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# Valuation, Optimal Asset Allocation and Retirement Incentives of Pension Plans

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*We provide a framework in which we link the valuation and asset allocation policies of defined benefits plans with the lifetime marginal productivity schedule of the worker and the pension plan formula. In turn, we examine the retirement policies that are implied by the primitives of the model and the value of pension obligations. Our model provides an explicit valuation formula for a stylized defined benefits plan. The optimal asset allocation policies consist of the replicating portfolio of the pension liabilities and the growth optimum portfolio independent of the pension liabilities. We show that the worker will retire when the ratio of pension benefits to current wages reaches a critical value which depends on the parameters of the pension plan and the discount rate. Using numerical techniques we analyze the feedback effect of retirement policies on the valuation of plans and on the asset allocation decisions.*

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Pension plans in the private and public sectors have become a key institution in the functioning of financial markets. These plans provide a mechanism for consumers to save and can influence the retirement incentives of labor. Given the size of pension assets, it is not surprising that pension funds as a class are the dominant institutional investors in capital markets: a significant percentage of equities and fixed income securities are held by pension funds. These observations suggest that the valuation and the financial policies (funding and asset allocation) of pension funds should be of great interest to policy makers and researchers.

There are two types of pension plans: defined benefits (DB) plans and defined contribution (DC) plans. In DB plans the employer promises to pay a certain amount of benefits. These plans typically promise to pay an amount which depends on the number of years of service that the employee has at the time of retirement, as well as the history of wages over the employment period. Hence the funding and investment risks are borne by the sponsoring employer. In a DC plan, the employer agrees only to contribute a certain amount in each period to the employee's pension. The investment risk is borne by the employee.<sup>1</sup> This article is only about DB plans, which as of 1989 accounted for nearly 70% of the financial assets in private pension plans. In a DC plan, the valuation is simple—at any time, the value of such a plan is equal to the market value of the portfolio held by the fiduciary on behalf of the employee.

This article investigates the relationship between the employee's marginal productivity schedule over her lifetime, her current wages, and deferred wages or pensions. This link is examined in the context of an exogenously specified DB plan sponsored by the employer. The value of the pension plan is derived under "no arbitrage" conditions. We provide an objective function for the sponsor in a utility-maximizing framework in which they are assumed to be risk averse (and therefore we model their objective as a concave utility function), which leads them to pursue optimal asset allocation policies that will meet the promised pension obligations at any time. The value of pension assets accumulated by the employee also influences her retirement decision. For the specified pension plan, we characterize the employee's optimal retirement policy. In so doing, we have tried to synthesize two important strands of research in one unified setting. First, the labor economist's view of pensions as a tool to induce op-

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<sup>1</sup> The relative merits of these two classes of pension plans are discussed in detail by Bodie, Marcus and Merton (1988).

timal retirement is typically modeled in settings where the value of pension obligations is specified exogenously. We endogenize the valuation of pensions. On the other hand, the financial economist's view of pensions is typically one of valuation, asset allocation, and the design of efficient pension contracts without regard for the retirement incentives or the characteristics of pension liabilities. We establish a bridge between the previous two views by endogenizing the retirement incentives of pensions in a valuation setting and linking the valuation and asset allocation to the pension liabilities. The optimal retirement problem is a free boundary problem, and we identify a utility maximizing formulation that captures the feedback effects of optimal retirement decisions on valuation and allocation using numerical techniques.

In Section 2 we define the basic characteristics of our model. Here, we model a stylized DB plan and we derive a wage process that relates the lifetime marginal product profile of the worker to both current wage and deferred compensation (or pensions) offered by the employer. In Section 3, we derive a valuation rule for the obligations of the pension plan. We use a no-arbitrage argument that is guaranteed by our assumption that wages are perfectly correlated with some risky asset portfolio. The valuation of pension obligations is the cornerstone of the issues in the pension literature: to determine whether a plan is underfunded or overfunded, to establish funding and asset allocation policies, etc., one needs to establish the present value of pension obligations. In fact, the valuation of pension plans is the major focus of our article. Surprisingly, this issue has received only limited formal modeling. In Section 4, we use the valuation formula to identify the asset allocation policies that will ensure that the pension obligations will be met at all times. The associated asset allocation policies will be called the *replicating asset allocation policies*. We characterize the asset allocation policies with a conservative objective function which implies infinite costs to underfunding. It is straightforward to modify the objective function to introduce finite penalties for underfunding and extend our results.

In Section 5, we recognize the fact that the pension wealth of the worker, under a DB plan, creates an incentive for the worker to optimally retire early. The retirement decision is linked to the lifetime marginal productivity schedule: given our assumption about the evolution of marginal productivity, increasing at the beginning of the working life and then decreasing, wages will start to decrease after some point (while pension benefits might still be increasing). The worker has the "option value to work" [Stock and Wise (1990)] and we characterize the optimal exercise aspects of this option and explain

the intuition of the result. Using numerical techniques, we examine the feedback effects of optimal retirement policies on asset allocation. We close the article with some conclusions in Section 6.

## 1. Model and Assumptions

In this section we describe our model and introduce the assumptions that we will use in our analysis in the next two sections. In Section 5 we relax some of the current assumptions to derive the optimal retirement rules. We consider a stylized economy consisting of one firm and an employee.

We assume that all participants are competitive and price takers. Our model is cast in real terms.<sup>2</sup> Uncertainty in the economy is described by the evolution of  $B$ , a single standard Brownian motion process. We consider a finite time span that starts at 0 and finishes at a fixed and known date  $\tau$ , at which the single worker of the economy dies. The assumption of a known lifetime precludes us from examining the insurance aspects of pension plans. We also define a date  $T$ ,  $T < \tau$  that represents the date at which the worker retires. Throughout Sections 3 and 4 we assume that  $T$  is constant and known. In Section 5 we endogenize the retirement date  $T$ . The following assumptions characterize the employee.

### 1.1 The employee

The employee is characterized by her age at time  $t$ , that we denote by  $a(t)$ . Explicitly,  $a(t) = a_0 + t$ , where the employee is assumed to start work at date 0, at an age of  $a_0$ . Marginal product is assumed to be given by  $f(a)m$  where  $m$  follows a stochastic process as shown in Equation (1):

$$dm_t = m_t[\alpha dt + \sigma dB_t]; \quad m_0 = 1, \quad (1)$$

where  $\alpha$  and  $\sigma$  are assumed to be constants. Therefore, the marginal product  $f(a)m$  is a stochastic process with drift  $f'(a) + \alpha$ . The drift of the stochastic process followed by the marginal product exhibits systematic effects that arise due to the fact that the worker gets older with time. We now describe that function  $f(\cdot)$ .

The function  $f(a(t))$  is a function of the age  $a(t)$  of the employee. The employee is assumed to work for the firm until her retirement,

<sup>2</sup> It would be possible with minor modifications to incorporate nominal features as well: assuming a simple Fisherian relationship between nominal and real rates will allow us to map the results of our article to a nominal setting. We could also explicitly model the price level in addition to the marginal productivity and obtain similar conclusions provided the markets are complete. In order to keep the exposition simple, we have not explicitly modeled a nominal economy.

whereupon she will receive the pension and die at date  $\tau$ . We assume that the function  $f(a(t))$  is increasing in age up to a threshold age  $a^*$  and then begins to decline and gradually levels off. This pattern is consistent with one's intuition about the productivity pattern in practice. In exchange for her contribution to production the worker receives compensation from two sources: a salary continuously paid (that we denote by  $x$ ) and pension benefits that accumulate over life but are paid at retirement  $T$  (whose value we denote by  $P$ ). In Sections 3 and 4 the worker is simply the beneficiary of this exogenously fixed compensation: an objective function for the worker is not pertinent, as will become clear. In Section 5 the worker takes into account the effect of pensions on her retirement decision and we will introduce her objective function explicitly to characterize that decision. The firm is characterized by the following assumptions.

### 1.2 The firm

The firm offers a contract  $(x_t, P_T)$  to the worker, where  $x_t$  is the current wage rate and  $P_T$  is the deferred wages or pension benefits to be received upon retirement at  $T$ .<sup>3</sup> The pension contract, which is a function of the worker's wage history and the number of years of her service, is prespecified next. The exogenous specification of the pension contract is a limitation of our article.

### 1.3 Pension plan

We will consider a defined benefit plan as described next. The pension plan pays at date  $T$  an amount that depends on the weighted average  $z_T$  of wages, which we represent by  $x$  (but whose dynamics we do not make explicit yet), and an exogenous constant  $\lambda$  as shown below:<sup>4</sup>

$$P_T = \lambda z_T, \quad (2)$$

where

$$z_T = \beta \int_0^T e^{-\beta(T-s)} x_s ds. \quad (3)$$

We may write this in differential form as

$$dz_t = \beta(x_t - z_t)dt \quad (4)$$

<sup>3</sup> Note that it is generally not possible to get the worker to commit to a retirement date  $T$ . The firm must determine the wages without knowing precisely when the worker might choose to retire. In Sections 2 and 3, however, we assume that  $T$  is exogenously given and known by both sides. We will be more specific about our assumption in Section 4.

<sup>4</sup> Later, we show that the existence of Equation (3) is guaranteed.

We assume that the employee behaves as if the benefits are fully vested and are not subject to any default. The presence of default may lead to perverse investment behavior on the part of plan sponsors. This, in part, has motivated some pension regulations concerning minimum funding requirements, and the creation of guaranteeing agencies such as the Pension Benefits Guarantee Corporation (PBGC). Such issues, while important, will take us far away from the main inquiry of our article. The assumption that pension benefits are treated as default-free is not critical in the case of mandatory retirement. It is, however, crucial to allow tractability when the worker can choose the optimal retirement date. Together, Equations (2) and (3) capture parsimoniously the features of a defined benefit plan. In Equation (3) we have assumed  $z_0 = 0$ . If  $z_0 > 0$  and  $\beta$  is small, then  $z_t$  is approximately  $z_0$ , and hence there is little that is uncertain about future benefits. This special case approximates the "dollar amounts formula" used in the industry in which the pension is based on the years of service to the firm multiplied by a fixed dollar amount. On the other hand, as  $\beta$  increases to intermediate values, the history of wages becomes important and we capture various career-averaging pension plans, referred to as fixed benefit plans. Finally, when  $\beta$  becomes large, pension benefits depend only on the terminal wages, and this approximates the widely used "terminal earnings formula" or unit benefit plans. Together, these formulas are the most extensively used ones in the DB plans. [See the Employee Benefits Research Institute (EBRI) databook on employee benefits.] In what follows, we assume  $z_0 = 0$  for simplicity.

#### **1.4 Endogenous wage process**

We recall our assumption about the marginal productivity profile over lifetime introduced above. In this section, we endogenize wages so that the total compensation package (which includes both current wages and changes in the value of pensions) will be equal to marginal product. This condition is similar to the one used by Nalebuff and Zeckhauser (1985). The resulting equilibrium wage rate depends not only on the marginal productivity profile during the worker's tenure, but also on the parameters of the pension plan. Thus, we assume that the sponsoring firm equates the expected value of the compounded marginal productivity over the lifetime to the expected value of the compounded lifetime payments of current wages and deferred wages, as shown next. Hence, Equation (5) is a "fairness rule" that equates the expected (compounded) total compensation to the expected (compounded) total marginal product. The precise formulation in Equation

(5) is necessary to get tractable results:

$$\begin{aligned} E \left[ \int_0^T e^{\gamma(T-s)} x_s ds + \lambda \beta \int_0^T e^{-\beta(T-s)} x_s ds \right] \\ = E \int_0^T f(a(s)) e^{\gamma(T-s)} m_s ds \end{aligned} \quad (5)$$

In Equation (5), we have used a subjective and constant compounding rate  $\gamma$ . This compounding rate might be equal to the interest rate (that we introduce later) if the worker and the firm are risk neutral. The discount rate represents the (common) time preference. The firm essentially adjusts the current wage rate by comparing the total compensation with the age-dependent productivity of the employee. The wage  $x$  is the unknown of Equation (5). We assume that the equilibrium wage schedule satisfies Equation (6) and the pension formula [Equations (2) and (3)], where the parameters satisfy the lifetime “fairness constraint” specified in Equation (5). The wages are below the marginal product during the early years of employment and tend to increase with tenure as marginal productivity increases (but they will be less than the marginal product, as we explain below). The specific wage process that we consider is

$$x_t = m_t \left( \frac{f(a(t))}{1 + \lambda \beta e^{-(\beta+\gamma)(T-t)}} \right). \quad (6)$$

The reason why we choose this wage process is that this is the only objective rule that guarantees that the stochastic terms within the expectations in Equation (6) are equal path by path, that is,

$$\int_0^T e^{\gamma(T-s)} x_s ds + \lambda \beta \int_0^T e^{-\beta(T-s)} x_s ds = \int_0^T f(a(s)) e^{\gamma(T-s)} m_s ds,$$

regardless of  $T$ . There will be other wage processes that also satisfy Equation (5). But only the process chosen by us guarantees the following: when the worker retires the (compounded) compensation and (compounded) marginal product will be identical.

As expected, the current wage rate depends on the parameters of the pension plan, age of the employee, and discount rates used by the firm in determining the total lifetime productivity and compensation. Note that the equilibrium wage rate  $x$  is less than the marginal product  $m_t f(a(t))$  of the worker. The bonding aspect of this property has been discussed by Lazear (1979). When  $\lambda = 0$ ,  $x_t = f(a(t)) m_t$ , implying that the instantaneous wage rate adjusts to the marginal product, which is the standard result. Given our assumptions regarding the



marginal product, we can readily verify from Equation (6) that the process  $\{x_t\}$  is well defined. The worker, if fired, gets  $\lambda z_t$  by way of pension benefits, which are assumed to be fully vested. It is also easy to verify that the expected cost savings from firing the worker and paying him  $\lambda z_t$ , the accrued pension, is always less than the expected productivity loss and hence the firm will have no incentive to fire the worker. Formally, the expected savings is

$$E[\lambda z_T - e^{\gamma(T-t)} \lambda z_t \mid \mathcal{F}_t] + E\left[\int_t^T e^{\gamma(T-s)} x_s ds \mid \mathcal{F}_t\right].$$

The expected loss of marginal productivity is

$$E\left[\int_t^T e^{\gamma(T-s)} f(a) m_s ds \mid \mathcal{F}_t\right].$$

It is easy to show that the expected productivity loss is greater than the expected savings by firing the worker by the amount  $(e^{\gamma(T-t)} - e^{-\beta(T-t)})\lambda z_t$  which is positive. Note that the equilibrium wage rate is adjusted by the firm to reflect both the productivity of the employee and the relative attractiveness of the deferred compensation plan. It is useful to note that the wage rate is a declining function of  $\lambda$ . It may be either increasing or decreasing in  $\beta$ : namely,  $\frac{dx}{d\beta} > 0$  if  $\beta > \frac{1}{T-t}$  and  $\frac{dx}{d\beta} < 0$  if  $\beta < \frac{1}{T-t}$ . When  $T-t$  is large, wages are essentially unaffected by the choice of  $\beta$ . For large values of  $\beta$ , it is clear that the wage rate will be higher for a given level of productivity. The wage-tenure profile is of considerable interest to economists. In our model, wages are uncertain and their expected growth rate over time depends both on the parameters of the pension plan and the marginal productivity schedule. The dynamics of the wage process may then be written as

$$dx_t = x_t[\alpha^x(t)dt + \sigma dB_t]. \quad (7)$$

The drift term  $\alpha^x$  is the expected growth rate in wages and it captures all the interactions in the wage contract and productivity pattern. We may identify  $\alpha^x$  to be the following deterministic function

$$\alpha^x(t) = \left( \alpha - \frac{\lambda\beta(\beta + \gamma)e^{-(\beta+\gamma)(T-t)}}{[1 + \lambda\beta e^{-(\beta+\gamma)(T-t)}]} + \frac{f_a(a(t))}{f(a(t))} \right). \quad (8)$$

Note that the expected growth rate in wages may be greater than or less than the expected growth rate in marginal product. When  $f_a(a(t))$  is positive and the employee has many years to retirement so that  $T-t$  is large, we find that  $\alpha^x(t)$  is always positive. This is consistent with the rising wage rates over time that have been documented in the

literature. The expected growth rate in wages will have a point of inflection as marginal product falls with age and when the employee has only a few years to retirement. When  $T - t$  is large, the second term on the right-hand side is negligible and hence the expected growth rate of wages is independent of the pension plan. This is, however, subject to the qualification that if  $\beta$  is sufficiently small, then the effect of  $T - t$  is considerably weakened and the expected growth rate of wages captures the impact of pension plans. For small  $\beta$  the expected growth rate is higher and for large  $\beta$  the expected growth rate is lower. These observations become crucial in understanding the impact of  $\beta$  on the value of pension obligations.<sup>5</sup>

### 1.5 Capital markets

Capital markets are assumed to be frictionless with no taxes, transaction costs, or informational asymmetries. We focus on two assets in the capital market: a risky portfolio and the riskless asset. Prices of these assets are determined in the financial markets and are exogenous to our model. Prices are expressed in units of our numeraire, the single consumption good of this (real) economy. The riskfree asset is a bond whose price  $V^B$  satisfies

$$dV_t^B = V_t^B r dt; \quad V_0^B = 1. \quad (9)$$

The interest rate  $r$  is assumed to be constant. The risky asset (stock) is assumed to be perfectly correlated with aggregate wages (that is, it is driven by the single Brownian motion process that explains all the uncertainty in this economy) and its price  $V^M$  is assumed to satisfy

$$dV_t^M = V_t^M [\alpha^M dt + \sigma^M dB_t]; \quad V_0^M = v^M, \quad (10)$$

where  $\alpha^M$  and  $\sigma^M$  are constants.<sup>6</sup> Markets are, therefore, complete. Technically speaking, the crucial assumption of our model is that wages are perfectly correlated with a basket of securities (not necessarily the whole market portfolio). Assuming a single Brownian motion process provides a sparse setting for our results.<sup>7</sup> Some empirical

<sup>5</sup> It is worth restating that the wage process that we have derived may be readily modified to capture stochastic inflation, provided the markets are complete. Propositions 1 and 2 that follow are robust with respect to more general specifications of the wage process. For more general wage schedules (nominal) that are exogenously specified, Propositions 1 and 2 may be readily derived.

<sup>6</sup> Clearly, modeling the problem with more than one state variable is an agenda item for future research.

<sup>7</sup> The previous parametrization of the financial markets is standard in intertemporal models. Constant parameters  $r$ ,  $\alpha^M$ , and  $\sigma^M$  is not a desirable assumption, but is required for analytical tractability. The assumptions embodied in Equations (9) and (10) are consistent with a general equilibrium production economy in which a representative investor has power utility and faces a

studies have shown that there is an association between stock returns and growth in average real compensation. Black (1989) cites this as “clear evidence of the relation between stock returns and growth in pension liabilities.”<sup>8</sup>

## 2. Pension valuation

As we explained in the previous section, under our assumptions, markets are complete. Wages (and therefore pensions) are driven by the unique Brownian motion process that induces uncertainty in the economy and so does the risky security (“stock”). We can replicate the value of the pension plan  $P$  by constructing a dynamic portfolio consisting of the risky security and the riskless asset that perfectly replicates the dynamics of the value of the pension plan. The valuation is based on no arbitrage arguments—no explicit notion of an equilibrium is required. Therefore, we do not have to specify an objective function for the agents of our model. The valuation procedure provides us with two positive implications: first, we are able to measure the value of pension obligations. This step is essential in measuring the financial health of any pension plan. Using the value of pension obligations from this model and comparing it to the market value of pension assets, we may determine the degree to which a pension plan is underfunded or overfunded. Second, the model also provides us with constructive recipes as to how the pension fund asset allocation is to be managed so as to meet the promised obligations in the future. Next, we state the valuation formula of the DB pension plan as a proposition.

**Proposition 1.** *Consider the model described in Section 2. Assume that the retirement date of the worker  $T$  is a known constant. Define the constant price of risk  $\theta$  as*

$$\theta = \frac{\alpha^M - r}{\sigma^M} \geq 0. \quad (11)$$

*The pension value function is of the form*

$$P(x, z, t) = z_t \lambda e^{-(r+\beta)(T-t)} + x_t b(t), \quad (12)$$

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random opportunity set described by technologies that follow processes as in Equation (10).

<sup>8</sup> We also point that, from Equations (4) and (7), current wages paid upon the historical average  $z$  will be positively correlated with past portfolio performance.

where  $b(t)$  is

$$b(t) = \lambda\beta e^{-(r+\beta)(T-t)} \int_t^T e^{\int_t^s (\alpha_u^x + \beta - \theta\sigma) du} ds. \quad (13)$$

*Proof.* The proof of this proposition is in the Appendix. ■

The value of pension obligations consists of two components. The first component is  $z_t \lambda e^{-(r+\beta)(T-t)}$ , which may be thought of as the present value of future benefits due to past wages earned by the employee or accrued (accumulated) benefit obligations (ABOs). The second component is  $x_t b(t)$  and is the value created due to current wages and expected future wages: this latter point is better understood by realizing that the function  $b(t)$  depends on the expected growth rate of wages. The sum of both terms approximates the projected benefit obligations (PBOs).

The structure of our solution is fairly robust—for a richer specification of the underlying stochastic process, we will still be able to split the valuation of pension obligations into two components. It is also useful to note that the result does not require us to specify any additional structure on  $f(a)$ . To verify this claim note that from Harrison and Kreps (1979) we can write

$$P_t = \tilde{E}[e^{-r(T-t)} \lambda z_T \mid \mathcal{F}_t],$$

where  $\tilde{E}$  means expectation with respect to the equivalent probability measure. Harrison and Kreps (1979) show that that measure is unique in complete markets (the setting we consider) and that there will be infinite equivalent measures in incomplete markets (in this case the measure will be undetermined and we cannot perform the valuation).

It is straightforward to show that

$$\begin{aligned} & \tilde{E}[e^{-r(T-t)} \lambda z_T \mid \mathcal{F}_t] \\ &= e^{-(\beta+r)(T-t)} \lambda z_t + \tilde{E}\left[e^{-(\beta+r)(T-t)} \lambda \beta \int_t^T e^{-\beta(t-s)} x_s ds \mid \mathcal{F}_t\right]. \end{aligned}$$

In our complete markets setting we can solve for the expectation in the second term of the previous equation. In an incomplete markets setting, the second term cannot be explicitly solved, but the split between the two terms (with the first term as in Proposition 1) still holds. This is robust for very general wage processes in which the wages are driven by many factors.

Note that the value of the pension obligations will be dominated by the second term during the early part of the employment: this is due to the fact that little has been accumulated in the pension obligations

due to past service, and current wages and expected future wages dominate the value. The present value of pension obligations is an increasing function of  $z_t$ , the average wages as of date  $t$ ,  $x_t$ , the wage rate at date  $t$ , and the expected growth rate in wages. On the other hand, the value of pension obligations is a decreasing function of the discount rate  $r$ . As  $\beta$  increases, the pension value begins to capture wage history. In the limit as  $\beta$  takes on very high values, pension value is dependent only on the terminal wages.

The value of pension obligations depends in an intricate way on the marginal productivity schedule and its effect on equilibrium wage rates. These effects are subsumed in the function  $b(t)$ . To understand these effects a little better, we need to characterize this function which is closely related to the elasticity of pension value to current wages. Note that the elasticity of pension values to changes in the current wage rate may be defined as

$$\frac{\partial P}{\partial x} \frac{x}{P} \equiv b(t) \frac{x}{P}.$$

It is easy to verify that the elasticity is positive. As the tenure  $T - t$  that is remaining decreases, elasticity declines. In order to further characterize the value of pensions and its elasticity to current wages, we consider a simple marginal productivity schedule over the lifetime of the worker as given by

$$f(a(t)) = e^{-b_1 a(t)} - e^{-b_2 a(t)}, \quad (14)$$

where  $b_1$  and  $b_2$  are constant and  $b_2 > b_1$ .

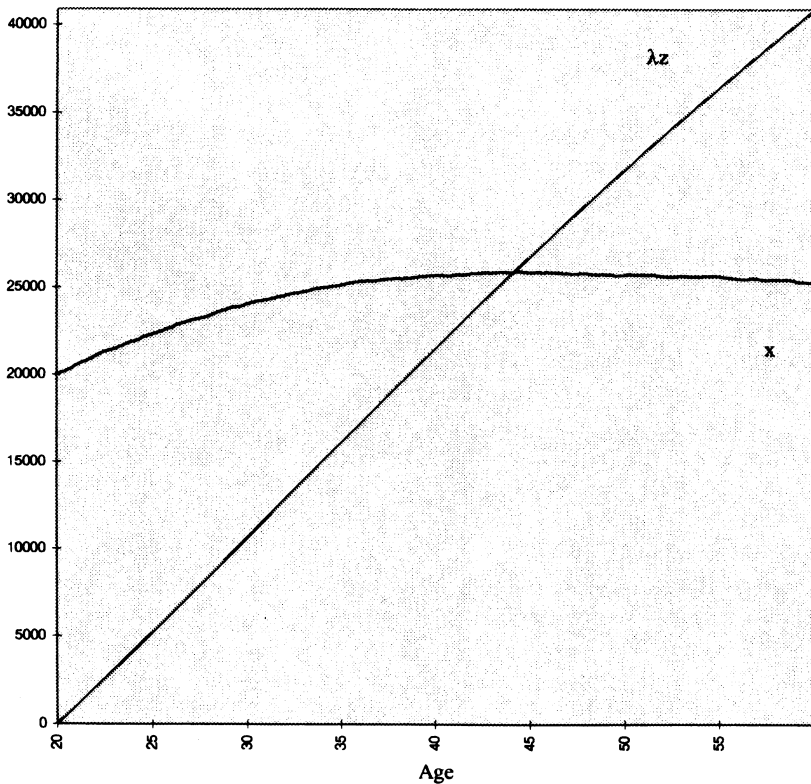
For example, by choosing  $b_1 = 0.005$  and  $b_2 = 0.05$  we obtain one schedule in which the worker reaches her peak productivity at an age of 51. This functional form affords us some flexibility: by setting  $b_2 = 0.06$ , the peak productivity age changes to 45. By varying  $b_2$  we can vary the peak productivity age. Note also that around this function  $f(a(t))$  there are stochastic variations that will govern the realized productivity.

The value of pension obligations is an increasing function of  $b(t)$ . Through this function, the value of pension obligations depends on factors such as  $\beta$ ,  $\theta$ ,  $\lambda$ , and the marginal productivity-age schedule. The behavior of the wage elasticity of the pension value depends on the averaging rule specified in the defined benefits plan. We examined the wage elasticities for three averaging rules:  $\beta = 0.05, 0.10$ , and  $0.50$ . DB plans which differ in the payment formula produce pension obligations with varying wage elasticities. When the employee has only a few years to retire, pension values tend to be less wage elastic, in general. But in DB plans with a large  $\beta$ , the sensitivity is high. Note

that DB plans with  $\beta = 0.50$  are more wage elastic than DB plans with a  $\beta = 0.05$ , when the tenure left is 15 years or less. When the employee has many years of tenure left with the firm, pension values are moderately elastic with respect to current wages. This happens because the first term in the valuation formula is close to zero. After the tenure left shrinks to about 10 years, pension value becomes less sensitive to current wages because it is primarily driven by the averaging formula. The averaging formula reduces the sensitivity sooner for smaller values of  $\beta$ : this is intuitive because when  $\beta$  is large, the averaging formula is less important and pensions tend to be much more elastic to current wages. These factors consequently affect the asset allocation policies. The greater the wage elasticity, the greater will be the exposure to wage uncertainty and hence the greater will be the allocation in the risky asset portfolio. The elasticity peaks as the tenure of the employee decreases and then begins to fall as the retirement date approaches. For DB plans with a high  $\beta$  ( $\beta = 0.50$ ), the peak elasticity occurs much closer to the retirement date. For DB plans with a small beta ( $\beta = 0.05$ ), the elasticity peaks much sooner. The model can also be used to study the effects of peak productivity age on pension values. In a well-functioning pension plan, we would expect an employee with higher lifetime marginal productivity to receive a more generous pension. This turns out to be the case in our model. The model implied that the employee with a higher peak productivity age will have a higher pension value everywhere during the tenure compared to an employee with a lower peak productivity age, given the respective equilibrium wage levels. The effect of discount rate changes on pension valuation is both direct and significant. Pension values were inversely related to discount rates. This outcome is hardly surprising given the long duration of pension obligations. In Figure 1 we illustrate some of the previous issues with an example where the evolution of wages and pension benefits are shown for a set of parameter values.

### 3. Asset Allocation Policies

Pension funds are managed by professional money managers. For example, Employee Benefits Research Institute reports that nearly \$1.5 trillion of pension assets were managed by private trustee pension funds as of 1989 [see EBRI, quarterly pension investment report (1990)]. As established in the previous section, a straightforward replication of the argument of Black and Scholes (1973) would verify that there exists an asset allocation policy that will guarantee the pension obligations under full funding. Such a policy requires an investment of  $\frac{P_x x \sigma}{\sigma_M}$  in the risky asset and the remainder in the riskless asset. The



**Figure 1**

We graph the wage  $x$  and vested pension benefits  $\lambda z$  for a worker that starts working at age 20 and retires at  $T = 60$ . Parameter values are  $\beta = 0.1$ ,  $\gamma = 0.03$ ,  $\alpha = 0$ ,  $\sigma = 0.06$ ,  $\lambda = 5$ ,  $b_1 = 0.005$ , and  $b_2 = 0.05$ . The plots are the result of simulating numerically the evolution of wage and pension benefits and averaging over the simulations. As expected, the wage starts to drop at about age 40 (when marginal productivity reaches a peak). Pension benefits grow at a decreasing rate.

dollar value of the pension assets must be allocated between equity and riskless assets in accordance with the replication rule.

In this section we wish to determine the optimal investment strategy of the pension fund sponsor. In order to fully specify our partial equilibrium setting, we need to specify the objective function of the pension fund. We assume that the cost of not producing a surplus relative to the value of the pension obligations that flow from the valuation rule in Section 3 is infinite. While we recognize that this assumption is extreme, it is conservative and is in the spirit of the fiduciary responsibility that the sponsoring firm has for its employee. We therefore assume that the sponsor maximizes the expected growth rate of sur-

plus, which is consistent with an infinite penalty on underfunding.<sup>9</sup> Our approach is also in the spirit of portfolio insurance policies followed by many pension funds' money managers—we provide an objective function that implies that the pension assets will always be managed so as to remain above the pension liabilities at all times. The next proposition summarizes our asset allocation results. Similar results are derived in the portfolio insurance literature [see, e.g., Black and Perold (1992)].

Consider the arbitrage-free value of the pension plan denoted by  $P$  which satisfies Equation (12). Denote by  $V_t$  the market value of the assets (at date  $t$ ) of the pension fund. Define surplus  $S_t$  as<sup>10</sup>

$$S_t = V_t - P_t.$$

The sponsor of the pension assets is assumed to maximize

$$\text{Max}_{\{V_T\}} E \left[ \frac{(V_T - \lambda z_T)^\gamma}{\gamma} \right]. \quad (15)$$

The sponsor has free access to the financial markets described in Section 2. Assume also that the sponsor is endowed with an initial level of funding  $V$  equal to  $V_0 = P_0 + \epsilon$ ,  $\epsilon > 0$ . Finally, define the following stochastic variable,<sup>11</sup>

$$\eta_t \equiv \exp \left[ - \int_0^t \theta dB_s - \frac{1}{2} \int_0^t \theta^2 ds \right], \quad (16)$$

where  $\theta$  is defined in Equation (11).

**Proposition 2.** *Assume the objective of the sponsor of the pension plan is given by Equation (15); the optimal dollar investment in the risky asset (that we denote by  $\pi$ ) is*

$$\pi_t = \frac{e^{rt} \epsilon \theta}{(1 - \gamma) \sigma^M (\eta_t)^{\frac{1}{1-\gamma}}} \exp \left[ - \frac{1}{2} \frac{\gamma}{(\gamma - 1)^2} \theta^2 t \right] + \frac{\frac{\partial P_t}{\partial x_t} x_t \sigma}{V^M \sigma^M}. \quad (17)$$

where,  $\eta$  and  $x$  are stochastic,  $\frac{\partial P_t}{\partial x_t}$  is a deterministic function, and  $\theta$ ,  $\sigma^M$ , and  $\sigma$  are constants as defined earlier.

<sup>9</sup> It is possible to extend our results by admitting a finite cost to underfunding as indicated later.

<sup>10</sup> By defining  $S_t = V_t - \delta P_t$  where  $\delta$  is a positive constant less than 1, we can admit finite penalties for underfunding.

<sup>11</sup> The meaning of this stochastic variable is explained in the Appendix.



*Proof.* This proposition is quite general and goes through for utility functions defined over the surplus of assets over liabilities. The full solution to this problem (including admissible strategies of the sponsor) is discussed in the Appendix. ■

Note that as  $\gamma$  tends toward zero, the objective of the sponsor is to maximize the log of terminal surplus. It is well known that this is equivalent to maximizing the rate of growth of the surplus,  $g(t, T)$ :

$$g(t, T) = \frac{\ln S_T - \ln S_t}{T - t}. \quad (18)$$

The expected growth rate of surplus may then be expressed as

$$E[g(t, T) \mid \mathcal{F}_t] = E\left[\frac{\ln S_T - \ln S_t}{T - t} \mid \mathcal{F}_t\right]. \quad (19)$$

The general structural form of the asset allocation policy which neatly splits into two components affords a very nice interpretation. Note that for small values of  $\epsilon$ , the optimal asset allocation policy approaches  $\frac{P_x \chi \sigma}{\sigma^M}$ , the benchmark policy that was derived in Section 3. The intuition for this outcome is straightforward: the penalty for policies that cause the market value of pension assets to become equal to the present value of pension obligations (or zero surplus) is infinite under growth optimum policy (that is, under the strategy that maximizes the expression in Equation (15)). As a result, when the market value approaches this floor ( $P$ ) the asset allocation approaches the benchmark policies derived earlier. If the plan is overfunded by the amount  $\epsilon$ , then Proposition 2 provides the recipe for asset allocation policies associated with overfunding. Note that the amount invested in the risky asset in excess of the liability is directly proportional to the amount of overfunding  $\epsilon$ . The proportionality factor  $\frac{e^{r^* t} \epsilon \theta}{\sigma^M(\eta_t)^{1-\gamma}} \exp[-\frac{1}{2} \frac{\gamma}{(\gamma-1)^2} \theta^2 t]$  is increasing in the price of risk and decreasing in the volatility of the risky asset's return. It is interesting to note that the amount overfunded is allocated based on the parameters that define the dynamics of the market. This amount is allocated independent of the characteristics of the pension liabilities of the firm. In fact, it is the growth optimum portfolio. The properties of the growth optimum portfolio are discussed in detail by Merton (1990). In this strategy the investor maximizes the rate of growth [as shown in Equation (19)] of the portfolio's assets.

The replicating asset allocation, on the other hand, is entirely pinned down by the idiosyncratic features of pension liabilities.

In a nutshell, Proposition 2 says that the optimal asset allocation consists of the replicating portfolio of the pension liability plus the growth optimum portfolio for the surplus.

With a long tenure to go, as seen from the valuation formula of Equation (12), pensions become more elastic to current wages as  $\beta$  decreases. Consequently, for such long-dated pension obligations, the smaller the  $\beta$ , the greater is the proportion allocated to risky assets to span or "hedge" the uncertainty in future wages. As seen from Equation (13), the sensitivity of pensions to current wages is captured by  $h(t)$  which varies with  $\beta$ . This implies a different asset allocation policy for defined benefits plans with different averaging rules. When  $T - t$  is very small, so that the employee is about to retire within a few years, pensions are driven by the averaging rule in the benefits formula. For the employee who is to retire within 15 years, if  $\beta$  is small (say, 0.05), then the sensitivity to wages is low and hence the riskless asset will be used more actively in the asset allocation process. But for the same employee, if  $\beta = 0.50$ , the exposure to current wages is high and more equity will be used in the asset allocation policy. The sensitivity of asset allocation policies to the wage uncertainty parameter  $\sigma$  is easily derived in our model. As the uncertainty in wages increases, the asset allocation emphasizes the equity sector more, so as to track the behavior of pension liabilities. The proportion of equity held declines as the employees get older since the uncertainty gets quickly resolved in the final years of tenure. Of course, our conclusions here are strongly influenced by the fact that we have only one state variable.

The asset allocation policies that we have characterized are precisely the ones that provide portfolio insurance to pension assets with a floor equal to the total value of pension liabilities. Many of the implications of our asset allocation policies are consistent with the observations made in Black (1989). In that article, Black argues that if a narrow view of pension liability is taken (whereby pension liabilities are viewed to be equal to the termination value of the plan), then asset allocation policies should emphasize the fixed-income sector. On the other hand, if a broader view is taken (whereby pension liabilities are treated as those associated with an ongoing plan and hence take into account projected growth rate in wages), then asset allocation policies should emphasize the equity sector more. Our model's predictions confirm these observations. Our asset allocation policy was based on treating the total liability (we take a broader view of pension liabilities) as the floor in the pension portfolio. Our analysis is easily modified to treat the current vested liability as the floor, as suggested by Black.

As we have pointed out,  $\frac{e^{rt}\epsilon\theta}{\sigma^M(\eta_t)^{\frac{1}{1-\gamma}}} \exp[-\frac{1}{2}\frac{\gamma}{(\gamma-1)^2}\theta^2 t]$  represents the proportion of the level of overfunding  $\epsilon$  invested in equity. At moment 0, when our one time funding takes place, that proportion is equal to  $\frac{\theta}{\sigma^M}$ . The proportion is obviously independent of the level of overfunding  $\epsilon$ , since the utility function we have chosen for the money manager exhibits constant relative risk aversion.

An interesting issue (not examined in this article) is the implications when the sponsor is unable to hedge completely in the capital market the risk of not meeting the liabilities. A power utility would not be compatible with such a possibility: there will be a positive probability of default and the penalty associated with it is infinity (marginal utility at zero is infinity). We suspect that in an incomplete markets setting with power utility the sponsor would become very conservative [as is the case with the utility specification in (15)] and would invest an increasing amount of the surplus in the riskfree security. That seems to contradict empirical evidence which suggests that an increasing proportion of the overfunding is actually placed in equity. A potential explanation of that fact consistent with our model might be that wages are in fact highly correlated with some of the securities in the stock market and money managers are able to hedge most of the risk. In addition, penalties for not meeting the liabilities may be finite so that an increased investment in equity might provide upside potential without large penalties on the downside. Their tendency to invest an increasing proportion of the excess of funding might also be due to the fact that they have decreasing relative risk aversion (as opposed to constant relative risk aversion as in our model).

It could be argued that the equity holders may be indifferent about the firm hedging its pension risk: presumably, they could unwind any positions that the firm may take. In the presence of frictions and penalties, the firm still should pursue a policy such as the one proposed here. We have not treated the tax implications of asset allocation explicitly: clearly, tax incentives will provide greater incentives for the firm to emphasize the riskless asset in the asset allocation.<sup>12</sup> Corporate bonds are priced to reflect some tax premium. Given the tax-exempt nature of the pension plan, it may be optimal to place some investment in corporate debt securities. But the surplus should still be placed in equity as indicated by Proposition 2.

<sup>12</sup> Black (1980) argues that there are tax incentives to replace stocks by bonds in the pension plan and simultaneously issue more debt by the company: interest payments to the pension plan are tax exempt and additionally the company benefits from the tax deduction of the interest are to be paid to the debt.

#### 4. The Worker's Retirement Decision

Sections 2 and 3 examined the valuation and asset allocation decisions assuming that the worker will retire at a known future date  $T$ . In this important sense, the problem was partial equilibrium in nature. Still, the specifications in Sections 2 and 3 allowed us to examine the link between pension liabilities, DB rules, valuation, and asset allocation. Clearly, the DB pension plan provides retirement incentives.

Since we do not model the disutility of work in our model, the "optimal retirement" that we consider here is simply the wealth maximizing one: the worker will retire when retirement is optimal for the maximization of the expected wealth; the interaction between falling wages and pension benefits is such that continuing to work will have smaller marginal utility than retiring and collecting the benefits. The explicit introduction of disutility of work and the determination of endogenous labor supply significantly complicate the analysis contained in this article. A mitigating aspect is the fact that the retirement policy derived in this section is robust with respect to all utility functions increasing in consumption and final wealth.<sup>13</sup> (The proof is valid for all utility functions of this class.)

In this section, we examine the retirement incentives induced by the DB plan. In Section 2 we described the compensation package to be received by the worker. It included pension benefits to be received upon retirement at a given constant date  $T$ . In this section we specify an objective function for the worker. The worker will try to achieve that objective by choosing optimal consumption and investment strategies and a retirement date. Retirement will be one of the controls of the worker of our model. We thus relax the assumption introduced in Section 2 of a constant retirement date  $T$ . Recall, however, that the wage dynamics, as described in Section 2, depend as well on the retirement date  $T$ . For now, we assume that wages satisfy Equations (7) and (8) for a constant estimated retirement date, but the worker is free to choose the retirement date.<sup>14</sup> This decision turns on incentive effects and "the option value to work" that have been discussed in the literature [see, e.g., Diamond and Mirrlees (1985), Nalebuff and Zeckhauser (1985), and Stock and Wise (1990)].

The worker in general will be unable to commit to a fixed retirement date. Indeed, she will attempt to decide the retirement date based on all available information. We characterize this problem next.

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<sup>13</sup> We thank the referee and the editor for raising these issues.

<sup>14</sup> We will relax this assumption later by endogenizing the parameter  $T$  that drives the dynamics of the wage process in Equation (8).

Our treatment of this problem is similar to Stock and Wise (1990) but differs from it in two respects. First, we solve the dynamic programming problem explicitly. Second, our utility specifications differ from their work which is calibrated to empirical observations (they conjecture also constant relative risk-aversion utility and calibrate the parameters). Our article differs both in the functional forms that we have chosen for the utility function as well as the parameters used.

The objective of the worker is to maximize lifetime utility. We let  $\delta$  represent her subjective discount rate. The retirement date  $T$  is also a decision variable for the worker. Between  $[T, \tau]$  (period 2), the consumer solves a fixed-horizon problem having collected her pension wealth and accumulated wealth. This optimization program leads to her optimal consumption policy and her value function. Between  $[0, T]$  (period 1) she solves a stochastic consumption-portfolio problem where the time horizon is a control, leading to her optimal consumption-portfolio choice and value function. Recall from Equation (8) that  $\alpha^x$  is a function of the retirement day. However, in this section the retirement date is a control variable and, therefore, stochastic. In that case, the "fairness rule" defined by Equation (5) is no longer valid. We will consider in this section that  $\alpha^x$  is set by the firm based on its best estimate of the worker's retirement date, which is treated as fixed by the firm. We thus suppose that the corporation estimates the retirement date at the beginning of the contract and this is the constant parameter that will drive the expected growth rate in wages until actual retirement. In our model, pensions perform a role in inducing the employee to retire when her marginal product falls. This is due to the fact that as the marginal product falls, the wage rate begins to fall. This has the effect of lowering the average lifetime wages ( $z_t$ ). The precise fall will depend on the  $\beta$  factor. So, by continuing to work, one can earn wages but must accept the falling average wage. The trade-off between these two considerations will determine the precise timing of an individual's retirement. Note that this is quite distinct from the retirement considerations that will arise in a market where pension benefits are accessible only upon retirement.<sup>15</sup> We state our result as a proposition.

**Proposition 3.** *Define a constant  $\kappa$  as the estimated retirement date that the firm uses to define the dynamics of the salary  $x$ . Assume that the worker has monotonic preferences and will live up to  $\tau$ . The worker is endowed with an initial wealth  $X_0$ ,  $X_0 \geq 0$ , until retirement at  $T$ ,*

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<sup>15</sup> We regard this approach as a first step in analyzing the incompleteness due to nontraded human capital. This latter issue has been studied by Merton (1983) in a welfare context.

$T \leq \tau$  she receives a salary  $x$  that satisfies Equation (7), with

$$\alpha^x(t) = \left( \alpha - \frac{\lambda\beta(\beta + \gamma)e^{-(\beta+\gamma)(\kappa-t)}}{[1 + \lambda\beta e^{-(\beta+\gamma)(\kappa-t)}]} + \frac{f_a(a(t))}{f(a(t))} \right) \quad (20)$$

and upon retirement at  $T$  she receives her pension benefits as described by Equations (2) and (3). Furthermore, the worker has free access to the financial markets described in Section 2.

Define  $\phi$ , the return from employment adjusted for the riskiness of marginal product  $\phi(t) \equiv \frac{\alpha^x(t)-r}{\sigma}$ . Assume further that<sup>16</sup>  $\phi(t) < \theta - \frac{r+\beta}{(1+\lambda\beta)\sigma}$ .

Then the worker retires the first time  $T$  at which  $\frac{\lambda z_T}{x_T} = \frac{1+\lambda\beta}{(r+\beta)}$ .

*Proof.* The proof of this proposition is in the Appendix. ■

Proposition 3 specifies an optimal exercise policy for the option value associated with continuing to work. This option value has the simplest possible interpretation when  $\beta$  is close to (but not equal to) 0. In this case, the rule is to retire when the pension benefits approximately equal the capitalized value of wages.

The results of our proposition shed some light on how DB plans may influence voluntary retirement. Under our assumptions, the present value of the labor income (wages and pension benefits) of the worker is endogenous, since she will choose the retirement date that maximizes her value function. The retirement date will vary according to the state variables  $x_t$  and  $z_t$ .

Since we are facing a free-boundary problem, we cannot compute in explicit form the value function. We denote by  $T^*(t)$  the retirement date as perceived at date  $t$  and express the optimal consumption and indirect utility as functions of it. However, as we show in the Appendix, for a large number of cases we are able to provide the rule that governs the optimal stopping time. The intuition behind this proposition is summarized next. Given our assumptions about the productivity curve of Equation (14), it is clear from Equation (20) that  $\alpha^x$ , the drift of the wages, is decreasing and eventually will even become negative. Therefore, wages will have a point of inflection. It is reasonable to expect that the pension benefits, evolving as expressed by Equation (4), will eventually reach a proportion  $\frac{1+\lambda\beta}{(r+\beta)\lambda}$  of wages. If this happens, given our assumption the worker will retire, because

<sup>16</sup> This assumption says that the risk-adjusted return from employment  $\phi(t)$  is less than the price of risk  $\theta$  adjusted by a factor  $\frac{r+\beta}{(1+\lambda\beta)\sigma}$ , which depends on the pension plan parameters and the discount rate. It is easy to check that this condition is satisfied for most parameter configurations of interest.

a strictly positive time horizon results in a smaller value of the expected labor income and, therefore, of the value function. However, should the pension benefits reach that proportion of wages before  $\phi(t)$  falls below the market price of risk adjusted by the factor as in the assumption above, such a rule is not valid anymore and we cannot provide an alternative optimal policy. This may occur because the worker finds that her return from employment is more valuable and finds that retirement is not wealth maximizing.

#### 4.1 Retirement and state of the economy

Note that the firm is in a position to induce the worker to quit voluntarily by choosing the parameters of the plan suitably. The firm may select the parameters of the pension plan to minimize potential losses that may be imposed by the worker.<sup>17</sup> Although we do not address the design of such pension plans in our analysis, it is clear that our framework may be used to design a simple policy for the firm. Note also that the critical value  $\frac{1+\lambda\beta}{(r+\beta)\lambda}$  (we assume  $\lambda r > 1$ ) is decreasing in  $\lambda$ , increasing in  $\beta$ , and decreasing in  $r$ . Therefore, increases in  $\lambda$  and  $r$  will induce earlier retirement, while increases in  $\beta$  will typically induce later retirement.

To get additional insights on the retirement decision, we characterize the critical ratio  $\frac{\lambda z}{x}$  further. By applying Ito's lemma, we get

$$d\left(\frac{\lambda z_t}{x_t}\right) = (\beta + \alpha - \sigma^2) \left[ \frac{\lambda\beta}{(\beta + \alpha - \sigma^2)} - \frac{\lambda z_t}{x_t} \right] dt - \frac{\lambda z_t}{x_t} \sigma dB.$$

The stochastic process above indicates that the ratio of ABOs to current wages follows a mean reverting process. The long-term mean rate is given by  $\frac{\lambda\beta}{(\beta + \alpha - \sigma^2)}$ . Note that this is increasing in  $\sigma$  so that increases in the volatility of wages causes this ratio in the long term to increase. This implies that the worker will tend to retire later, *ceteris paribus*. Note also that the critical ratio  $\frac{\lambda z}{x}$  is negatively correlated with the stock market/wage processes. In "good states" (high value of the stock market/high wage) the critical ratio will be lower. The opposite holds when the economy does not perform well. The worker will optimally retire earlier in "bad states" and later in "good states".

In general, pension funds are overfunded, that is, the value of the assets is greater than the actuarial present value of the expected obligations of the pension fund. (The status of funding tends to change with levels of interest rates, as with lower interest rates the pension liabilities tend to be valued more.) It is argued that the tax break (con-

<sup>17</sup> As pointed out earlier, for a given plan, there is no incentive for the firm to fire the worker.

tributions to the fund are tax deductible and the pension earnings are tax exempt) is one of the driving forces of this fact [see, e.g., Black (1980) and Tepper (1981)]. Along with taxes that obviously induce an overfunding policy, there are disincentives to over- or underfund, as discussed earlier. It is possible to close our model by introducing taxes and penalty. But a more satisfactory treatment of funding would require the modeling of firm's investment and financing decisions. In this context, pensions as a "financial slack variable" would play a key part. Unfortunately, these interesting issues are beyond the scope of this article.

#### 4.2 Feedback effects of voluntary retirement

We study in this section the valuation and optimal asset allocation policies derived at the beginning of this section when the retirement decision is endogenous. As we pointed out, there is no closed form solution for the value of the pension plan and we cannot derive an optimal investment strategy as we were able to do when the retirement date was mandatory. Intuitively, since we are in a complete market setting, the value of the pension plan when the worker can choose her retirement date will be higher than when the retirement date is mandatory. However, this does not give us any hint about the change in the composition of the optimal portfolio. We address this issue numerically.

The baseline is the optimal investment strategy with mandatory retirement as given by Equation (17). The first term is the optimal growth portfolio for the surplus  $\epsilon$ . Obviously this part of the optimal portfolio will be the same for both types of retirement. The second term is the replicating portfolio. The key component is the volatility of the pension plan. In order to study the replicating portfolio, we need to study the volatility of the value of the pension plan. In order to do this, we perform Monte Carlo simulations.

From our analysis in Section 2, the value of the pension plan is given by

$$P_t = e^{(r+\beta)t} \left( \lambda z_t E \left[ \frac{\eta_T}{\eta_t} \bar{e}^{(r+\beta)T} \mid \mathcal{F}_t \right] + E \left[ \frac{\eta_T}{\eta_t} \bar{e}^{(r+\beta)T} \lambda \beta \int_t^T e^{\beta(s-t)} x_s ds \mid \mathcal{F}_t \right] \right).$$

We compute the previous expression for each path of the Brownian motion process. Two cases are treated: mandatory retirement and voluntary retirement. The difference is that in the case of voluntary retirement  $T$ , the retirement date is stochastic. For each path of the



**Table 1**  
**Value and volatility of pension plans under the different regimes ( $\lambda = 5, r = 0.02$ )**

$\beta$	$\alpha^M$	$P^M$	$\sigma^M = 0.15$		$\nu^V$	$P^M$	$\sigma^M = 0.18$		$\nu^V$
			$P^V$	$\nu^M$			$P^V$	$\nu^M$	
0.03	0.05	65,134	65,450	0.02174	-0.01519	67,002	66,422	0.02261	-0.00610
	0.06	60,756	63,295	0.02044	-0.03686	63,773	64,765	0.02123	-0.02192
0.04	0.05	72,792	85,151	0.02359	-0.01212	74,969	84,917	0.02446	-0.00791
	0.06	67,744	85,701	0.02229	-0.01860	71,215	85,323	0.02308	-0.01460

We consider a worker who starts working at  $a_0 = 20$ . The parameters of the marginal productivity function of Equation (14) are  $b_1 = 0.005$  and  $b_2 = 0.05$  with starting salary  $x_0 = 20,000$ . We set  $\alpha = 0, \sigma = 0.06$  (these are the parameters that explain the path of the marginal productivity; pension plan values always increase with them), and  $\gamma = 0.03$ . We take one random path of the Brownian motion process that describes uncertainty in this model up to the time  $a = 40$ . We then estimate the value and volatility of the pension plan under each regime. We repeat this exercise for different values of the parameters of the model. The notation for the parameters is as in the article:  $P^M$  is the value of the pension plan under the mandatory retirement regime and  $P^V$  is the value of the pension plan under the voluntary retirement regime. Numbers have been rounded to the unit.  $\nu^M$  represents the volatility of the Pension Plan under the mandatory retirement regime and  $\nu^V$  is the volatility under the voluntary retirement regime. The relationship between these two equals the relationship between the proportions of the plans under the different regimes that should be invested in the risky stock. For the case when the worker chooses retirement optimally, the value  $T$  that drives the dynamics of  $\alpha^x$ , the drift of the wage process, is exogenous and unrelated to the actual age of retirement of the worker. The corresponding values of  $x$  and  $z$  are given in Table 2. We keep  $r$  fixed because its effect is the opposite of that of  $\alpha^M$ . The effect of  $\lambda$  is unclear; it seems that as  $\lambda$  increases  $\nu^M$  is unaffected, while  $\nu^V$  seems to go down slightly.

Brownian motion process we will select the value of  $T$  at which the condition expressed in Proposition 3 is first satisfied. This will give us a value for the pension plan under each regime. We then estimate the volatility of each value. Straightforward algebra shows that the previous expression can be rewritten as

$$P_t = \lambda \beta e^{rt} \tilde{E} \left[ e^{-(r+\beta)T} \int_0^T e^{\beta s} x_s ds \mid \mathcal{F}_t \right].$$

Differentiating,

$$dP_t = P_t r dt + P_t \nu_t d\tilde{W}_t,$$

where  $\nu$  is the volatility we are trying to estimate. From the definition of  $\tilde{W}$ , we can solve for the volatility in the previous expression and we get

$$\nu_t = \frac{dP_t - P_t(r + \theta)dt}{P_t \tilde{W}_t}.$$

The feedback effect of retirement decisions leads to the following qualitative changes in the valuation of DB plans and in the asset allocation policies. These are summarized in Table 1.

Table 1 provides a basis to analyze the effects of the design of the pension plan on both the value of the plan (depending on whether it

is mandatory or voluntary) and the optimal portfolio that corresponds to it.

Overall (and this seems to be the key observation) the plan under voluntary retirement requires, in general, a smaller investment in the risky security. When the worker optimally chooses her retirement date, the value of the pension plan (for most parameter values) is less responsive to shocks to the economy and, therefore, the hedging component of the funds optimal investment strategy is smaller. Furthermore, the value of the plan under the mandatory retirement regime is positively correlated with  $\sigma^M$ , the volatility of the risky security, and negatively correlated with  $\alpha^M$ , the drift of the risky security, while the value of the plan under the voluntary regime does not seem to show a clear pattern. The same type of relationship as with the value of the pension plan seems to hold between the parameters that describe the dynamics of the risky security and the volatilities (and therefore optimal investment strategies) of the values of the different plans. As a result, the higher  $\sigma^M$ , the higher the proportion of the fund of the pension plan under mandatory regime to be invested in the risky security, and the higher  $\alpha^M$ , the smaller that proportion. However, this relationship does not seem to apply to the plan under the voluntary retirement regime. Finally, increases in  $\beta$  result in an increase in the optimal proportion of the plan under mandatory retirement to be invested in the risky security, however, the correlation of the value of the plan under voluntary retirement seems to drop in absolute value.

A drawback of the previous analysis is the fact that for the case when the worker chooses retirement optimally, the value  $T$  that drives the dynamics of  $\alpha^x$ , the drift of the wage process, is exogenous and unrelated to the actual age of retirement of the worker. One way to improve upon this assumption in our numerical setting is to study the distribution of the retirement age over the different runs of the simulation and try to find a  $\tilde{T}$  for the dynamics of  $\alpha^x$  that will correspond to the expected value of the distribution of the age of retirement.<sup>18</sup> It turns out it is possible to find a “fixed point” to the previous problem. We present the results in Table 2.

We first observe that at  $a = 40$ , the salary  $x$  and vested pension  $z$  are smaller than for their corresponding values in Table 1, while the value of the pension plan  $P^V$  is higher, since we expect the worker to retire sooner. Both facts aim at the satisfaction of the “fairness rule” in an expected value sense. The difference with the values in

<sup>18</sup> We are indebted to Chester Spatt (the editor) for suggesting this approach. In a true rational expectations, fixed-point setting, one would want to specify a distribution of  $T$  and solve the problem and verify that at equilibrium the actual distribution of  $T$  is precisely the one assumed to solve the problem. We have not solved this more general problem in this article.

**Table 2**  
**Value and volatility of pension plan with voluntary retirement and average retirement age versus mandatory fixed retirement age ( $\lambda = 5, r = 0.02$ )**

		$T = 60$				$\bar{T}$					
$\beta$		$z$		$x$		$z$		$x$		$T$	
0.03		13,354		30,855		13,334		30,399		59.25763	
0.04		16,070		29,688		15,273		26,428		50.30551	
		$\sigma^M = 0.15$				$\sigma^M = 0.18$					
$\beta$	$\alpha^M$	$P^M$	$P^V$		$v^M$	$v^V$	$P^M$	$P^V$		$v^M$	$v^V$
0.03	0.05	65,134	65,824		0.02174	0.01204	67,002	66,644		0.02261	0.01496
	0.06	60,756	64,027		0.02044	0.00419	63,773	65,250		0.02123	0.00976
0.04	0.05	72,792	79,257		0.02359	-0.00432	74,969	79,030		0.02446	-0.00262
	0.06	67,744	79,697		0.02229	-0.00666	71,215	79,407		0.02308	-0.00525

In Table 2 we have recomputed the value and volatility (and, therefore, optimal investment in the risky security) of the pension plan with voluntary retirement for a new  $\bar{T}$  that satisfies the condition described below. The notation is as before, but we add a column with  $\bar{T}$ , the parameter that drives the dynamics of  $\alpha^*$  and corresponds to the expected age of retirement—where the expectation comes from the simulation—for such  $\bar{T}$ . In the upper panel we compare the values of  $z$  and  $x$  under mandatory retirement with  $T = 60$  and compare them to the resulting values when the driving parameter of the wage dynamics is the expected—over the paths of the numerical simulation—retirement date. The expected retirement date  $\bar{T}$  is provided in the upper column of the upper panel. Values of  $z$ ,  $x$ , and  $\bar{T}$  do not depend on  $\alpha^M$  and  $\sigma^M$ .  $\bar{T}$  is computed in the following way: first the wage process is calculated for a retirement age of 60 years, then the voluntary age of retirement that corresponds to the wage process is computed; a new wage process is calculated using this age of retirement in order to compute the new voluntary age of retirement; this process is repeated until a fixed point is reached according to a five-digit after the decimal point convergence criteria. In the lower panel the  $f$  of the pension plan and its volatility  $v$  are computed under both regimes:  $P^M$  and  $v^M$  are as before, while  $P^V$  and  $v^V$  refer to the values under voluntary retirement when the dynamics of wages are given by  $\bar{T}$ .

Table 1 is larger, the larger the difference between the  $\bar{T}$  of Table 2 and 60. With respect to the optimal investment strategy, in general the volatility  $v^V$  is lower than the corresponding  $v^V$  for the case of fixed  $T = 60$ . Therefore, this approach seems to reinforce that conclusion of Table 1.

5. Conclusion

We have provided a unified framework for valuing pension obligations and finding asset allocation policies which depend on the life-time marginal productivity schedule and the type of DB plan that the sponsoring firms have. This part of our work took the retirement decision of the worker as given. We then recognize that the worker's retirement decision is influenced by the nature of the DB plan. We explicitly characterize the resulting retirement policy, taking advantage of the valuation of pension obligations. The analysis was conducted in an arbitrage-free, partial equilibrium setting. We model the feedback

effects of retirement policies on valuation and asset allocation using numerical techniques.

There are several open questions that are not addressed here. We do not model the disutility of work and the labor supply decision. The form of the pension contracts and their optimality is another important question that deserves scrutiny. For example, why are DB plans extensively used? The retirement incentives provided by DB plans might be the key, as has been pointed out in the literature. The trade-offs between the retirement incentives and the risk-sharing properties (lack of diversification) is a key issue in the design of pension contracts. DB plans, while forcing workers to hold an ill-diversified pension portfolio, provide a unique claim on the time path of the future wage stream. The resulting trade-offs is an issue that is not addressed here. The endogenous determination of the pension contract might shed some light on the efficiency properties of DB plans.

## Appendix

*Proof of Proposition 1.* From Equations (2) and (3) it is obvious that the value of the pension plan  $P$  at any time  $t$  is a function of  $(x_t, z_t)$ . Two alternative and equivalent approaches based on no arbitrage arguments show that Proposition 1 obtains. First, we apply Ito's lemma and derive the dynamics of  $P$  according to Equations (4) and (7). We now use the no-arbitrage argument of Black and Scholes (1973). By taking a position in  $\frac{P_x x \sigma}{V_M \sigma_M}$  of the risky asset and placing  $(P - \frac{P_x x \sigma}{V_M \sigma_M})$  in the riskless asset and rebalancing the position at each instant, we may replicate the pension plan. This leads to the familiar valuation equation which is shown next (we drop the time indicator; subscripts represent partial derivatives),

$$P_t + P_x x(\alpha^x - \theta \sigma) + P_z \beta(x - z) + \frac{1}{2} P_{xx} x^2 \sigma^2 - Pr = 0, \quad (21)$$

with boundary condition

$$P(x, z, T) = \lambda z_T. \quad (22)$$

It can be easily checked that the value stated in the proposition satisfies the partial differential equation above.

Alternatively, Harrison and Kreps (1979) show that the solution to Equations (21) and (22) is equivalent to

$$\tilde{E}[\lambda z_T \mid \mathcal{F}_t], \quad (23)$$

where the expectation is taken with respect to the “equivalent martingale measure” [Cox and Huang (1989) and Karatzas et al. (1987)].

It is straightforward to check that Equation (12) is a solution to the previous expression. ■

*Proof of Proposition 2.* The second component of the optimal strategy in Equation (17),  $\frac{\frac{\partial P_t}{\partial x_t} x_t \sigma}{\sigma^M}$  is the investment in the risky asset that replicates the dynamics of the liabilities.

The first component is the optimal investment of an agent that maximizes  $E[\frac{(\epsilon_T)^{\frac{1}{1-\gamma}}}{\gamma}]$ . It is standard [Merton (1971)] that the optimal investment in the risky asset (that we denote by  $\rho$ ) corresponding to such an objective function is

$$\rho_t = \frac{\theta}{(1-\gamma)\sigma^M} \epsilon_t, \quad (24)$$

with  $\frac{e^{r_t} \epsilon}{(\eta_t)^{\frac{1}{1-\gamma}}} \exp[-\frac{1}{2} \frac{\gamma}{(\gamma-1)^2} \theta^2 t]$  as the solution to the stochastic differential equation

$$d\epsilon_t = \rho_t \frac{dV_t^M}{V_t^M} + (\epsilon_t - \rho_t) r dt, \quad (25)$$

with initial value  $\epsilon_0$  and  $\rho$  as expressed in Equation (24). ■

*Proof of Proposition 3.* The worker will choose her optimal retirement day so as to maximize  $L_t(T)$ ,

$$L_t(T) = P_t + \tilde{E} \left[ \int_t^T e^{-r(s-t)} x_s ds \mid \mathcal{F}_t \right],$$

where  $P_t$  is as defined in Equation (12). Our problem is now to find the optimal stopping time  $T^*$ . Suppose that we knew such value. Then we would have

$$\begin{aligned} L_t(T^*) &= \lambda z_t e^{-(r+\beta)(T^*-t)} \\ &\quad + x_t \left( \int_t^{T^*} e^{-r(s-t) + \int_t^s \alpha_v^x dv - \sigma \theta(s-t)} ds \right. \\ &\quad \left. + e^{-r(T^*-t)} \lambda \beta \int_t^{T^*} e^{-\beta(T^*-s) + \int_t^s \alpha_v^x dv - \sigma \theta(s-t)} ds \right). \end{aligned}$$

It is obvious that when the optimal retirement day arises,  $\frac{\partial L_t(T)}{\partial T} (T = t) = 0$  and  $\frac{\partial^2 L_t(T)}{\partial T^2} (T = t) \leq 0$ . From the first derivative we get

$$z_{T^*} = x_{T^*} \frac{1 + \lambda \beta}{(r + \beta) \lambda}.$$

To get a negative second derivative, given the previous condition, we need

$$\alpha_{T^*}^x \leq r + \sigma\theta - \frac{r + \beta}{1 + \lambda\beta}.$$

These two conditions guarantee that we have reached a “local” optimum. It is easy to check that when they are satisfied  $\frac{\partial L(T)}{\partial T} (T > t) < 0$  and the previous point is also a “global” optimum.

Therefore, we can conceive a situation where the accrued pension benefits are a fraction of the current wages smaller than  $\frac{1+\lambda\beta}{(r+\beta)\lambda}$ . After a given  $t$  predetermined from Equations (8) and (14),  $\alpha^x$  becomes smaller than  $r + \sigma\theta - \frac{r+\beta}{1+\lambda\beta}$  and will always stay below that level: the marginal productivity is approaching its peak. The first time after that date that  $z$  hits the mentioned proportion of the wages it will be optimal for the worker to retire. Indeed, for most reasonable parameters the second order condition is trivially satisfied from the outset. However, if this is not the case, it is conceivable that  $z$  reaches the critical value shown above before  $\alpha^x$  reaches the critical level. In that case, it will be optimal to continue. ■

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