Futures Prices on Yields, Forward Prices, and Implied Forward Prices from Term Structure

Suresh Sundaresan*

Abstract

When futures contracts are settled with respect to underlying asset prices, received theory suggests that the differences between futures prices and implied forward prices (from the term structure) are strictly due to marking to market, *ceteris paribus*. Empirical evidence appears to indicate that such differences are small for contracts with short maturities. What happens when the futures contract settles to yields implied by future prices of underlying assets? The Eurodollar futures contract, which is the most actively traded futures contract in the United States, settles to yield as opposed to prices. This unique settlement feature is shown to imply that the implied forward prices from the LIBOR term structure should differ from the futures prices even in the absence of marking to market. Differences due to marking to market effect are small: they are shown to vary between 2 to 45 basis points (less than one-half percent of futures prices). On the other hand, differences between implied forward prices and futures prices are shown to be relatively large.

I. Introduction

The literature on the pricing of futures and forward contracts has focused on contracts that settle to the future price of a prespecified underlying asset. The paper by Cox, Ingersoll, and Ross (1981), for instance, shows that forward prices and futures prices should not differ in the absence of the marking to market feature that is usually present in the futures market. This implication holds true for implied forward prices from the term structure as well, ignoring differences due to the possibility of default or lack of liquidity.[1] Empirical evidence also has focused on such contracts. Empirical research in this area has concluded that the differences between forward prices and futures prices are small for contracts with short maturities.

The Eurodollar futures contract introduced by the Chicago Mercantile Exchange is currently the most actively traded futures contract in the United States. This contract settles to the 90-day London Interbank Offered Rate (LIBOR),

*Graduate School of Business, Columbia University, New York, NY 10027. The author thanks Warren Bailey, Philippe Jorion, Avraham Kamara, and an anonymous *JFQA* referee for their constructive comments. Any remaining errors are the author's responsibility.

[1]The effects of default and liquidity differentials are not addressed in this paper.

which is the yield implied by the underlying asset (the 90-day Eurodollar time deposit). This method of computing the futures price is rather unique and is a departure from the futures pricing formulations in the literature. Since the price to yield transformation is not linear, the pricing of futures contracts on yields does not follow directly from the pricing of corresponding futures on prices. More importantly, it also renders the standard implied forward rates calculations from the LIBOR term structure to differ from the futures price: such forward rates appy only to situations where forward and futures contracts are on asset prices and not to situations where contracts are on yields.

Currently, Eurodollar futures contracts with virtually identical specifications are traded at the International Monetary Market (IMM) at Chicago, Singapore International Monetary Exchange (SIMEX), and London International Financial Futures Exchange (LIFFE). Proposals are currently well under way to list Eurodollar futures contracts at Tokyo as well. IMM and SIMEX also have common clearing systems whereby Eurodollar futures positions that were established in one exchange may be offset in the other. Currently, it is the most liquid futures contract ever to be traded in the United States. For example, the open interest in the nearby maturities (the two nearest contracts) exceeded a \$50 billion face amount in August 1989.

A. Futures Settlement to Yields

A key feature of Eurodollar futures contracts is the manner in which they are settled at maturity. The futures contract settles by cash on its maturity date without any delivery or timing flexibilities to either the investor who is "short" or the investor who is "long." On the expiration date, which is the second London business day before the third Wednesday of the maturity month, the contract settles by cash to LIBOR as per the following procedure. The clearing house, on the expiration date, determines the LIBOR for three-month Eurodollar deposits on two different times: the time of termination of trading, and at a randomly selected time within 90 minutes of termination of trading. The method of determination of LIBOR is via a sampling procedure. The clearing house selects, at random, 12 reference banks from a list of no less than 20 participating banks. Each bank provides a quotation to the clearing house on LIBOR applied to three-month Eurodollar time deposits. The clearing house eliminates the two highest quotes and the two lowest quotes and computes an arithmetic average of the remaining eight LIBOR quotes. This is regarded by the clearing house as LIBOR for that time. The final settlement price of the Eurodollar futures contract is obtained by performing this LIBOR computation at the two times as indicated above, and then subtracting the arithmetic mean of computed LIBOR (rounded to the nearest basis point) from 100. Thus, at maturity, the futures price (by design) "converges" to $100 \times (1 - \text{LIBOR})$, where LIBOR is expressed in decimals with a resolution of a basis point.[2] Equivalently, 100 *minus* the Eurodollar futures price will converge to three-month LIBOR.

This paper makes the following contributions: first, the general pricing principles developed by Cox, Ingersoll, and Ross (CIR hereafter) (1981) are

[2]For example, an 8.01 percent (annualized) LIBOR will be expressed as 0.0801.

modified for futures contracts on yields. Using a minimal set of internally consistent assumptions, we provide valuation formulas for futures prices, correctly defined forward prices, and implied forward prices from the spot LIBOR curve. This step provides some basic insights into Eurodollar futures and forward prices. Second, using the simple arbitrage-free term structure model of CIR (1985), the paper computes a forward rate in closed form that is consistent with the existing contractual design of futures contracts on Eurodollar rates. These are, in turn, used for a comparative analysis of Eurodollar futures, forwards, and implied forwards from the spot LIBOR curve. Our simulations suggest that the effect of marking to market varies from about 2 basis points to 45 basis points for futures contracts with three to six months maturity. But the differences between implied forward rates and futures rates are shown to be relatively large, and they are due to the contractual features in the settlement of Eurodollar futures prices. Using data on LIBOR and Eurodollar futures prices, empirical evidence also is presented on the relationship between Eurodollar futures and implied forward prices.

The next section develops basic pricing principles to provide initial distinctions between Eurodollar futures and forward rates. It also defines some notations and conventions. In Section III, we develop the ideas further in the context of an arbitrage-free term structure model to further characterize the forward prices. Section IV provides a simulation analysis of futures prices, forward prices, and implied forward prices using the term structure model. In Section V, empirical evidence is presented on the futures and implied forward prices. The final section concludes.

II. Determinants of Futures and Forward Prices

A rigorous understanding of Eurodollar futures markets necessarily requires a careful study of the idiosyncratic aspects of the contractual specifications that were discussed in the previous section. We will now develop some concepts of pricing Eurodollar futures contracts that explicitly take into account the fact that they settle to yields as opposed to prices (as, for example, is the case with Treasury bill futures). These concepts are useful in giving us some intuition as to the differences between futures and forward rates. In order to get additional insights, it will be necessary to impose additional restrictions on the manner in which the underlying spot LIBOR evolves over time. Such restrictions will be discussed in Section III.

A. Notations and Conventions

To identify the factors that determine the futures price, $H_t(s)$, quoted at date t for maturity date s, we begin by making some simplifying assumptions.

Assumption 1. The futures contract matures on a specific date s. There are no timing flexibilities to either the "short" or the "long." This assumption is valid for the Eurodollar futures contract, which settles by cash at a precise date as explained in Section I.

Assumption 2. The futures contract settles to Eurodollar deposit rates, which trade in a competitive market free of any restrictions. The spot price of the time deposit may not be manipulated. In effect, we will assume that, on date s, settlement at competitive prices is possible. This is not an unreasonable assumption, given the number and size of banks that participate in these markets.
Assumption 3. Futures contracts are marked to market daily. This assumption is consistent with the institutional practice.

The present value at date t of a random cash flow X_s received at date s will be denoted by $PV_t[X_s, s]$. One plus the daily financing rate in the Eurodollar market will be denoted by R_j from day j to day $j+1$. These rates may be random. For future reference, we will denote by $R_t^+(s)$ the following quantity,

$$R_t^+(s) \equiv \Pi_{i=t}^{s-1}(R_i) \equiv R_t \times R_{t+1} \times \ldots \times R_{s-1}.$$

Clearly, $R_t^+(s)$ is the amount earned at date s by rolling 1 dollar in a sequence of overnight LIBOR assets from date t until date s. It follows that $PV_t[R_t^+(s), s] = 1$.

The price at date t of a time deposit that pays 1 dollar at date s will be denoted by $b(t, s)$. $l_s(90)$ will denote the three-month LIBOR (annualized) at date s. LIBOR on a Eurodollar time deposit with a maturity of τ days is defined in money market convention according to

$$l_s(\tau) = \frac{360}{\tau} \times \left(\frac{1}{b(s, s+\tau)} - 1\right). \tag{1}$$

The Eurodollar futures price, $H_s(s)$, at maturity date s is

$$H_s(s) \equiv 100 \times [1 - l_s(90)]. \tag{2}$$

This follows from the "add on" settlement feature of the Eurodollar futures contract that was described earlier. The resulting Eurodollar futures price is in percentages of 1 million times the face amount of a 90-day time deposit. The market resolution is a basis point worth $1{,}000{,}000 \times \frac{1}{100} \times \frac{1}{100} \times \frac{90}{360} = \$25$. If the futures contract were to settle to price rather than yield, the resulting futures price, $\hat{H}_s(s)$, would have converged to

$$\hat{H}_s(s) = 100 \times b(s, s+90) = \frac{100}{1 + l_s(90)\frac{90}{360}}. \tag{3}$$

This, for instance, is the settlement feature for the T-bill futures contracts, where the rates are expressed as discount rates. To clarify matters further, consider the T-bill futures. The T-bill rates are discount rates, d, shown as

$$d = \frac{F-M}{F} \times \frac{360}{90},$$

where M is the market price of the T-bill and F is its face amount. The T-bill futures price at maturity is $H_s(s) = 100 \times [1 - d \times (90/360)]$, which is a linear function of the discount rate d and converges to the price P.[3]

[3]A direct comparison of (2) and (3) may be a bit misleading; Equation (2) uses annualized LIBOR, and, in Equation (3), the prices are computed based on quarterly rates. But, it should be kept in mind that the futures price changes in (2) will be multiplied by 90/360, as shown earlier, so that each basis point move is worth $25.

B. Results on Futures and Forward Prices

The next few propositions will characterize Eurodollar forward and futures prices in some detail. These propositions are fairly general in the sense that their validity is not dependent on a specific model of futures and forward prices.
Proposition 1. The Equilibrium Eurodollar futures price is

$$H_t(s) \equiv 100 - PV_t[\{l_s(90) \times R_t^+(s)\}, s]. \tag{4}$$

This follows from the traditional futures pricing result shown by CIR (1981). The proposition says that the futures price today is 100 minus the present value of rolling a dollar into a sequence of one period interest rates until the maturity date, s, of the futures contract and earning *on date* s, an amount equal to the three-month LIBOR that will prevail on date s. Note that the three-month LIBOR on date s is assumed to be earned on the same date. Thus, there is *no lag between the time LIBOR is established and the time the payment is made.* If one were to buy a three-month deposit on date s, one would be able to earn LIBOR only by holding it to maturity, which implies a lag of three months between the time LIBOR was established and the time the deposit matured. A long position in the futures contract allows an investor to lock in a *futures rate*, $h_t(s)$, defined as

$$h_t(s) = 100 - H_t(s). \tag{5}$$

Note that if we ignored marking to market, then the futures rate, $h_t(s)$, would be given by

$$h_t(s) = \frac{PV_t[l_s(90), s]}{b(t, s)}. \tag{6}$$

Thus, the futures rate is the present value of receiving date s LIBOR on date s. The discount factor in the denominator reminds us that the payment is made only at date s. Let us now define a forward contract on LIBOR that is identical to the futures contract except that the forward contract is not marked to market.
Definition. The Eurodollar forward contract is an agreement (established at date $t < s$) to pay or receive on maturity date s an amount equal to $100 \times [1 - l_s(90)]$ at a currently agreed upon forward price $G_t(s)$. The contract is not marked to market and is settled by cash on date s.

Note that the Eurodollar forward price bears a close relationship to the *swap rate* in which a single 90-day floating LIBOR is exchanged for a fixed payment on the reset date, which is coincident with the maturity date of the forward contract.

The next proposition characterizes such a forward price, which will be referred to as the Eurodollar forward price.
Proposition 2. The equilibrium forward price is

$$G_t(s) \equiv 100 - \frac{1}{b(t, s)} \times PV_t[l_s(90), s]. \tag{7}$$

Intuitively, this proposition says that the Eurodollar forward price is 100 minus the present value of earning *on date* s, the then prevailing three-month LIBOR, on the proceeds from investing a dollar at date t in a time deposit that matures on date s. As a result of the first two propositions, we have the following result that characterizes the difference between Eurodollar futures and forward rates.

Proposition 3. The difference between the futures and forward price is

$$H_t(s) - G_t(s) \equiv PV_t\left[l_s(90) \times \left\{\frac{1}{b(t,s)} - R_t^+(s)\right\}, s\right]. \tag{8}$$

When will the Eurodollar futures price be lower than the forward price? Intuitively, as short-term rates R_i go up, we may expect LIBOR to go up. This should cause futures and forward prices to fall. But this loss will be marked to market in the futures position. Thus, a person with a long position will have to meet this margin call by liquidating assets that earn the prevailing high rates. This would cause the investor to lower the futures price at which he is willing to enter into a trade. Thus, under normal circumstances, we may expect the Eurodollar futures prices to be lower than the Eurodollar forward price. This bias is due to the marking to market feature of Eurodollar futures contracts. Later, in the context of a term structure model, we verify this intuition through simulations. The conclusions in this section are obtained from the basic pricing propositions of CIR (1981). *Notice, however, a key implication: in both cases (forward and futures prices), it is necessary to earn the three-month LIBOR that will prevail on date s on the same date.* This rules out the use of the implied forward rate from the spot LIBOR curve as shown next.

C. Implied Forward Prices

The concept of the implied forward rate from the spot curve is standard in the term structure literature. The existence of a spot market in Eurodollar time deposits implies that investors can synthetically create forward rates. If the synthetically created forward rates differ from futures rates after accounting for transactions costs and risks, then arbitrage profits should be possible, *provided the cash flow distribution and its timings are the same in the futures market.* To examine this, we first illustrate how the Eurodollar time deposit markets may be used to borrow at forward rates. Table 1 illustrates a strategy to borrow at a future date s at a currently known *forward rate*, $f_t(s, s+90)$. The forward rate is given by

$$f_t(s, s+90) = \frac{1}{\tau}\left[\frac{b(t,s)}{b(t,s+90)} - 1\right], \tag{9}$$

where $\tau = 90/360$. The bid-offer spread in the Eurodollar time deposits market is of the order of one-sixteenth to one-eighth of a percent. But, for the participating banks, the costs are probably much less. We will ignore the bid-offer spreads in the analysis that follows.

Based on the implied forward rate, we may define a forward price, $F_t(s)$, as

$$F_t(s) = \frac{100}{1 + f_t(s, s+90)\tau}. \tag{10}$$

TABLE 1

To Borrow at Forward Rate

Transaction on Current Date t	Investment on Current Date t	Maturity Date of First Deposit s	Maturity Date of Second Deposit $s+90$
Buy the deposit that matures on s	$b(t,s)$	1	0
Sell the deposit that matures on $s+90$	$-b(t,s+90)$		-1
Borrow the difference until $s+90$	$-[b(t,s)-b(t,s+90)]$		$-\frac{b(t,s)-b(t,s+90)}{b(t,s+90)}$
Total Investment	zero		
Total Cash Flow		1	$-\frac{b(t,s)}{b(t,s+90)}$

Note that in Proposition 2, if the forward contract settled to price instead of yield, we would have the resulting price, $\hat{G}_t(s)$, as

$$\hat{G}_t(s) = \frac{PV_t(b(s,s+90),s)}{b(t,s)}. \tag{11}$$

This follows from standard cash and carry arguments or Proposition 1 of CIR (1981). Utilizing the fact that $PV_t[b(s,s+90),s] = b(t,s+90)$, and Equations (9) and (10) we get

$$\hat{G}_t(s) = F_t(s). \tag{12}$$

Given these observations, *a comparison of the Eurodollar futures price, $H_t(s)$, in Equation (4), the Eurodollar forward rate, $G_t(s)$, in Equation (7) should clearly demonstrate why these are fundamentally different prices.*

In a similar manner, a comparison of the futures rate, $h_t(s)$, with the implied forward rate, $f_t(s)$, also should clearly demonstrate why they should be different. To make this point transparent, we will compute the difference $h_t(s)-f_t(s,s+90)$ under the assumption that the futures contracts are not marked to market,

$$h_t(s)-f_t(s,s+90) = b(t,s)PV_t[l_s(90),s]-\frac{1}{T}\left[\frac{b(t,s)}{b(t,s+90)}-1\right]. \tag{13}$$

Note that the futures rate may differ from the implied forward unless the RHS is zero.

Eurodollar futures prices differ from the Eurodollar forward due to the marking to market feature. Eurodollar futures rates differ from the implied forward rates because they represent different timing of cash flows. In a Eurodollar futures contract, LIBOR at date s is paid on the same date. In an implied forward transaction, the forward rate is earned at date s by purchasing the deposits and holding them until the maturity dates. One of the deposits must be held until date $s+90$ to lock in the forward rate on date s. Since the effective payment is delayed by the maturity of the deposit, we may expect the forward rate to differ from the futures rate.

III. A Model of Eurodollar Futures and Forward Prices

The purpose of this section is to further characterize the differences between futures, forward, and implied forward rates in a quantitative manner using a term structure model. The term structure model of CIR (1985) appears to be a reasonable choice, given its arbitrage-free property and the extent of empirical research that is currently being carried out to test the model (see Brown and Dybvig (1985), for instance). This model is used because it is an internally consistent framework that enables us to get tractable results. The conclusions based on this model still need to be examined in broader contexts.

The model of CIR (1985) is cast in a continuous time, continuous state framework, grounded on the following assumptions.

Assumption 4. The uncertainty is fully summarized by the short rate that follows a mean-reverting stochastic process,

$$dr = \kappa(\mu - r)dt + \sigma\sqrt{r}dz, \tag{14}$$

where κ is the speed of adjustment, μ is the long run mean rate of interest, and $\sigma^2 r$ may be viewed as the variance of changes in interest rates. Furthermore, $[z(t), t > 0]$ is a standard Wiener process.

In the context of our application, care must be exercised in the interpretation of the variable r, which is the short rate. Eurodollar rates are subject to credit risk as well as sovereign risk. An explicit account of these risks will render the problem intractable. The state variable r, which will be interpreted as the instantaneous LIBOR, essentially is assumed to manifest these risks. A multi-state variable formulation with explicit account of credit risk and bankruptcy condition will be relevant for explaining the spread between T-bill yields and Eurodollar time deposit yields (LIBOR). This, however, is *not* the focus of our study. The model admits a factor risk premium of λr. Loosely speaking, by selecting a high (negative) value for this factor, we may proxy the higher liquidity premium in the LIBOR market relative to the T-bill market. In the simulations that are presented later, we document the effect of this factor. Prices of deposits in the CIR (1985) model were obtained in closed form and are reproduced here for ready reference and further use.

Let the current time be denoted by t, then a deposit paying 1 dollar at time s should be priced as

$$b(t,s) = A(s-t)e^{-B(s-t)r_t} \tag{15}$$

where the functions $A(\tau)$ and $B(\tau)$ are defined as,

$$A(\tau) = \left(\frac{2\gamma e^{(\kappa+\gamma+\lambda)\tau/2}}{2\gamma + (\kappa+\gamma+\lambda)(e^{\gamma\tau}-1)}\right)^{2\kappa\mu/\sigma^2}$$

$$B(\tau) = \frac{2(e^{\gamma\tau}-1)}{2\gamma + (\kappa+\gamma+\lambda)(e^{\gamma\tau}-1)},$$

$$\gamma = \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}, \text{ and}$$

$$\tau = s - t.$$

In the formula above, λ may be interpreted as factor risk premium. Given this solution, the implied forward prices are obtained simply by substituting these formulas in Equation (10). In order to quantitatively characterize the differences between Eurodollar futures prices, forward prices, and implied forward prices, it is necessary to solve these prices in terms of the underlying state variable r and other parameters κ, λ, μ, and σ. We proceed to do this next.

A. Eurodollar Futures

The Eurodollar futures price, H, will satisfy the valuation equation,

$$H_t + H_r(\kappa(\mu - r) - r\lambda) + \frac{1}{2}H_{rr}r\sigma^2 \quad = \quad 0. \tag{16}$$

The futures price also satisfies the condition that at maturity, the futures price must be equal to 100 minus LIBOR at maturity date,

$$H_s(s) \quad \equiv \quad 100 \times [1 - l_s(90)]. \tag{17}$$

This valuation equation was solved using numerical procedures (implicit finite difference method). In the numerical procedures, a mesh size of 500 is used for interest rate and time dimensions. The lower boundary for interest rate was set at $r = 0$, and the upper boundary was set at $r = 50$ percent. We provide results for futures contracts with 90 days and 180 days to maturity: much of the open interest is usually concentrated in the nearer maturity futures contracts.

The numerical procedure was first calibrated to accurately solve the known discount bond prices of the CIR (1985) model given in Equation (15). The prices matched the closed form solution up to a basis point. Since the resolution in the market for the Eurodollar futures price is a basis point, the calibrated numerical procedure should provide sufficiently accurate results. The resulting futures prices are shown in Tables 2 through 7. To study the effect of marking to market, it is necessary to compute the forward price under the same assumptions. This is done next.

B. Eurodollar Forwards

The Eurodollar forward price, G, will satisfy the valuation equation,

$$G_t + G_r(\kappa(\mu - r) - r\lambda) + \frac{1}{2}G_{rr}r\sigma^2 - rG \quad = \quad 0. \tag{18}$$

The forward price also satisfies the condition that at maturity, the forward price must be equal to 100 minus LIBOR at maturity date,

$$G_s(s) \quad \equiv \quad 100 \times [1 - l_s(90)]. \tag{19}$$

Note that $l_s(90)$ was defined earlier. The Eurodollar forward price under these assumptions may be shown as

$$G_t(s) \quad = \quad 100 - 100\frac{g(\tau_f)e^{-h(\tau_f)r_t} - b(t,s)}{\tau b(t,s)},$$

where

$$
\begin{aligned}
g(\tau_f) &= \frac{1}{A(\tau)}\left(\frac{a_1 e^{\theta_1 \tau_f} + e^{(\theta_1 - \gamma)\tau_f}}{1 + a_1}\right)^{-2\kappa\mu/\sigma^2} \\
h(\tau_f) &= \frac{2}{\sigma^2}\left(\frac{a_1 \theta_1 e^{\gamma \tau_f} + (\theta_1 - \gamma)}{1 + a_1 e^{\gamma \tau_f}}\right), \\
\theta_1 &= \frac{1}{2}[-(\kappa + \lambda) + \gamma], \\
\gamma &= ((\kappa + \lambda)^2 + 2\sigma^2)^{1/2} > 0, \text{ and} \\
a_1 &= -\frac{\frac{\sigma^2}{2}B(\tau) + (\theta_1 - \gamma)}{\theta_1 + \frac{\sigma^2}{2}B(\tau)}.
\end{aligned}
$$

τ represents the three-month maturity period, and τ_f denotes the time to maturity of the Eurodollar futures contract. The forward rates solved above are, in fact, related to the swap rates in the interest rate swap markets as shown in Sundaresan (1991). To complete the analysis, we need to compute the implied forward rates. The procedures for this are shown next.

C. Implied Forward Rates

Since the discount functions are available in closed form, it is easy to compute the implied forward prices from the formulation provided in Equation (10). Note that the key advantage of the approach presented here is the *internal consistency* of the assumptions used in computing these three distinct rates. In the next section, we provide a comparison of these rates for various shapes of the LIBOR term structure.

IV. A Comparison of Futures, Forwards, and Implied Forwards

We begin by setting forth some parameter values for the stochastic process used in obtaining these rates. Later, we vary these parameters to check the robustness of our initial conclusions. For Tables 2 and 3, the following values were used in the simulation: $\kappa = 0.5$, $\mu = 8$ percent, $\lambda = 0$, and $\sigma^2 = 0.25$.

We also performed additional simulations for varying levels of κ, λ, and σ^2, but the qualitative results remained unchanged. Thus, we have chosen to report only a limited set of results to conserve space. For all tables, we report values for rates that are +100 basis points and −100 basis points from the long run mean rate of 8 percent. Intervals of 10 basis points are used for reporting purposes.

In all the tables, futures prices and forward prices are presented. For ready reference, the difference between futures prices and forward prices also is reported in basis points. These results correspond to 90-day maturity and 180-day maturity futures contracts.

As may be seen from Table 2, the difference between futures and forward prices never exceeds 6 basis points. The table corresponds to a maturity of three months. This result suggests that the marking to market feature accounts for less than 6 basis points in the price of the futures contract. As may be expected, the difference is higher as the futures contract's maturity increases. In Table 3, the effect of marking to market is examined for a futures contract with six months to maturity. The difference now is of the order of 20 basis points. Expressed in percentage terms, this difference on a futures price of 90.00 is just 0.22 percent.

TABLE 2

Effect of Marking to Market[a]
(90-day maturity)
$\kappa = 0.5, \mu = 8\%, \lambda = 0,$ and $\sigma^2 = 0.25; s - t = 90$ days

Interest in % r	LIBOR in % $l_t(90)$	ED Futures Price $H_t(s)$	ED Forward Price $G_t(s)$	Difference in Basis Points (Futures minus Forward) $H_t(s) - G_t(s)$ ×100
7.00%	7.11%	92.74	92.78	−4.66
7.10%	7.20%	92.65	92.70	−4.72
7.20%	7.30%	92.57	92.62	−4.79
7.30%	7.39%	92.48	92.53	−4.85
7.40%	7.49%	92.40	92.45	−4.92
7.50%	7.58%	92.31	92.36	−4.98
7.60%	7.68%	92.23	92.28	−5.05
7.70%	7.77%	92.14	92.19	−5.11
7.80%	7.87%	92.06	92.11	−5.18
7.90%	7.97%	91.97	92.03	−5.24
8.00%	8.06%	91.89	91.94	−5.31
8.10%	8.16%	91.80	91.86	−5.37
8.20%	8.25%	91.72	91.77	−5.44
8.30%	8.35%	91.63	91.69	−5.50
8.40%	8.44%	91.55	91.60	−5.56
8.50%	8.54%	91.46	91.52	−5.63
8.60%	8.64%	91.38	91.44	−5.69
8.70%	8.73%	91.29	91.35	−5.76
8.80%	8.83%	91.21	91.27	−5.82
8.90%	8.92%	91.12	91.18	−5.89
9.00%	9.02%	91.04	91.10	−5.95

[a] κ is the speed of adjustment, μ is the long-run mean, σ^2 is the volatility parameter, and λ is the factor risk premium.

To test the robustness of our conclusions in this regard, we decreased the volatility variable σ^2 to 0.1: this decreases the difference between futures and forward prices significantly. For three-month maturity futures contracts, the difference between forward and futures prices was about 2 basis points. The difference is about 7 to 8 basis points for contracts with six months to maturity.

In all cases, futures prices are lower than forward prices. This is consistent with one's intuition: in a long position, futures prices go down when the rates go up. This would produce adverse variation margin calls, when assets are earning high returns. In a similar way, futures prices go up when the rates go down. As a result, favorable variation margin cash flows may be invested only at low

TABLE 3

Effect of Marking to Market[a]
(180-day maturity)
$\kappa = 0.5, \mu = 8\%, \lambda = 0$, and $\sigma^2 = 0.25; s - t = 180$ days

Interest in % r	LIBOR in % $l_t(90)$	ED Futures Price $H_t(s)$	ED Forward Price $G_t(s)$	Difference in Basis Points (Futures minus Forward) $H_t(s) - G_t(s)$ ×100
7.00%	7.11%	92.60	92.77	−16.57
7.10%	7.20%	92.53	92.70	−16.77
7.20%	7.30%	92.45	92.62	−16.98
7.30%	7.39%	92.38	92.55	−17.18
7.40%	7.49%	92.30	92.48	−17.39
7.50%	7.58%	92.23	92.40	−17.69
7.60%	7.68%	92.15	92.33	−17.80
7.70%	7.77%	92.08	92.26	−18.01
7.80%	7.87%	92.00	92.18	−18.21
7.90%	7.97%	91.92	92.11	−18.42
8.00%	8.06%	91.85	92.04	−18.73
8.10%	8.16%	91.77	91.96	−18.93
8.20%	8.25%	91.70	91.89	−19.14
8.30%	8.35%	91.62	91.82	−19.35
8.40%	8.44%	91.55	91.74	−19.55
8.50%	8.54%	91.47	91.67	−19.76
8.60%	8.64%	91.40	91.60	−20.07
8.70%	8.73%	91.32	91.52	−20.27
8.80%	8.83%	91.24	91.45	−20.48
8.90%	8.92%	91.17	91.38	−20.69
9.00%	9.02%	91.09	91.30	−20.80

[a] κ is the speed of adjustment, μ is the long-run mean, σ^2 is the volatility parameter, and λ is the factor risk premium.

rates. As a result, the equilibrium futures price should be bid down compared to the equilibrium forward price.

The effect of the factor risk premium on futures, forwards, and implied forward prices also was examined for 90-day and 180-day maturities, respectively. Increase in the factor risk premium has two effects: (i) it *reduces the levels* of futures prices, and (ii) the difference between futures and forward prices *increases* to about 7.5 to 9.5 basis points for contracts with 90 days to maturity and to about 36.5 to 46.5 basis points for contracts with 180 days to maturity.

Thus, our results suggest the following two conclusions.

1. Differences between futures prices and forward prices as defined in this paper are usually not too large and fall near or within bid-offer spreads for futures contracts with 90-day maturity or less.

2. Differences between futures prices and forward prices sometimes may be large enough to be outside the bid-offer spreads for futures contracts with 180-day maturity or more.

These differences are not driven by differences in liquidity or tax considerations. They are purely the artifacts of the marking to market feature that is present in the Eurodollar futures contract. We analyzed the difference between the futures rate $h_t(s)$ and the implied forward rates, $f_t(s, s + 90)$.

TABLE 4

Futures on Yields versus Implied Forwards[a]
(90-day maturity)
$\kappa = 0.5, \mu = 8\%, \lambda = 0,$ and $\sigma^2 = 0.25; s - t = 90$ days

Interest in % r	LIBOR in % $l_t(90)$	Futures Rate $h_t(s)$	Implied Forward $f_t(s, s+90)$	Difference in Basis Points $h_t(s) - f_t(s, s+90)$ ×100
7.00%	7.11%	7.13%	7.26%	−13.32
7.10%	7.20%	7.21%	7.35%	−13.50
7.20%	7.30%	7.29%	7.43%	−13.68
7.30%	7.39%	7.38%	7.52%	−13.86
7.40%	7.49%	7.46%	7.60%	−14.04
7.50%	7.58%	7.54%	7.69%	−14.22
7.60%	7.68%	7.63%	7.77%	−14.40
7.70%	7.77%	7.71%	7.86%	−14.59
7.80%	7.87%	7.79%	7.94%	−14.77
7.90%	7.97%	7.88%	8.03%	−14.95
8.00%	8.06%	7.96%	8.11%	−15.13
8.10%	8.16%	8.04%	8.20%	−15.31
8.20%	8.25%	8.13%	8.28%	−15.50
8.30%	8.35%	8.21%	8.37%	−15.68
8.40%	8.44%	8.29%	8.45%	−15.86
8.50%	8.54%	8.38%	8.54%	−16.04
8.60%	8.64%	8.46%	8.62%	−16.22
8.70%	8.73%	8.54%	8.71%	−16.41
8.80%	8.83%	8.63%	8.79%	−16.59
8.90%	8.92%	8.71%	8.88%	−16.77
9.00%	9.02%	8.79%	8.96%	−16.95

[a] κ is the speed of adjustment, μ is the long-run mean, σ^2 is the volatility parameter, and λ is the factor risk premium.

Tables 4 and 5 correspond to the parameter values chosen for Tables 2 and 3, respectively. The following conclusions may be drawn from these tables.

1. Differences between the futures rate and the implied forward rate become more negative as rates increase. In all cases, futures rates were lower than implied forward rates.
2. The differences between futures rates and implied forward rates are much greater than the differences between futures prices and forward prices.
3. In most cases, the differences exceed the bid offer spreads that we tend to see in the LIBOR market.

The impact of factor risk premium was very similar to that reported earlier.

In the next section, we examine the extent to which our results are supported by the data on Eurodollar futures prices and LIBOR.

V. Empirical Evidence

The daily data on Eurodollar futures prices were obtained from the statistics and surveillance division of the Commodity Futures Trading Commission for the period 1985–1988. The daily data on LIBOR was based on Data Resources Incorporated. These are closing prices in these two distinct markets. The data set presents the obvious problem of *nonsimultaneity* between the futures price

TABLE 5

Futures on Yields versus Implied Forwards[a]
(180-day maturity)
$\kappa = 0.5$, $\mu = 8\%$, $\lambda = 0$, and $\sigma^2 = 0.25$; $s - t = 180$ days

Interest in % r	LIBOR in % $l_t(90)$	Futures Rate $h_t(s)$	Implied Forward $f_t(s, s+90)$	Difference in Basis Points $h_t(s) - f_t(s, s+90)$ ×100
7.00%	7.11%	7.08%	7.40%	−31.67
7.10%	7.20%	7.15%	7.47%	−32.07
7.20%	7.30%	7.22%	7.55%	−32.48
7.30%	7.39%	7.29%	7.62%	−32.88
7.40%	7.49%	7.36%	7.70%	−33.29
7.50%	7.58%	7.44%	7.77%	−33.69
7.60%	7.68%	7.51%	7.85%	−34.10
7.70%	7.77%	7.58%	7.92%	−34.51
7.80%	7.87%	7.65%	8.00%	−34.91
7.90%	7.97%	7.72%	8.08%	−35.32
8.00%	8.06%	7.79%	8.15%	−35.73
8.10%	8.16%	7.86%	8.23%	−36.13
8.20%	8.25%	7.94%	8.30%	−36.54
8.30%	8.35%	8.01%	8.38%	−36.95
8.40%	8.44%	8.08%	8.45%	−37.35
8.50%	8.54%	8.15%	8.53%	−37.76
8.60%	8.64%	8.22%	8.60%	−38.17
8.70%	8.73%	8.29%	8.68%	−38.57
8.80%	8.83%	8.37%	8.76%	−38.98
8.90%	8.92%	8.44%	8.83%	−39.39
9.00%	9.02%	8.51%	8.91%	−39.80

[a] κ is the speed of adjustment, μ is the long-run mean, σ^2 is the volatility parameter, and λ is the factor risk premium.

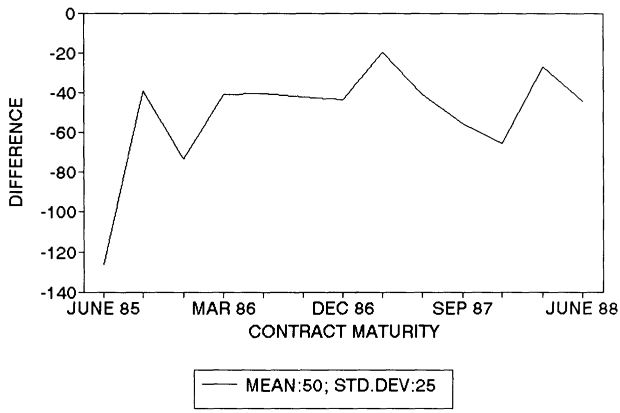
series and the series on LIBOR. In addition to this problem, there is a subtler issue: the bid-offer spreads in the LIBOR cash market run from one-sixteenth to one-eighth of a percent. On the other hand, in the futures market, the resolution is up to a basis point.[4] In addition, the two markets (futures and cash LIBOR) represent different organizations with respect to contract performance: in the futures market, the clearing house guarantees the integrity of the contracting process. On the other hand, in the cash market, the credit reputation of the members of the interbank market establishes the integrity. These observations should serve as a backdrop in assessing the empirical evidence that is presented in this section.

As we move up the maturity spectrum, the liquidity of both the futures market and the cash market diminishes rapidly. The open interest in futures contracts maturing beyond six months is much smaller than the futures maturing within six months. In the cash market, the depth of the market is the greatest for LIBOR with three and six months maturity. Beyond six months maturity, the depth diminishes and the bid-offer spreads widen. For this reason, we have chosen to work with three and six months LIBOR in this study, although it is possible to compute the three-month forward rates using LIBOR with higher maturities.

[4]Having made this observation, it should be noted that the settlement is always with respect to cash market and, hence, on the maturity date, the resolution in the two markets is the same.

The structure of LIBOR is such that the quotes are for fixed time to maturities: 30 days, 90 days, 180 days, etc. Thus, we may only be able to compute implied forward rates for specific maturities: implied 90-day rates will be available only for maturities starting from 90 days from the quote date. On the other hand, futures prices are quoted for fixed maturity dates. Therefore, for each futures contract, we may only be able to compute one matched implied forward rate. During the sample period, we were able to gather 13 matched forward and futures prices that gave us 90-day forward 90-day rates.

Using the daily three and six months LIBOR data, the implied forward price was calculated first. This constructed forward price was compared with the futures price. Figure 1 plots the differences in basis points for these two prices with 90 days maturity. Note that the implied forward prices are systematically higher than the futures prices. In Figure 1, the mean difference was 50 basis points with a standard deviation of 25 basis points. This is in broad agreement with our theory and the simulations.



Futures–Implied Forward

VI. Conclusion

In this paper, the unique settlement feature of the Eurodollar futures contract was shown to imply that the implied forward price from the spot LIBOR term structure is inappropriate for the purposes of comparison with the Eurodollar futures prices. An appropriate measure of the forward price with the same settlement feature as the Eurodollar futures price was defined and computed. The futures price was compared with both the implied forward price and the Eurodollar forward price. The biases were quantified and analyzed for a variety of term structure scenarios. Some empirical evidence was provided comparing the Eurodollar futures prices with Eurodollar forward prices that are implied by the LIBOR term structure.

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