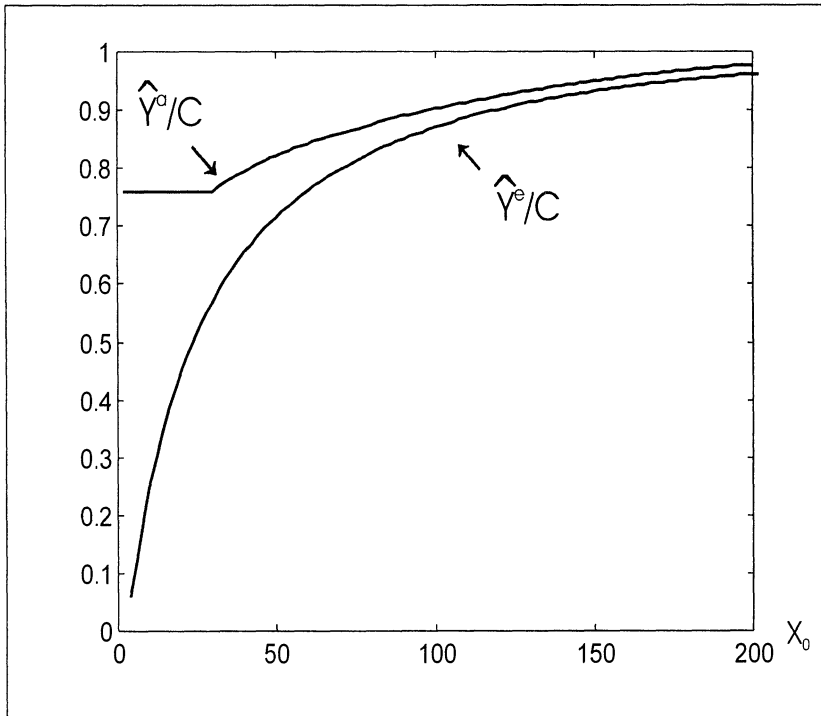
Figure 2
Liquidity effect
Parameter values: $\mu = .08$, $\sigma = .3$, $r = .05$, $R = 3$, $k = 80$, $S_0 = 100$, $T = 1$, $N = 6$. Wealth between 2 and 200.

in Figure 3 the American ESO value may be at a substantial discount to the unconstrained value for moderate risk aversion levels even if there is no discount for risk aversions less than or equal to 1.

The ESO also exhibits high sensitivity to the drift of the underlying asset. An increase in drift raises the American ESO value since the probability of a binding constraint decreases (see Figure 4).

Contrary to conventional wisdom an increase in volatility may reduce the ESO value. In the context of our model a higher volatility has two effects. On the one hand, it increases the upside potential of the ESO and this increases its CE value. On the other hand, it may reduce the demand for the stock, thereby increasing the probability of a binding constraint. This second effect reduces the CE value. As Figure 5 illustrates, the negative impact due to the failure to smooth terminal consumption perfectly dominates over certain regions of parameter values. This behavior emerges, in particular, when the manager's liquidity is low. When liquid wealth is sufficiently high the
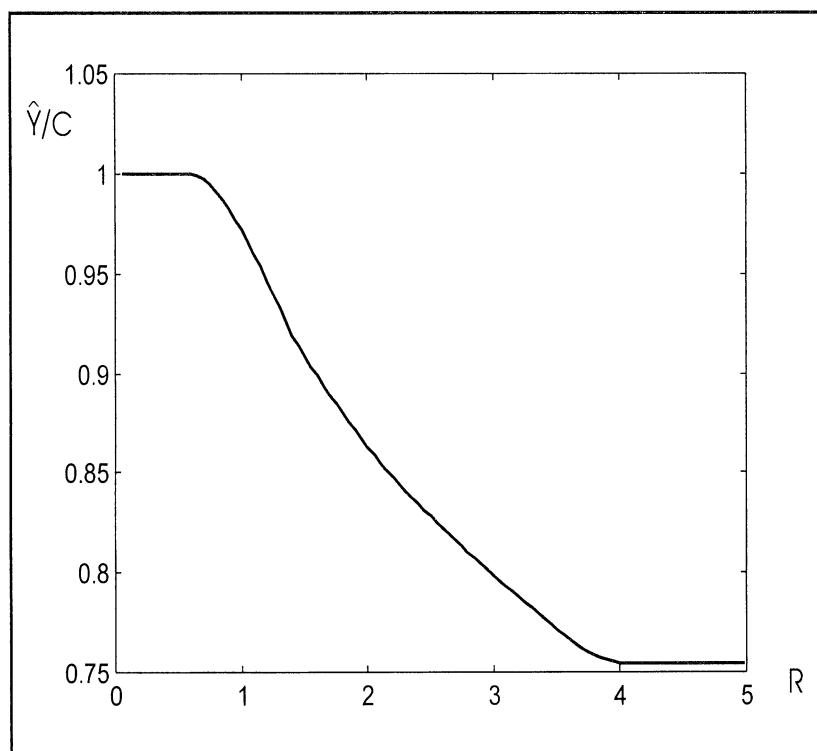
Figure 3
Risk aversion effect
Parameter values: $\mu = .08$, $\sigma = .3$, $r = .05$, $X_0 = 40$, $k = 80$, $S_0 = 100$, $T = 1$, $N = 7$. Risk aversion values between .1 and 5.

probability of a binding constraint decreases and the American ESO value mimicks the behavior of an unconstrained American call option value over typical ranges of volatility values.

Finally, we note that time to maturity has the usual effect on the American ESO: value increases with time to maturity since a longer maturity implies an increased set of feasible exercise policies.

4. The Effects of Imperfect Correlation

We now consider an extension of our model to a situation in which the nontraded payoff depends on a price that is imperfectly correlated with the price of the asset in which the investor can trade. Let S^1 be the price of the traded asset and S^2 be the price of the asset underlying the nontraded payoff. We consider a nontraded European call option with payoff $(S^2 - k)^+$. The
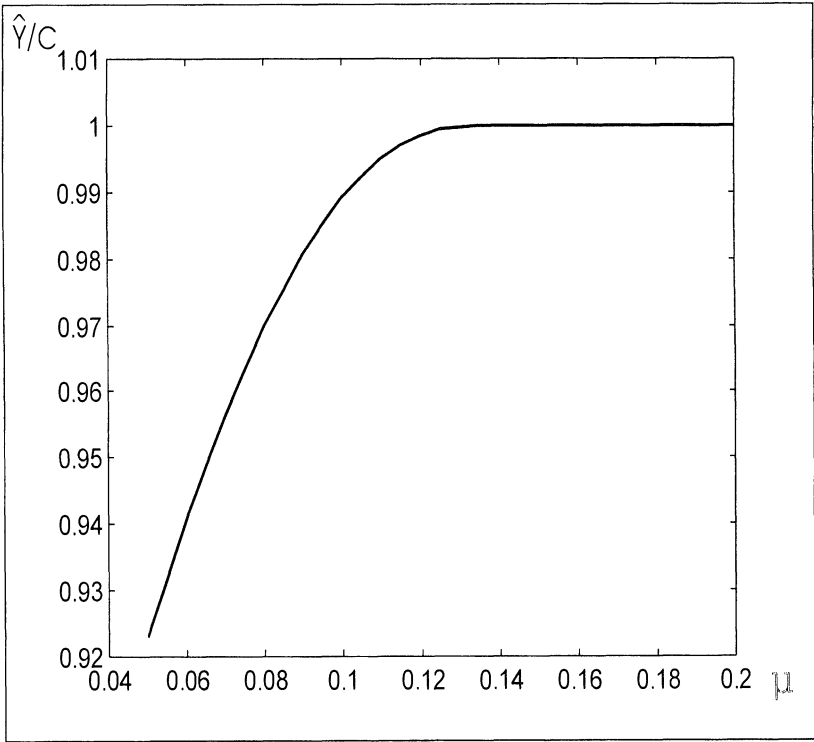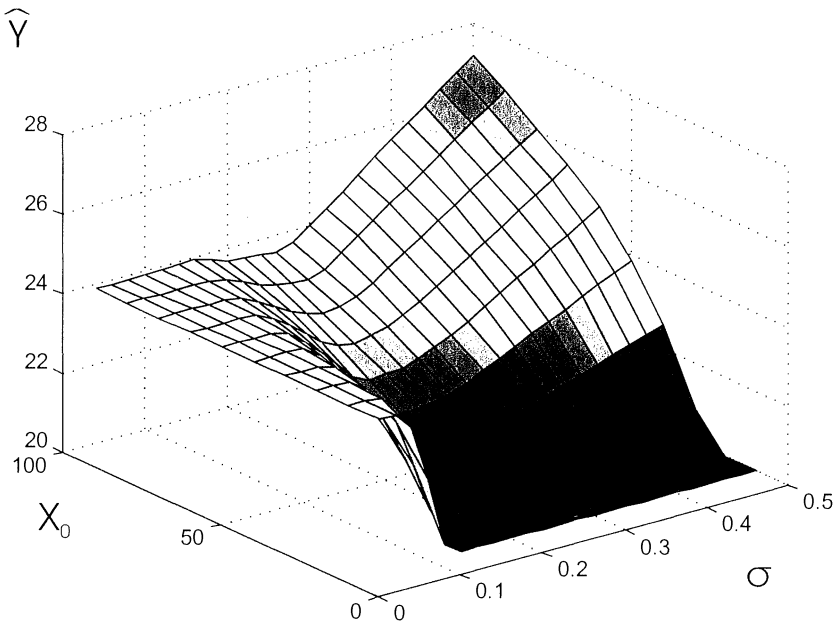
Figure 4
Drift effect
Parameter values: $\sigma = .3$, $r = .05$, $R = 1$, $X_0 = 40$, $k = 80$, $S = 100$, $T = 1$, $N = 6$. Drift μ between .04 and .2.

model of the previous sections corresponds to the case of perfectly correlated assets. Our objective is to examine the structure of the certainty equivalent in this more general context; in particular we are interested in the effect of correlation between the two assets.

4.1 Dynamic programming for the multiasset case
In order to model correlated assets we consider a trinomial model with three possible states of nature following each node. The tree profile is as follows (at date 0):

$$
(S_0^1, S_0^2) \;\rightarrow\;
\begin{array}{ll}
(S_0^1 u_1, S_0^2 u_2) & \text{w.p. } p_1 \\[4pt]
(S_0^1 m_1, S_0^2 m_2) & \text{w.p. } p_2 \\[4pt]
(S_0^1 d_1, S_0^2 d_2) & \text{w.p. } p_3
\end{array}
$$

Figure 5
Volatility and liquidity effects
Parameter values: $\mu = .08$, $r = .05$, $R = 3$, $k = 80$, $S_0 = 100$, $T = 1$, $N = 6$. Volatility ranges from .04 to .50; wealth from 10 to 100.

where $p_1 + p_2 + p_3 = 1$. The initial asset values are S_0^1, S_0^2 and the tree has N steps. The riskfree asset has a return equal to r.

The solution of our problem is given by the same set of equations as in Sections 1 and 2 and in Appendix A, with the proviso that we must now account for three possible states following each node of the tree. Furthermore, since the investor cannot trade in the asset underlying the nontraded payoff, we have an additional constraint on his investment policy. For power utility these considerations lead to a set of first-order conditions described in Appendix D. We present some numerical results next.

4.2 Numerical results
We calibrate the trinomial tree using the parametrization of He (1990).[8] The backward numerical algorithm of Section 1.6 is used to solve the equations characterizing the solution (see Appendix D).

---

[8] The model is calibrated as follows
$$u_1 = \exp\left(\mu_1 h + \sigma_1 \sqrt{3h/2}\right)$$
$$m_1 = \exp(\mu_1 h)$$

Consider a nontraded European call option written on the price S^2 of a non-dividend-paying asset. Figure 6 displays the correlation effect on the certainty equivalent expressed as a fraction of the unrestricted call option value (the ratio $\widehat{Y}_2/C(S^2)$). When the underlying asset prices (S^1, S^2) are negatively correlated the nontraded option hedges fluctuations in the traded asset S^1. The investor values this hedging function and prices the nontraded derivative above its unrestricted value. As correlation increases, its usefulness as a hedging vehicle diminishes. In the limit the nontraded option behaves more and more like an option on asset 1: its private value converges to the certainty equivalent of a call option written on the first asset.[9]

Note also that the certainty equivalent falls below the immediate exercise value when the correlation coefficient ρ is sufficiently large [max($S^2 - k, 0$)$/C(S^2) = .9403$]. If the contract were American style it would be optimally exercised prior to maturity. Early exercise would be optimal even in the absence of dividend payments on the underlying asset.

5. Conclusion

In this article we have provided a simple framework to value derivative assets subject to trading restrictions. The approach, which is based on the binomial model, is computationally tractable and easy to implement numerically. The methodology is also flexible: it accommodates any type of derivative contract as well as any type of utility function for the holder of the nontraded asset. In particular it enables us to characterize the optimal portfolio and exercise decisions for nontraded American-style derivatives.

In the case of a no-short-sales constraint, we have shown that the certainty-equivalent value of a nontraded derivative is bounded above by the unconstrained value of the asset. The constraint is in fact equivalent to the presence of an implicit dividend yield in the risk neutralized underlying asset price

---

$$d_1 = \exp\left(\mu_1 h - \sigma_1\sqrt{3h/2}\right)$$

$$u_2 = \exp\left(\mu_2 h + \sigma_2\left(\rho\sqrt{3/2} + \sqrt{1-\rho^2}\sqrt{1/2}\right)\sqrt{h}\right)$$

$$m_2 = \exp\left(\mu_2 h - \sigma_2\sqrt{1-\rho^2}\sqrt{4/2)}\sqrt{h}\right)$$

$$d_2 = \exp\left(\mu_2 h - \sigma_2\left(\rho\sqrt{3/2} - \sqrt{1-\rho^2}\sqrt{1/2}\right)\sqrt{h}\right)$$

where $h = T/N$. States have equal probabilities: $p_1 = p_2 = p_3 = 1/3$.

[9] In the calibration of He (1990) the returns on the second asset (u_2, m_2, d_2) depend on the correlation coefficient. In fact the distribution of asset 2's return is symmetric with respect to correlation and has less favorable outcomes when correlation is closer to zero. This payoff effect complements the hedging effect and explains the mildly humped (decreasing-increasing) shape of the CE. When the option is deeper in the money the shape can exhibit multiple humps.
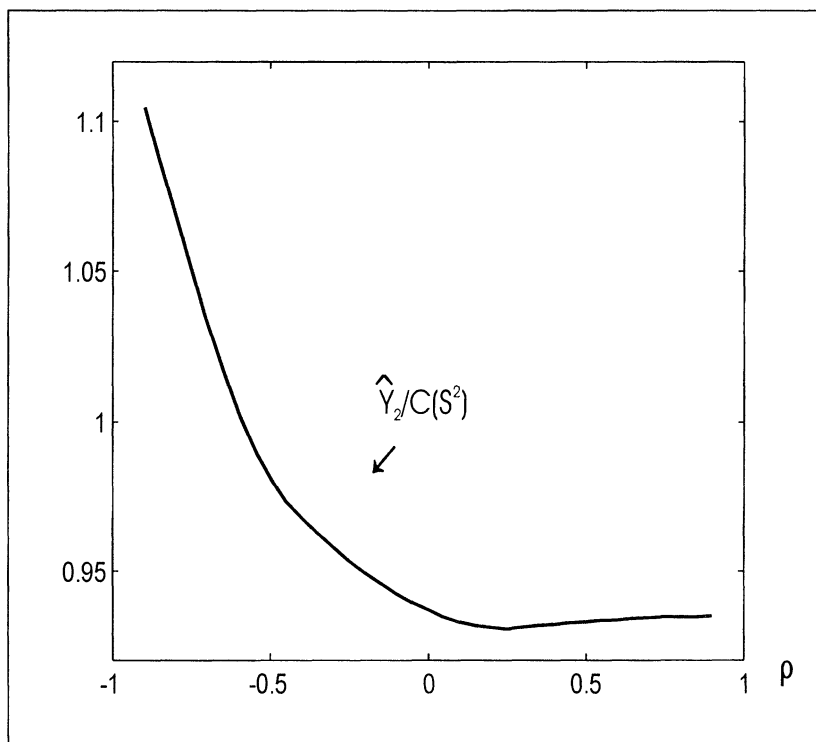
Figure 6
Correlation effect on certainty equivalent for call option on asset 2
Parameter values: $\mu_1 = \mu_2 = .1$, $\sigma_1 = \sigma_2 = .3$, $r = .1$, $R = 3$, $X_0 = 100$, $S_0^1 = S_0^2 = 100$, $k = 40$,
$T = 1$, $N = 3$. Correlation varies between $-.9$ and $+.9$.

process. This implicit dividend yield leads to qualitatively different pre-
dictions for the exercise policies of American options. The most notable
property is that an American call option may be optimally exercised prior
to maturity even when the underlying asset pays no dividends.

When applied to the case of an executive compensation option, our model
shows that the private value of such an option is bounded above by the
Black–Scholes value (in the absence of dividend payments) or the standard
American option valuation formula (with dividend payments). The model
also suggests that early exercise may take place even when the underlying
asset pays no dividends. This property is consistent with empirical and a pri-
ori puzzling facts. Naturally the private valuation of an ESO and the optimal
exercise decision of the manager are influenced by additional factors such as
incentive effects or provisions of the contract (reload options, vesting restric-
tions). These aspects can be easily incorporated in our setting and analyzed.

The framework that we propose can be used to value any nontraded derivative with an underlying asset subject to trading restrictions. Besides ESOs other claims in this category include forward contracts with thin market. In this context it is possible to show that the convenience yield which arises in the forward's valuation is related to trading restrictions impacting the underlying asset. This endogenous convenience yield is easily characterized and its structure in terms of the deep parameters of the economy can be examined.

Appendix A: Some Dynamic Programming Results

This appendix details some of the steps taken in the resolution of the intertemporal portfolio problem in the body of the article. These results could also be used to show the equivalence with the Cox and Huang (1989) martingale approach.

A.1 The unconstrained case

Let $J_n(X_n)$ denote the value function at date n. The unconstrained dynamic programming problem is (here π_n represents the amount of wealth invested in the stock)

$$J_n(X_n) = \max_{\pi_n} E_n \left[J_{n+1}(X_{n+1}) \right] \quad s.t.$$

$$X_{n+1} = X_n r + \pi_n \begin{cases} u - r \\ d - r \end{cases}, \quad X_0 = x$$

for $n = 0, \ldots, N - 1$, subject to the boundary condition $J_N(X_N) = u(X_N + Y_N)$.

Since the market is dynamically complete we can at each date n optimize state by state over wealth in the next period X_{n+1} and then compute the portfolio policy which supports optimal wealth. Using the definition of the SPD in Equation (1) enables us to write the budget constraint at date n as $X_n = E_n \left[\xi_{n,n+1} X_{n+1} \right]$. Thus the optimization problem can be reformulated as

$$J_n(X_n) = \max_{X_{n+1}} E_n \left[J_{n+1}(X_{n+1}) \right]$$

$$= \max_{(X_{n+1}^u, X_{n+1}^d)} \left[p J_{n+1}(X_{n+1}^u) + (1 - p) J_{n+1}(X_{n+1}^d) \right] \text{s.t.}$$

$$X_n = E_n \left[\xi_{n,n+1} X_{n+1} \right] = \frac{1}{r} \left[q X_{n+1}^u + (1 - q) X_{n+1}^d \right]$$

for $n = 0, \ldots, N - 1$. The corresponding optimal portfolio is uniquely (by complete markets) given by

$$\pi_n = \frac{X_{n+1}^u - X_n r}{u - r} = \frac{X_{n+1}^d - X_n r}{d - r}.$$

The first-order conditions for the program above are

$$\begin{cases} J'_{n+1}(X_{n+1}^u) = y_n \xi_{n+1}^u = y_n \frac{1}{r}(q/p) \\ J'_{n+1}(X_{n+1}^d) = y_n \xi_{n+1}^d = y_n \frac{1}{r}((1 - q)/(1 - p)) \\ X_n = E_n \left[\xi_{n,n+1} X_{n+1} \right], \quad y_n > 0 \end{cases}$$

for $n = 0, \ldots, N - 1$. Standard arguments show that the value function $J_n(\cdot)$ is strictly increasing and concave (thus the first-order conditions are also sufficient). It follows that there is a unique solution (X_{n+1}^*, y_n^*) for $n = 0, \ldots, N - 1$.

A.2 The constrained case with European-style nontraded assets

Suppose that the nontraded asset pays off at time N only (European-style claim). The dynamic programming algorithm for the constrained portfolio problem is

$$J_n(X_n) = \max_{\pi_n} E_n [J_{n+1}(X_{n+1})] \quad s.t.$$

$$X_{n+1} = X_n r + \pi_n \begin{cases} u - r \\ d - r \end{cases}$$

$$\pi_n \geq 0$$

for $n = 0, \ldots, N - 1$, subject to the boundary condition $J_N(X_N) = u(X_N + Y)$.

Due to the presence of the portfolio constraint the market is not dynamically complete. It follows that the choice of wealth in any state is a constrained choice problem. More precisely, for any date n since

$$\pi_n = \frac{X_{n+1}^u - X_n r}{u - r} = \frac{X_{n+1}^d - X_n r}{d - r} \geq 0$$

the portfolio constraint is equivalent to the wealth constraint

$$X_{n+1}^u \geq X_n r$$

$$X_{n+1}^d \leq X_n r$$

$$\left(X_{n+1}^u - X_n r\right)(d - r) = \left(X_{n+1}^d - X_n r\right)(u - r).$$

Note that the last constraint is redundant and can be eliminated. Indeed

$$0 = \left(X_{n+1}^u - X_n r\right) \frac{d - r}{u - d} - \left(X_{n+1}^d - X_n r\right) \frac{u - r}{u - d}$$

$$\Leftrightarrow 0 = \left(X_{n+1}^u - X_n r\right) q + \left(X_{n+1}^d - X_n r\right)(1 - q)$$

$$\Leftrightarrow X_n = E_n \left[\xi_{n,n+1} X_{n+1}\right]$$

where the last line follows upon dividing by r and using the definition of $\xi_{n,n+1}$ in Equation (1). The constrained dynamic problem is then equivalent to

$$J_n(X_n) = \max_{X_{n+1}} E_n [J_{n+1}(X_{n+1})] \quad s.t.$$

$$X_n = E_n \left[\xi_{n,n+1} X_{n+1}\right]$$

$$0 \leq X_{n+1}^u - X_n r; \quad X_{n+1}^d - X_n r \leq 0$$

for $n = 0, \ldots, N - 1$.

The Kuhn–Tucker conditions for the dynamic program are, for $n = 0, \ldots, N - 1$

$$
\begin{cases}
J'_{n+1}(X^u_{n+1}) = y_n \xi^u_{n,n+1} - \gamma^u_n / p \\[2mm]
J'_{n+1}(X^d_{n+1}) = y_n \xi^d_{n,n+1} + \gamma^d_n / (1 - p) \\[2mm]
E_n \left[\xi_{n,n+1} X_{n+1} \right] = X_n, \quad y_n > 0 \\[2mm]
X^u_{n+1} - X_n r \geq 0, \quad \gamma^u_n \geq 0 \\[2mm]
X^d_{n+1} - X_n r \leq 0, \quad \gamma^d_n \geq 0 \\[2mm]
\gamma^u_n [X^u_{n+1} - X_n r] = 0 \\[2mm]
\gamma^d_n [X^d_{n+1} - X_n r] = 0.
\end{cases}
$$

Here γ^u_n and γ^d_n are the Kuhn–Tucker multipliers associated with the inequality constraints and the last two conditions are the complementary slackness conditions.

Next note that the two constraints are linked through the budget constraint. When $\gamma^u_n = 0$ then $\gamma^d_n = 0$ as well, and conversely. Now suppose that $\gamma^u_n > 0$. It must then be the case that $X^u_{n+1} - X_n r = 0$. The multiplier y_n ensures that the budget constraint $E_n \left[\xi_{n,n+1} X_{n+1} \right] = X_n$ is satisfied (and this for any arbitrary choice of $\gamma^d_n > 0$). Combining these two equalities yields $X^d_{n+1} - X_n r = 0$ for any $\gamma^d_n > 0$. In other words, we can set $\gamma^d_n = \gamma^u_n$ without loss of generality.

Using the change of variables,

$$
\gamma^u_n = y_n \frac{\delta_n}{r(u - d)} \text{ and } \gamma^d_n = y_n \frac{\delta_n}{r(u - d)}
$$

enables us to rewrite the Kuhn-Tucker conditions as

$$
\begin{cases}
J'_{n+1}(X^u_{n+1}) = y_n \frac{1}{r}(q - \frac{\delta_n}{u-d})/p \\[2mm]
J'_{n+1}(X^d_{n+1}) = y_n \frac{1}{r}(1 - q + \frac{\delta_n}{u-d})/(1 - p) \\[2mm]
E_n \left[\xi_{n,n+1} X_{n+1} \right] = X_n, \quad y_n > 0 \\[2mm]
X^u_{n+1} - X_n r \geq 0, \quad \delta_n \geq 0 \\[2mm]
\delta_n [X^u_{n+1} - X_n r] = 0.
\end{cases}
$$

Defining

$$
q^\delta_n = \frac{r - d - \delta_n}{u - d} = q - \frac{\delta_n}{u - d}
$$

we obtain the sequence of equalities

$$
X_n = E_n \left[\xi_{n,n+1} X_{n+1} \right] = \frac{1}{r} \left[q(X^u_{n+1} - X_n r) + (1 - q)(X^d_{n+1} - X_n r) \right] + X_n
$$

$$= \frac{1}{r}\left[\left(q - \frac{\delta_n}{u-d}\right)(X_{n+1}^u - X_n r) + \left(1 - q + \frac{\delta_n}{u-d}\right)(X_{n+1}^d - X_n r)\right] + X_n$$

$$= \frac{1}{r}\left[q_n^\delta(X_{n+1}^u - X_n r) + (1 - q_n^\delta)(X_{n+1}^d - X_n r)\right] + X_n = E_n\left[\xi_{n,n+1}^\delta X_{n+1}\right],$$

where in the third line we use the complementary slackness conditions $\delta_n[X_{n+1}^u - X_n r] = 0$ and $\delta_n[X_{n+1}^d - X_n r] = 0$. Substituting in the Kuhn–Tucker conditions leaves us with

$$\begin{cases} J_{n+1}'(X_{n+1}^u) = y_n \frac{1}{r}(q_n^\delta/p) \\[2mm] J_{n+1}'(X_{n+1}^d) = y_n \frac{1}{r}((1 - q_n^\delta)/(1 - p)) \\[2mm] E_n\left[\xi_{n,n+1}^\delta X_{n+1}\right] = X_n, \quad y_n > 0 \\[2mm] X_{n+1}^u - X_n r \geq 0, \quad q - q_n^\delta \geq 0 \\[2mm] (q - q_n^\delta)[X_{n+1}^u - X_n r] = 0, \end{cases} \tag{A.1}$$

for $n = 0, \ldots, N - 1$. Equivalently, if $I(\cdot, n + 1)$ denotes the inverse of the marginal value of wealth at $n + 1$, we can write

$$\begin{cases} X_{n+1} = I(y_n \xi_{n,n+1}^\delta, n + 1) \\[2mm] E_n\left[\xi_{n,n+1}^\delta I(y_n \xi_{n,n+1}^\delta, n + 1)\right] = X_n, \quad y_n > 0 \\[2mm] X_{n+1}^u - X_n r \geq 0, \quad q - q_n^\delta \geq 0 \\[2mm] (q - q_n^\delta)[X_{n+1}^u - X_n r] = 0. \end{cases} \tag{A.2}$$

The first three conditions in Equation (A.1) [equivalently, the first two conditions in Equation (A.2)] correspond to an unconstrained portfolio problem in an auxiliary economy in which the stock price follows a binomial model with coefficients $(u + \delta_n, d + \delta_n)$. Let $\pi_n(\delta_n)$ be the solution of this unconstrained problem. The last two conditions are equivalent to $\pi_n(\delta_n) \geq 0$, $\delta_n \geq 0$, and $\delta_n \pi_n(\delta_n) = 0$.

This system of first-order conditions characterizes the solution of the constrained problem and underlies the discussions in Sections 1 and 2. This characterization is similar to the one obtained using a martingale approach [see Cvitanic and Karatzas (1992)].

Appendix B: Proofs

Proof of Theorem 1. The Kuhn–Tucker conditions for the constrained problem are given by Equation (A.1). They imply $X_{n+1}^* = I(y_n^* \xi_{n,n+1}^*, n + 1)$, where $\xi_{n,n+1}^*$ is the constrained SPD. Let $\delta_n^* = (q - q_n^*)(u - d)$. Using Equation (3) we obtain

$$\pi_n^* = \frac{1}{u - r}\frac{\left[X_{n+1}^{*u} - X_n^* r\right]}{X_n^*} = \frac{1}{u - r}\left(\frac{u + \delta_n^* - r}{u + \delta_n^* - r}\right)\left[\frac{X_{n+1}^{*u}}{X_n^*} - r\right]$$

$$= \frac{1}{u-r}\left(\frac{u-r}{u+\delta_n^*-r}\right)\left[\frac{X_{n+1}^{*u}}{X_n^*}-r\right] + \frac{1}{u-r}\left(\frac{\delta_n^*}{u+\delta_n^*-r}\right)\left[\frac{X_{n+1}^{*u}}{X_n^*}-r\right]$$

$$= \frac{1}{u+\delta_n^*-r}\left[\frac{X_{n+1}^{*u}}{X_n^*}-r\right] = \frac{1}{(1-q_n^*)(u-d)}\left[\frac{\xi_{n+1}^{*u}X_{n+1}^{*u}}{\xi_n^*X_n^*}(\xi_{n,n+1}^{*u})^{-1}-r\right]$$

$$= \frac{1}{(1-q_n^*)(u-d)}\left[\frac{G_{n+1}^{*u}}{G_n^*}\left(\frac{1}{r}\frac{q_n^*}{p}\right)^{-1}-r\right] = \frac{r}{q_n^*(1-q_n^*)(u-d)}\left[\frac{G_{n+1}^{*u}}{G_n^*}p-q_n^*\right].$$

The first equality above follows from Equation (3), the fourth uses the complementary slackness condition in Equation (A.2), $\delta_n^*(X_{n+1}^{*u}-X_n^*r) = (u-d)(q-q_n^*)(X_{n+1}^{*u}-X_n^*r) = 0$, the fifth the relation $u+\delta_n^*-r = (1-q_n^*)(u-d)$, and the sixth the definition of the constrained SPD which satisfies Equation (1) substituting q_n^* for q. ∎

Proof of Proposition 2. (i) Suppose that the no-short-sales constraint never binds. Applying a standard Cox–Huang (1989) methodology shows that the portfolio problem with initial wealth $x_0 + \widehat{Y}_0$ has solution

$$\begin{cases} X_N = I(\widehat{y}\xi_N) \\ x + \widehat{Y}_0 = E[\xi_N I(\widehat{y}\xi_N)], \end{cases}$$

where $I(\cdot)$ is the inverse of $u'(\cdot)$. The value function is $\widehat{J}(x+\widehat{Y}_0,0) = Eu(I(\widehat{y}\xi_N))$. On the other hand, the solution of the "constrained" problem with the NTA paying off at N is

$$\begin{cases} X_N = I(y^*\xi_N) - Y_N \\ x_0 = E[\xi_N[I(y^*\xi_N) - Y_N]]. \end{cases}$$

Equivalently, the static budget constraint can be written as

$$x_0 + V_0 = E[\xi_N I(y^*\xi_N)],$$

where $V_0 = E[\xi_N Y_N]$. The corresponding value function is $\widehat{J}(x+V_0,0) = Eu(I(y^*\xi_N))$.

It follows immediately from these expressions that $y^* = \widehat{y}$ and $\widehat{Y}_0 = V_0$, where $V_0 = E[\xi_N Y_N]$ is the unconstrained value of the claim.

(ii) Suppose now that the constraint binds with positive probability in the constrained problem with European-style claim. Assume that $\widehat{Y}_0 > V_0$. But then by Assumption (b) we must have $\widehat{J}(x+\widehat{Y}_0,0) > \widehat{J}(x+V_0,0)$, where the right-hand side is the unconstrained value function starting from initial wealth $x + V_0$. Since the left-hand side equals $J(x,0)$ by definition of the certainty equivalent it follows that $J(x,0) > \widehat{J}(x+V_0,0)$, that is, the individual is better off constrained than unconstrained. This cannot hold since the portfolio constraint reduces the feasible choice set. ∎

Appendix C: Backward Construction of Certainty Equivalents

C.1 A general recursive procedure

In order to construct the sequence of certainty equivalents we need to solve the pure portfolio problem without nontraded asset but starting from an adjusted wealth level. This problem can be solved by using the method of Cvitanic and Karatzas (1992). This leads to the following result.

Theorem 7. *Consider the pure portfolio problem over $\{n+1, \ldots, N\}$ with initial wealth $\widehat{X}_{n+1} = X_{n+1} + \widehat{Y}_{n+1}$ and subject to a no-short-sales constraint. Let $I(\cdot)$ denote the inverse of the marginal utility function $u'(\cdot)$. Optimal terminal wealth is*

$$\widehat{X}_N = I(\widehat{y}_{n+1}\widehat{\xi}_{n+1,N})$$

where $\widehat{y}_{n+1}$ solves $X_{n+1} + \widehat{Y}_{n+1} = E_{n+1}[\widehat{\xi}_{n+1,N} I(\widehat{y}_{n+1}\widehat{\xi}_{n+1,N})]$. The value function, wealth process and portfolio policy are, for $m \geq n + 1$,

$$\widehat{J}(X_{n+1} + \widehat{Y}_{n+1}, n + 1) = E_{n+1}[u(I(\widehat{y}_{n+1}\widehat{\xi}_{n+1,N}))]$$

$$\widehat{X}_m = E_m[\widehat{\xi}_{m,N} I(\widehat{y}_{n+1}\widehat{\xi}_{n+1,N})]$$

$$\widehat{\pi}_m = \frac{r}{\widehat{q}_m(1 - \widehat{q}_m)(u - d)}\left(\frac{\widehat{G}_{m+1}}{\widehat{G}_m}p - \widehat{q}_m\right),$$

where $\widehat{G}_m = E_m[\widehat{\xi}_N I(\widehat{y}_{n+1}\widehat{\xi}_{n+1,N})]$ and $\widehat{q}_m$ satisfies $q - \widehat{q}_m \geq 0, \widehat{\pi}_m \geq 0$ and $(q - \widehat{q}_m)\widehat{\pi}_m = 0$.

By definition the certainty equivalent $\widehat{Y}_{n+1}$ solves

$$J(X_{n+1}, n + 1) = \widehat{J}(X_{n+1} + \widehat{Y}_{n+1}, n + 1).$$

The dynamic problem [Equations (6) and (7)] can then be written

$$J(X_n, n) = \max_{\pi_n \geq 0} E_n \widehat{J}(\widetilde{X}_{n+1} + \widehat{Y}_{n+1}, n + 1) \qquad \text{s.t.} \qquad X_n = E_n[\xi^\delta_{n,n+1}\widetilde{X}_{n+1}]$$

for $n = 0, \ldots, N - 1$. Taking account of the fact that the certainty-equivalent payoff depends on liquid wealth (i.e., $\widehat{Y}_{n+1} = \widehat{Y}_{n+1}(X_{n+1})$) leads to the first-order conditions

$$\begin{cases} \widehat{J}'(X_{n+1} + \widehat{Y}_{n+1}, n + 1)(1 + K_{n+1}(X_{n+1})) = y_n \xi^\delta_{n,n+1} \\[2mm] X_n = E_n[\xi^\delta_{n,n+1}\widetilde{X}_{n+1}]; \quad y_n > 0 \\[2mm] X''_{n+1} - X_n r \geq 0, q - q^\delta_n \geq 0, \text{ and } (q - q^\delta_n)[X''_{n+1} - X_n r] = 0, \end{cases}$$

where

$$K_{n+1}(X_{n+1}) = \frac{\partial \widehat{Y}_{n+1}(X_{n+1})}{\partial X_{n+1}}$$

is the derivative of the certainty equivalent at $n + 1$. The structure of the first-order conditions is similar to the conditions in Appendix A. Let $H_{n+1}(\cdot)$ be the inverse of $\widehat{J}'(X_{n+1} + \widehat{Y}_{n+1}, n + 1)$ with respect to the first argument, $X_{n+1} + \widehat{Y}_{n+1}$. We can write

$$\begin{cases} X_{n+1} = H_{n+1}\left(\frac{y_n \xi^\delta_{n,n+1}}{1 + K_{n+1}(X_{n+1})}\right) - \widehat{Y}_{n+1}(X_{n+1}) \\[3mm] X_n = E_n\left[\xi^\delta_{n,n+1}\left[H_{n+1}\left(\frac{y_n \xi^\delta_{n,n+1}}{1 + K_{n+1}(X_{n+1})}\right) - \widehat{Y}_{n+1}(X_{n+1})\right]\right]; \quad y_n > 0 \qquad \text{(C.1)} \\[3mm] X''_{n+1} - X_n r \geq 0, q - q^\delta_n \geq 0, \text{ and } (q - q^\delta_n)[X''_{n+1} - X_n r] = 0 \end{cases}$$

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and

$$\pi_n = \frac{X_{n+1}^u - r X_n}{(1 - q_n^\delta)(u - d) X_n} = \frac{r}{q_n^\delta (1 - q_n^\delta)(u - d)} \left(\frac{G_{n+1}^u}{G_n} p - q_n^\delta \right), \qquad \text{(C.2)}$$

where $G_n \equiv \xi_n^\delta X_n$ and $G_{n+1}^u = (\xi_{n+1}^\delta X_{n+1})^u$. In the event that the constraint is not binding, $q_n^\delta = q$ and π_n satisfies Equation (C.2) evaluated at q.

Solving Equation (C.1) for (y_n, q_n^δ) gives the solution of the constrained portfolio problem at date n. Let $(X_{n+1}^*, \widehat{Y}_{n+1}^*, y_n^*, q_n^*)$ denote the solution. The value function is $J(X_n, n) = E_n \widehat{J}(X_{n+1}^* + \widehat{Y}_{n+1}^*, n + 1)$. The certainty-equivalent payoff at date n is then

$$\widehat{Y}_n(X_n) = \widehat{J}^{-1}(J(X_n, n), n) - X_n,$$

where $\widehat{J}^{-1}$ is the inverse of the date n value function $\widehat{J}(\cdot, n)$ of the pure portfolio problem with initial wealth $X_n + \widehat{Y}_n$.

C.2 Power utility function

In the case of the power utility function we have $u'(x) = x^{-R}$ and $I(y) = y^{-\frac{1}{R}}$, where R denotes the relative risk-aversion coefficient. The solution of the portfolio problem with initial wealth $\widehat{X}_{n+1} = X_{n+1} + \widehat{Y}_{n+1}$ is

Corollary 8. *Consider the pure portfolio problem over $\{n+1, \ldots, N\}$ with initial wealth $\widehat{X}_{n+1} = X_{n+1} + \widehat{Y}_{n+1}$ and subject to a no-short-sales constraint. Suppose that u exhibits constant relative risk aversion. Optimal terminal wealth is*

$$X_N = \left(X_{n+1} + \widehat{Y}_{n+1} \right) \widehat{\xi}_{n+1,N}^{-1/R} \widehat{g}_{n+1,N}^{-1},$$

where $\widehat{g}_{n+1,N} = E_{n+1}[\widehat{\xi}_{n+1,N}^{1-1/R}]$. The value function, wealth process, and portfolio policy are, for $m \geq n + 1$,

$$\widehat{J}(X_{n+1} + \widehat{Y}_{n+1}, n + 1) = \frac{1}{1 - R} \left(X_{n+1} + \widehat{Y}_{n+1} \right)^{1-R} \widehat{g}_{n+1,N}^{R}$$

$$\widehat{X}_m = \left(X_{n+1} + \widehat{Y}_{n+1} \right) \widehat{\xi}_{n+1,m}^{-1/R} \frac{\widehat{g}_{m,N}}{\widehat{g}_{n+1,N}}$$

$$\widehat{\pi}_m = \frac{r}{\widehat{q}_m (1 - \widehat{q}_m)(u - d)} \left(\frac{\widehat{G}_{m+1}^u}{\widehat{G}_m} p - \widehat{q}_m \right),$$

where $\widehat{G}_m = E_m[\widehat{\xi}_N^{1-1/R}]$ and $\widehat{q}_m$ is such that $q - \widehat{q}_m \geq 0, \widehat{\pi}_m \geq 0$, and $(q - \widehat{q}_m)\widehat{\pi}_m = 0$.

The constrained portfolio problem with NTA at date n can now be written as

$$\max_{\pi_n \geq 0} E_n \frac{1}{1 - R} \left(X_{n+1} + \widehat{Y}_{n+1} \right)^{1-R} \widehat{g}_{n+1,N}^{R} \qquad \text{s.t.} \qquad X_n = E_n[\xi_{n,n+1}^\delta X_{n+1}].$$

The first-order conditions are

$$
\begin{cases}
(X_{n+1} + \widehat{Y}_{n+1}))^{-R}(1 + K_{n+1}(X_{n+1}))\widehat{g}^R_{n+1,N} = y_n \xi^\delta_{n,n+1} \\[2mm]
X_n = E_n[\xi^\delta_{n,n+1} X_{n+1}]; \quad y_n > 0 \\[2mm]
X^u_{n+1} - X_n r \geq 0, q - q^\delta_n \geq 0, \text{ and } (q - q^\delta_n)[X^u_{n+1} - X_n r] = 0.
\end{cases} \tag{C.3}
$$

At date N we have $\widehat{Y}_N(X_N) = Y_N$ and $K_N(X_N) = 0$. At an arbitrary date n we can write the solution of the first-order conditions as

$$
X^u_{n+1} = (X_n + W_n) \frac{(\xi^{\delta u}_{n,n+1})^{-1/R}(1 + K^u_{n+1}(X^u_{n+1}))^{1/R}\widehat{g}^u_{n+1,N}}{E_n\left[(\xi^\delta_{n,n+1})^{1-1/R}(1 + K_{n+1}(X_{n+1}))^{1/R}\widehat{g}_{n+1,N}\right]} - \widehat{Y}^u_{n+1}(X^u_{n+1})
$$

$$
W_n = \frac{1}{r}[q^\delta_n \widehat{Y}^u_{n+1} + (1 - q^\delta_n)\widehat{Y}^d_{n+1}]
$$

and

$$
K_{n+1}(X_{n+1}) = \frac{\partial \widehat{Y}_{n+1}}{\partial X_{n+1}}.
$$

Defining $F_n(a, b) = E_n[\xi^{1-1/R}_{n,n+1}(1 + K_{n+1}(a))^b \widehat{g}_{n+1,N}]$ we can then write the value function, the certainty equivalent, and its derivative as

$$
J(X_n, n) = \begin{cases}
\dfrac{1}{1-R}(X_n + W_n)^{1-R}\dfrac{F_n(X_{n+1}, 1/R-1)}{(F_n(X_{n+1}, 1/R))^{1-R}} & \text{if } X^u_{n+1} > rX_n \\[4mm]
\dfrac{1}{1-R}E_n\left[(rX_n + \widehat{Y}_{n+1}(rX_n))^{1-R}\widehat{g}^R_{n+1,N}\right] & \text{if } X^u_{n+1} = rX_n
\end{cases}
$$

$$
\widehat{Y}_n(X_n) = \begin{cases}
(X_n + W_n)\dfrac{(F_n(X_{n+1}, 1/R-1))^{1/(1-R)}}{F_n(X_{n+1}, 1/R)}(\widehat{g}_{n,N})^{-\frac{R}{1-R}} - X_n \\[2mm]
\quad \text{if } X^u_{n+1} > rX_n \\[4mm]
\left(E_n\left[(rX_n + \widehat{Y}_{n+1}(rX_n))^{1-R}\widehat{g}^R_{n+1,N}\right]\right)^{\frac{1}{1-R}}(\widehat{g}_{n,N})^{-\frac{R}{1-R}} - X_n \\[2mm]
\quad \text{if } X^u_{n+1} = rX_n
\end{cases}
$$

$$
1 + K_n = rE_n\left[(X_{n+1} + \widehat{Y}_{n+1})^{-R}(1 + K_{n+1})\widehat{g}^R_{n+1,N}\right](X_n + \widehat{Y}_n)^R \widehat{g}^{-R}_{n,N}.
$$

Appendix D: The Trinomial Model

For the power utility function the first-order conditions at date n are

$$
\begin{cases}
(X_{n+1} + \widehat{Y}_{n+1}))^{-R}(1 + K_{n+1}(X_{n+1}))\widehat{g}^R_{n+1,N} = y_n \xi^\delta_{n,n+1} \\[2mm]
X_n = E_n[\xi^\delta_{n,n+1} X_{n+1}]; \quad y_n > 0 \\[2mm]
\dfrac{1}{u_1+\delta_1-r}(X^u_{n+1} - rX_n) \geq 0, \delta_1 \geq 0, \text{ and } \dfrac{\delta_1}{u_1+\delta_1-r}[X^u_{n+1} - X_n r] = 0 \\[2mm]
(X^m_{n+1} - rX_n) = \dfrac{m_1+\delta_1-r}{u_1+\delta_1-r}(X^u_{n+1} - rX_n),
\end{cases} \tag{D.1}
$$

where $\delta_1 = (u_1 - d_1)(q_1^\delta - q_1) + (m_1 - d_1)(q_2^\delta - q_2) - (d_1 - r)$. The first two conditions in Equation (D.1) parallel the corresponding conditions for the one asset case in Appendix C. To derive the next two conditions note that optimal wealth satisfies

$$\begin{bmatrix} X_{n+1}^u \\ X_{n+1}^m \end{bmatrix} = r X_n + \begin{bmatrix} u_1^\delta - r & u_2^\delta - r \\ m_1^\delta - r & m_2^\delta - r \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}.$$

Solving for the optimal portfolio yields

$$\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \frac{1}{\det} \begin{bmatrix} (m_2^\delta - r)(X_{n+1}^u - rX_n) - (u_2^\delta - r)(X_{n+1}^m - rX_n) \\ -(m_1^\delta - r)(X_{n+1}^u - rX_n) + (u_1^\delta - r)(X_{n+1}^m - rX_n) \end{bmatrix},$$

where $\det = (m_2^\delta - r)(u_1^\delta - r) - (u_2^\delta - r)(m_1^\delta - r)$. The constraint $\pi_2 = 0$ is then equivalent to

$$(X_{n+1}^m - rX_n) = \frac{m_1^\delta - r}{u_1^\delta - r}(X_{n+1}^u - rX_n),$$

provided $u_1^\delta - r > 0$ (this is automatically satisfied if $u_1 - r > 0$ and $\delta_1 \geq 0$). Substituting in the equation for π_1 gives

$$\pi_1 = \frac{1}{\det} \left[(m_2^\delta - r) - (u_2^\delta - r)\frac{m_1^\delta - r}{u_1^\delta - r} \right] (X_{n+1}^u - rX_n) = \frac{1}{u_1^\delta - r}(X_{n+1}^u - rX_n).$$

The last two conditions in Equation (D.1) follow from these expressions.

At date N we get $\widehat{Y}_N(X_N) = Y_N$ and $K_N(X_N) = 0$. At an arbitrary date n the quadruple $(q_1^\delta, q_2^\delta, X_{n+1}^u, X_{n+1}^m)$ solves the system of equations

$$X_{n+1}^u = (X_n + W_n)\frac{(\xi_{n,n+1}^{\delta u})^{-1/R}(1 + K_{n+1}^u(X_{n+1}^u))^{1/R}\widehat{g}_{n+1,N}^u}{E_n\left[(\xi_{n,n+1}^\delta)^{1-1/R}(1 + K_{n+1}(X_{n+1}))^{1/R}\widehat{g}_{n+1,N}\right]} - \widehat{Y}_{n+1}^u(X_{n+1}^u)$$

$$X_{n+1}^m = (X_n + W_n)\frac{(\xi_{n,n+1}^{\delta m})^{-1/R}(1 + K_{n+1}^m(X_{n+1}^m))^{1/R}\widehat{g}_{n+1,N}^m}{E_n\left[(\xi_{n,n+1}^\delta)^{1-1/R}(1 + K_{n+1}(X_{n+1}))^{1/R}\widehat{g}_{n+1,N}\right]} - \widehat{Y}_{n+1}^m(X_{n+1}^m)$$

$$(X_{n+1}^m - rX_n) = \frac{m_1 + \delta_1 - r}{u_1 + \delta_1 - r}(X_{n+1}^u - rX_n)$$

$$X_{n+1}^u \geq rX_n,$$

where

$$W_n = \frac{1}{r}[q_1^\delta \widehat{Y}_{n+1}^u + q_2^\delta \widehat{Y}_{n+1}^m + (1 - q_1^\delta - q_2^\delta)\widehat{Y}_{n+1}^d]$$

and

$$K_{n+1}(X_{n+1}) = \frac{\partial \widehat{Y}_{n+1}}{\partial X_{n+1}}.$$

The optimal portfolio is

$$X_n \pi_{1n.} = \frac{1}{u_1 + \delta_1 - r}(X_{n+1}^u - rX_n).$$

The value function, the certainty-equivalent, and its derivative are

$$
J(X_n, n) = \begin{cases}
\frac{1}{1-R}(X_n + W_n)^{1-R}\, \frac{F_n(X_{n+1}, 1/R-1)}{\left(F_n(X_{n+1}, 1/R)\right)^{1-R}} & \text{if } X^u_{n+1} > rX_n \\[2ex]
\frac{1}{1-R} E_n\left[(rX_n + \widehat{Y}_{n+1}(rX_n))^{1-R}\widehat{g}^R_{n+1,N}\right] & \text{if } X^u_{n+1} = rX_n
\end{cases}
$$

$$
\widehat{Y}_n(X_n) = \begin{cases}
(X_n + W_n)\frac{F_n(X_{n+1}, 1/R-1)^{1/(1-R)}}{F_n(X_{n+1}, 1/R)}\left(\widehat{g}_{n,N}\right)^{-\frac{R}{1-R}} - X_n \\[1ex]
\quad \text{if } X^u_{n+1} > rX_n \\[2ex]
\left(E_n\left[(rX_n + \widehat{Y}_{n+1}(rX_n))^{1-R}\widehat{g}^R_{n+1,N}\right]\right)^{\frac{1}{1-R}}\left(\widehat{g}_{n,N}\right)^{-\frac{R}{1-R}} - X_n \\[1ex]
\quad \text{if } X^u_{n+1} = rX_n
\end{cases}
$$

$$
1 + K_n = rE_n\left[(X_{n+1} + \widehat{Y}_{n+1})^{-R}(1 + K_{n+1})\widehat{g}^R_{n+1,N}\right](X_n + \widehat{Y}_n)^R\widehat{g}^{-R}_{n,N}.
$$

with $F_n(a, b) = E_n[(\xi^\delta_{n,n+1})^{1-1/R}(1 + K_{n+1}(a))^b\widehat{g}_{n+1,N}]$. Solving these equations recursively from $n = N - 1, \ldots, 0$ leads to the certainty equivalent of the nontraded asset at the initial date.

For the case of an American-style NTA it suffices to replace the CE in the dynamic programming algorithm above by $\widehat{Y}_{n+1} = \widehat{Y}^c_{n+1} \vee Y_{n+1}$ in the manner of Section 2.4.

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