Optimal Pricing of Services with Switching Costs

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Abstract

Customer switching costs are an important factor in account-based services such as telecommunications, financial, insurance and brokerage services. In these businesses, existing customers incur significant costs if they switch to another provider. Such costs include physical configuration and installation costs, contractual costs (e.g. termination fees) and cognitive costs of learning. These switching costs enable a firm to extract more revenue from incumbent customers by charging them higher prices. However, higher prices disproportionately deter new customers from buying, because, ex ante, they face similar set-up and learning costs with all providers and hence are more price sensitive. This raises an important question of how best to balance the tradeoff between short-term revenue gains and long-run account growth. We develop an optimal control model to study this tradeoff. We show that a simple target market share policy is the optimal strategy. In particular, there exists a target price and target market share such that the firm should price to reach the target market share as fast as possible, at which point it should switch to the target price. We also examine how these targets change with the competitive outside price, shopping frequency of customers, firm's discount rate and market growth rate. In addition, we extend the basic monopoly model to the duopoly case in which two firms compete for market share and maximize their expected discounted revenue. We also look at the situation in which a firm is able to charge a lower introductory price to attract new customers.

1 Introduction

Switching costs are the costs associated with changing from one alternative product or service to another. On a broad level, they include physical costs such as setups and installation, artificial and contractual costs such as cancelation fees and loyalty rebates, cognitive costs such as learning to use a new technology or service, and even psychological costs, e.g., the pain of giving up a benefit is more pronounced than the pleasure of gaining a new benefit for loss-averse customers. [7] Indeed, notions of "brand loyalty" and "customer relationships" are pervasive in strategic pricing analyses, and there is extensive research on switching costs and their effects in the economics, marketing and behavioral science communities. As Shapiro and Varian [16] note: "You just cannot compete effectively unless you know how to identity, measure, and understand switching costs and map strategy accordingly."

The economics literature studies the role of switching costs in turning ex ante homogenous goods into ex post heterogeneous ones. They focus on fundamental switching cost economics, including inter-temporal pricing and its impact on social welfare and industrial organization. See for example Klemperer [8] [10] [11] [12] [13], Fudenberg and Tirole [3], Shapiro and Varian [16]. Marketing researchers have examined search and cognitive lock-in behavior in e-commerce sites and explain this behavior with the power law of practice. See Johnson et al. [4] [5].

Yet there is a more limited understanding of the impact of switching costs on operational pricing decisions. Current revenue management methods, in particular, largely ignore how short-term pricing decisions affect customers' willingness to switch to a competing firm and the resulting long-term consequences for revenues. For instance, executives at a major low-cost airline have commented to us that they do not price as aggressively as they could in several markets in which they have a dominant position because they feel high prices could damage their long-run perception as a "value-based" airline. They therefore override the recommendations of their revenue management system, setting upper bounds on prices. This is just one case of short-run pricing decisions being tempered by long-run strategic considerations. There are many others. Yet the approaches used to manage such important trade-off in practice are largely ad hoc.

Similar factors are important in financial, communication and information services, where customer retention and loyalty are widely regarded as critically important to sustaining and growing profits. To take one example, British Telecom launched a package that slashed prices for six months to undercut Internet telephony provider Skype (Best[1]). The strategy seems sensible, yet the impact of such a move on profits is ambiguous; on the one hand, lower prices attract new customers because new customers incur similar setup costs with all service providers and hence they are quite price sensitive. So offering lower, more competitive prices helps insure that the firm's customer base grows over time. However, lowering prices creates an immediate revenue loss from the base of existing, locked-in customers, who will typically tolerate higher prices due to the switching costs they incur if they defect to another firm. Therefore, there is a tradeoff between profiting from the existing customer base through high prices and attracting new customers to grow the customer base through low prices. How best to balance the tradeoff is an important question for both researchers and practitioners.

1.1 Literature Review

As mentioned, the economics literature has extensively addressed switching costs and their effects. A series of papers by Klemperer [8], [10], [9], [11], [12], investigate a setting in which there are price wars in an entry period, price increases in a post-entry period, and deterrence of new entrants into the market. They also analyze the competitiveness of markets with and without switching costs. The main results are that in a market with switching costs, a firm can earn monopoly profits from repeated purchasers even though it faces oligopolistic competition. As a result, firms often follow a bargains-then-ripoffs price pattern. That is, they vie fiercely for early adopters and then exploit them later by charging higher prices (a so-called "penetration pricing" strategy). This result helps explain practices such as the low introductory price offers commonly used in cable TV and Internet services. They also explore the role that switching costs play in deterring new entrants. They show that a market with high switching costs may be socially inefficient because a large amount of social surplus is dissipated by the resulting barriers to entry. For more details see Klemperer [13] and Farrell and Klemperer [2].

Recently, Lewis and Yildirim [14] explore procurement in multiple periods when buyers strategically manage switching costs to reduce procurement costs. They find that the bargains-then-ripoffs price pattern may not hold in a multiple period procurement setting when customers anticipate switching costs and respond strategically.

The behavioral science and marketing literature has empirically and experimentally investigated switching costs. For example, online shoppers, contrary to popular belief, exhibit significant loyalty and stickiness due to efficiencies gained from learning through repeated experience. Experimental results by Johnson etc. [5] suggest that customers surprisingly visit very few shopping sites online, even though other sites “just a mouse click away” could save them money. On average, households visit only 1.2 book sites, 1.3 CD sites, and 1.8 travel sites during a month for each category before they make online purchases. Johnson etc. [4] explain such cognitive lock-in behavior in e-commerce environments by the “power law of practice,” a cognitive psychology concept; efficiency gains resulting from learning reduce the number of sites visited and strengthen customer “stickiness” or loyalty to familiar sites. Jones et al. [6] propose a multidimensional conceptual framework to measure various switching cost perceptions. They conceptualize services switching costs along several dimensions including lost performance costs, pre-switching and evaluation costs, post-switching behavioral and cognitive costs. They examine various relationships among switching cost dimensions, both conceptually and empirically, and discuss potential cross-industry differences in perceptions of service switching costs.

To date little work exists in the revenue management literature on incorporating switching costs into tactical pricing optimization. A notable exception is the work of Phillips [15], who considers a fixed-term service contract in which repricing may occur at various times (e.g. home insurance). Using a Markov chain model, he analyzes the optimal initial price (acquisition price) and subsequent re-pricing policy over time.

While such a personalized pricing policy is feasible in industries like insurance, for industries like telecommunications and financial services, prices are typically advertised broadly and are applied uniformly (e.g. mobile phone rate plans, flat-fees for brokerage trades, Internet service provider (ISP) monthly rates, etc.). In such settings, it is difficult to discriminate among customers when pricing or repricing or to charge new customers a different price than incumbent customers. In fact, our research on this topic was motivated by a consulting engagement with a major on-line brokerage firm that was struggling with just such a pricing decision. The firm competed based on a flat fee-per-trade pricing scheme. This fee was an integral part of the firm’s marketing and was heavily advertised. A competitor had recently lowered it's comparable fixed fee-per-trade. The firm needed to to respond but faced a difficult choice: keep their fixed fee-per-trade at the current level and maintain short-term revenues, but risk losing new accounts and thus face a dwindling customer base over time; or cut prices and absorb an immediate – and quite substantial – short-term revenue drop, but maintain growth in their account base. This is the precisely trade-off we analyze.

1.2 Overview

The remainder of the paper is organized as follows. We first study a monopoly firm that offers an account-based service and chooses prices over time to maximize its discounted revenue over an infinite horizon. In section 2.2, we show the firm's optimal pricing strategy in this situation is what we call a *a target market share policy*. Namely, there exists a target price and target market share such that the optimal strategy of the firm is to price to reach this target market share as quickly as possible, at which point the firm switches to the target price. This means pricing as high as possible if the market share is above the target and pricing as low as possible if the market share is below the target.

In section 2.3, we examine how the target price and target market share are affected by the firm's discount rate, the market growth rate, the outside market price, and customers' shopping frequency. These quasi-statics provide some interesting insights. For example, we show that when the competitive outside price drops, the initial optimal response of the firm can be to increase price. The reason is that with a lower competitive price, the future market is less promising (i.e. has a lower target share and lower target price); as a result, the firm is better off extracting revenue from its current customers than building market share for the (now bleaker) future, and so it increases its price.

We then extend the analysis in section 3 to a duopoly market in which two firms compete for market share and each firm maximizes its own expected discounted revenue, and in section 4 we analyze the case where the firm is able to charge a lower introductory price to new customers. Finally, concluding remarks are provided in section 5.

2 The Monopoly Model

We first consider a single firm that offers an account-based service, such as telecommunications, insurance, banking or brokage service. We assume that the potential market size is deterministic and grows at rate θ. The potential market size at time t is denoted by $Ne^{\theta t}$. Note that $\theta > 0$ implies the market is expanding while $\theta < 0$ corresponds to a shrinking market; $\theta = 0$ indicates the market size is constant over time. The number of accounts the firm has at time t is denoted by $x(t)$. The firm charges $p_1(t)$ for its service at time t. There are competitors which charge a prevailing market price p_0, which we assume is exogenously given. We interpret p_0 as a proxy for the prevailing price of competitive alternatives. Based on the firm's price $p_1(t)$ and an outside alternative price p_0, new customers decide whether to sign up for service or not, and current customers of the firm decide whether to continue service or switch to the outside alternative. We assume these purchase decisions are made at rate α, called the *shopping frequency*, which reflects the frequency with which customers renew or review their service purchase options. The average time between purchase or re-purchase decisions is then $1/\alpha$. For example, most mobile companies offer one to two year wireless contracts, insurance is normally renewed annually, etc.

We assume that the fraction of customers willing to purchase or defect is a linear function of prices. In particular, the fraction of new customers willing to purchase is defined by $d(p_0, p_1, t) = B + b_0 p_0 - b_1 p_1(t)$ where $0 < b_0 < b_1$, which implies that demand is more sensitive to a change in the firm's own price than it is to a simultaneous change in the outside price. The fraction of current customers willing to defect from the firm is defined by $g(p_0, p_1, t) = A - a_0 p_0 + a_1 p_1(t)$ where $0 < a_0 < a_1$. Again, this implies that defection is more sensitive to the firm's price than to the competitive price. The rate at which new customers purchase service (the acquisition rate) is then given by $(Ne^{\theta t} - x(t))\alpha(B + b_0 p_0 - b_1 p_1(t))$; that is, the acquisition rate is a multiplicative function of three components: the available external market size $Ne^{\theta t} - x(t)$, the shopping frequency α at which customers make purchase decisions, and the fraction of new customers willing to purchase, $B + b_0 p_0 - b_1 p_1(t)$. Similarly, the rate at which current customers defect to the outside market (the defection rate) is $x(t)\alpha(A - a_0 p_0 + a_1 p_1(t))$. We assume that $A - a_0 p_0 \geq 0$. This is a reasonable assumption; it simply says even when the firm offers free service, there are still customers who defect for exogenous reasons other than price. Intuitively, the acquisition rate decreases in the firm's own price p_1 and increases in the outside price p_0; and the defection rate increases in p_1 and decreases in p_0. To guarantee $0 \le d(p_0, p_1, t) \le 1$ and $0 \le g(p_0, p_1, t) \le 1$, we require the firm's price p_1 satisfies

$$\max\left\{\frac{B + b_0 p_0 - 1}{b_1}, \frac{a_0 p_0 - A}{a_1}\right\} \le p_1 \le \min\left\{\frac{B + b_0 p_0}{b_1}, \frac{a_0 p_0 - A + 1}{a_1}\right\}. \tag{1}$$

Current and new customers have different price sensitivities since current customers incur switching costs when they defect while new customers do not. Most of the economics literature on switching costs assumes that switching costs are exogenously given or can be directly expressed as a function of location (e.g., distance). In contrast, we implicitly incorporate switching costs by postulating that current and new customers have different responses to price. In particular, let $\epsilon_d(p_i)$ be the price elasticity of demand from new customers at price p_i; $\epsilon_g(p_i)$ be the price elasticity of defections of current customers at price p_i, and $i = 0, 1$. That new customers are more price sensitive implies that $\epsilon_d(p_i) > \epsilon_g(p_i)$, $i = 0, 1$, which requires

$$p_1 > \max\left\{\frac{a_1 B - b_1 A + (a_0 b_1 + a_1 b_0) p_0}{2 a_1 b_1}, \frac{a_0 B - b_0 A + 2 a_0 b_0 p_0}{a_1 b_0 + a_0 b_1}\right\}. \tag{2}$$

One sufficient condition under which $\epsilon_d(p_i) > \epsilon_g(p_i)$, $i = 0, 1$ for any price p_1 is the following:

$$\frac{b_i}{a_i} > \frac{B + b_0 p_0}{A - a_0 p_0}, \quad i = 0, 1.$$

We can easily see that the ratio of price elasticities of acquisition to defection, $\frac{\epsilon_d(p_i)}{\epsilon_g(p_i)}$, increases in b_i and decreases in a_i, $i = 0, 1$. A large ratio of these price elasticities can be interpreted as high switching costs. Hence, a larger b_i and (or) a smaller a_i implies that existing customers incur higher switching costs when they defect. Note also that

$$\frac{\epsilon_d(p_1)}{\epsilon_g(p_1)} \Big/ \frac{\epsilon_d(p_0)}{\epsilon_g(p_0)} = \frac{b_1}{a_1} \Big/ \frac{b_0}{a_0}. \tag{3}$$

When $\frac{b_1}{a_1} / \frac{b_0}{a_0} > 1$, (3) says that the ratio of price elasticities of acquisition to defection is more pronounced for a change in the firm's own price than for the same amount of change in the competitor's price (outside market price).

2.1 Model Formulation

We assume that future revenue is discounted at rate δ. The firm's initial customer base is denoted x_0. We assume $\underline{p}$ and $\bar{p}$ are, respectively, the lowest and highest prices the firm could charge. For example, prices may be constrained by historical norms, competitors' prices, internal policies, etc. At a more abstract level, $\underline{p}$ and $\bar{p}$ simply represent the lowest and highest prices that the firm would consider using. Particularly, these bounds on price are constrained by (1) and (2). The firm's problem is to choose prices at each point in time to maximize its discounted revenue over an infinite horizon, namely

$$
\begin{aligned}
& \max \int_0^\infty e^{-\delta t} p_1(t) x(t) dt \qquad (4) \\
& \text{s.t.} \\
& \dot{x}(t) = (Ne^{\theta t} - x(t))\alpha(B + b_0 p_0 - b_1 p_1(t)) - x(t)\alpha(A - a_0 p_0 + a_1 p_1(t)), \\
& x(0) = x_0, \\
& \underline{p} \le p_1(t) \le \bar{p}, \\
& 0 \le x(t) \le Ne^{\theta t}.
\end{aligned}
$$

To simplify the analysis, we reformulate the problem (4) by a change of variable. Replacing $x(t)$ by $y(t)Ne^{\theta t}$, then $y(t)$ can be interpreted as the firm's market share at time t; that is, the fraction of the total market potential acquired by the firm at time t. We denote the firm's initial market share by $y_0 = x_0/N$. The optimization problem (4) can then be rewritten as

$$
\begin{aligned}
& N \max \int_0^\infty e^{-(\delta-\theta)t} p_1(t) y(t) dt \qquad (5) \\
& \text{s.t.} \\
& \dot{y}(t) = \alpha(B + b_0 p_0 - b_1 p_1(t)) - y(t)(\theta + \alpha(A + B) + \alpha(b_0 - a_0)p_0 - \alpha(b_1 - a_1)p_1(t)), \\
& y(0) = y_0, \\
& \underline{p} \le p_1(t) \le \bar{p}, \\
& 0 \le y(t) \le 1.
\end{aligned}
$$

We assume $\delta > \theta$ to ensure that total discounted revenue is finite.

2.2 Analysis of the Optimal Pricing Policy

To solve the optimization problem defined by (5), we use Pontryagin's maximum principle. We first show that there exists an optimal long-run stationary equilibrium price and market-share state, which can be uniquely determined in closed form. Further, we characterize the optimal control policy (price trajectory) and associated market share trajectory leading to this optimal long-run stationary equilibrium.

The current-value Hamiltonian for the optimization problem (5) is

$$H(y, p_1, \lambda) = p_1 y + \lambda\alpha(B + b_0 p_0 - b_1 p_1) - \lambda y(\theta + \alpha(A + B) + \alpha(b_0 - a_0)p_0 - \alpha(b_1 - a_1)p_1). \quad (6)$$

The associated Lagrangian when the state constraint $0 \leq y(t) \leq 1$ is taken into account is

$$L(y, p_1, \lambda, \mu) = H + \mu_1 y + \mu_2(1 - y),$$

where $\mu = (\mu_1, \mu_2)$. The optimal long-run equilibrium, denoted by the quadruple $(y_e, p_e, \lambda_e, \mu_e)$, therefore satisfies

$$\alpha(B + b_0 p_0 - b_1 p_e) - y_e(\theta + \alpha(A + B) + \alpha(b_0 - a_0)p_0 - \alpha(b_1 - a_1)p_e) = 0, \quad (7)$$

$$(\delta - \theta)\lambda_e = L_y(y_e, p_e, \lambda_e, \mu_e), \quad (8)$$

$$\mu_1^e y_e = 0, \quad \mu_2^e(1 - y_e) = 0, \quad \mu_e \geq 0, \quad (9)$$

$$H(y_e, p_e, \lambda_e) \geq H(y_e, p, \lambda_e), \quad \forall \underline{p} \leq p \leq \bar{p}. \quad (10)$$

We call the solution to the above system of equations and inequalities, y_e and p_e, the *target market share* and *target price*, respectively. The existence and uniqueness of such an equilibrium target market share and target price is established in Proposition 1.

Proposition 1 *For the problem (5), there exists a unique stationary equilibrium target market share y_e and target price p_e, defined as follows:*

1. *When $a_1 \neq b_1$, p_e and y_e are given by:*

$$p_e = \frac{\delta + \theta + 2\alpha(A + B) + 2\alpha(b_0 - a_0)p_0 - \sqrt{\Delta}}{2\alpha(b_1 - a_1)}, \quad (11)$$

$$y_e = \frac{\delta + \theta + 2\alpha(A + B) + 2\alpha(b_0 - a_0)p_0 - \sqrt{\Delta}}{2(1 - \frac{a_1}{b_1})(\delta + \alpha(A + B) + \alpha(b_0 - a_0)p_0)}, \quad (12)$$

where

$$\Delta = (\delta+\theta+2\alpha(A+B)+2\alpha(b_0-a_0)p_0)^2 - 4\alpha(1-\frac{a_1}{b_1})(B+b_0p_0)(\delta+\alpha(A+B)+\alpha(b_0-a_0)p_0).$$

2. *When $a_1 = b_1$, p_e and y_e are determined by:*

$$p_e = \frac{(B+b_0p_0)(\delta+\alpha(A+B)+\alpha(b_0-a_0)p_0)}{b_1(\delta+\theta+2\alpha(A+B)+2\alpha(b_0-a_0)p_0)}, \quad (13)$$

$$y_e = \frac{\alpha(B+b_0p_0)}{\delta+\theta+2\alpha(A+B)+2\alpha(b_0-a_0)p_0}. \quad (14)$$

We next characterize the optimal control policy leading to the long-run stationary equilibrium. It turns out that the optimal pricing policy is simply to price so as to reach the target market share as fast as possible. This important property is stated in Proposition 2. To facilitate presentation, we define $R(p)$ and $S(p)$ as follows:

$$R(p) = \alpha(B+b_0p_0-b_1p) \ ,$$

$$S(p) = \theta+\alpha(A+B)+\alpha(b_0-a_0)p_0-\alpha(b_1-a_1)p \ .$$

It is easy to check that $\frac{R(p)}{S(p)}$ decreases in p and $y_e = \frac{R(p_e)}{S(p_e)}$ at the long-run stationary state.

Proposition 2 *Assuming $\underline{p} \le p_e \le \bar{p}$, then the optimal price trajectory $p_1^*(t)$ and associated market share trajectory $y^*(t)$ depend on the initial market share y_0 as follows:*

1. *When $y_0 \ge y_e$, the firm prices as high as possible until the market share reaches the target market share, at which point the optimal price switches to the target price. Namely,*

$$p_1^*(t) = \begin{cases} \bar{p} & \text{if } t < t_h \\ p_e & \text{otherwise} \end{cases}$$

$$y^*(t) = \begin{cases} \left(y_0 - \frac{R(\bar{p})}{S(\bar{p})}\right)e^{-S(\bar{p})t} + \frac{R(\bar{p})}{S(\bar{p})} & \text{if } t < t_h \\ y_e & \text{otherwise} \end{cases}$$

where t_h is the time it takes for the firm to reach the target market share and determined by

$$t_h = \frac{1}{S(\bar{p})} \ln \frac{y_0 - \frac{R(\bar{p})}{S(\bar{p})}}{y_e - \frac{R(\bar{p})}{S(\bar{p})}}$$

2. *When $y_0 < y_e$, the firm prices as low as possible until the market share reaches the target market share, at which point the firm prices at the target price. Namely,*

$$p_1^*(t) = \begin{cases} \underline{p} & \textit{if } t < t_l \\ p_e & \textit{otherwise} \end{cases}$$

$$y^*(t) = \begin{cases} \left(y_0 - \frac{R(\underline{p})}{S(\underline{p})}\right) e^{-S(\underline{p})t} + \frac{R(\underline{p})}{S(\underline{p})} & \textit{if } t < t_l \\ y_e & \textit{otherwise} \end{cases}$$

where

$$t_l = \frac{1}{S(\underline{p})} \ln \frac{y_0 - \frac{R(\underline{p})}{S(\underline{p})}}{y_e - \frac{R(\underline{p})}{S(\underline{p})}}$$

We call the policy described in Proposition 2 *a target market share policy.* It simply says that it is optimal for the firm to price to reach the target market share as fast as possible and then switch to the target price. In particular, when the firm starts with a market share smaller than the target level y_e, it should price as low as possible in order to attract new customers and build market share quickly. In this case, pricing to attract customers for the future is more important than pricing for short-term profits. This suggests, for example, that a new entrant into an industry with low initial share should offer a low price in an attempt to gain and grow market share. Conversely, when the initial market share is greater than the target level y_e, the firm should price as high as possible to extract revenue from its current customer base until its market share is reduced to y_e. This suggests, for example, when a firm has a legacy dominant market share in an industry with new competitors, it is more profitable to price for short-term profits rather than to maintain market share.

In proposition 2, we assume that the target price p_e can be achieved; that is, p_e is within the feasible range of prices $[\underline{p}, \bar{p}]$. If the target price p_e falls outside this feasible range, the optimal pricing policy becomes considerably more complex. We present the results informally and without proofs in the Appendix B. But an equilibrium price outside the feasible range is more likely an indication that the bounds on price are not well specified.

2.3 Comparative Statics of the Target State

In this section, we investigate the comparative statics of the target price and target market share. Specially, we examine how the target price and target market share are affected by the discount rate δ, the market growth rate θ, the shopping frequency α, the outside competitive price p_0, and the coefficients of the willingness-to-pay function coefficients a_i, b_i, $i = 0, 1$.

Proposition 3 *The target price p_e and the target market share y_e defined in (11) and (12) vary with the problem parameters as follows:*

1. *The target price p_e increases in p_0, δ and a_0, and decreases in θ, a_1, b_0 and b_1; p_e decreases in α if $p_e \leq \left(\frac{B+b_0p_0}{b_1}\right)\frac{\delta}{\delta+\theta}$ and increases in α otherwise.*

2. *The target market share y_e increases in a_0, b_1 and α, and decreases in θ, δ and a_1; y_e decreases in p_0 if $y_e \geq \frac{1-\sqrt{1-\frac{1-\frac{a_1}{b_1}}{1-\frac{a_0}{b_0}}}}{1-\frac{a_1}{b_1}}$ and $\frac{a_1}{b_1} \geq \frac{a_0}{b_0}$, and it increases in p_0 otherwise; y_e decreases in b_0 if $y_e \geq \frac{1-\sqrt{\frac{a_1}{b_1}}}{1-\frac{a_1}{b_1}}$ and increases in b_0 otherwise.*

Some results here are quite intuitive. For example, as the discount rate δ increases, future profits become less important than current profits, leading to more extraction from existing customers and less market share building. Hence, higher discount rates lead to a higher target price and a smaller target market share.

The quasi-statics with respect to the purchase frequency α are more subtle and reflect two main effects. As the shopping frequency increases, the rate at which potential new customers consider the firm's offering increases, but so does the rate at which current customers reconsider whether to stay or defect. A higher α will therefore increase both the rate of acquisition of new customers as well as the rate of defections of current customers. Which of these two effects dominates as α increases is determined by the condition

$$p_e \leq \left(\frac{B + b_0p_0}{b_1}\right)\frac{\delta}{\delta + \theta}. \tag{15}$$

The important term here is the ratio $\frac{\delta}{\delta+\theta}$, which depends on how the discount rate δ compares to the market growth rate θ (which can be negative). If the discount rate is large relative to the growth rate, then $\frac{\delta}{\delta+\theta}$ is close one (when $\theta \leq 0$, this ratio is always greater than 1) and the condition (15) will be satisfied due to the price constraint (1). In this case the present value of future new customer acquisitions is less important than defections of current customers. So an increase in shopping frequency impacts defection costs more than future acquisition benefits, which leads to a lower equilibrium price to avoid defections. Conversely, if the market growth rate is much larger than the discount rate, $\frac{\delta}{\delta+\theta}$ is close to $\frac{1}{2}$ (we require $\theta < \delta$ to ensure a finite total discounted revenue) and the inequality (15) may not be satisfied. In this case, the present value of new customer acquisitions becomes more significant than defections from current customers. So an increase in shopping frequency impacts future acquisition benefits more than defection costs, leading to a higher equilibrium price to profit from the higher rate of new customer purchases.

Some of the quasi-statics in Proposition 3 lead to somewhat counter-intuitive conclusions. Specifically, when the outside competitive price p_0 declines, under certain conditions the results imply that the optimal response for the firm (at least initially) is to increase its price. To see why, note when $\frac{a_1}{b_1} < \frac{a_0}{b_0}$ (implying the ratio of price elasticity of acquisition to elasticity of defection is greater for the firm's own price change than for the outside price; see (3)) the target market share is decreasing in p_0. This means if the firm is operating in equilibrium at its target market share and the outside competitive price p_0 suddenly declines (e.g. due to a competitive price cut), the new target market share will fall below the current market share. In this case, the firm's optimal response is to move to the new target market share as quickly as possible, i.e. increase its price as much as possible to $\bar{p}$. Only when the market share drops to the new lower target will the firm reduce price to the new (lower) target price. So we get the counter-intuitive result that the initial optimal response to a competitive price cut can be a price increase. The intuition is that lower competitive prices reduce the value of the market in the future, since lower prices and lower market share will eventually be the new equilibrium. With this more dismal future to look forward to, the firm finds itself suddenly less concerned with maintaining growth of new accounts and switches to generating profits from exploiting its (initially relatively large) current base of customers.

Lastly, some special cases of the quasi-static results are also worth noting. When $a_1 = b_1$ (i.e., the absolute change of acquisition rate is equal to the absolute change of defection rate given a change of price), one can easily check that the target price p_e given in (13) increases in δ, p_0, and b_0, while it decreases in θ, α and a_0; the target market share y_e given in (14) increases in p_0, α, a_0, and b_0, while decreases in δ and θ. Furthermore when $\alpha \to +\infty$ (corresponding to continuous reviewing of purchase decisions), the target price becomes $\frac{B+b_0p_0}{2b_1}$, which is exactly the optimal price when maximizing the revenue rate below pointwise:

$$\max \int_0^\infty e^{-(\delta-\theta)t} N\alpha(B + b_0p_0 - b_1p_1(t))p_1(t)dt \ .$$

That is, with infinite shopping frequency and no price sensitivity difference between current and new customers, the firm prices to maximize revenue myopically.

3 A Duopoly Market

In this section, we consider the duopolistic version of the model (4). The salient feature of this model is that, for each firm, the outside market price is no longer static but is the result of a competitor's price optimization over time. We model this as a differential game between the two competing firms.

3.1 Model Formulation

For compactness, we use the same notation as in the monopoly case wherever possible, though some new notation is introduced for the duopoly case. In particular firms are indexed by $i = 1, 2,$; $p_i(t)$ is the price charged by firm i at time t; $y_i(t)$ is the market share of firm i at time t; $d_i(p_1, p_2)$ is the willingness-to-purchase function of new customers for firm i; $g_i(p_1, p_2)$ is the willingness-to-defect function of current customers for firm i. Specifically,

$$d_1(p_1, p_2) = B_1 - b_{11}p_1 + b_{12}p_2, \quad d_2(p_1, p_2) = B_2 + b_{21}p_1 - b_{22}p_2.$$

$$g_1(p_1, p_2) = A_1 + a_{11}p_1 - a_{12}p_2, \quad g_2(p_1, p_2) = A_2 - a_{21}p_1 + a_{22}p_2.$$

For analytical convenience, we assume a perfectly symmetric market; that is, the coefficients of the willingness-to-purchase and willing-to-defect functions for each firm are the same,

$$d_1(p_1, p_2) = B - b_1p_1 + b_2p_2, \quad d_2(p_1, p_2) = B + b_2p_1 - b_1p_2.$$

$$g_1(p_1, p_2) = A + a_1p_1 - a_2p_2, \quad g_2(p_1, p_2) = A - a_2p_1 + a_1p_2.$$

We assume that $b_1 > b_2$ and $a_1 > a_2$. This implies that the willingness-to-purchase and willingness-to-defect functions for each firm are more sensitive to a change in their own price than they are to a change in the competitor's price. Let $\epsilon_{d_i}(p_j)$ denote the price elasticity of acquisition rate for new customers at firm i relative to price p_j, and $\epsilon_{g_i}(p_j)$ be the price elasticity of defection rate for existing customers at firm i relative to price p_j, $i, j = 1, 2$. That new customers are more price sensitive than existing customers requires that the price elasticity of the acquisition rate be larger than the price elasticity of the defection rate at both the firm's own price and the competitor's price, that is, $\epsilon_{d_i}(p_j) \geq \epsilon_{g_i}(p_j)$, $i, j = 1, 2$. We assume that the allowable price range for each firm is the same, that is, $\underline{p} \leq p_1, p_2 \leq \bar{p}$. These price elasticity assumptions then require that $b_1A - a_1B \geq (a_1b_2 + a_2b_1)\bar{p} - 2a_1b_1\underline{p}$ and $b_2A - a_2B \geq 2a_2b_2\bar{p} - (a_1b_2 + a_2b_1)\underline{p}$.

Note that the ratio of elasticity of acquisition rate to elasticity of defection rate, $\frac{\epsilon_{d_i}(p_i)}{\epsilon_{g_i}(p_i)}$, increases in b_1 while decreases in a_1, $i = 1, 2$. Therefore, a larger b_1 and (or) a smaller a_1 leads to a larger ratio of demand elasticities to the firm's own price. The ratio $\frac{\epsilon_{d_i}(p_j)}{\epsilon_{g_i}(p_j)}$ ($i, j = 1, 2$ and $i \neq j$) increases in b_2 while decreases in a_2. This implies a larger b_2 and (or) a smaller a_2 result in a larger ratio of demand elasticities to the competitor's price. In any case, a large ratio of $\frac{\epsilon_{d_i}(p_j)}{\epsilon_{g_i}(p_j)}$ ($i, j = 1, 2$) corresponds to the case of high switching costs. Notice also that

$$\frac{\epsilon_{d_i}(p_i)}{\epsilon_{g_i}(p_i)} \Big/ \frac{\epsilon_{d_i}(p_j)}{\epsilon_{g_i}(p_j)} = \frac{b_1}{a_1} \Big/ \frac{b_2}{a_2}, \quad i, j = 1, 2.$$

When $\frac{b_1}{a_1} / \frac{b_2}{a_2} > 1$, the ratio of elasticity of acquisition to elasticity of defection is larger for the firm's own price change than for the competitor's price.

Each firm chooses prices simultaneously at each point in time to maximize its own discounted revenue over an infinite horizon. In particular, given the price of its opponent, each firm optimally selects its own price such that this pair of prices constitute a Nash equilibrium. Given firm 2's price $p_2(t)$, firm 1's decision problem can be written as follows:

$$\max J_1 = N \int_0^\infty e^{-(\delta-\theta)t} p_1(t) y_1(t) dt,$$

s.t.

$$\dot{y}_1(t) = (1 - y_2(t))\alpha(B - b_1 p_1(t) + b_2 p_2(t)) - y_1(t)(\theta + \alpha(A + B - (b_1 - a_1)p_1(t) + (b_2 - a_2)p_2(t))),$$

$$\dot{y}_2(t) = (1 - y_1(t))\alpha(B + b_2 p_1(t) - b_1 p_2(t)) - y_2(t)(\theta + \alpha(A + B + (b_2 - a_2)p_1(t) - (b_1 - a_1)p_2(t))),$$

$$y_1(0) = y_1^0, \quad 0 \le y_1(t) \le 1,$$

$$y_2(0) = y_2^0, \quad 0 \le y_2(t) \le 1,$$

$$\underline{p} \le p_1(t), p_2(t) \le \bar{p}.$$

Similarly, firm 2 maximizes its discounted revenue, denoted J_2, given firm 1's price $p_1(t)$:

$$\max J_2 = N \int_0^\infty e^{-(\delta-\theta)t} p_2(t) y_2(t) dt$$

subject to the same constraints as in firm 1's problem.

If there exist $p_1^*(t)$ and $p_2^*(t)$ such that $p_1^*(t)$ maximizes J_1 given $p_2^*(t)$ and $p_2^*(t)$ maximizes J_2 given $p_1^*(t)$, then $(p_1^*(t), p_2^*(t))$ is a Nash equilibrium.

Because the state differential equation for each firm depends on the market share state of the other firm, it is difficult to derive closed-form equilibrium solutions. Hence, to simplify the problem somewhat we consider next a special case of an infinitely large potential market.

3.2 Solution for a Large-Market Case

Here we address a special case of the duopoly model in which we assume the potential market size is large enough that we can neglect the saturation effect of the existing customers of the two competing firms. That is, the size of the external market is not affected by each firm's current market share. In this case, firm 1's decision problem given firm 2's price $p_2(t)$ is the following:

$$\max J_1 = N \int_0^{\infty} e^{-(\delta-\theta)t} p_1(t) y_1(t) dt \tag{16}$$

s.t.

$$\dot{y}_1(t) = \alpha(B - b_1 p_1(t) + b_2 p_2(t)) - y_1(t)(\theta + \alpha(A + a_1 p_1(t) - a_2 p_2(t))$$

$$\dot{y}_2(t) = \alpha(B + b_2 p_1(t) - b_1 p_2(t)) - y_2(t)(\theta + \alpha(A - a_2 p_1(t) + a_1 p_2(t))$$

$$y_1(0) = y_1^0, \quad y_2(0) = y_2^0$$

$$\underline{p} \leq p_1(t), p_2(t) \leq \bar{p}.$$

Firm 2 maximizes its discounted revenue given firm 1's price $p_1(t)$:

$$\max J_2 = N \int_0^{\infty} e^{-(\delta-\theta)t} p_2(t) y_2(t) dt \tag{17}$$

subject to the same constraints as in firm 1's problem (16). Again, if $p_1^*(t)$ maximizes (16) given $p_2^*(t)$, and $p_2^*(t)$ maximizes (17) given $p_1^*(t)$, then $(p_1^*(t), p_2^*(t))$ constitutes an open-loop Nash equilibrium.

We first show in Proposition 4 that there exists a stationary Nash equilibrium under certain conditions. We then characterize the optimal price trajectories and associated market share trajectories for each firm before the stationary equilibrium is reached in Proposition 5.

Proposition 4 *When $\frac{a_1}{a_2} + \frac{b_2}{b_1} \geq 2$, there exists a symmetric, stationary Nash equilibrium for the control problems defined in (16) and (17). The symmetric stationary equilibrium target price, denoted p^e, is the solution to:*

$$\alpha(2a_2b_1 - a_1b_1 - a_2b_2)p^2 - (b_1(\delta + \theta + 2\alpha A) - b_2(\delta + \alpha A) + a_2\alpha B)p + B(\delta + \alpha A) = 0,$$

and the symmetric target market share, y_e, for each firm is determined by:

$$y_1^e = y_2^e := y^e = \frac{\alpha(B - b_1 p^e + b_2 p^e)}{\theta + \alpha(A + a_1 p^e - a_2 p^e)}.$$

We are also interested in the optimal control policies before the stationary Nash equilibrium is reached. Similar to the monopoly case, the firm prices either at the highest allowable price or at the lowest allowable price, depending on the market share of each firm. However, unlike the monopoly case, the firm may price at some intermediate level for part of the trajectory because of interdependent optimal control problems. To streamline the exposition of this result, we define

$$\begin{aligned}
\hat{p}_1 &= \frac{B + b_2\underline{p} - y^e(\frac{\theta}{\alpha} + A - a_2\underline{p})}{b_1 + a_1 y^e} \ , \\
\hat{p}_2 &= \frac{B + b_2\bar{p} - y^e(\frac{\theta}{\alpha} + A - a_2\bar{p})}{b_1 + a_1 y^e} \ ,
\end{aligned}$$

and

$$\begin{aligned}
R(p_1, p_2) &= \alpha(B - b_1 p_1 + b_2 p_2) \ , \\
S(p_1, p_2) &= \theta + \alpha(A + a_1 p_1 - a_2 p_2), \\
T(p_1, p_2, y) &= \frac{1}{S(p_1,p_2)} \ln \frac{y - \frac{R(p_1,p_2)}{S(p_1,p_2)}}{y^e - \frac{R(p_1,p_2)}{S(p_1,p_2)}} \ , \\
Y(y_1, y_2, p_1, p_2) &= \frac{(y_1 - \frac{R(p_1,p_2)}{S(p_1,p_2)})(y^e - \frac{R(p_1,p_2)}{S(p_1,p_2)})}{y_2 - \frac{R(p_1,p_2)}{S(p_1,p_2)}} + \frac{R(p_1,p_2)}{S(p_1,p_2)} \ .
\end{aligned}$$

We assume $\underline{p} \leq p^e \leq \bar{p}$. It is then easy to check that

$$\underline{p} \leq \hat{p}_1 \leq p^e \leq \hat{p}_2 \leq \bar{p},$$

and

$$\frac{R(\bar{p},\underline{p})}{S(\bar{p},\underline{p})} \leq \frac{R(\bar{p},\hat{p}_2)}{S(\bar{p},\hat{p}_2)} \leq \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})} \leq y^e \leq \frac{R(\underline{p},\underline{p})}{S(\underline{p},\underline{p})} \leq \frac{R(\underline{p},\hat{p}_1)}{S(\underline{p},\hat{p}_1)} \leq \frac{R(\underline{p},\bar{p})}{S(\underline{p},\bar{p})}.$$

Without loss of generality, we assume $y_1^0 \geq y_2^0$. The optimal price trajectories and associated market share trajectories are characterized in the next proposition.

Proposition 5 *The optimal pricing policies $p_i^*(t)$ and associated market share trajectories $y_i^*(t)$ $(i = 1, 2)$ depend on each firm's initial market share y_1^0 and y_2^0 as follows:*

1. *If $y_1^0 \geq y_2^0 \geq y^e$, define $t_1 = T(\bar{p}, \bar{p}, y_2^0)$ and $t_2 = T(\bar{p}, \hat{p}_2, \tilde{y})$ where $\tilde{y} = Y(y_1^0, y_2^0, \bar{p}, \bar{p})$. The optimal pricing trajectory for each firm is then:*

$$p_1^*(t) = \begin{cases} \bar{p} & \text{if } t \leq t_1 + t_2; \\ p^e & \text{if } t > t_1 + t_2. \end{cases}$$

$$p_2^*(t) = \begin{cases} \bar{p} & \text{if } t \leq t_1; \\ \hat{p}_2 & \text{if } t_1 < t \leq t_1 + t_2; \\ p^e & \text{if } t > t_1 + t_2. \end{cases}$$

The optimal market share trajectory for each firm follows:

$$y_1^*(t) = \begin{cases} (y_1^0 - \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})})e^{-S(\bar{p},\bar{p})t} + \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})} & \textit{if } t \le t_1; \\ (\tilde{y} - \frac{R(\bar{p},\hat{p}_2)}{S(\bar{p},\hat{p}_2)})e^{-S(\bar{p},\hat{p}_2)(t-t_1)} + \frac{R(\bar{p},\hat{p}_2)}{S(\bar{p},\hat{p}_2)} & \textit{if } t_1 < t \le t_1 + t_2; \\ y^e & \textit{if } t > t_1 + t_2. \end{cases}$$

$$y_2^*(t) = \begin{cases} (y_2^0 - \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})})e^{-S(\bar{p},\bar{p})t} + \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})} & \textit{if } t \le t_1; \\ y^e & \textit{if } t > t_1. \end{cases}$$

2. *If $y_1^0 \ge y^e \ge y_2^0$, define $t_1^c = T(\underline{p}, \bar{p}, y_2^0)$ and $t_2^c = T(\bar{p}, \underline{p}, y_1^0)$ as the time it takes for each firm to reach the target market share given that firm 1 prices at $\bar{p}$ and firm 2 prices at $\underline{p}$, respectively.*

   (a) *When $t_1^c \ge t_2^c$, define $\tilde{y} = Y(y_1^0, y_2^0, \bar{p}, \underline{p})$ and $t_1 = t_2^c$, $t_2 = T(\bar{p}, \hat{p}_2, \tilde{y})$. Then each firm's optimal price and market share trajectories are characterized by:*

$$p_1^*(t) = \begin{cases} \bar{p} & \textit{if } t \le t_1 + t_2; \\ p^e & \textit{if } t > t_1 + t_2. \end{cases}$$

$$p_2^*(t) = \begin{cases} \underline{p} & \textit{if } t \le t_1; \\ \hat{p}_2 & \textit{if } t_1 < t \le t_1 + t_2; \\ p^e & \textit{if } t > t_1 + t_2. \end{cases}$$

$$y_1^*(t) = \begin{cases} (y_1^0 - \frac{R(\bar{p},\underline{p})}{S(\bar{p},\underline{p})})e^{-S(\bar{p},\underline{p})t} + \frac{R(\bar{p},\underline{p})}{S(\bar{p},\underline{p})} & \textit{if } t \le t_1; \\ (\tilde{y} - \frac{R(\bar{p},\hat{p}_2)}{S(\bar{p},\hat{p}_2)})e^{-S(\bar{p},\hat{p}_2)(t-t_1)} + \frac{R(\bar{p},\hat{p}_2)}{S(\bar{p},\hat{p}_2)} & \textit{if } t_1 < t \le t_1 + t_2; \\ y^e & t > t_1 + t_2. \end{cases}$$

$$y_2^*(t) = \begin{cases} (y_2^0 - \frac{R(\underline{p},\bar{p})}{S(\underline{p},\bar{p})})e^{-S(\underline{p},\bar{p})t} + \frac{R(\underline{p},\bar{p})}{S(\underline{p},\bar{p})} & \textit{if } t \le t_1; \\ y^e & \textit{if } t > t_1. \end{cases}$$

   (b) *When $t_1^c < t_2^c$, define $\tilde{y} = Y(y_2^0, y_1^0, \underline{p}, \bar{p})$ and $t_1 = t_1^c$, $t_2 = T(\underline{p}, \hat{p}_1, \tilde{y})$. The optimal price and market share trajectories for each firm are:*

$$p_1^*(t) = \begin{cases} \bar{p} & \textit{if } t \le t_1; \\ \hat{p}_1 & \textit{if } t_1 < t \le t_1 + t_2; \\ p^e & \textit{if } t > t_1 + t_2. \end{cases}$$

$$p_2^*(t) = \begin{cases} \underline{p} & \text{if } t \le t_1 + t_2; \\ p^e & \text{if } t > t_1 + t_2. \end{cases}$$

$$y_1^*(t) = \begin{cases} (y_1^0 - \frac{R(\bar{p},\underline{p})}{S(\bar{p},\underline{p})})e^{-S(\bar{p},\underline{p})t} + \frac{R(\bar{p},\underline{p})}{S(\bar{p},\underline{p})} & \text{if } t \le t_1; \\ y^e & \text{if } t > t_1. \end{cases}$$

$$y_2^*(t) = \begin{cases} (y_2^0 - \frac{R(\underline{p},\bar{p})}{S(\underline{p},\bar{p})})e^{-S(\underline{p},\bar{p})t} + \frac{R(\underline{p},\bar{p})}{S(\underline{p},\bar{p})} & \text{if } t \le t_1; \\ (\tilde{y} - \frac{R(\underline{p},\hat{p}_1)}{S(\underline{p},\hat{p}_1)})e^{-S(\underline{p},\hat{p}_1)(t-t_1)} + \frac{R(\underline{p},\hat{p}_1)}{S(\underline{p},\hat{p}_1)} & \text{if } t_1 < t \le t_1 + t_2; \\ y^e & \text{if } t > t_1 + t_2. \end{cases}$$

3. *If $y^e \ge y_1^0 \ge y_2^0$, define $t_1 = T(\underline{p}, \underline{p}, y_1^0)$ and $t_2 = T(\underline{p}, \hat{p}_1, \tilde{y})$ where $\tilde{y} = Y(y_2^0, y_1^0, \underline{p}, \underline{p})$. The optimal price and market share trajectories for each firm are given by:*

$$p_1^*(t) = \begin{cases} \underline{p} & \text{if } t \le t_1; \\ \hat{p}_1 & \text{if } t_1 < t \le t_1 + t_2; \\ p^e & \text{if } t > t_1 + t_2. \end{cases}$$

$$p_2^*(t) = \begin{cases} \underline{p} & \text{if } t \le t_1 + t_2; \\ p^e & \text{if } t > t_1 + t_2. \end{cases}$$

$$y_1^*(t) = \begin{cases} (y_1^0 - \frac{R(\underline{p},\underline{p})}{S(\underline{p},\underline{p})})e^{-S(\underline{p},\underline{p})t} + \frac{R(\underline{p},\underline{p})}{S(\underline{p},\underline{p})} & \text{if } t \le t_1; \\ y^e & \text{if } t > t_1. \end{cases}$$

$$y_2^*(t) = \begin{cases} (y_2^0 - \frac{R(\underline{p},\underline{p})}{S(\underline{p},\underline{p})})e^{-S(\underline{p},\underline{p})t} + \frac{R(\underline{p},\underline{p})}{S(\underline{p},\underline{p})} & \text{if } t \le t_1; \\ (\tilde{y} - \frac{R(\underline{p},\hat{p}_1)}{S(\underline{p},\hat{p}_1)})e^{-S(\underline{p},\hat{p}_1)(t-t_1)} + \frac{R(\underline{p},\hat{p}_1)}{S(\underline{p},\hat{p}_1)} & \text{if } t_1 < t \le t_1 + t_2; \\ y^e & \text{if } t > t_1 + t_2. \end{cases}$$

Note that in contrast to the monopoly model in which the firm always price at either the maximum, minimum or equilibrium level, when two firms compete, one of them may price at an intermediate level before both firms reach the target market share. In particular, when both firms start with a larger market share than the target level, both price high so as to extract more revenues from their large customer base until one firm's market share is reduced to the target level (the firm with smaller initial share). At this point, the "smaller" firm switches to an intermediate price (still higher than the target price) and maintains its customer base at the target level, while the "larger" firm continues to price high until its market share reaches the target. The intuition is that the firm that first reaches the target share faces a competitor with a high price, so it can afford to price higher than the target price and still maintain its target share. Once its larger competitor reaches the target share, however, it must lower its price to the target price to continue to maintain the target share. Conversely, when both firms start with a lower market share than the target level, then both price low initially to build up their base of customers. The one with a higher market share reaches the target level first and, as a result, it prices at an intermediate value (lower than the target price) because it wants to maintain the target share yet faces a low-price competitor. The large firm then switches to the higher target price once its smaller competitor reaches the target share and raises its price.

4 Repricing Opportunity

In this section, we extend the basic model to the case in which a firm can charge an introductory price for new customers, as is often observed in practice. For example, in cable TV service, a promotional package is often offered to new customers only for the first year of service, while a regular (full) price is charged to regular customers. To model this setting, we need to use different state variables to describe introductory and regular customers. Let $x_1(t)$ and $x_2(t)$ denote, respectively, the number of introductory and regular customers the firm has at time t. The prices charged for introductory customers and regular customers at time t are denoted by $p_1(t)$ and $p_2(t)$. Again, there is an external market with a size of N and an outside market price p_0.

New customers decide to enter into service or not based on the outside market price p_0 and the prices charged for introductory and regular customers, $p_1(t)$ and $p_2(t)$, which we call the *introductory* and *regular* price, respectively. The acquisition rate of new customers is, again, comprised of their shopping frequency, denoted α_1, the external market size N, and the fraction of customers willing to purchase, denoted $B + b_0p_0 - b_1p_1(t) - b_2(p_2(t) - p_1(t))$ where $b_0, b_1, b_2 \geq 0$.

A few comments are in order. First, we assume customers behave strategically in the sense that they take into account both the introductory and regular prices when making their purchase decisions. When customers anticipate that future prices will be higher, they become more reluctant to buy service. This is reflected in the last term in the fraction of customers willing to purchase function, $b_2(p_2(t)-p_1(t))$, which represents the decrease in the willingness-to-purchase when the regular price is higher than the introductory price, where b_2 characterizes the sensitivity of customers to this price difference. Also, to avoid triviality, we assume $b_1 > b_2$, else demand is increasing in the regular price. Lastly, as in the duopoly analysis, we assume that the external market size is large enough to ignore the impact of the firm's customer base on the number of potential new customers (no saturation effects).

Introductory customers become regular customers at a rate of μ, so $1/\mu$ can be interpreted as the duration of the introductory price offer. Regular customers decide to stay with the firm or defect based on the outside price, p_0, and the regular price, $p_2(t)$. The rate at which regular customers defect from the firm is determined by their shopping frequency, denoted α_2, the number of accounts $x_2(t)$, and the fraction of customers willing to defect, denoted $A - a_0p_0 + a_1p_2(t)$.

The firm decides its introductory and regular prices at each point in time so as to maximize its total discounted revenue over time. The revenue is discounted at a rate of δ. Namely,

$$\max \int_0^\infty e^{-\delta t}(p_1(t)x_1(t) + p_2(t)x_2(t))dt \tag{18}$$

s.t.

$$\dot{x}_1(t) = N\alpha_1(B + b_0p_0 - b_1p_1(t) - b_2(p_2(t) - p_1(t))) - \mu x_1(t),$$

$$\dot{x}_2(t) = \mu x_1(t) - \alpha_2 x_2(t)(A - a_0p_0 + a_1p_2(t)),$$

$$x_1(0) = x_0^1, \quad x_2(0) = x_0^2,$$

$$\underline{p}_1 \le p_1(t) \le \bar{p}_1, \quad \underline{p}_2 \le p_2(t) \le \bar{p}_2,$$

The following proposition shows that there exists a long-run stationary equilibrium state under certain conditions.

Proposition 6 *If the long-run stationary state exists for the optimal control problem (18), then* *denote it by $(p_1^e, x_1^e, p_2^e, x_2^e)$. These equilibrium prices and account levels are determined by:*

$$
\begin{aligned}
p_1^e &= \frac{\delta+\mu}{(b_1-b_2)(\delta+2\mu)}(B+b_0p_0+\frac{b_2}{a_1}(A-a_0p_0)-\frac{b_2}{2a_1\alpha_2}(\Delta-\delta)) \\
&\quad -\frac{\mu^2(\Delta-\delta-2\alpha_2(A-a_0p_0))}{a_1\alpha_2(\delta+2\mu)(\Delta+\delta)}, && (19) \\
p_2^e &= \frac{\Delta-\delta}{2a_1\alpha_2}-\frac{A-a_0p_0}{a_1}, && (20) \\
x_1^e &= \frac{N\alpha_1}{\mu}(B+b_0p_0-b_1p_1^e-b_2(p_2^e-p_1^e)), && (21) \\
x_2^e &= \frac{N\alpha_1(B+b_0p_0-b_1p_1^e-b_2(p_2^e-p_1^e)}{\alpha_2(A-a_0p_0+a_1p_2^e)}, && (22)
\end{aligned}
$$

where $\Delta = \sqrt{\delta^2+4\mu(\frac{b_1}{b_2}-1)(\delta+\alpha_2(A-a_0p_0))}$.

One can easily show that the regular price p_2^e increases in μ and b_1, while it decreases in b_2. The introductory price p_1^e decreases in μ and b_1, while it increases in b_2. Note that there exists a $\hat{\mu}$ such that $p_1^e = p_2^e$. Since p_2^e strictly increases in μ and p_1^e strictly decreases in μ, for any $\mu > \hat{\mu}$, it must be that $p_1^e < p_2^e$. Since $1/\mu$ is the duration of the introductory price, this implies that, ceteris paribus, reducing the duration of the introductory price period leads to a lower introductory price and a higher regular price. The reason is that with a shorter introductory period, the firm loses less revenue when offering low introductory prices. Hence, the cost of acquiring customers with a low introductory price declines, which leads to both more aggressive (lower) introductory prices, but also to higher regular price, since defecting customer can be replaced at lower cost.

When $b_2 = 0$, the acquisition rate at which new customers purchase service depends only on the firm's introductory price p_1 and the outside market price p_0. In this case, customers' purchase decisions are affected by the introductory price only. We call such customers myopic. The next proposition characterizes the long-run stationary introductory and regular prices when customers are myopic.

Proposition 7 *When $b_2 = 0$, the long-run stationary equilibrium introductory and regular prices, denoted $\hat{p}_1^e$ and $\hat{p}_2^e$, are determined by:*

$$
\begin{aligned}
\hat{p}_e^1 &= \frac{(\delta+\mu)(\frac{B+b_0p_0}{b_1})-\frac{\mu^2\bar{p}_2}{\delta+\alpha_2(A-a_0p_0+a_1\bar{p}_2)}}{\delta+2\mu}, \\
\hat{p}_2^e &= \bar{p}_2.
\end{aligned}
$$

Furthermore, compared with the case of strategic customers (i.e., $b_2 > 0$), when customers are myopic (i.e., $b_2 = 0$), the optimal equilibrium introductory price is lower while the optimal equilibrium regular price is higher.

Proposition 7 implies that the firm charges a higher introductory price for new customers and a lower regular price for long term customers when they are strategic as opposed to myopic. This is intuitive, since myopic customers are easy to acquire with low introductory prices. In then makes sense to use a "bait-and-switch" policy of offering very low introductory prices followed by high regular prices. However, such a strategy is less likely to attract customers who are forward looking, so the introductory and regular price are more similar.

5 Conclusion

Our optimal control model with linear demand functions enables us to investigate the important tradeoff between extracting revenue from existing customers with higher prices versus growing market share with lower prices in account-based services. We show that a simple target market share policy is optimal; the firm should price in order to reach the target market share as fast as possible. In other words, pricing at the upper bound when the market share is above the target, pricing at the lower bound if the market share is below the target, and pricing at the equilibrium target price when the market share is at the target. These results can also be extended to a duopoly market for a large potential market size and to a differentiated market case in which customers strategically respond to introductory and regular prices.

The model proposed here is quite stylized and could be extended in a number of ways. For one, we assume linear demand functions, which leads to the simple "bang-bang" result of the target market share policy. A nonlinear demand function would likely lead to smoother behavior. It would be worth analyzing the optimal pricing policy for more general classes of demand functions to see if a similar target-market-share policy is optimal. We also assume there is no time variability or uncertainty in demand, and these assumptions too would be worth relaxing. Lastly, it would be worthwhile investigating implementation issues in real-world settings.

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Appendix A: Proofs

Proof of Proposition 1

First, using (7), (8), (9) and (10), it is easy to show that $0 < y_e < 1$. Therefore, $\mu_e^1 = \mu_e^2 = 0$. The optimal long-run stationary state is then determined by the following system of equations:

$$\alpha(B + b_0 p_0 - b_1 p_e) - y_e(\theta + \alpha(A + B) + \alpha(b_0 - a_0)p_0 - \alpha(b_1 - a_1)p_e) = 0,$$
$$\lambda_e(\delta + \alpha(A + B) + \alpha(b_0 - a_0)p_0 - \alpha(b_1 - a_1)p_e) - p_e = 0,$$
$$y_e - \alpha b_1 \lambda_e + \alpha(b_1 - a_1)\lambda_e y_e = 0.$$

Solving the above system of equations when $a_1 \neq b_1$, note that $0 < y_e < 1$, we have

$$\begin{aligned} p_e &= \frac{\delta + \theta + 2\alpha(A + B) + 2\alpha(b_0 - a_0)p_0 - \sqrt{\Delta}}{2\alpha(b_1 - a_1)}, \\ y_e &= \frac{\delta + \theta + 2\alpha(A + B) + 2\alpha(b_0 - a_0)p_0 - \sqrt{\Delta}}{2(1 - \frac{a_1}{b_1})(\delta + \alpha(A + B) + \alpha(b_0 - a_0)p_0)}, \\ \lambda_e &= \frac{\delta + \theta + 2\alpha(A + B) + 2\alpha(b_0 - a_0)p_0 - \sqrt{\Delta}}{\alpha(b_1 - a_1)(\delta - \theta + \sqrt{\Delta})}, \end{aligned}$$

where

$$\Delta = (\delta + \theta + 2\alpha(A + B) + 2\alpha(b_0 - a_0)p_0)^2 - 4\alpha(1 - \frac{a_1}{b_1})(B + b_0 p_0)(\delta + \alpha(A + B) + \alpha(b_0 - a_0)p_0).$$

Recall that we assume $A - a_0 p_0 \geq 0$, so it must be that $\Delta > 0$.

When $a_1 = b_1$, solving the above system of equations yields:

$$\begin{aligned} y_e &= \frac{\alpha(B + b_0 p_0)}{\delta + \theta + 2\alpha(A+B) + 2\alpha(b_0 - a_0)p_0}, \\ p_e &= \frac{(B + b_0 p_0)(\delta + \alpha(A+B) + \alpha(b_0 - a_0)p_0)}{b_1(\delta + \theta + 2\alpha(A+B) + 2\alpha(b_0 - a_0)p_0)}, \\ \lambda_e &= \frac{B + b_0 p_0}{b_1(\delta + \theta + 2\alpha(A+B) + 2\alpha(b_0 - a_0)p_0)}. \end{aligned}$$

■

Proof of Proposition 2

We first ignore the state constraint $0 \leq y(t) \leq 1$ in the optimization problem (5), then later verify the resulting state trajectory indeed satisfies this constraint. The current-value Hamiltonian function is given by

$$H(y, p_1, \lambda) = p_1 y + \lambda\alpha(B + b_0 p_0 - b_1 p_1) - \lambda y(\theta + \alpha(A+B) + \alpha(b_0 - a_0)p_0 - \alpha(b_1 - a_1)p_1),$$

and the current-value adjoint equation is

$$\dot{\lambda} = (\delta - \theta)\lambda - \frac{\partial H}{\partial y} = \lambda(\delta + \alpha(A+B) + \alpha(b_0 - a_0)p_0 - \alpha(b_1 - a_1)p_1) - p_1.$$

The optimal control is obtained by maximizing the Hamiltonian function with respect to p_1, that is

$$\begin{aligned} p_1^*(t) = \arg\max_{p_1(t)} \Big\{ \quad & (y^*(t) - \alpha b_1 \lambda(t) + \alpha(b_1 - a_1)\lambda(t) y^*(t)) p_1(t) \\ & + \alpha(B + b_0 p_0)\lambda(t) - (\theta + \alpha(A+B) + \alpha(b_0 - a_0)p_0)\lambda(t) y^*(t) \Big\}. \end{aligned}$$

Hence, the optimal policy is a bang-bang control; namely

$$p_1^*(t) = \begin{cases} \underline{p} & \text{if } y^*(t) - \alpha b_1 \lambda(t) + \alpha(b_1 - a_1)\lambda(t) y^*(t) < 0; \\ \bar{p} & \text{if } y^*(t) - \alpha b_1 \lambda(t) + \alpha(b_1 - a_1)\lambda(t) y^*(t) > 0; \\ \underline{p} \leq p \leq \bar{p} & \text{if } y^*(t) - \alpha b_1 \lambda(t) + \alpha(b_1 - a_1)\lambda(t) y^*(t) = 0. \end{cases} \tag{23}$$

Pontryagin's maximum principle requires that the control trajectory $p_1^*(t)$ and the state trajectory $y^*(t)$, characterized in Proposition 2 indeed satisfy (23). In the case of $y_0 > y_e$, the trajectory of the adjoint variable $\lambda(t)$ is determined by:

$$\lambda(t) = \begin{cases} C_1 e^{(\delta-\theta+S(\bar{p}))t} + \frac{\bar{p}}{\delta-\theta+S(\bar{p})} & \text{if } t < t_c^1; \\ \lambda_e & \text{otherwise.} \end{cases}$$

where

$$C_1 = \left(\lambda_e - \frac{\bar{p}}{\delta-\theta+S(\bar{p})}\right)\left(\frac{y_e - \frac{R(\bar{p})}{S(\bar{p})}}{y_0 - \frac{R(\bar{p})}{S(\bar{p})}}\right)^{\frac{\delta-\theta+S(\bar{p})}{S(\bar{p})}}.$$

Since $\frac{p}{\delta-\theta+S(p)}$ is an increasing function in p, we have $\frac{\underline{p}}{\delta-\theta+S(\underline{p})} \le \lambda_e = \frac{p_e}{\delta-\theta+S(p_e)} \le \frac{\bar{p}}{\delta-\theta+S(\bar{p})}$. Therefore, $C_1 \le 0$ and thus $\lambda(t)$ decreases in t. Note also $\frac{R(\bar{p})}{S(\bar{p})} \le y_e = \frac{R(p_e)}{S(p_e)} \le \frac{R(\underline{p})}{S(\underline{p})}$ because $\frac{R(p)}{S(p)}$ decreases in p.

Denote $G(t) = y^*(t) - \alpha b_1\lambda(t) + \alpha(b_1 - a_1)\lambda(t)y^*(t)$, then

$$\begin{aligned} G(t) &= \left(y_0 - \frac{R(\bar{p})}{S(\bar{p})}\right)\left(1 + \frac{\alpha(b_1-a_1)\bar{p}}{\delta-\theta+S(\bar{p})}\right) e^{-S(\bar{p})t} - \left(b_1 - (b_1-a_1)\frac{R(\bar{p})}{S(\bar{p})}\right)\alpha C_1 e^{(\delta-\theta+S(\bar{p}))t} \\ &\quad + \alpha(b_1-a_1)C_1\left(y_0 - \frac{R(\bar{p})}{S(\bar{p})}\right)e^{(\delta-\theta)t} + \frac{R(\bar{p})}{S(\bar{p})}\left(1 + \frac{\alpha(b_1-a_1)\bar{p}}{\delta-\theta+S(\bar{p})}\right) - \frac{\alpha b_1\bar{p}}{\delta-\theta+S(\bar{p})}. \end{aligned}$$

It is easy to show that $G(t)$ is strictly convex (i.e., $G''(t) > 0$); $G(t_h) = 0$. We now show that $G'(t)|_{t=t_h} \le 0$.

$$\begin{aligned} G'(t_h) &= (R(\bar{p}) - S(\bar{p})y_e)\left(1 + \frac{\alpha(b_1-a_1)\bar{p}}{\delta-\theta+S(\bar{p})}\right) \\ &\quad + \alpha\left(\frac{\bar{p}}{\delta-\theta+S(\bar{p})} - \lambda_e\right)\Big(b_1(\delta-\theta+S(\bar{p})) - (b_1-a_1)(R(\bar{p}) + (\delta-\theta)y_e)\Big) \\ &= -y_e(\theta + \alpha(A+B) + \alpha(b_0-a_0)p_0) + \alpha\lambda_e y_e(b_1-a_1)(\delta-\theta) \\ &\quad - \alpha\lambda_e(\delta b_1 + \alpha b_1(A - a_0p_0) + \alpha a_1(B + b_0p_0)) \\ &= -y_e(\delta + \alpha(A+B) + \alpha(b_0-a_0)p_0) - \alpha\lambda_e(\theta b_1 + \alpha b_1(A - a_0p_0) + \alpha a_1(B + b_0p_0)) \end{aligned}$$

The last equation follows from $\alpha(b_1-a_1)\lambda_e y_e = -y_e + \alpha b_1\lambda_e$. Therefore, $G'(t)|_{t=t_h} \le 0$; together with $G''(t) > 0$ and $G(t_h) = 0$, we conclude that $G(t) > 0$ when $t < t_h$.

We have verified that $y^*(t) - \alpha b_1\lambda(t) + \alpha(b_1-a_1)\lambda(t)y^*(t) > 0$ when $y_0 > y_e$ and $t < t_h$. Along the same line of arguments, we can show that $y^*(t) - \alpha b_1\lambda(t) + \alpha(b_1-a_1)\lambda(t)y^*(t) < 0$ when $y_0 < y_e$ and $t < t_l$. Last, we can check that the state constraint $0 \le y^*(t) \le 1$ is indeed valid. This completes the proof. ∎

Proof of Proposition 3

According to the proof of Proposition 1, the optimal long-run stationary state satisfies:

$$\alpha(B + b_0p_0 - b_1p_e) - y_e(\theta + \alpha(A + B) + \alpha(b_0 - a_0)p_0 - \alpha(b_1 - a_1)p_e) = 0,$$
$$\lambda_e(\delta + \alpha(A + B) + \alpha(b_0 - a_0)p_0 - \alpha(b_1 - a_1)p_e) - p_e = 0,$$
$$y_e - \alpha b_1\lambda_e + \alpha(b_1 - a_1)\lambda_e y_e = 0.$$

Rearranging the above system of equations, one can show that y_e and p_e are separated as follows:

$$(1-\frac{a_1}{b_1})(\delta+\alpha(A+B)+\alpha(b_0-a_0)p_0)y_e^2-(\delta+\theta+2\alpha(A+B)+2\alpha(b_0-a_0)p_0)y_e+\alpha(B+b_0p_0) = 0, \quad (24)$$

$$\alpha(b_1-a_1)p_e^2-(\delta+\theta+2\alpha(A+B)+2\alpha(b_0-a_0)p_0)p_e+\frac{1}{b_1}(B+b_0p_0)(\delta+\alpha(A+B)+\alpha(b_0-a_0)p_0) = 0. \quad (25)$$

Applying the Implicit Function Theorem to (24) and (25), the results follow from straightforward algebraic calculations. We omit the details for conciseness. ■

Proof of Proposition 4

For the optimal control problems defined in (16) and (17), the current-value Hamiltonian functions for each firm are given by:

$$\begin{aligned}
H_1(p_1,p_2,y_1,y_2,\lambda_1,\lambda_2) = \quad & p_1y_1 + \lambda_1\Big(\alpha(B - b_1p_1(t) + b_2p_2(t)) - y_1(t)(\theta + \alpha(A + a_1p_1(t) - a_2p_2(t))\Big) \\
& +\lambda_2\Big(\alpha(B + b_2p_1(t) - b_1p_2(t)) - y_2(t)(\theta + \alpha(A - a_2p_1(t) + a_1p_2(t))\Big),
\end{aligned}$$

$$\begin{aligned}
H_2(p_1,p_2,y_1,y_2,\gamma_1,\gamma_2) = \quad & p_2y_2 + \gamma_1\Big(\alpha(B - b_1p_1(t) + b_2p_2(t)) - y_1(t)(\theta + \alpha(A + a_1p_1(t) - a_2p_2(t))\Big) \\
& +\gamma_2\Big(\alpha(B + b_2p_1(t) - b_1p_2(t)) - y_2(t)(\theta + \alpha(A - a_2p_1(t) + a_1p_2(t))\Big).
\end{aligned}$$

The stationary Nash equilibrium, denoted $(p_1^e, p_2^e, y_1^e, y_2^e, \lambda_1^e, \lambda_2^e, \gamma_1^e, \gamma_2^e)$, is defined by:

$$\alpha(B - b_1p_1^e + b_2p_2^e) - y_1^e(\theta + \alpha(A + a_1p_1^e - a_2p_2^e) = 0,$$

$$\alpha(B + b_2p_1^e - b_1p_2^e) - y_2^e(\theta + \alpha(A - a_2p_1^e + a_1p_2^e) = 0,$$

$$(\delta - \theta)\lambda_1^e = \frac{dH_1(p_1^e, p_2^e, y_1^e, y_2^e, \lambda_1^e, \lambda_2^e)}{dy_1}, \quad (\delta - \theta)\lambda_2^e = \frac{dH_1(p_1^e, p_2^e, y_1^e, y_2^e, \lambda_1^e, \lambda_2^e)}{dy_2},$$

$$(\delta - \theta)\gamma_1^e = \frac{dH_2(p_1^e, p_2, y_1^e, y_2^e, \gamma_1^e, \gamma_2^e)}{dy_1}, \quad (\delta - \theta)\gamma_2^e = \frac{dH_2(p_1^e, p_2, y_1^e, y_2^e, \gamma_1^e, \gamma_2^e)}{dy_2},$$

$$H_1(p_1^e, p_2^e, y_1^e, y_2^e, \lambda_1^e, \lambda_2^e) \geq H_1(p_1, p_2^e, y_1^e, y_2^e, \lambda_1^e, \lambda_2^e), \quad \forall \underline{p} \leq p_1 \leq \bar{p},$$

$$H_2(p_1^e, p_2^e, y_1^e, y_2^e, \gamma_1^e, \gamma_2^e) \geq H_2(p_1^e, p_2, y_1^e, y_2^e, \gamma_1^e, \gamma_2^e), \quad \forall \underline{p} \leq p_2 \leq \bar{p}.$$

After some algebraic rearrangement of the above system of equations and inequalities, one can show that the stationary equilibrium prices for both firms are the same, that is, $p_1^e = p_2^e$. Again, we call this stationary equilibrium price the *target price*, denoted p^e. It is the solution to:

$$\alpha(2a_2b_1 - a_1b_1 - a_2b_2)p^2 - (b_1(\delta + \theta + 2\alpha A) - b_2(\delta + \alpha A) + a_2\alpha B)p + B(\delta + \alpha A) = 0. \tag{26}$$

When $\frac{a_1}{a_2} + \frac{b_2}{b_1} \geq 2$, there exists only one positive solution to (26). Accordingly, the target market share for each firm is determined by:

$$y_1^e = y_2^e := y^e = \frac{\alpha(B - b_1p^e + b_2p^e)}{\theta + \alpha(A + a_1p^e - a_2p^e)}.$$

■

Proof of Proposition 5

For the optimal control problems defined in (16) and (17), the current-value Hamiltonian functions for each firm are given by:

$$\begin{aligned} H_1(p_1, p_2, y_1, y_2, \lambda_1, \lambda_2) = \quad & p_1y_1 + \lambda_1\Big(\alpha(B - b_1p_1(t) + b_2p_2(t)) - y_1(t)(\theta + \alpha(A + a_1p_1(t) - a_2p_2(t))\Big) \\ & + \lambda_2\Big(\alpha(B + b_2p_1(t) - b_1p_2(t)) - y_2(t)(\theta + \alpha(A - a_2p_1(t) + a_1p_2(t))\Big), \end{aligned}$$

$$\begin{aligned} H_2(p_1, p_2, y_1, y_2, \gamma_1, \gamma_2) = \quad & p_2y_2 + \gamma_1\Big(\alpha(B - b_1p_1(t) + b_2p_2(t)) - y_1(t)(\theta + \alpha(A + a_1p_1(t) - a_2p_2(t))\Big) \\ & + \gamma_2\Big(\alpha(B + b_2p_1(t) - b_1p_2(t)) - y_2(t)(\theta + \alpha(A - a_2p_1(t) + a_1p_2(t))\Big). \end{aligned}$$

We denote the adjoint variables associated with the optimization problems (16) and (17) by λ_1, λ_2,γ_1 and γ_2. The adjoint equations are

$$\begin{aligned}
\dot{\lambda}_1 &= \lambda_1(\delta + \alpha(A + a_1 p_1 - a_2 p_2)) - p_1, \\
\dot{\lambda}_2 &= \lambda_2(\delta + \alpha(A - a_2 p_1 + a_1 p_2)), \\
\dot{\gamma}_1 &= \gamma_1(\delta + \alpha(A + a_1 p_1 - a_2 p_2)), \\
\dot{\gamma}_2 &= \gamma_2(\delta + \alpha(A - a_2 p_1 + a_1 p_2)) - p_2.
\end{aligned}$$

We claim that $\lambda_2(t) = 0$ and $\gamma_1(t) = 0$, $\forall t \geq 0$. This is due to the fact that the adjoint function is continuous and piecewise continuously differentiable, and the fact that $\lambda_2 = 0$ and $\gamma_1 = 0$ at the stationary equilibrium. The optimal control policies p_1^* and p_2^* are then determined by:

$$p_1^* = \arg \max_{\underline{p} \leq p_1 \leq \bar{p}} \left\{ (y_1^* - \alpha b_1 \lambda_1 - \alpha a_1 \lambda_1 y_1^*) p_1 + (\alpha b_2 \lambda_1 + \alpha a_2 \lambda_1 y_1^*) p_2 + \alpha B \lambda_1 - \lambda_1(\theta + \alpha A) y_1^* \right\},$$

$$p_2^* = \arg \max_{\underline{p} \leq p_2 \leq \bar{p}} \left\{ (y_2^* - \alpha b_1 \gamma_2 - \alpha a_1 \gamma_2 y_2^*) p_2 + (\alpha b_2 \gamma_2 + \alpha a_2 \gamma_2 y_2^*) p_1 + \alpha B \gamma_2 - \gamma_2(\theta + \alpha A) y_2^* \right\}.$$

Therefore, the optimal prices for each firm are:

$$p_1^*(t) = \begin{cases} \underline{p} & \text{if } y_1^*(t) - \alpha b_1 \lambda_1(t) - \alpha a_1 \lambda_1(t) y_1^*(t) < 0; \\ \bar{p} & \text{if } y_1^*(t) - \alpha b_1 \lambda_1(t) - \alpha a_1 \lambda_1(t) y_1^*(t) > 0; \\ \underline{p} \leq p \leq \bar{p} & \text{if } y_1^*(t) - \alpha b_1 \lambda_1(t) - \alpha a_1 \lambda_1(t) y_1^*(t) = 0. \end{cases} \tag{27}$$

$$p_2^*(t) = \begin{cases} \underline{p} & \text{if } y_2^*(t) - \alpha b_1 \gamma_2(t) - \alpha a_1 \gamma_2(t) y_2^*(t) < 0; \\ \bar{p} & \text{if } y_2^*(t) - \alpha b_1 \gamma_2(t) - \alpha a_1 \gamma_2(t) y_2^*(t) > 0; \\ \underline{p} \leq p \leq \bar{p} & \text{if } y_2^*(t) - \alpha b_1 \gamma_2(t) - \alpha a_1 \gamma_2(t) y_2^*(t) = 0. \end{cases} \tag{28}$$

Using Pontryagin's maximum principle, we need to verify that the control trajectories $p_1^*(t)$ and $p_2^*(t)$, and the state trajectories $y_1^*(t)$ and $y_2^*(t)$, characterized in Proposition 5, indeed satisfy (27) and (28). Here we give the proof for the case of $y_1^0 \geq y_2^0 \geq y^e$ only. The other cases follow the same line of argument.

We now show that $y_2^*(t) - \alpha b_1 \gamma_2(t) - \alpha a_1 \gamma_2(t) y_2^*(t) > 0$ when $t < t_1$ and $y_2^*(t) - \alpha b_1 \gamma_2(t) - \alpha a_1 \gamma_2(t) y_2^*(t) = 0$ when $t = t_1$.

Note that $\gamma_2^e = \frac{p^e}{\delta-\theta+S(p^e,p^e)}$ and $\gamma_2(t_1) = \gamma_2^e$, we then have

$$\gamma_2(t) = \begin{cases} \left(\frac{p^e}{\delta-\theta+S(p^e,p^e)} - \frac{\bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right)\left(\frac{y^e - \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}}{y_2^0 - \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}}\right)^{\frac{\delta-\theta+S(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}} e^{(\delta-\theta+S(\bar{p},\bar{p}))t} + \frac{\bar{p}}{\delta-\theta+S(\bar{p},\bar{p})} & \text{if } t < t_1; \\ \frac{p^e}{\delta-\theta+S(p^e,p^e)} & \text{if } t \geq t_1. \end{cases}$$

When $t < t_1$,

$$\begin{aligned} G_2(t) &= y_2^*(t) - \alpha b_1 \gamma_2(t) - \alpha a_1 \gamma_2(t) y_2^*(t) \qquad (29) \\ &= \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}\left(1 - \frac{\alpha a_1 \bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right) - \frac{\alpha b_1 \bar{p}}{\delta-\theta+S(\bar{p},\bar{p})} + \left(y_2^0 - \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}\right)\left(1 - \frac{\alpha a_1 \bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right)e^{-S(\bar{p},\bar{p})t} \\ &\quad -\alpha a_1 \left(y_2^0 - \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}\right)\left(\frac{p^e}{\delta-\theta+S(p^e,p^e)} - \frac{\bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right)\left(\frac{y^e - \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}}{y_2^0 - \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}}\right)^{\frac{\delta-\theta+S(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}} e^{(\delta-\theta)t} \\ &\quad -\alpha\left(b_1 + a_1 \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}\right)\left(\frac{p^e}{\delta-\theta+S(p^e,p^e)} - \frac{\bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right)\left(\frac{y^e - \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}}{y_2^0 - \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}}\right)^{\frac{\delta-\theta+S(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}} e^{(\delta-\theta+S(\bar{p},\bar{p}))t}. \end{aligned}$$

Because $\frac{p^e}{\delta-\theta+S(p^e,p^e)} < \frac{\bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}$ and $y_2^0 \geq y^e$, we can easily show that $G_2(t)$ is strictly convex by $G_2''(t)$. Also at $t = t_1$,

$$\begin{aligned} G_2(t_1) &= y^e\left(1 - \frac{\alpha a_1 \bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right) - \frac{\alpha b_1 \bar{p}}{\delta-\theta+S(\bar{p},\bar{p})} + \alpha\left(\frac{p^e}{\delta-\theta+S(p^e,p^e)} - \frac{\bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right)(-a_1 y^e - b_1) \\ &= y^e - \frac{\alpha a_1 p^e y^e}{\delta-\theta+S(p^e,p^e)} - \frac{\alpha b_1 p^e}{\delta-\theta+S(p^e,p^e)} = 0. \end{aligned}$$

$$\begin{aligned} G_2'(t_1) &= \left(1 - \frac{\alpha a_1 \bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right)(R(\bar{p},\bar{p}) - S(\bar{p},\bar{p})y^e) \\ &\quad -\alpha\left(\frac{p^e}{\delta-\theta+S(p^e,p^e)} - \frac{\bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right)(a_1(R(\bar{p},\bar{p}) + (\delta-\theta)y^e) + b_1(\delta-\theta+S(\bar{p},\bar{p}))) \\ &= -y^e(\delta + \alpha(A - a_2\bar{p}) + \alpha(B + b_2\bar{p})) - \alpha\gamma_2^e(b_1 S(\bar{p},\bar{p}) + a_1 R(\bar{p},\bar{p})) \end{aligned}$$

The last equation uses the result of $\gamma_2^e = \frac{p^e}{\delta-\theta+S(p^e,p^e)}$ and $y^e - \alpha b_1 \gamma_2^e - \alpha a_1 \gamma_2^e y^e = 0$. Therefore, $G_2'(t_1) < 0$. It then follows $G_2(t) > 0$ when $t < t_1$.

We next show that $y_1^*(t) - \alpha b_1 \lambda_1(t) - \alpha a_1 \lambda_1(t) y_1^*(t) > 0$ when $t < t_1 + t_2$ and $y_1^*(t) - \alpha b_1 \lambda_1(t) - \alpha a_1 \lambda_1(t) y_1^*(t) = 0$ when $t = t_1 + t_2$. Note that $\lambda_1^e = \frac{p^e}{\delta-\theta+S(p^e,p^e)}$ and $\lambda_1(t_1+t_2) = \lambda_1^e$. The adjoint function $\lambda_1(t)$ is given by:

$$\lambda_1(t) = \begin{cases} \left(\lambda_1^c - \frac{\bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right) e^{(\delta-\theta+S(\bar{p},\bar{p}))(t-t_1)} + \frac{\bar{p}}{\delta-\theta+S(\bar{p},\bar{p})} & \text{if } t \leq t_1; \\ \left(\lambda_1^e - \frac{\bar{p}}{\delta-\theta+S(\bar{p},\hat{p}_2)}\right) e^{(\delta-\theta+S(\bar{p},\hat{p}_2))(t-t_1-t_2)} + \frac{\bar{p}}{\delta-\theta+S(\bar{p},\hat{p}_2)} & \text{if } t_1 < t \leq t_1+t_2; \\ \lambda_1^e & \text{if } t > t_1+t_2. \end{cases}$$

where $\lambda_1^c = \frac{\bar{p}}{\delta-\theta+S(\bar{p},\hat{p}_2)} + \left(\lambda_1^e - \frac{\bar{p}}{\delta-\theta+S(\bar{p},\hat{p}_2)}\right) e^{-(\delta-\theta+S(\bar{p},\hat{p}_2))t_2}$.

When $t \leq t_1$,

$$\begin{aligned} G_1(t) &= y_1^*(t) - \alpha b_1 \lambda_1(t) - \alpha a_1 \lambda_1(t) y_1^*(t) \\ &= \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})} - \frac{\alpha b_1 \bar{p}}{\delta-\theta+S(\bar{p},\bar{p})} - \frac{\alpha a_1 \bar{p} R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})(\delta-\theta+S(\bar{p},\bar{p}))} + \left(y_1^0 - \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}\right)\left(1 - \frac{\alpha a_1 \bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right) e^{-S(\bar{p},\bar{p})t} \\ &\quad -\alpha\left(\lambda_1^c - \frac{\bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right)\left(b_1 + a_1 \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}\right) e^{(\delta-\theta+S(\bar{p},\bar{p}))(t-t_1)} \\ &\quad -\alpha a_1 \left(\lambda_1^c - \frac{\bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}\right)\left(y_1^0 - \frac{R(\bar{p},\bar{p})}{S(\bar{p},\bar{p})}\right) e^{-(\delta-\theta+S(\bar{p},\bar{p}))t_1 + (\delta-\theta)t} . \end{aligned}$$

Since $\lambda_1^c < \frac{\bar{p}}{\delta-\theta+S(\bar{p},\bar{p})}$ and $y_1^0 > y^e$, thus $G_1(t)$ is strictly convex. We can also check that $G(t_1) > 0$ and $G'(t_1) < 0$. Therefore, $G_1(t) > 0$ when $t \leq t_1$.

When $t_1 < t \leq t_1 + t_2$,

$$\begin{aligned} G_1(t) &= y_1^*(t) - \alpha b_1 \lambda_1(t) - \alpha a_1 \lambda_1(t) y_1^*(t) \\ &= \frac{R(\bar{p},\hat{p}_2)}{S(\bar{p},\hat{p}_2)} - \frac{\alpha b_1 \bar{p}}{\delta-\theta+S(\bar{p},\hat{p}_2)} - \frac{\alpha a_1 \bar{p} R(\bar{p},\hat{p}_2)}{S(\bar{p},\hat{p}_2)(\delta-\theta+S(\bar{p},\hat{p}_2))} \\ &\quad +\left(\tilde{y} - \frac{R(\bar{p},\hat{p}_2)}{S(\bar{p},\hat{p}_2)}\right)\left(1 - \frac{\alpha a_1 \bar{p}}{\delta-\theta+S(\bar{p},\hat{p}_2)}\right) e^{-S(\bar{p},\hat{p}_2)(t-t_1)} \\ &\quad -\alpha\left(\lambda_1^e - \frac{\bar{p}}{\delta-\theta+S(\bar{p},\hat{p}_2)}\right)\left(b_1 + a_1 \frac{R(\bar{p},\hat{p}_2)}{S(\bar{p},\hat{p}_2)}\right) e^{(\delta-\theta+S(\bar{p},\hat{p}_2))(t-t_1-t_2)} \\ &\quad -\alpha a_1 \left(\lambda_1^e - \frac{\bar{p}}{\delta-\theta+S(\bar{p},\hat{p}_2)}\right)\left(\tilde{y} - \frac{R(\bar{p},\hat{p}_2)}{S(\bar{p},\hat{p}_2)}\right) e^{-(\delta-\theta+S(\bar{p},\hat{p}_2))t_2 + (\delta-\theta)(t-t_1)} . \end{aligned}$$

Again, we can show that $G_1''(t) > 0$, $G_1(t_2) = 0$ and $G_1'(t_2) < 0$. Hence, $G_1(t) > 0$ when $t_1 < t < t_1 + t_2$. ∎

Proof of Proposition 6

The current-value Hamiltonian function is given by:

$$
\begin{aligned}
H(x_1, x_2, p_1, p_2, \lambda_1, \lambda_2) = \quad & p_1x_1 + p_2x_2 + \lambda_1 N\alpha_1(B + b_0p_0 - b_1p_1 - b_2(p_2 - p_1)) - \lambda_1\mu x_1 \\
& + \lambda_2\mu x_1 - \lambda_2\alpha_2 x_2(A - a_0p_0 + a_1p_2).
\end{aligned}
$$

The optimal control policies are obtained by maximizing the Hamiltonian function with respect to p_1 and p_2, respectively; that is,

$$
\begin{aligned}
p_1^*(t) = \arg\max_{p_1(t)} \quad & \Big\{(x_1(t) - N\alpha_1b_1\lambda_1(t) + N\alpha_1b_2\lambda_1(t))p_1(t) + (x_2(t) - N\alpha_1b_2\lambda_1(t) - \alpha_2a_1x_2(t)\lambda_2(t))p_2(t) \\
& + N\alpha_1\lambda_1(t)(B + b_0p_0) - \mu\lambda_1(t)x_1(t) + \mu\lambda_2(t)x_1(t) - \alpha_2x_2(t)\lambda(t)(A - a_0p_0)\Big\}. \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
p_2^*(t) = \arg\max_{p_2(t)} \quad & \Big\{(x_2(t) - N\alpha_1b_2\lambda_1(t) - \alpha_2a_1x_2(t)\lambda_2(t))p_2(t) + (x_1(t) - N\alpha_1b_1\lambda_1(t) + N\alpha_1b_2\lambda_1(t))p_1(t) \\
& + N\alpha_1\lambda_1(t)(B + b_0p_0) - \mu\lambda_1(t)x_1(t) + \mu\lambda_2(t)x_1(t) - \alpha_2x_2(t)\lambda(t)(A - a_0p_0)\Big\}. \qquad (31)
\end{aligned}
$$

The optimal control policy is then a bang-bang control; namely,

$$
p_1^*(t) = \begin{cases} \bar{p}_1 & \text{if } x_1^*(t) - N\alpha_1b_1\lambda_1(t) + N\alpha_1b_2\lambda_1(t) > 0; \\ \underline{p}_1 & \text{if } x_1^*(t) - N\alpha_1b_1\lambda_1(t) + N\alpha_1b_2\lambda_1(t) < 0; \\ \underline{p}_1 \le p_1 \le \bar{p}_1 & \text{if } x_1^*(t) - N\alpha_1b_1\lambda_1(t) + N\alpha_1b_2\lambda_1(t) = 0. \end{cases} \qquad (32)
$$

$$
p_2^*(t) = \begin{cases} \bar{p}_2 & \text{if } x_2^*(t) - N\alpha_1b_2\lambda_1(t) - \alpha_2a_1x_2^*(t)\lambda_2(t) > 0; \\ \underline{p}_2 & \text{if } x_2^*(t) - N\alpha_1b_2\lambda_1(t) - \alpha_2a_1x_2^*(t)\lambda_2(t) < 0; \\ \underline{p}_2 \le p_2 \le \bar{p}_2 & \text{if } x_2^*(t) - N\alpha_1b_2\lambda_1(t) - \alpha_2a_1x_2^*(t)\lambda_2(t) = 0. \end{cases} \qquad (33)
$$

The adjoint equations are:

$$
\begin{aligned}
\dot{\lambda}_1(t) &= (\delta + \mu)\lambda_1 - \lambda_2\mu - p_1, \\
\dot{\lambda}_2(t) &= \delta\lambda_2 + \alpha_2(1 - a_0p_0 + a_1p_2)\lambda_2 - p_2.
\end{aligned}
$$

The long-run stationary equilibrium, denoted by $(p_1^e, x_1^e, p_2^e, x_2^e)$, if it exists, satisfies:

$$\begin{cases} N\alpha_1(B + b_0p_0 - b_1p_1^e - b_2(p_2^e - p_1^e)) - \mu x_1^e = 0, \\ \mu x_1^e - \alpha_2(A - a_0p_0 + a_1p_2^e)x_2^e = 0, \\ \delta\lambda_1^e = H_{x_1}(x_1^e, x_2^e, p_1^e, p_2^e, \lambda_1^e, \lambda_2^e), \\ \delta\lambda_2^e = H_{x_2}(x_1^e, x_2^e, p_1^e, p_2^e, \lambda_1^e, \lambda_2^e), \\ H(x_1^e, x_2^e, p_1^e, p_2^e, \lambda_1^e, \lambda_2^e) \geq H(x_1^e, x_2^e, p_1, p_2, \lambda_1^e, \lambda_2^e), \quad \forall \underline{p}_1 \leq p_1 \leq \bar{p}_1,\ \underline{p}_2 \leq p_2 \leq \bar{p}_2. \end{cases}$$

This is equivalent to solving the following system of equations:

$$\begin{cases} N\alpha_1(B + b_0p_0 - b_1p_1 - b_2(p_2 - p_1)) - \mu x_1 = 0, \\ \mu x_1 - \alpha_2(A - a_0p_0 + a_1p_2)x_2 = 0, \\ (\delta + \mu)\lambda_1 - \mu\lambda_2 - p_1 = 0, \\ \delta\lambda_2 + \alpha_2(A - a_0p_0 + a_1p_2)\lambda_2 - p_2 = 0, \\ x_1 - N\alpha_1b_1\lambda_1 + N\alpha_1b_2\lambda_1 = 0, \\ x_2 - N\alpha_1b_2\lambda_1 - \alpha_2a_1x_2\lambda_2 = 0. \end{cases}$$

Solving the above system of equations, we can show that the long-run stationary equilibrium state is determined by (19) - (22). ■

Proof of Proposition 7

First, applying transversality conditions for an infinite horizon control problem, we can show that the adjoint equation corresponding to the state $x_2(t)$ satisfies:

$$\lambda_2(t) = \frac{p_2}{\delta + \alpha_2(1 - a_0p_0 + a_1p_2(t))}.$$

Note that the coefficient of p_2 in the Hamiltonian function (31) is:

$$1 - \alpha_2a_1\lambda_2 = 1 - \frac{a_1\alpha_2p_2}{\delta + \alpha_2(1 - a_0p_0 + a_1p_2)} > 0$$

According to the optimal control policy defined by (33), the optimal pricing policy p_2^* is equal to $\bar{p}_2$ all the time.

The long-run stationary equilibrium introductory price, denoted $\hat{p}_e^1$, is determined by the following system of the equations:

$$\begin{cases} x_1 - N\alpha_1 b_1 \lambda_1 = 0, \\ (\delta + \mu)\lambda_1 - \mu\lambda_2 - p_1 = 0, \\ N\alpha_1(1 + b_0 p_0 - b_1 p_1) - \mu x_1 = 0. \end{cases} \tag{34}$$

Therefore,

$$p_e^1 = \frac{(\delta + \mu)(\frac{B + b_0 p_0}{b_1}) - \frac{\mu^2 \bar{p}_2}{\delta + \alpha_2(A - a_0 p_0 + a_1 \bar{p}_2)}}{\delta + 2\mu}.$$

One can easily check $\hat{p}_e^1 < p_e^1$. ■

Appendix B

In proposition 2, we assume that the target price p_e can be achieved; that is, p_e is within the feasible range of prices $[\underline{p}, \bar{p}]$. The following is an informal argument describing the optimal policy when the target price p_e falls outside this feasible range.

To start, when the target price is greater than the highest allowable price (i.e., $p_e > \bar{p}$), the firm will eventually price at $\bar{p}$; when the target price is smaller than the lowest allowable price (i.e., $p_e < \underline{p}$), the firm will eventually price at $\underline{p}$. But the optimal pricing policy before switching to $\bar{p}$ or $\underline{p}$ depends on the firm's market share.

Denote $\bar{y} = \frac{\alpha b_1 \bar{p}}{\delta + \alpha(A+B) + \alpha(b_0 - a_0)p_0}$ and $\underline{y} = \frac{\alpha b_1 \underline{p}}{\delta + \alpha(A+B) + \alpha(b_0 - a_0)p_0}$. When the firm has a market share greater than $\bar{y}$, it prices at the highest allowable price $\bar{p}$; when the firm has a market share less than $\underline{y}$, it prices at the lowest allowable price $\underline{p}$; when its market share lies within $[\underline{y}, \bar{y}]$, the firm prices at some intermediate value between the lowest price $\underline{p}$ and the highest price $\bar{p}$. Specially, when $p_e > \bar{p}$ and $\underline{y} < y_0 < \bar{y}$, the optimal price trajectory $p_1^*(t)$ and associated market share trajectory $y^*(t)$ both increase in time t and satisfy $y^*(t) - \alpha b_1 \lambda(t) + \alpha(b_1 - a_1)\lambda(t) y^*(t) = 0$ until the market share level reaches $\bar{y}$. When $p_e < \underline{p}$ and $\underline{y} < y_0 < \bar{y}$, the optimal price trajectory $p_1^*(t)$ and associated market share $y^*(t)$ again satisfy $y^*(t) - \alpha b_1 \lambda(t) + \alpha(b_1 - a_1)\lambda(t) y^*(t) = 0$, and both decrease in time t before the firm's market share is reduced to $\underline{y}$.

An informal argument for why such a policy is optimal is as follows: consider the case of $p_e > \bar{p}$ and $\underline{y} < y_0 < \bar{y}$ for illustration. We can check that the coefficient of the control variable p_1 in the Hamiltonian function (6), that is, $y^*(t) - \alpha b_1 \lambda(t) + \alpha(b_1 - a_1)y^*(t)$, becomes $-\frac{\alpha b_1 p_1 - y_0(\delta+\alpha(A+B)+\alpha(b_0-a_0)p_0)}{\delta+\alpha(A+B)+\alpha(b_0-a_0)p_0-\alpha(b_1-a_1)p_1}$ given the firm's price p_1 at $t = 0$. If the firm prices at $\bar{p}$, this coefficient is negative, which contradicts the optimal pricing policy described in (23). If the firm prices at $\underline{p}$, the coefficient is positive, which again contradicts (23). Hence, the optimal price at $t = 0$ must be equal to $p_1^*(0) = \frac{y_0(\delta+\alpha(A+B)+\alpha(b_0-a_0)p_0)}{\alpha b_1}$ and $\underline{p} < p_1^*(0) < \bar{p}$. We can easily verify that $\frac{dy^*}{dt}|_{t=0} > 0$. Therefore, the associated market share is increasing in $[0, \delta t]$ where δt is an infinitesimal time interval. Suppose the market share increases to $y^*(\delta t)$ at time δt. Using the same argument as in the case of $t = 0$, we can show that $p_1^*(\delta t) = \frac{y^*(\delta t)(\delta+\alpha(A+B)+\alpha(b_0-a_0)p_0)}{\alpha b_1}$, which is greater than $p_1^*(0)$ because $y^*(\delta t) > y_0$. Repeat this process until the market share reaches $\bar{y}$, and the optimal price increases to $\bar{p}$. Hence, the optimal price and the associated market share continually increase in time until the market share reaches $\bar{y}$ when the firm's initial market share is within $[\underline{y}, \bar{y}]$ and $p_e > \bar{p}$. This implies that the firm may use continually increasing or decreasing (when $p_e < \underline{p}$) prices for part of the optimal control trajectory when the target price is not attainable.