The Valuation of Options on Futures Contracts

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ABSTRACT

Rational restrictions are derived for the values of American options on futures contracts. For these options, the optimal policy, in general, involves premature exercise. A model is developed for valuing options on futures contracts in a constant interest rate setting. Despite the fact that premature exercise may be optimal, the value of this American feature appears to be small and a European formula due to Black serves as a useful approximation. Finally, a model is developed to value these options in a world with stochastic interest rates. It is shown that the pricing errors caused by ignoring the location of the interest rate (relative to its long-run mean) range from −5% to 7%, when the current rate is ±200 basis points from its long-run value. The role of interest rate expectations is, therefore, crucial to the valuation. Optimal exercise policies are found from numerical methods for both models.

IN RECENT YEARS, THERE has been a steady growth in the number of financial assets, which one might properly call derivative assets, that are available for trading on the organized exchanges. Among these, the most recent and notable are options contracts written on available futures contracts: options are now traded on the futures contracts on stock market indexes; on the futures contracts on Treasury instruments; on the futures contracts on foreign exchange rates; and on the futures contracts on some metals. In design, these options contracts do not differ substantially from the well-known options contracts on common stock, except that the underlying asset is a futures contract. The received theory of the valuation of these options suggests that such contracts provide a direct vehicle for investors to alter the terminal payoffs on their portfolios to a desired distribution. In the absence of these contracts, investors would be forced to pursue a dynamic portfolio policy at the expense of considerable transactions in order to achieve a similar distribution. Indeed, the availability of put options on the futures contracts on stock market indexes now enables well-diversified investors to purchase portfolio insurance.

In his seminal work, Black [1] provided a complete description of forward and futures contracts, as well as the description and valuation of European options on forward contracts and on spot commodities. Breeden [3] has examined the use of commodity options in a theoretical framework, and Courtadon [6] has studied the valuation of options on Treasury bond futures. French [13], Cox,

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Ingersoll, and Ross [8], and Richard and Sundaresan [17] studied relationships between forward and futures contracts and examined procedures for valuing these contracts.

The valuation of options on futures contracts differs from the valuation of options on common stocks in two important ways. First, the futures price must be found as a function of the underlying state variables. Except in the case of a flat term structure, when the state is fully described by the underlying asset's value, this is a difficult task. Second, as we show below, even when the underlying asset does not pay any dividends, it may be optimal to exercise the option on the futures contract prematurely. Therefore, the valuation problem must characterize endogenously the critical *region* of the underlying asset values and the interest rates where exercise is optimal. In this paper, we provide a framework to value the options on futures contracts with and without interest rate uncertainty. The futures prices and the critical regions are found endogenously in both settings, as part of the option valuation problem.

In Section I, we provide a description of options on futures contracts and discuss rational restrictions on their prices. In Section II, we develop a model for pricing American options on futures contracts in a world with constant interest rates. To illustrate, we use a stock index as the underlying asset. Under the assumptions that the dividend yield is constant and that the stock index follows a lognormal diffusion, we characterize the call option pricing function and the optimal exercise policies. In Section III, we extend the analysis to a setting in which both the stock index value and the interest rates are random. Option values and critical regions are derived for call options on stock indexes and for call options on stock index futures contracts. We compare the properties of options on the index with those of options on futures contracts on the index.

I. Options on Futures and Rational Option Prices

In this section, a brief description of options on futures is provided, and rational restrictions on their prices are derived.

A *futures* contract, whose price at date t is quoted as $H(t)$, commits the buyer (and seller) to consummate the purchase (sale) of the commodity at the maturity of the contract, date T_2. The buyer will pay (or receive from) the seller the full amount of the change in the futures price from the previous day, whenever this price change is negative (positive). The payment to be made by the buyer in exchange for the commodity at the maturity of the futures contract is simply the commodity price at that time. In the so-called "cash settlement" contracts, such as the futures contracts on stock market indexes, there is no exchange of the commodity and the cash price on the maturity date. A complete characterization of forward and futures contracts is given in Black [1], and a thorough treatment of the relationships between these contracts is given in Cox, Ingersoll, and Ross [8].

An American call (respectively, put) option on a futures contract gives the holder the right to purchase (sell) a futures contract on or before a prespecified date at a prespecified *futures* price, called the exercise price, K. Note that the option may expire at date T_1 prior to the maturity of the underlying futures

contract ($T_1 \leqq T_2$). The holder of an option on a futures contract, upon exercise at date s, receives $\$H(s) - K$ in cash and opens immediately a long position in a futures contract at a futures price, $H(s)$. That is, by exercising the call option at the *futures* exercise price of K, the buyer of the option has a futures contract that is immediately "marked-to-market." The call-option writer will put up the amount $\$H(s) - K$ in cash and open a short position in a futures contract at the futures price, $H(s)$. Since the newly opened futures contract at the futures price, $H(s)$, has zero value (see, e.g., Black [1]), the wealth in an investor's portfolio is altered only by the cash inflow or outflow upon exercise, although the future dynamic properties of the value of the portfolio could be altered significantly by the newly opened futures position.

To fix matters, let

$C(H(t), t)$ = value at date t of an American call option on a futures contract with the futures price, $H(t)$, where the option expires in $\tau = T_1 - t$ periods, the underlying futures contract matures at date T_2, and the option exercise price is K;

$P(H(t), t)$ = value of an American put option on a futures contract; and

$b(t, T)$ = the price at date t of a unit discount bond paying $\$1$ at date T.

The terminal conditions of options on futures contracts are

$$P(H(T_1), T_1) = \text{Max}\{0, K - H(T_1)\},$$

$$C(H(T_1), T_1) = \text{Max}\{0, H(T_1) - K\}. \tag{1}$$

The values of the options upon rational exercise at date s will be as in the equations above, with s replacing T_1.

It is well-known that with nonstochastic interest rates and for contracts of equal maturity, the futures price will equal the forward price. If interest rates are stochastic, these prices will, in general, differ (see, e.g., Cox, Ingersoll, and Ross [8]). Cornell and Reinganum [5] have shown empirically that the difference is rather small, and French [14] argues that available models are useful in discriminating between these prices. It turns out, however, that the values of options on forwards will differ from the values of options on futures contracts, depending on the definition of the payoff (upon exercise) to the holder of the option on the forward contract.[1]

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[1] Options on forward contracts are not traded assets. However, options on forward contracts are useful theoretical constructs in the pricing of currency option bonds (see, e.g., Feiger and Jacquillat [12] and Garman and Kohlhagen [15]). There are, at least, two ways in which options on forward contracts may be defined.

Under the first definition, the owner of a call option on a forward contract receives from the option writer, upon exercise at date, s, the difference between the forward price, $G(s)$, and the strike price, K, and opens a long position in a newly created forward contract. The newly created forward contract has zero value, so the implication is that the owner receives a cash inflow equal to the difference between the current forward price and the exercise price, K. This definition is consistent with the view that the option buyer speculates on the *forward* price, being given an exercise price, K; and, in a world with nonstochastic rates, this definition leads to option values that are identical to the values of options on futures contracts. Under the second definition, the owner of a call option on a forward contract receives from the option writer, upon exercise at date s, a forward contract with the forward

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A. Rational Option Prices

It is possible to develop (as has been done in Merton [16]) rational restrictions on option prices. We assume: (a) that investors are price takers in frictionless markets; (b) that they prefer more wealth to less; and (c) that there is a market for pure discount bonds of every maturity. No assumptions (strictly speaking, rather weak assumptions) on the stochastic process of options or their underlying assets (futures and forward contracts) are employed. Most of the restrictions that can be derived in this framework are similar to those for options on common stocks, and they are stated below without proof.[2]

Upper and lower bounds for these options are given in the relationships below:

$$K \geqq P(H, t) \geqq \text{Max}\{0, K - H\}$$
$$H \geqq C(H, t) \geqq \text{Max}\{0, H - K\}. \tag{2}$$

In comparison to the established bounds for American options on stocks, the inequalities in (2) are identical: formally, the upper bound for American calls on futures contracts, when violated, requires the use of the "rollover" strategy outlined in Proposition 2 of Cox, Ingersoll, and Ross [7].[3]

It is easy to show that put (respectively, call) option values are increasing (decreasing) and convex functions of the exercise prices, and that the difference in the values of two otherwise similar options is bounded above by the difference in their exercise price. It is also easy to verify that American option values are increasing functions of the maturity $(T_1 - t)$ of the options, keeping the maturity date (T_2) of the underlying futures contract fixed.[4]

We now turn our attention to the relationship between call-option and put-option values. Unlike options on common stocks which obey the put-call parity

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price, K. No money changes hands at date s. The value of the forward contract received at date s is the *present value of* $\$G(s) - K$, payable at date T_2, where $G(s)$ is the current forward price. This definition is consistent with the view that the option buyer speculates on the *value* of the forward contract whose price is $G(t)$, being given a "forward" exercise price, K; and it is consistent with the fact that forward contracts are not marked-to-market.

It can be shown, under the second definition, that options on forward contracts will not be exercised, even though the options on futures will be. This conclusion holds even if $G(s) = H(s)$ for all s.

[2] Merton ([16], Theorem 9 and Appendix) has shown that if the distribution of the rate of return per dollar on the underlying stock is independent of stock price level, then the option value must be homogeneous of degree one in the stock price and the exercise price. This is a useful statement in that it permits the use of option valuations normalized to an exercise price (of unity, say) to value otherwise similar options at different exercise prices. A similar statement for options on futures contracts can only be made with much stronger (and hence, less plausible) restrictions. We would require here that the distribution of future interest rates, as well as the correlation between future interest rates and changes in futures prices, be independent of futures price levels. For options on the futures contracts on fixed income securities, for example, this would be an undesirable assumption.

[3] This strategy is described in this section in the proof to the Proposition.

[4] The options on the futures contract on Standard and Poor's 500 as well as the futures contract expire on the same day. Note that if $T_1 = T_2$, then European options on forwards and European options on futures contracts are both equivalent to European options on the spot commodity. This follows from the fact that at maturity, the forward price, the futures price, and the spot price are all equal.

relationship, one can only demonstrate, without additional restrictions on the stochastic processes for the futures price and interest rates, an upper bound for a call in terms of the put and other variables. This is shown in the proposition below:

PROPOSITION

$$C(H, t) \leq P(H, t) + H(t) - Kb(t, T_1)$$

Proof: Compare the payoffs to the following two portfolios:
  (A) a long position in a call option;
  (B) a long position in an otherwise similar put option, and the dynamic strategy (hereafter "*H*-strategy") outlined in Cox, Ingersoll, and Ross [7], Proposition 2: following their notation, invest $\$H(t)$ in one-period bonds and roll them over until T_1, earning random one-period interest rates, $R_t - 1$, $R_{t+1} - 1$, $\cdots$, $R_{T_1-1} - 1$. At each $j, j = t, t + 1, \cdots, T_1 - 1$, take a long position in $\Pi_{l=t}^{j} R_l$ futures contracts, liquidating these contracts and reinvesting the proceeds or borrowing the deficit at the one-period rates. The value of this strategy at date s is

$$H(s)\Pi_{l=t}^{s-1} R_l$$

which is, with nonnegative interest rates, no less than $H(s)$. Futhermore, borrow $\$Kb(t, T_1)$ to be repaid at T_1. The following table summarizes the cash flows to these portfolios.

Portfolio	Cost at t	Value at T_1	
		$H(T_1) > K$	$H(T_1) \leq K$
(A) call	$C(H, t)$	$H(T_1) - K$	0
(B) put	$P(H, t)$	0	$K - H(T_1)$
"*H*-strategy"	$H(t)$	$H(T_1)\Pi_{l=t}^{T_1-1}R_l$	$H(T_1)\Pi_{l=t}^{T_1-1}R_l$
Borrowing	$-Kb(t, T_1)$	$-K$	$-K$
Totals (B)	$P + H - Kb(t, T_1)$	$H(T_1)\Pi_{l=t}^{T_1-1}R_l - K$	$H(T_1)\{\Pi_{l=t}^{T_1-1}R_l - 1\}$

The payoffs to (B) dominate (A), so that $C \leq H + P - Kb(t, T_1)$. It may be verified that if the call is exercised at $s < T_1$, the payoffs to (B) would still dominate (A) at s. Q.E.D.

Analogous to a similar argument in Merton [16], one would like to demonstrate using these assumptions that $C \geq P + H - K$, which is a lower bound for the value of the call option. But this inequality leads to absurd conclusions due to the possibility of premature exercise. For example, if the call is optimally exercised, this inequality leads to $P \leq 0$ which is absurd. It will be shown below that premature exercise may indeed be optimal for call options, so that this lower bound for calls will not hold.

The possibility of premature exercise is best understood if one recognizes that options on a futures contract are isomorphic to options on a portfolio whose stochastic properties are identical to that of the futures price but which pays a continuous dividend at the (perhaps stochastic) riskless rate of interest. Consider a portfolio, Q, which at date t contains one futures contract at the futures price,

$H(t)$, and $\$H(t)$ invested in a one-period loan. At $t + 1$, the gain or loss, $H(t + 1) - H(t)$, augments the now matured loan, and the one-day interest from the loan at date t equal to $\{R_t - 1\}H(t)$ is paid out as a dividend. Thus, the ex-dividend value of Q at any date s is simply $\$H(s)$, the amount that is invested in the one-period loan; and the stochastic properties of the ex-dividend value of Q are identical to the stochastic properties of the futures price. Define an American call or put option written on the portfolio, Q, with $\$K$ payable in cash in exchange for Q upon exercise. This call or put option on Q will have the *same payoff* as the traded call or put options on the futures contract, both upon premature exercise and at expiration. However, the call option on Q may be exercised prematurely, using the arguments from Cox and Rubinstein [10], when the present value of the future dividends from Q is less than the interest that can be earned on the exercise price. Therefore, the call option on the futures contract may be exercised prematurely. This result is fairly general. However, in order to value these options, one needs to characterize the optimal exercise policies in detail. In Sections II and III, we develop models for the valuation of options on futures contracts with constant interest rates and with stochastic interest rates. It should be clear that the use of a model with constant interest rates will preclude application to options on the futures contracts on fixed income securities. The use of a stochastic interest model permits such application, but it would require us to develop the valuation for the underlying fixed income security. We have chosen to focus the application on options on stock index futures contracts, where the models fit somewhat more directly with the contingent claims view of these options.

II. The Valuation of Options on Futures Contracts

In this section, we provide a framework to value American options on a stock index and on its associated futures contract. The valuation of options on the underlying index is discussed in order to compare them with options on futures contracts.

The approach to the valuation of options on futures contracts begins by specifying a process for the evolution of the underlying asset's value and a process for the evolution of interest rates. By doing so, we avoid bringing preference assumptions *explicitly* into the valuation framework, and we are able to employ contingent claims valuation techniques using observable and traded assets. This approach can be made consistent within a general equilibrium context by suitably specifying preferences and the technology. For example, by assuming that preferences are represented by isoelastic utility functions that are additive over time, that the opportunity set is stationary, and that investment technologies are linear and exhibit constant stochastic returns to scale, we can derive (the well-known) implications that spot prices will follow a lognormal diffusion and that the interest rate will be constant. These considerations underlie the assumptions below.

(A1) Investors prefer more wealth to less and act as price takers in frictionless

markets. There are no taxes, and all margin requirements can be met by posting interest-bearing securities.

(A2) The dynamics of the spot price, denoted $S(t)$, are given by the stochastic differential equation

$$dS(t) = [\alpha(S,\ t) - \delta S]\ dt + \sigma_1 S\ dz_1, \tag{3}$$

where σ_1^2 represents the variance rate of proportional price changes, $\alpha(S,\ t)$ is the *cum* dividend expected change in the price, $\{z_1(t),\ t > 0\}$ represents a standard Wiener process, and δ represents the (flow rate) continuous dividend yield. It is clear that one can accommodate the lognormal diffusion as a special case.

(A3) The rate of interest on default-free securities is a constant, r.

In this framework, it is easy to show that the futures price, $H(S,\ t;\ T_2)$, at time t is given by

$$H(S,\ t;\ T_2) = S(t)e^{(r-\delta)(T_2-t)}. \tag{4}$$

Relation (4) says that the futures price will be at a premium relative to the spot price throughout the contract's life for $r > \delta$ and will be at a discount for $r < \delta$. Furthermore, as t approaches T_2, the futures price approaches the spot value. This second property has an interesting implication for the dynamic behavior of the futures price: by applying Ito's lemma to (4), we find that the futures price evolves as

$$dH = \left[\frac{\alpha(S,\ t)}{S} - r\right]H\ dt + \sigma_1 H\ dz_1. \tag{5}$$

This says that the expected rate of change in the futures price is equal to the expected rate of change in the spot index price minus the risk-free rate. Thus, the dynamics of the futures price involves an "implicit dividend" flow at the riskless rate. This suggests that, in general, it may be optimal to exercise the option on a futures contract prematurely. This conclusion holds whether or not the underlying asset pays dividends, because the evolution of the futures price in (5) is independent of the dividend yield.

We focus on the valuation of a call option on a futures contract; the treatment for puts proceeds similarly. Since the underlying state variable is the spot price, S, we represent the value of a call option on the futures contract as $C(S,\ \tau;\ T_2)$, with[5] option maturity $\tau = T_1 - t$, and the futures contract maturing at date T_2. By employing standard arbitrage arguments, it is possible to show that $C(S,\ \tau;\ T_2)$ satisfies

$$\tfrac{1}{2}\sigma_1^2 S^2 C_{SS} + (r - \delta)SC_S - rC - C_\tau = 0, \tag{6}$$

for values of $S(t) < \bar{S}(\tau)$, where $\bar{S}(\tau)$ is the critical boundary at which early exercise is optimal and is endogenous to the valuation problem. The terminal

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[5] Note that the state space consists of a single variable, S; the functional form, $C(S,\ \tau;\ T_2)$, used here differs from and should not be confused with the form, $C(H,\ t)$, employed in Section I.

condition is

$$C(S, 0; T_2) = \text{Max}\{0, Se^{(r-\delta)(T_2-T_1)} - K\}. \tag{7}$$

The boundary conditions are fairly standard.[6] Black [1] provides the European option's value in this setting.[7]

Note that an equivalent problem is one which solves for the option value as a function of the prevailing futures price option and maturity. This is consistent with the contingent claims view of the option on the futures contract. A continuously rebalanced self-financing portfolio of the underlying futures contracts and the riskless asset can be constructed to duplicate the payoff to the option on the futures contract. Since we expect in the short run that the call (respectively, put) option on the futures contract will increase (decrease) in value as the futures price rises, we expect that the duplicating portfolio for a call (put) will contain some futures contracts held long (short). However, the futures contracts cost us nothing to initiate, whereas the put and call options require a positive investment. As a result, the duplicating portfolio for the call *and* the put option will require a positive investment in the riskless asset, and the amount loaned will always equal the value of the respective option. The dynamic rebalancing will require that the gains and losses on the futures positions be additions or withdrawals from the loan account, since adjustments to the futures position do not require any net new investment.[8] Proceeding along these lines, and noting that the dynamic evolution of the futures price in (5) is independent of the dividend yield, δ (although the *level* of the futures price is affected), one could provide a valuation equation for the option price in a completely analogous manner, and in this case, the *critical futures price* will be independent of the dividend yield, δ.

The value of an American call option (with similar contractual terms) *on the spot* will also satisfy the valuation equation in (6). However, the value at the terminal date and upon premature exercise will reflect the spot value, S, rather than the "compounded" futures price, $Se^{(r-\delta)(T_2-t)}$. As a result, with $r > \delta$, the

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[6] For the assumed stochastic process on S, zero is an absorbing boundary. Thus, at $S = 0$, the option is worthless. Since the option is American, as S approaches $\bar{S}(\tau)$, the option is exercised optimally. Thus, the two boundary conditions are:

$$C(S = 0, \tau; T_2) = 0,$$

and

$$\text{Lim}_{S(t)\uparrow\bar{S}(\tau)}C_S(S, \tau; T_2) = e^{(r-\delta)(T_2-t)}.$$

[7] Black [1] provides the European solution for the case $T_1 = T_2$. His solution, which is stated in terms of the futures price, holds even when $T_1 < T_2$. To express it in terms of the spot price, we need to substitute $H = S \exp\{(r - \delta)(T_2 - t)\}$ from Equation (4).

[8] Under current institutional rules, a long or short position in a futures contract requires the investor to post a "performance bond" as initial margin. If the gains and losses on the futures position are added to or taken from this interest-bearing cash account, then one can view the overall futures position as similar (in local behavior) to a position in options on the futures contract. Indeed, if the initial margin is 10% of the futures price, then the performance bond posted together with one long futures contract may be less than the outlay for a deep-in-the-money call option, which has an equivalent *futures* position underlying it. Note also that in duplicating the local behavior of a put option, we would sell short some futures contracts and lend money.

option on the futures contract will be worth more than the option on the spot. For $r = \delta$, the options will be equal in value, and for $r < \delta$, the option on the spot will be worth more.

There are no known analytical solutions to the valuation Equation (6) for the American option on the stock index futures. We employ implicit finite difference methods to find the option values and the optimal exercise boundary.[9] The parameter values chosen are $\sigma_1 = 0.25$ and a dividend yield of 5%. To be consistent with the treatment of options on the S&P500 futures contract, we assume $T_1 = T_2$.

Table I provides values of American and European calls on futures contracts for maturities of 3, 6, and 9 months, together with the prices of the underlying futures contracts. It can be seen that unless the stock index value rises to 120% of the exercise price, premature exercise is not optimal. Therefore, the value added by the "American" feature is rather small, especially for options that are at-the-money. Black [1] provides a European formula which gives very close answers for at-the-money options in the constant interest rate case. For options that are deep in-the-money, the value added due to the "American" feature is obviously greater.

Figures 1 and 2 provide critical regions for call options on stock index futures contracts and on the stock index at three interest rate levels: $r = 3\%$; 5%; and 7%. For both options, the critical stock index value is an increasing function of the maturity. On the other hand, the interest rate level has a markedly different impact on the critical regions: for stock index options, the critical region is everywhere higher at $r = 5\%$ than its level at $r = 3\%$ as shown in Figure 2. In the case of options on stock index futures, the effect is precisely the opposite, and the critical region is everywhere *lower* at the higher interest rate levels. The intuition for this result is straightforward: in the case of options on the index, higher levels of the interest rate enable the option holder to earn higher interest on the exercise price. For the owner of a call option on a futures contract, increases in r are equivalent to a higher "implicit dividend" on the futures price.

It is easy to verify that the critical exercise boundary is a decreasing function of the dividend yield for the stock index option and an increasing function of the dividend yield for options on stock index futures contracts. Note that for put options on futures contracts, these results are reversed. Premature exercise of the put option on a futures contract occurs if the stock price *falls* to a critical boundary which is a decreasing function of the option's maturity, a decreasing function of the interest rate level, and a decreasing function of the dividend yield.

The comparative statics properties of these options with respect to σ_1^2 and r are similar to the familiar options on common stocks. Increases in either of these parameters, *ceteris paribus*, will lead to increases in call option values. Figure 3 shows the difference between stock index futures option value and the stock index option value for $r = 3\%$, $r = 5\%$, and $r = 7\%$. At $r = 5\%$, these options have the same value. At an interest rate of $r = 3\%$, the option on the index sells

[9] The discrete time approach taken by Cox and Rubinstein [10] can also be used to price American options and characterize optimal exercise policies. In this approach, one could treat the futures price or the spot price as the underlying state variable.

Table I

Values of Call Options on Futures Contracts ($\times 10^{-2}$)

Spot Price	Option Maturity (Days)		
	90	180	270
80.00	0.0054	0.1073	0.3683
	0.0054	0.1070	0.3659
	80.9924	81.9971	83.0143
85.00	0.0577	0.4220	0.9953
	0.0577	0.4202	0.9865
	86.0544	87.1219	88.2027
90.00	0.3542	1.2341	2.2075
	0.3535	1.2265	2.1808
	91.1165	92.2468	93.3911
95.00	1.3577	2.8403	4.1760
	1.3535	2.8419	4.1076
	96.1785	97.3716	98.5795
100.00	3.5692	5.4089	6.9619
	3.5510	5.3396	6.8089
	101.2405	102.4964	103.7679
105.00	7.0703	8.9023	10.5127
	7.0107	8.7408	10.2063
	106.3025	107.6212	108.9563
110.00	11.4961	13.1367	14.7079
	11.3411	12.8054	14.1475
	111.3646	112.7460	114.1447
115.00	16.4266	17.9011	19.4167
	16.0870	17.2874	18.4649
	116.4266	117.8709	119.3331
120.00	21.4886	22.9957	24.5215
	20.9772	21.9931	23.0169
	121.4886	122.9957	124.5215

Notes: The first row at each spot index value contains the values of an Amerian call option on a futures contract. The second row contains the values of a European call on the futures contract, with similar contractual terms. The third row contains the price of the underlying futures contract. The futures contract matures on the same day as the option on the futures contract expires. The following parameters have been used: interest rate = 0.10; dividend yield = 0.05; spot asset volatility = 0.25; and exercise price = 100.

at a premium and this premium increases with option maturity. At an interest rate of 7%, the option on the stock index sells at a discount, and this discount increases with option maturity. This result follows from the fact that, with constant interest rates, the yield-to-maturity of a pure discount bond (maturing
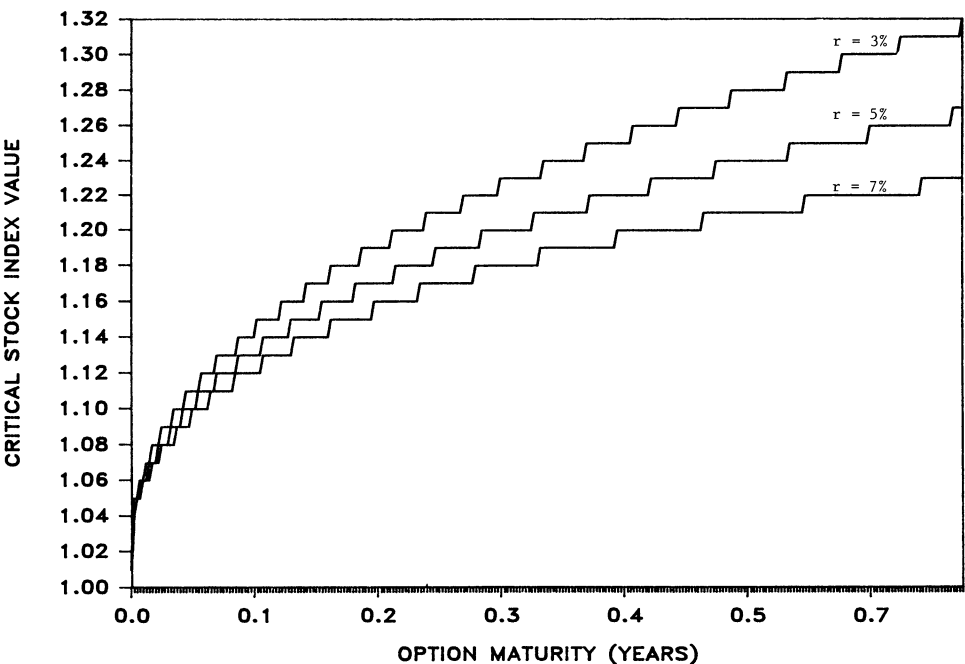
Figure 1. Critical Region for Call Options on Futures Contracts — Constant Interest Rate (r). Exercise Price = 1, Volatility of Underlying Index Asset (σ) = 0.25, Dividend Yield = 5%.
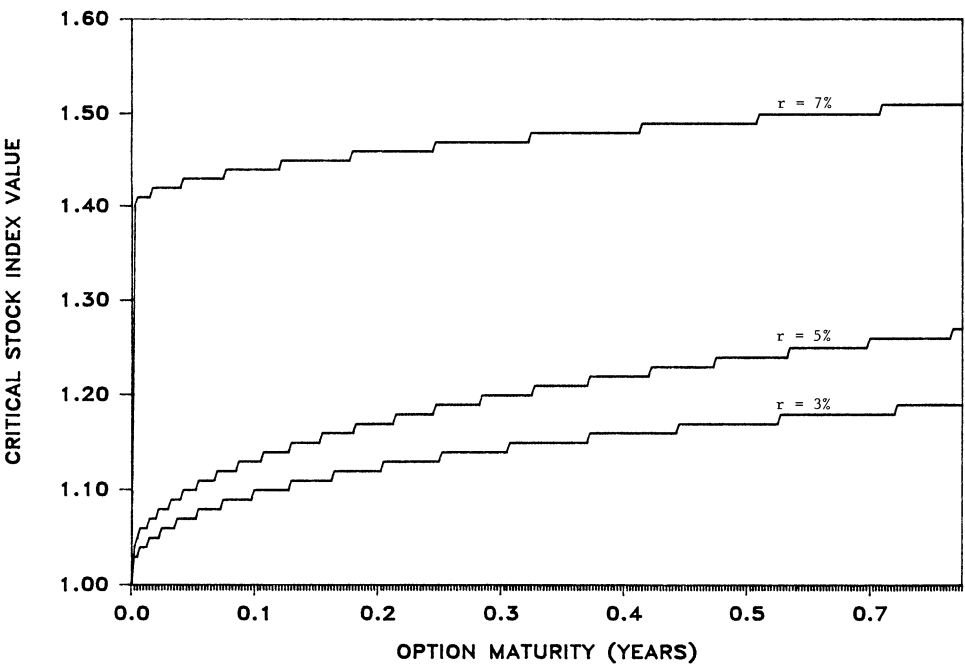

Figure 2. Critical Region for Call Options on Stock Index — Constant Interest Rate (r). Exercise Price = 1, Volatility of Underlying Index Asset (σ) = 0.25, Dividend Yield = 5%.
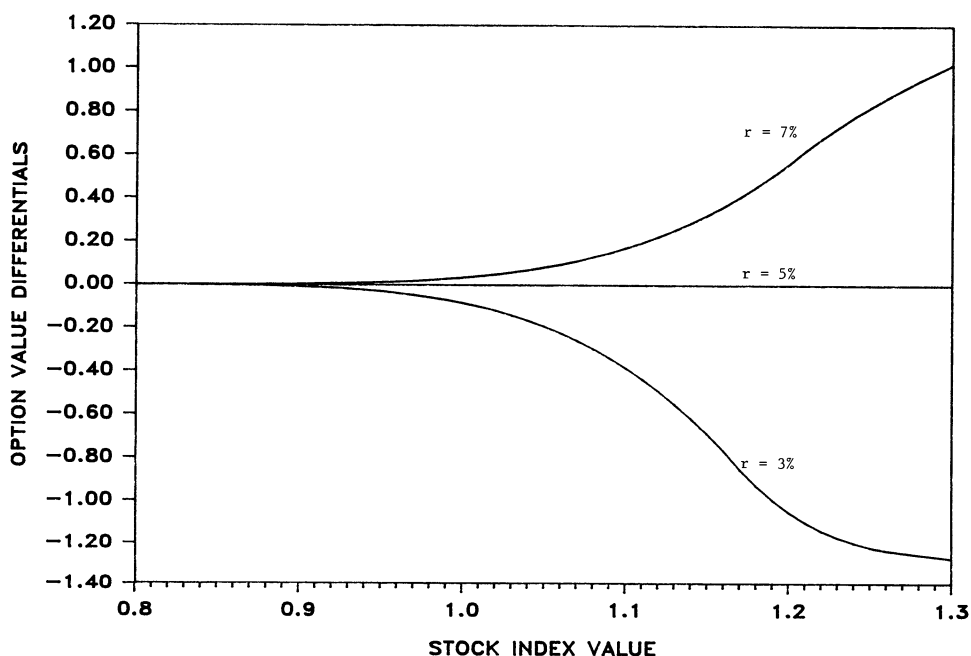
Figure 3. Option Value Differentials (Futures Minus Spot) − Constant Interest Rate (r). Exercise Price = 1, Volatility of Underlying Index Asset (σ) = 0.25, Dividend Yield = 5%.

at date T_2) is constant through time, and hence the futures contract will sell at a premium or a discount relative to the spot if this yield is greater than or less than the dividend yield. If the term structure is nonstochastic but rising or falling, the yield-to-maturity of a pure discount bond maturing at date T_2 will rise or fall over time, and as a result, the discount or premium that prevails in the spot and futures prices can reverse itself. Therefore, one cannot conclude that the option on the index will be dominated by the option on the futures contract for all future dates based upon a current comparison of the current yield of a bond maturing at T_2 and the dividend yield on the stock.[10]

III. Valuation of Options on Stock-Index Futures Contracts with Stochastic Interest Rates

In this section, we extend the model presented in Section II to a world with stochastic interest rates. Numerical methods are employed to determine the values of options on futures contracts and to characterize the optimal exercise policy. Courtadon [6] has presented a similar analysis on the valuation of options on Treasury bond futures contracts in a single state variable setting.

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[10] With discrete dividends, the option on the spot index may be exercised just prior to the ex-dividend date. If the futures price at that time is higher than the spot price, then the call option on the futures will be worth more: this condition will depend on the size of remaining dividends (until T_2) which affect the futures price.

In order to develop a tractable model that preserves the essential features, we retain assumptions (A1) and (A2) and replace (A3) by (A3').

(A3') All uncertainty in the term structure of interest rates (and hence in the valuation of default-free bonds) is captured by the movements of the instantaneously riskless rate, $r(t)$. Its dynamics are given by:

$$dr(t) = \kappa(\mu - r) \, dt + \sigma_2 \sqrt{r} \, dz_2. \tag{8}$$

According to (8), the instantaneously riskless rate is expected to drift towards the long-run mean level μ, with a speed of adjustment, κ, and the instantaneous variance of changes in r is proportional to its level. $\{z_2(t), \ t \geqq 0\}$ is a standard Wiener process. We assume that $\text{Cov}(dz_1, dz_2) = \rho \, dt$, where ρ is the correlation coefficient;

and we employ an additional assumption:

(A4) Default-free discount bonds are priced according to the Local Expectations Hypothesis (henceforth LEH; for a discussion, see Cox, Ingersoll, and Ross [7]). That is, we assume that the expected instantaneous holding period return on any default free bond is equal to the prevailing risk-free rate, r:

$$E_t[dB/B] = r(t) \, dt. \tag{9}$$

The process for the riskless rate in (8) has been employed by Cox, Ingersoll, and Ross [7], who discuss its properties.[11] The motivation for (A4) should be clear. The value of a contingent claim with a stochastic term structure will depend (in general) on investor preferences. To see this, note that in order to construct a locally riskless hedge, an interest-rate-based hedging instrument such as a default-free bond is necessary. This instrument's price dynamics will depend on liquidity premiums, which depend on preferences and are present in the fundamental valuation equation for the contingent claim. The LEH permits us to avoid explicitly modelling preferences, and leads to tractable solutions with potential for empirical testing. Further, it precludes arbitrage and it is consistent with equilibrium models in which preferences induce liquidity premiums proportional to the level of the interest rate, $r(t)$.

The underlying state variables in this model are the index price, $S(t)$, and the interest rate, $r(t)$, so we now represent the value of an American call option on the futures contract as $C(S, r, \tau; T_2)$, where $\tau = T_1 - t$ is the option's maturity; we represent by $H(S, r, \tau_2)$ the date t price of a futures contract which matures in $\tau_2 = T_2 - t$ periods, i.e., at date T_2. Assumptions (A1), (A2), (A3'), and (A4) permit the development of a continuous hedging argument as in Black and

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[11] Alternative processes can be employed for the short rate, and, together with assumption (A4), used to find futures prices and option values. The futures price, H, is a function of the spot value, (S), the interest rate, (r), and maturity, (τ_2), and must be solved for first. Given complicated processes for r, this is a difficult task. Considerable simplification is achieved by the assumption that the spot price process exhibits constant stochastic returns to scale, for in this case, the futures price function is linear in the spot price, so that a single state variable problem emerges and can be solved numerically. The option valuation then proceeds as shown in Section III.

Scholes [2]. Having written (or purchased) one call option on the futures contract, we can design a dynamic, self-financing strategy that involves investment in the underlying stock index and a default-free discount bond such that the evolution of the portfolio's value is locally riskless. This portfolio must earn the locally riskless rate, $r(t)$, at time t. These arguments lead to the following fundamental valuation equation for options on stock index futures:

$$\tfrac{1}{2}\sigma_2^2 r C_{rr} + \tfrac{1}{2}\sigma_1^2 S^2 C_{SS} + \rho\sigma_1\sigma_2 S\sqrt{r}\, C_{rS} + \kappa(\mu - r)C_r$$
$$+ (r - \delta)SC_S - rC = C_\tau \quad (10)$$

with the terminal condition

$$C(S, r, 0; T_2) = \text{Max}\{0, H(S(T_1), r(T_1), T_2 - T_1) - K\} \quad (11)$$

where $H(S(T_1), r(T_1), T_2 - T_1)$ is the value at date T_1 of a futures contract with maturity, $T_2 - T_1$. The American call option on the futures contract will satisfy (10) and (11) for values of the pair $\{S(t), r(t)\}$ that lie below the critical boundary at date t, $\{\bar{S}(t), \bar{r}(t)\}$ at which exercise is optimal. This boundary is endogenous to the valuation problem. If the index value and the rate of interest rise to this "high-contact" boundary, then the option will be exercised, and

$$C(\bar{S}(t), \bar{r}(t), \tau) = H(\bar{S}(t), \bar{r}(t), \tau_2) - K \quad (11a)$$

will denote its value.[12]

The solution to (10) and (11) requires as an input the price of the futures contract as a function of the state variables, r and S. This function is itself found as a solution to a partial differential equation in these variables, thus complicating the overall problem. For $\rho = 0$, the futures pricing function is given by

$$H(S(t), r(t), t; \tau_2) = S(t)a(\tau_2)\exp[b(\tau_2)r(t)], \quad (12)$$

where

$$a(\tau_2) = \left[\frac{2\gamma e^{(\gamma+\kappa)\tau_2/2}}{2\gamma + (\gamma + \kappa)\{e^{\gamma\tau_2} - 1\}}\right]^{2\kappa\mu/\sigma_2^2} [\exp\{-\delta\tau_2\}],$$

$$b(\tau_2) = \frac{2(e^{\gamma\tau_2} - 1)}{2\gamma + (\gamma + \kappa)\{e^{\gamma\tau_2} - 1\}},$$

and

$$\gamma \equiv \sqrt{\kappa^2 - 2\sigma_2^2} > 0 \quad \text{(by assumption)}$$

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[12] As S approaches $\bar{S}$ and r approaches $\bar{r}$, the option will be exercised. These two conditions serve as upper boundaries for S and r, respectively. At $S = 0$, the process for S is absorbed so that the option is worthless. The process specified for r admits 0 as an accessible boundary when $\sigma_2^2 > 2\kappa\mu$. At $r = 0$, the valuation Equation (10) becomes:

$$1/2\sigma_1^2 S^2 C_{SS} + \kappa\mu C_r - \delta SC_S = C_r.$$

The equation shown above serves as the lower boundary condition for r.

For nonzero[13] values of ρ, there is no known closed-form solution for $H(S, r, \tau_2)$. We have solved for the futures price numerically for $\rho = 0.2$ and $\rho = -0.2$, and not found significant differences from the known solution for $\rho = 0$. The computational problem for the option is greatly simplified by using the solution in (12), which is done below. Note that the futures price in (12) is an increasing function of S, and an increasing and convex function of r. The futures contract can be at a discount or a premium relative to the index price, depending on the parameter values.

One could equivalently have posed the valuation problem by specifying the state space in terms of $(H(t), r(t))$. In developing that valuation equation, one must ensure that relationship (12) between the state variables, $H(t)$ and $r(t)$, is always satisfied. With the state space described in terms of $(H(t), r(t))$, the fundamental valuation equation and the terminal condition will *not* explicitly depend on the dividend yield. The call option price will be affected by the dividend yield only through the level of futures prices given by (12).[14]

The partial differential Equation (10) is solved numerically subject to the terminal condition (11) and the boundary conditions. We have solved for values of options on index futures by using the method of alternating directions described in Brennan and Schwartz [4]. The parameter values chosen are $\kappa = 2$, $\sigma_2 = 0.09$, and $\mu = 10\%$. The stock index volatility, σ_1, was set at 0.25, the exercise price, K, at 1, and both the option and its associated futures contract were assumed to expire on the same date $(T_2 = T_1)$. In the numerical procedure, we used a maximum stock index value of 2, and the maximum interest rate of 0.50; the numbers of mesh points along these axes were chosen to be 400 and 100, respectively. The maximum option maturity $(T_1 - t)$ was 9 months.

Tables II to IV show values of American call options and the values of associated futures contracts for maturities of 3, 6, and 9 months respectively, for a range of values of S and r. The call option values increase as the index value, S, and interest rate, r, increase. The option values in Table I, the constant interest rate case with $r = 10\%$, are marginally higher when compared to option values in these tables under the column $r = 0.10$, which corresponds to the long-run mean value (μ) for the interest rate. When the current rate, r, is located away from its long-run mean value, the term structure is not flat, and the option values can differ substantially from the corresponding values for the constant interest rate model. For 6-month call options, the relative error in using the constant interest rate model in place of the stochastic rate model varies from 7% (for $S = 100$ and $r = 8\%$) to -5% (for $S = 100$ and $r = 12\%$). We computed the option values for two alternative values of the interest rate volatility parameter, $\sigma_2 = 0.02$ and $\sigma_2 = 0.04$. The resulting option values differed only marginally from those presented in Tables II to IV. Our findings suggest that it is the location of the interest rate relative to its long-run mean value rather than the

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[13] Fama and Schwert [11] document for six-month and three-month holding periods, respectively, significant negative correlation between stock market returns and Treasury bill returns. The proof of (12) has been omitted to conserve space, but can be obtained from the authors.

[14] These assertions are valid only when the stock index value follows a lognormal diffusion.

Table II

Values of American Call Options on Futures Contracts ($\times 10^{-2}$)

(Stochastic Interest Rates)
Dividend Yield = 0.05

OPTION MATURITY = 3 MONTHS

Interest Rate (r)	Spot Index Value (S)				
	80.00	90.00	100.00	110.00	120.00
0.03	0.00	0.23	2.87	10.18	19.84
	79.90	89.88	99.87	109.86	119.84
0.05	0.00	0.26	3.06	10.54	20.31
	80.21	90.23	100.26	110.28	120.31
0.07	0.00	0.29	3.25	10.91	20.77
	80.52	90.58	100.65	110.71	120.77
0.08	0.00	0.31	3.36	11.10	21.01
	80.67	90.76	100.84	110.92	121.01
0.10	0.00	0.35	3.56	11.49	21.48
	80.99	91.11	101.23	111.35	121.48
0.12	0.01	0.39	3.78	11.88	21.95
	81.30	91.46	101.62	111.79	121.95

Notes: At each interest rate, r, the first row contains the value of an American call option on the futures contract, and the second row contains the underlying futures price. The spot index volatility (σ_1^2) = 0.0625, and the option's exercise price = 100.00. For the assumed stochastic process on interest rates, the speed of adjustment (κ) = 2.0, the long-run mean rate = 0.10, and the volatility (σ_2^2) = 0.0081.

volatility of interest rates which makes a sizable difference in the values.[15] Changes in the value of κ will affect the speed with which the interest rate is pulled toward its long-run mean, and hence impact option values. In Figure 4, we have shown call-option values against current interest rate levels for 6-month options on futures contracts, for values of $\kappa = \frac{1}{2}$, 2, and 5, keeping the value of S fixed at the striking price, K. When r is below its long-run mean level of 10%, increases in the value of κ will increase option values; when r is above the long-run mean value, decreases in κ are associated with increased option values, because the interest rate is expected to stay above its long-run mean for a larger interval of time.

In Figure 5, the critical regions of index and interest rate values $\{\bar{S}, \bar{r}\}$ are shown for three values of option maturity for call options on futures contracts.

[15] Thus, the deterministic term structure model [obtained from (8) by setting $\sigma_2 = 0$] can be used to incorporate one's prior about the term structure (e.g., as to whether it is upward sloping or downward sloping) in the valuation of options on futures contracts. The resulting option values differ considerably from those obtained using the flat term structure model reported in Table I. These conclusions are to be viewed in the context of the specific term structure model used in this study.

Table III

Values of American Call Options on Futures Contracts ($\times 10^{-2}$)

(Stochastic Interest Rates)
Dividend Yield = 0.05

OPTION MATURITY = 6 MONTHS

Interest Rate (r)	Spot Index Value (S)				
	80.00	90.00	100.00	110.00	120.00
0.03	0.06	0.84	4.24	11.17	20.36
	80.21	90.24	100.26	110.29	120.32
0.05	0.07	0.94	4.55	11.71	21.08
	80.71	90.80	100.89	110.98	121.07
0.07	0.08	1.05	4.88	12.26	21.83
	81.22	91.37	101.52	111.68	121.83
0.08	0.09	1.10	5.05	12.54	22.21
	81.47	91.66	101.84	112.02	122.21
0.10	0.10	1.22	5.39	13.12	22.97
	81.98	92.23	102.48	112.73	122.97
0.12	0.12	1.35	5.76	13.71	23.74
	82.50	92.81	103.12	113.43	123.74

Notes: At each interest rate, r, the first row contains the value of an American call option on the futures contract, and the second row contains the underlying futures price. The spot index volatility (σ_1^2) = 0.0625, and the option's exercise price = 100.00. For the assumed stochastic process on interest rates, the speed of adjustment (κ) = 2.0, the long-run mean rate = 0.10, and the volatility (σ_2^2) = 0.0081.

Four points are noteworthy. First, the futures price alone is *not* sufficient for characterizing the critical region. That is, the combinations $\{\bar{S}(\tau), \bar{r}(\tau)\}$ do not imply a constant $\bar{H}(\tau)$ at which exercise is optimal. Second, the critical price, $\bar{S}$, is a decreasing function of $\bar{r}$ at each maturity. Higher interest rates mean higher "implicit" dividends on the futures price, thereby lowering the critical futures price, and since the futures price itself is an increasing function of S and r, the index value at which exercise occurs at a given maturity falls as interest rates rise. Third, note that the critical boundary is relatively flat at high rates of interest. However, at low rates, for levels of r at and below μ, much smaller increases in r induce a given drop in the critical index value. The location of the current rate of interest relative to its long-run mean is thus crucial in characterizing the optimal exercise region. This underscores the importance of modelling stochastic variations in interest rates, for with deterministic rates, the critical index value is fixed for a given maturity. Finally, we note that for long maturities, the critical index value might be below the corresponding value for shorter maturities, especially at higher interest rate levels. Longer maturity options are written on futures contracts with correspondingly high maturity: at higher

Table IV
Values of American Call Options on Futures Contracts ($\times 10^{-2}$)

(Stochastic Interest Rates)
Dividend Yield = 0.05

OPTION MATURITY = 9 MONTHS

Interest Rate (r)	Spot Index Value (S)				
	80.00	90.00	100.00	110.00	120.00
0.03	0.22	1.56	5.48	12.36	21.32
	80.79	90.88	100.98	111.08	121.18
0.05	0.26	1.73	5.87	13.00	22.18
	81.41	91.59	101.76	111.94	122.12
0.07	0.30	1.90	6.29	13.66	23.07
	82.04	92.30	102.55	112.81	123.06
0.08	0.32	2.00	6.50	13.99	23.54
	82.36	92.65	102.95	113.24	123.54
0.10	0.36	2.19	6.94	14.68	24.49
	82.99	93.37	103.74	114.12	124.49
0.12	0.41	2.40	7.40	15.40	25.45
	83.63	94.09	104.54	115.00	125.45

Notes: At each interest rate, r, the first row contains the value of an American call option on the futures contract, and the second row contains the underlying futures price. The spot index volatility (σ_1^2) = 0.0625, and the option's exercise price = 100.00. For the assumed stochastic process on interest rates, the speed of adjustment (κ) = 2.0, the long-run mean rate = 0.10, and the volatility (σ_2^2) = 0.0081.

interest rates, there is an impact of maturity on the futures price that interacts with the option's maturity.

The properties of the critical regions of options on the stock index differ from those of options on stock index futures discussed so far. The critical index value increases with increases in interest rates for index options of all maturities. This implies that at high levels of interest rates, it is generally suboptimal to exercise call options on stock indexes. From Figure 5, it is clear that at high levels of interest rates, the probability of early exercise of call options on stock index futures contracts is very high. At higher dividend yields, it is easy to see that the critical region for call options on the stock index is everywhere lowered (implying higher probability of early exercise), whereas precisely the opposite is true for call options on the futures.[16]

The difference between the values of options on the index futures and options

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[16] While we have shown results for call options alone, the results for put options on futures contracts are reversed. Exercise occurs if the pair $\{S, r\}$ *falls* to a critical level given by a locus $\{\bar{S}, \bar{r}\}$; $\bar{S}$ is an increasing function of $\bar{r}$ for each maturity, τ; and the boundary is everywhere lower at distant maturities.
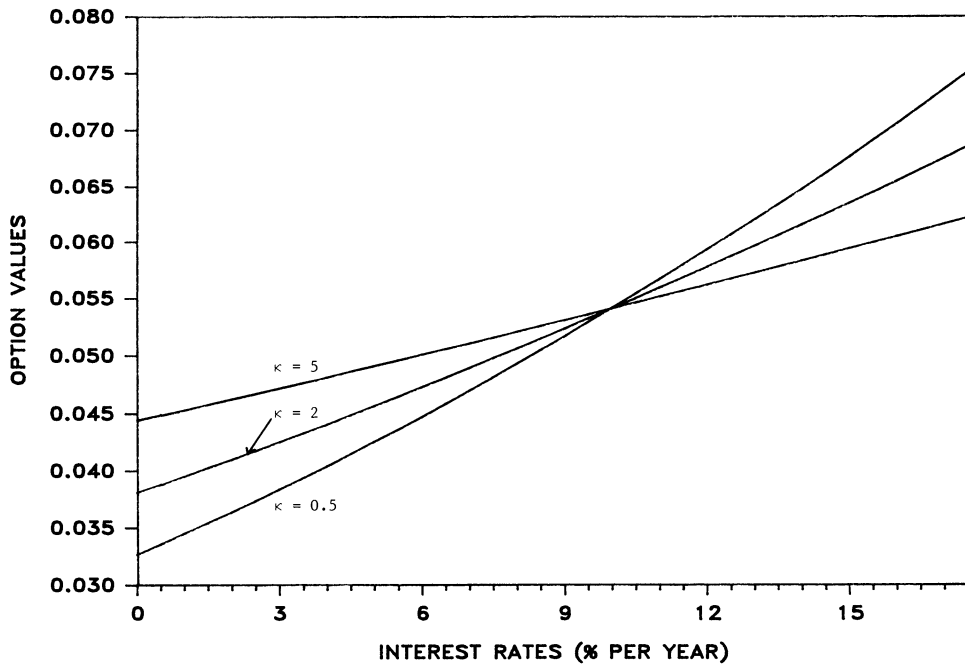
Figure 4. Values of Options on Futures — Stochastic Interest Rates. Exercise Price = 1, Volatility of Underlying Index Asset (σ_1) = 0.25, Dividend Yield = 5%, Long-run Mean Interest Rate (μ) = 10%, Volatility of Interest Rate (σ_2) = 0.09.
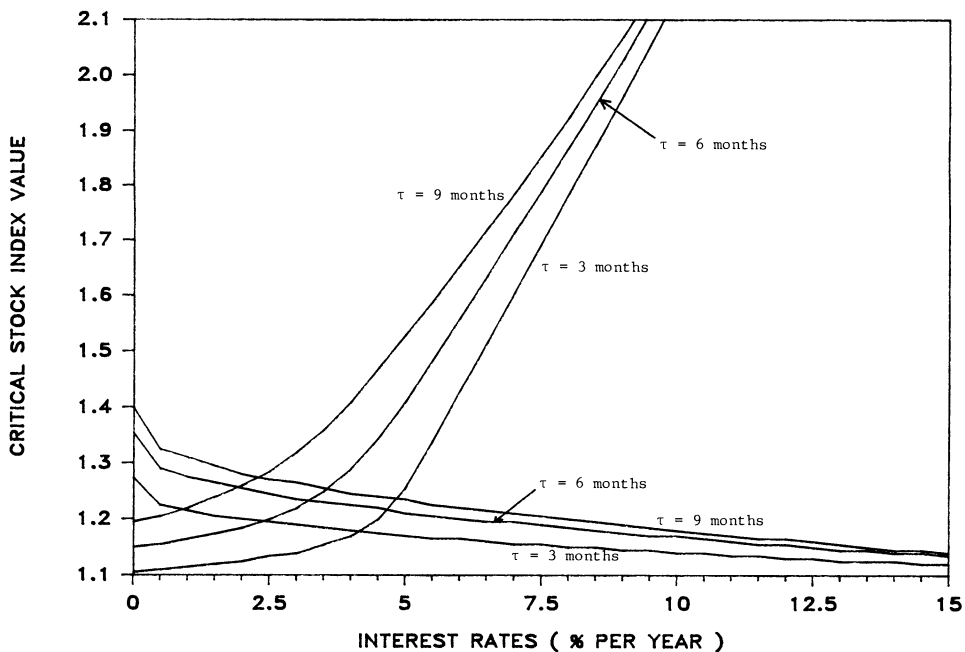

Figure 5. Critical Region for Options on Futures and Options on Stock Index — Stochastic Interest Rates. Exercise Price = 1, Volatility of Underlying Index Asset (σ_1) = 0.26, Dividend Yield = 5%, Long-run Mean Interest Rate (μ) = 10%, volatility of interest rate (σ_2) = 0.09.

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on the index are plotted in Figure 6 ($\tau = 6$ months) and Figure 7 ($\tau = 9$ months). The option on the futures always sells at a premium relative to the option on the index. Indeed, this is the case even when the current rate of interest is less than the dividend yield. This is because at low values of r it is expected that the interest rate will be pulled towards its long-run mean rate of 10% which is *higher* than the dividend yield. For options which have a life of 6 to 9 months, there is sufficient time for the rate of interest to revert to its mean value. As a result, call options contracts on futures sell at a premium relative to calls on the underlying index.

IV. Conclusions

We have examined rational restrictions for the values of options on futures contracts, and we have presented approaches to their valuation.

In presenting methods of valuation for options on futures contracts, we noted that one must solve first for the futures price as a function of the relevant state variables, and then employ this solution in the boundary or terminal condition for the valuation of the option. Except under special assumptions, as were employed here, the analytical characterization of the futures pricing function in the first step is difficult. We have presented two solution procedures for options on futures contracts, with and without interest rate uncertainty. We employed
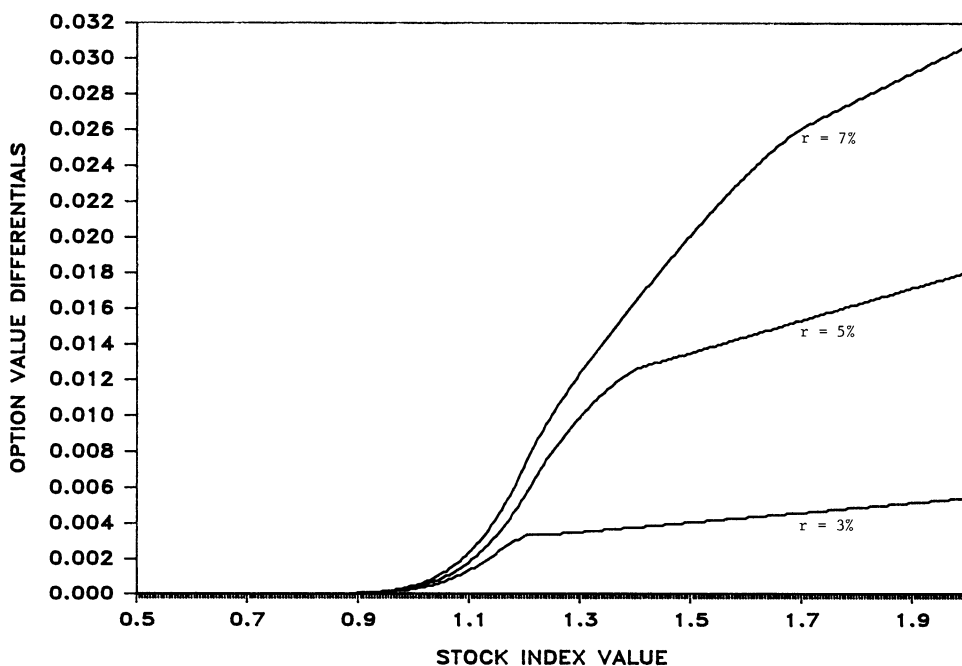

Figure 6. Option Value Differentials (Futures Minus Spot) − Stochastic Interest Rates. Exercise Price = 1, Volatility of Underlying Index Asset (σ_1) = 0.25, Dividend Yield = 5%, Long-run Mean Interest Rate (μ) = 10%, Volatility of Interest Rate (σ_2) = 0.09, Option Maturity = 6 Months.
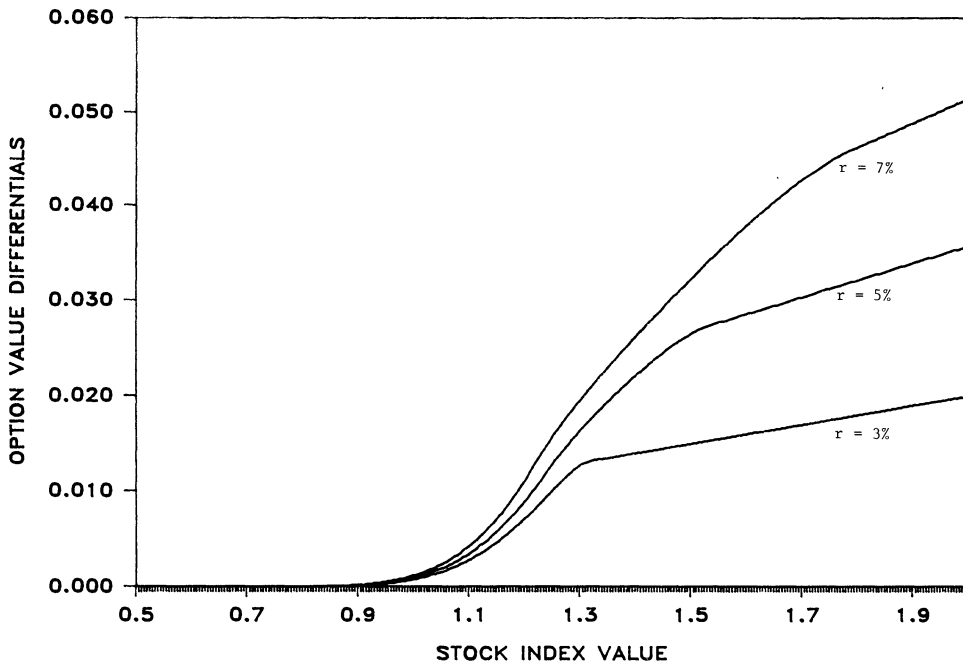
Figure 7. Option Value Differentials (Futures Minus Spot) – Stochastic Interest Rates. Exercise Price = 1, Volatility of Underlying Index Asset (σ_1) = 0.25, Dividend Yield = 5%, Long-run Mean Interest Rate (μ) = 10%, Volatility of Interest Rate (σ_2) = 0.09, Option Maturity = 9 Months.

numerical methods to examine options on a stock index and options on stock index futures contracts and characterized the optimal exercise policies. Sizable differences arose in option values and in the critical exercise boundaries between the two models. These differences are largely attributable to the location of the current rate of interest relative to its long-run value, or equivalently, on the expectations of rates through the option's life. Our analysis suggests that a deterministic term structure model which captures one's prior belief about the shape of the term structure can lead to significantly different option values relative to those obtained by using a constant interest rate model.

REFERENCES

1. F. Black. "The Pricing of Commodity Contracts." *Journal of Financial Economics* 3 (1976), 167–79.
2. —— and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (1973), 637–59.
3. D. T. Breeden. "Futures Markets and Commodity Options: Hedging and Optimality in Incomplete Markets." *Journal of Economic Theory* 32 (1984), 275–300.
4. M. J. Brennan and E. S. Schwartz. *Savings Bonds: Theory and Empirical Evidence*, Monograph Series in Finance and Economics, New York University, 1980.
5. B. Cornell and M. R. Reinganum. "Forward and Futures Prices: Evidence from the Foreign Exchange Markets." *Journal of Finance* 36 (1981), 1035–45.
6. G. Courtadon. "Options on Default Free Bonds and Options on Default Free Bond Futures: A

Comparison." Working Paper, Graduate School of Business Administration, New York University, 1983.

7. J. C. Cox, J. E. Ingersoll, and S. A. Ross. "A Re-examination of Traditional Hypotheses About the Term Structure of Interest Rates." *Journal of Finance* 36 (1981), 769–99.
8. ———. "The Relationship Between Forward Prices and Futures Prices." *Journal of Financial Economics*, 9 (1981), 321–46.
9. ———. "A Theory of the Term Structure of Interest Rates." *Econometrica* 53 (1985), 385–407.
10. J. C. Cox and M. Rubinstein. *Options Markets*. Engelwood Cliffs, NJ: Prentice-Hall, 1985.
11. E. F. Fama and G. W. Schwert. "Asset Returns and Inflation." *Journal of Financial Economics.* 15 (1977), 115–46.
12. G. Feiger and B. Jacquillat. "Currency Option Bonds, Puts and Calls on Spot Exchange and the Hedging of Contingent Foreign Earnings." *Journal of Finance* 34 (1979), 1129–43.
13. K. French "The Pricing of Futures and Forward Contracts." Working Paper, University of Rochester, 1982.
14. ———. "A Comparison of Futures and Forward Prices." *Journal of Financial Economics* 12 (1983), 311–42.
15. M. B. Garman and S. W. Kohlhagen. "Foreign Currency Option Values." *Journal of International Money and Finance* 2 (1983), 231–38.
16. R. C. Merton. "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4 (1973), 141–83.
17. S. F. Richard and M. Sundaresan. "A Continuous Time Equilibrium Model of Forward Prices and Futures Prices in a Multi-good Economy." *Journal of Financial Economics* 9 (1981), 347–71.