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Consumption and Equilibrium Interest Rates in Stochastic Production Economies

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ABSTRACT

In this paper, we analyze the behavior of equilibrium real interest rates in an identical consumer economy in which the preferences are represented by time additive logarithmic utility functions and production technologies are Cobb-Douglas with stochastic constant returns to scale. The following main results are established.

(i) When there is no relative price uncertainty, it is shown that the equilibrium interest rate exhibits a mean reverting tendency. A nontrivial steady state distribution is found to exist for the equilibrium interest rate. The properties of the equilibrium interest rate are also derived and discussed.

(ii) In a multigood economy, even with additive preferences across goods, the equilibrium interest rates depend explicitly on relative prices. The substitution possibilities in production technologies induce this result. This is in contrast to the findings of Richard and Sundaresan [11] who show that the analytical general equilibrium term structure of interest rates formula of Cox, Ingersoll, and Ross [5] is unaffected by the introduction of relative price uncertainty when the technologies are *linear* and hence involve no substitution.

Furthermore, we relate our results to those of Cox, Ingersoll, and Ross [5], Breeden [3], and Richard and Sundaresan [11] with special emphasis on stochastic production and realtive price uncertainty.

THE PURPOSE OF THIS PAPER is to present and discuss some new results on the determinants of equilibrium interest rates in stochastic production economies. The seminal contribution of Cox, Ingersoll, and Ross [5] provides the basic setting and serves as a frame of reference. This paper also analyzes the interrelation between equilibrium real interest rates and consumption behavior as examined by Breeden [3] and Cox and Ross [6]. Some of the key insights that these and other papers provide may be summarized as follows:[1] instantaneous interest rates and the time rate of preference are linked through the expected marginal rate of substitution for time additive preferences. Furthermore, the distributional properties of the consumption path chosen by individuals are

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[1] Papers by Barta [1], Cox, Ingersoll, and Ross [4], Radner [10], and Rubinstein [14], [15] have also addressed related questions. The relation between time rate of preference and interest rate has been recognized in the economic growth literature as well. Rubinstein [15] examines some of the issues addressed in this paper in a discrete time setting.

irrevocably linked to the equilibrium path taken by interest rates. In this context the studies of Breeden [3] and Cox and Ross [6] explicitly address the term structure implications of the distributional properties of the consumption path. One of the messages we glean from these studies is that models which deliver a rich yet potentially refutable optimal consumption behavior are likely to be good candidates for the study of equilibrium interest rates and their term structure. Many researchers have attempted to deliver empirically testable relationships between equilibrium interest rates and their determinants by making strong assumptions about preferences, technology, probability beliefs, and other related factors. The general equilibrium single good model of Cox, Ingersoll, and Ross [5], for instance, specifies identical consumers, logarithmic utility, linear technology, and a mean reverting productivity shock to obtain a rich analytical characterization of consumption function and term structure. In the same paper, they also consider many interesting partial equilibrium models of the term structure. Papers by Rubinstein [13] and Kraus and Litzenberger [8] explicitly obtain specifications for equilibrium interest rates in a world where preferences are represented by logarithmic utility functions. Such equilibrium models which simultaneously determine the consumption function (or savings function) and interest rates provide insight about their determinants and evolution. *This study which shares the spirit of analysis embodied in the papers cited above is a contribution to this growing literature with special emphasis on the stochastic nature of production technology and the role of relative price uncertainty.* Following the insights of Cox, Ingersoll, and Ross [5] and Merton [9], we examine the properties of equilibrium interest rates in stochastic Cobb-Douglas production economies.[2,3] Throughout the paper it is assumed that all investors are identical and that the preferences can be represented by time additive logarithmic utility functions.[4] Models of economy with one and multiple consumption goods are considered. *The results indicate that the instantaneous rate of interest has a tendency to revert to a mean in a single consumption good setting. This finding is consistent with the process used by Cox, Ingersoll, and Ross* [5].[5] The existence of a steady state distribution for the equilibrium interest rate is shown in a single consumption good setting under uncertainty. The properties of the interest rates are also discussed. The analysis is then extended to an economy with multiple consumption goods, wherein the production of a given consumption good not

[2] Merton [9] in his paper points out: "In addition, these analyses often provide "throw-offs" useful in other areas of research. For example, it is usually necessary to postulate some process for the short-rate over time. Using the model of this section, we can *derive an analytical description for the interest rate process ...*" (emphasis mine). Merton [9], however, assumed an exogenous savings function.

[3] Cox, Ingersoll, and Ross [5] note (see their footnote 3): "We consider a pure capital growth model by assuming that labor is unnecessary in production (or that there is a permanent labor surplus state). This provides a more streamlined setting for issues which we wish to stress. There is no essential difficulty in expanding the analysis to include labor inputs and nonlinear technologies." We pursue this direction in our paper.

[4] These assumptions were made by Cox, Ingersoll, and Ross [5] in deriving their analytical results.

[5] The mean reverting behavior exhibited by the instantaneously riskless rate in Cox, Ingersoll, and Ross [5] is attributable to their technology assumptions and *in particular to the exogenous mean reverting behavior postulated for the productivity shock.*

only requires that consumption good but also a common input, called the capital good.[6] *In this setting with endogenous relative price uncertainty, the equilibrium interest rates are directly affected by relative prices.* This conclusion is discussed in the context of the multiple good, *linear technology* analysis of Richard and Sundaresan [11], wherein it is shown that, under some assumptions, the analytical term structure results of Cox, Ingersoll, and Ross [5] are unaffected by the introduction of endogenous relative price uncertainty.

This paper is organized as follows. We describe a general identical consumer multiple good economy in the next section. Then in Section II we specialize to a partial equilibrium single consumption good model with a stochastic Cobb-Douglas technology. This technology requires a capital good (in addition to consumption good) for producing the consumption good. Under a simplifying assumption about the supply of the capital good, we derive and discuss the optimal consumption function and equilibrium interest rates. Section III extends the stochastic Cobb-Douglas technology analysis to a multiple consumption goods environment. The impact of relative price uncertainty on the equilibrium interest rate is explicitly identified. In Section IV we compare our results in the multiple goods framework to those of Richard and Sundaresan [11] and conclude.[7]

In order to focus on the results, all technical arguments and detailed proofs are relegated to the Appendix. Also, to simplify the exposition multiple goods analysis is carried out in a two goods world.

I. Description of the Economy

We consider an economy with a large number, N, of identical, infinitely-lived consumers. Each consumer seeks to maximize his or her lifetime expected utility

$$E\left[\int_0^\infty e^{-\rho t}u(c_1(t), c_2(t))\,dt\right], \tag{1}$$

where $c_i(t)$ is the stochastic process representing the time t rate of consumption of good i, ρ is the rate of impatience, $u(\cdot)$ is an instantaneous utility function, and $E[\cdot]$ is an expectation operator. Each consumer is identically endowed. Let $Q_i(0)$ denote the aggregate stock of consumption good i at time 0 in the economy. Then at time 0 each consumer has an endowment of $q_i(0) \equiv (1/N)Q_i(0)$ good i. The market value of endowments is denoted by $w(0)$. Throughout the paper we will assume that the preferences can be represented by the utility function

$$u(c_1, c_2) = \lambda_1 \ln c_1 + \lambda_2 \ln c_2. \tag{2}$$

Whenever we consider a single consumption good economy we will set $\lambda_2 = 0$ and without loss of generality $\lambda_1 = 1$.[8] Each consumption good is produced by a

[6] This introduces a common link across production technologies. The capital good sector is exogenous to the model. In other words, the rental payment made to hire capital good is a deadweight loss.

[7] The analytical results corresponding to the linear technology were reported in Richard and Sundaresan [11] but did not appear in Richard and Sundaresan [12].

[8] The additivity across time and goods is a critical assumption to most of our results.

stochastic constant returns to scale technology. The technology for consumption good i, in general, uses both good i and a capital good as inputs and produces an uncertain quantity of consumption good i. Each production technology changes over time due to random productivity shocks, which are specified exogenously. All consumers have free access to the production technologies. Let $k_i(t)$ be the quantity of capital good invested in the production technology of consumption good i at time t and $x(t)$ be a state variable which summarizes the state of the production technology at time t. Output of consumption good i net of consumption and rentals to capital $dq_i(t)$, is described by the stochastic differential equation[9]

$$dq_i = [\mu_i(q_i, k_i, x) - c_i - k_i y]\, dt + G_i(q_i, k_i, x)\, dZ \qquad \text{for } i = 1, 2. \tag{3}$$

In Equation (3) μ_i is the mean instantaneous gross output and $k_i y\, dt$ is the amount of rentals paid for the capital good investment. The equilibrium rental rate y will be determined endogenously. The total supply of capital good is fixed at K.[10] The capital good sector is exogenous to the model. In the model I am studying, all the risk is borne only by the identical investors and none by the absent owners of the capital good. If the capital good is owned by investors in equal proportions and the "rental rate" $y = 0$, then all our results will hold in such a setting as well.

In Equation (3) $G_i = (g_{i1}, g_{i2}, g_{i3})$ is a 1×3 vector of diffusion coefficient functions and $Z(t) = (z_1, z_2, z_3)'$ is a 3×1 vector of 3 independent Wiener processes. The functions μ_i and G_i are homogeneous of degree one in q_i and k_i, thereby making production stochastic constant returns to scale. Notice that (3) implies higher marginal net productivity with lower consumption. Each consumer knows that the dynamics of x are given by the stochastic differential equation

$$dx = \beta(x)\, dt + S(x)\, dZ \tag{4}$$

In (4) β is a function of x denoting the expected instantaneous change in x and $S(x)$ is a 1×3 diffusion matrix such that $SS' > 0$ is the instantaneous variance of the state variable. The state of the economy is fully captured by the triple (Q_1, Q_2, x).[11] The opportunity set facing consumers has several markets. A competitive spot market exists for both consumption goods. Consumption good 1 is assumed to be the numeraire good by convention and its price is always one. The relative price of consumption good 2 is denoted by P. By Ito's Lemma the dynamics of P are

$$dP = \beta_P\, dt + \sigma_P\, dZ, \tag{5}$$

where β_P and σ_P are defined in the Appendix. Consumers can also create and trade in competitive markets unit discount bonds on the numeraire good. The unit discount bond price, $F(t, T)$, is the price to be paid at time t in the numeraire good for default-free delivery of one unit of the numeraire good at $T > t$. We assume that consumers can instantaneously borrow or lend the numeraire good at the instantaneously risk-free interest rate r.[12] By Ito's Lemma the dynamics

[9] See Cox and Miller [7] for a lucid treatment of stochastic differential equations.

[10] I thank a referee for pointing this out to me. Alternative supply conditions and preferences have been explored in Sundaresan [16], [17].

[11] This assertion follows from the Markov property of the stochastic processes used in our analysis.

[12] The equilibrium instantaneous interest rate $r(t)$ is defined by $r(t) = -\dfrac{\partial F(t, t)}{\partial T}$.

of F are

$$dF = \beta_F\, dt + \sigma_F\, dZ, \tag{6}$$

where β_F and σ_F are defined in the Appendix.

We now proceed to sketch the consumer's optimization problem briefly. The consumer at each point in time must allocate his wealth among investments in the production technologies, unit discount bonds, and riskless borrowing or lending. In addition he must make his consumption choice. Let $q_i(t)$ be the investment of consumption good i and $m(t)$ be the number of unit discount bonds held. His total wealth, $w(t)$, must be fully invested at each time t. Therefore the wealth invested in riskless borrowing or lending is given by $[w - q_1 - Pq_2 - mF]$. The change in wealth over the next instant is the intertemporal budget dynamics and can be written as

$$dw = \beta_w\, dt + \sigma_w\, dZ, \tag{7}$$

where β_w is the instantaneous expected change in wealth and $\sigma_w \sigma_w'$ is the variance of this change. In the Appendix we have derived β_w and σ_w explicitly.

The consumer maximizes his lifetime expected utility (1) subject to his budget constraint (7). The price functions $P(\cdot)$, $r(\cdot)$, and $F(\cdot)$, the dynamics (4), and the dynamics of $Q_1(t)$ and $Q_2(t)$ are taken as given by the consumer. The Bellman equation for this problem and the first-order conditions are fairly standard. They are reported in the Appendix. The market clearing conditions are intuitive: *since all consumers are identical, there will be no lending or borrowing in the instantaneously riskless market. For the same reason no unit discount bonds will be held at equilibrium. Consequently, all the wealth must be allocated to investment in the production technologies.* Furthermore, the total amount of capital good demanded for investment purposes must be equal to the total fixed supply of the capital good. A further condition for market equilibrium is that consumers have rational expectations. In other words, the price functions and state dynamics assumed by consumers coincide with the actual price functions and state dynamics implied by the aggregation of the consumers' optimal decisions.[13] The formulas for the equilibrium price functions at this level of generality can be derived following the insights of Cox, Ingersoll, and Ross [5].[14] In order to get sharper insights we proceed to examine some special cases in the rest of this paper.

II. Single Consumption Good Cobb-Douglas Stochastic Production Economy

In this section, we specialize to a single consumption good so that there is no relative price uncertainty. Furthermore, we impose the following restrictions on (3):

$$\mu_1(q_1,\, k_1,\, x) \equiv q^{\alpha} k^{1-\alpha} \mu \qquad \text{where } 1 > \alpha > 0,$$

$$G_1(q_1,\, k_1,\, x) \equiv [\sigma q,\, 0,\, 0],$$

[13] Since all consumers are identical, there are no aggregation problems in this paper.

[14] See also Breeden [2], Rubinstein [14], and Richard and Sundaresan [12].

and

$$Z(t) \equiv [z_1, 0, 0]'.$$

These restrictions imply that there is only one source of uncertainty. Moreover with these restrictions the production technology for the consumption good can be written (gross of consumption and rental payments to capital good) as

$$dq = q^{\alpha}k^{1-\alpha}\mu\, dt + \sigma q\, dz_1, \tag{8}$$

It is instructive to rewrite (8) as follows:

$$\frac{dq}{q} = \left(\frac{q}{k}\right)^{\alpha-1} \mu\, dt + \sigma\, dz_1. \tag{9}$$

Equation (9) makes it transparent that the rate of return from the consumption good technology is additively uncertain. The expected return depends on the ratio of the capital to consumption good. The variance of the return is constant, and is independent of the quantities invested. An intutitive way to think about this technology is that the choice of consumption good investment to capital good investment ratio can be used to control the expected return from the technology. However, the variance of return from technology is always constant. This technology specification was chosen as it provides a simple generalization of linear technology to illustrate the point we wish to make. At this stage there is no essential difficulty in introducing many technologies with less than perfectly correlated returns, so long as the parameter α is the same across all technologies.[15] We retain the single production technology assumption to simplify exposition.

We now state our main result in this section as:

Theorem 1. *The equilbrium instantaneous interest rate is given by*

$$r = \alpha\mu\left(\frac{Q}{K}\right)^{(\alpha-1)} - \sigma^2.$$

If there are many technologies with $\mathbb{R}$ denoting the covariance matrix of instantaneous returns from M risky technologies, then

$$r = \alpha\mu\left(\frac{Q}{K}\right)^{\alpha-1} - [\underline{1}'\mathbb{R}^{-1}\underline{1}]^{-1}.$$

where $\underline{1}$ is an $M \times 1$ vector and $\mathbb{R}$ is assumed to be nonsingular. Furthermore, the equilibrium interest rate has a nontrivial steady state distribution and exhibits mean reversion.[16]

Before we proceed to discuss the results of Theorem 1, we note in passing that the aggregate consumption function is identical to the one found by Cox, Ingersoll, and Ross [5] and is given by

$$C = \rho Q.$$

[15] I thank a referee who suggested this extension.

[16] The result corresponding to the multiple technologies case indicates that there is a diversification effect, leading to lower overall variance.

The equilibrium instantaneous rate of interest is an increasing function of μ and a decreasing function of σ^2. In order to gain an intuitive feel for the relationship between r, Q, and their dynamics, the problem is first studied with $\sigma^2 = 0$. The consumption good stock dynamics under these conditions can be determined as

$$\frac{dQ}{Q} = (r - \rho)\, dt.$$

Thus the consumption good stock grows when $r > \rho$ and depletes when $r < \rho$. When $r = \rho$ there is no change in the consumption good stock. There is a market feedback mechanism that drives the economy to this steady state. For instance when $r > \rho$, the consumption good stock grows. As it grows, the interest rate tends to fall as can be verified from Theorem 1. Eventually r reaches a value equal to ρ, the rate of time preference and the consumption good stock reaches a steady state. Similarly, when $r < \rho$, the consumption good stock falls. This causes the interest rate to rise. This adjustment takes place until the interest rate equals the rate of time preference. Thus one can regard $r = \rho$ as the steady state of the economy.

Using the dynamics for Q at equilibrium in the expression for interest rate it is seen that interest rate dynamics follow

$$\frac{dr}{r} = (\alpha - 1)(r - \rho)\, dt.$$

Given that we know the interest rate r at time 0 to be $r(0)$ we may solve this differential equation to obtain interest rate at any time t to get

$$r(t) = \frac{r(0)e^{\rho(1-\alpha)t}}{1 + \dfrac{r(0)\{e^{\rho(1-\alpha)t} - 1\}}{\rho}}.$$

Clearly, the time rate of preference and the parameter α determine the relation between $r(t)$ and $r(0)$. It can be shown that the unit discount bond prices are given by

$$F(r, t, T) = \left[1 + \frac{r\{e^{(1-\alpha)\rho(T-t)} - 1\}}{\rho}\right]^{-\frac{1}{1-\alpha}}. \tag{10}$$

The bond price is a decreasing function of maturity and the spot (instantaneous) rate of interest.

Once again under a steady state $r = \rho$ and the bond prices reduce to

$$F(r, t, T) = e^{-\rho(T-t)}.$$

This has an intuitive present value interpretation. In general the unit discount bond prices inherit the parameters of the technology, time rate of preference of the individuals, the concurrent spot rate of interest, and the time of maturity. Since there is a one-to-one relation between the spot rate r and the aggregate physical stock of consumption good Q, the interest rate effectively serves as a state variable for the purposes of determining the yield curve.[17] In this particular case it has been possible to get an analytical characterization of the yield curve as,

$$y(t, T) = \frac{\ln\left(1 + \dfrac{r\{e^{(1-\alpha)\rho(T-t)} - 1\}}{\rho}\right)}{(T - t)(1 - \alpha)}.$$

When $\sigma^2 \neq 0$, the dynamics of the instantaneous rate of interest can be shown to follow the stochastic differential equation

$$dr = (\alpha - 1)(r + \sigma^2)\left[\left(r - \rho + \frac{\alpha\sigma^2}{2}\right) dt + \sigma\right] dz_1.$$

The distributional properties of instantaneous spot rate of interest are directly dependent upon the stochastic process followed by the consumption good stock at equilibrium in the economy. This follows from Theorem 1. It is now shown that a nontrivial steady state distribution exists for the aggregate consumption good stock. The dynamics for Q in this economy are

$$\frac{dQ}{Q} = [r + \sigma^2 - \rho]\, dt + \sigma\, dz_1. \tag{11}$$

The expected growth rate in the consumption good stock has an intuitive interpretation: with log utility, the relative risk aversion coefficient is unity so that σ^2 represents the risk premium and ρ represents the depletion rate due to consumption. Thus, as consumers become more impatient, current consumption tends to increase causing a reduction in the expected growth rate of consumption good stock. Compared to the certainty case, the expected growth rate tends to be higher for higher values of σ^2.

Let us define $\vartheta \equiv \ln Q$. Then (11) could be rewritten as

$$d\vartheta = h(\vartheta)\, dt + \sigma\, dz_1, \tag{12}$$

where

$$h(\vartheta) \equiv \alpha\mu K^{(1-\alpha)} e^{-(1-\alpha)\vartheta} - \left(\rho + \frac{\sigma^2}{2}\right). \tag{13}$$

Note that $h(\vartheta)$ is continuous in ϑ. Furthermore, $\lim_{\vartheta \to -\infty} h(\vartheta) = \infty$ and $\lim_{\vartheta \to +\infty} h(\vartheta) = -\left(\rho + \frac{\sigma^2}{2}\right) < 0$. Under these conditions it is demonstrated in the Appendix that

[17] Cox, Ingersoll, and Ross [5] have noted (see their footnote 30) that: "The most important circumstances sufficient for bond prices to depend only on the spot rate are: (i) the technology is nonstochastic and the interest rate is a monotone function of wealth, and (ii) individuals have constant relative risk aversion, uncertainty in the technology can be described by a single variable and the interest rate is a monotonic function of this variable." As pointed out by the referee, the technological environment must be nonstochastic for this result to obtain.

$-\infty$ and $+\infty$ are inaccessible for the process ϑ. This in turn implies that 0 and $+\infty$ are inaccessible for the process Q. Hence a nontrivial steady state distribution for Q exists. Note that there is a one-to-one correspondence between interest rate and consumption good stock as given by

$$r = \alpha\mu K^{(1-\alpha)}Q^{(\alpha-1)} - \sigma^2 \tag{14}$$

Equations (11) and (14) can be used in conjunction to study the behavior of interest rate in this economy. For instance, let us consider a state where r is very high to begin with. From Equation (11), this causes the consumption good stock to grow, on average. As the consumption good stock grows the spot rate tends to go down, as can be verified from Equation (14). Similarly, if r is very low to begin with, so much so that $r + \sigma^2 < \rho$, then the consumption good stock declines, on average. This causes the interest rate to go up. Thus, there is a correcting mechanism in the economy which causes the interest rate to fluctuate around a value dependent on ρ and σ^2. As shocks from the technology are realized, the consumption good stock fluctuates causing the interest rate to move. As described above there is a tendency for the interest rate to fluctuate around a mean.

What are the bounds for r? Notice that Q is bounded by the open interval $(0, \infty)$ and has a nontrivial steady state distribution in this interval. As Q tends to zero, r tends to ∞ but as Q tends to ∞, r tends to $-\sigma^2$. So the interest rate is bounded by the open interval $(-\sigma^2, \infty)$. This introduces the possibility that under certain states of the economy when the consumption good stock is very high, the real spot rate can be negative. As our analysis has shown, this situation gets corrected by reduction in the stock of consumption good due to higher consumption. As the consumption good stock falls, interest rates become positive. The distributional properties of equilibrium interest rates differ from those obtained by Cox, Ingersoll, and Ross [5]. *But our model also implies a mean reverting behavior for interest rates. In this sense our findings complement those of Cox, Ingersoll, and Ross* [5] . Our results in this context can also be related to the findings of Breeden [3]. Breeden [3] points out that with logarithmic utility function, the riskless rate is equal to the expected return on optimal risky investment, less its variance. This intuition is confirmed by our analysis. The expected rate of consumption growth in our model is $r + \sigma^2 - \rho$. Clearly expected rate of consumption growth is higher when the riskless rate is higher. The variance in the consumption growth rate σ^2 is inversely related to the interest rate. These findings are consistent with the analysis of Breeden [3]. In the approach taken by Breeden [3], only mild assumptions were needed as he was interested in obtaining some general relations. In the approach that we have taken, strong assumptions were made to get analytical characterization of equilibrium interest rate and its distributional properties. By imposing all equilibrium restrictions, we have also provided an explicit equilbrium setting in which our results confirm the findings of Breeden [3].

III. Multiple Consumption Goods Cobb-Douglas Stochastic Production Economy

In the previous section, relative price uncertainty was precluded. We admit explicit relative price uncertainty in this section by introducing a second consumption good whose price relative to the first consumption good is P and is generally stochastic. We impose the following restrictions on (3);

$$\mu_i(q_i, k_i, x) \equiv q_i^\alpha k_i^{1-\alpha}\mu_i \qquad \text{where } 1 > \alpha > 0,$$

$$G_1(q_i, k_i, x) \equiv [\sigma_1 q_1, 0, 0],$$

$$G_2(q_i, k_i, x) \equiv [0, \sigma_2 q_2, 0],$$

and

$$Z(t) \equiv [z_1, z_2, 0]'.$$

Again our assumption implies that $dx = 0$ or the technological environment is nonstochastic. Thus the production technology is given by

$$dq_i = q_i^\alpha k_i^{1-\alpha}\mu_i\, dt + \sigma_i q_i\, dz_i, \qquad (i = 1, 2).$$

These restrictions imply two important properties for the production technologies. First, each production technology is stochastic constant returns to scale. Second, there are substitution possibilities introduced by the fact that the capital good is needed to produce either of the two consumption goods. This will turn out to be an important factor in our subsequent analysis. *At each instant, the consumer must allocate a fixed supply of capital good between the production technologies of two consumption goods, in a regime of relative price uncertainty.* It is of interest to ask what specification will emerge for the equilibrium interest rate in this setting. This is addressed in the next theorem.

THEOREM 2. *The equilibrium interest rate is given by*

$$r = \alpha\mu^{1/\alpha}K^{1-\alpha}[Q_1\mu_1^{1/\alpha} + Q_2(P\mu_2)^{1/\alpha}]^{\alpha-1} - \sigma_1^2 \tag{15}$$

where the relative spot price P of consumption good 2 is given by

$$P = \frac{\lambda_2}{\lambda_1}\frac{Q_1}{Q_2}. \tag{16}$$

The proof of this theorem is in the Appendix. Two important observations may be made at this juncture. First, the equilibrium interest rate is directly influenced by the relative spot price P. Thus, shifts in the consumption opportunity set via relative price changes have a direct influence on the instantaneous spot rate of interest. This is likely to have interesting term structure implications. Second, the equilibrium interest rate can be expressed solely as a function of aggregate consumptions of good 1 and good 2. In the Appendix it is demonstrated that the aggregate consumption functions are given by

$$C_i = \rho Q_i. \tag{17}$$

Inspection of (15), (16), and (17) suggests that the interest rate can be expressed strictly in terms of C_1 and C_2 by substituting for P and Q_i. The key result that the equilibrium interest rate is directly affected by relative price should come as no surprise. Interest rates are anchored to marginal products from risky production technologies. When technologies are linear, marginal product depends only on exogenously specified productivity shocks. Thus with $\alpha = 1$, there is no substitution and $r = \mu_1 - \sigma_1^2$. Hence consumption goods stock do not influence interest rates. With Cobb-Douglas technologies, however, marginal products depend on consumption good stock and therefore on relative prices.

The properties of the equilibrium rate of interest are fairly intuitive. It is decreasing in σ_1^2 and increasing in the aggregate stock of capital good. The equilibrium interest rate is a decreasing function of Q_1 and an increasing function of Q_2. The dependence of equilibrium interest rate on the aggregate consumption is linear in aggregate stock as given by (17). The dynamics of the equilibrium interest rate depend in a fairly complicated way on the evolution of aggregate consumption goods stock. The interest rate specification (15) is fairly rich. Under certainty the equilibrium interest rate can be written as

$$\ln r = \beta_0 + \beta_1 \ln Kx + \beta_2 \ln[C_1 + \beta_3 C_1^{\beta_4} C_2^{(1-\beta_4)}] \tag{18}$$

where the coefficients β_i ($i = 0, \cdots, 4$) obey the following restrictions

$$\beta_0 = \ln\left[\frac{\alpha}{(1-\alpha)\rho}\right]; \qquad \beta_1 = 1; \qquad \beta_2 = -1;$$

$$\beta_3 = \left[\frac{\lambda_2 \mu_2}{\lambda_1 \mu_1}\right]^{1/\alpha} > 0; \qquad \text{and} \quad \beta_4 = \alpha^{-1} > 0.$$

Kx are the aggregate payments to the capital good sector. C_1 and C_2 are the aggregate physical consumption of good 1 and 2, respectively. Except for β_0 all the coefficients have refutable restrictions either in terms of values or signs. The sign of β_0 depends on the factor shares parameter, α, $(1 - \alpha)$ and their relation to the time rate of preference ρ. This specification places refutable restrictions on real rate of interest, capital expenditure, and aggregate consumption. The aggregate consumption function is robust and is linear in the aggregate stock of goods. The log utility assumption is essentially the main reason behind this result. With log utility individuals do not "hedge" movements in the opportunity set and hence their consumption decisions are myopic for a wide range of economic scenarios.

We proceed to relate our results to those of Richard and Sundaresan [11] and conclude in the next section.

IV. Conclusion

Richard and Sundaresan [11] demonstrated that the analytical term structure results obtained by Cox, Ingersoll, and Ross [5] in a general equilibrium setting are unaffected by the introduction of endogenous relative price uncertainty. In order to demonstrate this result they made certain assumptions which in the context of our paper imply the following restrictions on (3):

$$\mu_i(q_i, k_i, x) \equiv q_i \mu_i x,$$

$$G_i(q_i, k_i, x) \equiv [g_{i1}\sqrt{x}, g_{i2}\sqrt{x}, g_{i3}\sqrt{x}],$$

and

$$Z(t) \equiv [z_1, z_2, z_3]'.$$

Furthermore, they assumed that there is no capital good sector and that (4) is a mean reverting "square root" diffusion process used by Cox, Ingersoll, and Ross [5]. These assumptions were shown to be sufficient for the analytical term structure results of Cox, Ingersoll, and Ross [5] to hold under relative price uncertainty as well.[18] These assumptions imply lack of any complementarity or substitution across production technologies of different goods. In other words, only the output good is used as input to each good's technology. The assumption on preferences made by Richard and Sundaresan [11] was the same as (2). This implies additive preferences across goods. Our analysis in Section III suggests that the introduction of a common input across the production technologies of different goods is sufficient to cause the equilibrium interest rate to depend directly on relative prices. It may be possible to obtain similar conclusions by permitting substitutions or cross-elasticities of demand across goods. This will require relaxing the additivity (across goods) assumption made in (2).

Our analysis is intended to complement the research work on equilibrium interest rate determination by Cox, Ingersoll, and Ross [5] and Breeden [3]. By focusing on stochastic production technologies and relative price uncertainty this paper has highlighted the importance of these variables on the determination of equilibrium interest rates and their term structure. It appears that substitution or complementarity in either preferences and/or production technologies might lead to rich term structure specifications. Introduction of substitution possibilities across time and goods through preferences is a potentially interesting topic for future research. One of the lessons learned from this study is that the choice of assumptions about technology and the number of goods in the economy seems to be crucial in determining the properties of equilibrium interest rates.

Appendix

I. Derivation of Price Dynamics

Let $\pi(t) = \pi(Q_1, Q_2, X, t)$ be the price of any claim. Then,

$$d\pi(t) = (L\pi)\,dt + \pi_x S + \pi_Q G\,dZ. \tag{A}$$

In the equation above, $\pi_x \equiv \dfrac{\partial\pi}{\partial x}$; $\pi_Q = \left(\dfrac{\partial\pi}{\partial Q_1}, \dfrac{\partial\pi}{\partial Q_2}\right)$; $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$; and L is a differential operator defined by

$$L \equiv \frac{\partial}{\partial x}\beta(x) + \textstyle\sum_{i=1}^{2} \dfrac{\partial}{\partial Q_i}\phi_i + \dfrac{1}{2}\textstyle\sum_{i=1}^{2}\sum_{j=1}^{2} (GG')_{ij} \dfrac{\partial^2}{\partial Q_i \partial Q_j}$$
$$+ \frac{1}{2}(SS')\frac{\partial^2}{\partial x^2} + \textstyle\sum_{i=1}^{2} (GS')_i \dfrac{\partial^2}{\partial Q_i \partial x} + \dfrac{\partial}{\partial t}.$$

[18] In addition, it was assumed that all consumers are identical. The log utility assumption is critical to the result as well.

Using this general relationship (A), we can explicitly identify β_P, σ_P, β_F, and σ_F. (ϕ_i denotes the drift term in the dynamics of the aggregate stock of consumption good i.)

II. Derivation of Wealth Dynamics

$$dw = dq_1 + Pdq_2 + q_2\,dP + dP\,dq_2 + m\,dF + (w - q_1 - Pq_2 - mF)\,r\,dt.$$

Substituting from (3)–(6) and simplifying we get

$$dw = \beta_w\,dt + \sigma_w\,dz,$$

where

$$\beta_w \equiv [\mu_1 + P\mu_2 - c_1 - Pc_2 - k_1 y - k_2 y + q_2\beta_P + Pq_2\sigma_P G_2' + m(\beta_F - rF) + r(w - q_1 - Pq_2)]$$

and

$$\sigma_w \equiv (G_1 + PG_2 + q_2\sigma_P + m\sigma_F).$$

III. General Optimization Problem

The Bellman equation is

$$0 = \max_{\{c_1, c_2, q_1, q_2, k_1, k_2, m\}} [u(c_1, c_2) - \rho J + LJ + \beta_w J_w + \tfrac{1}{2} J_{ww}\sigma_w\sigma_w' + \sigma_w S' J_{wx} + \sigma_w G' J_{wQ}]. \quad \text{(A1)}$$

In the equation above, J is the optimum value function. Let us rewrite the Bellman equation as

$$0 = u(c_1, c_2) + \hat{L}J - \rho J,$$

where

$$\hat{L} = L + \beta_w \frac{\partial}{\partial w} + \frac{1}{2}\sigma_w\sigma_w' \frac{\partial^2}{\partial w^2} + \sigma_w S' \frac{\partial^2}{\partial w \partial x} + \sigma_w G' \frac{\partial^2}{\partial w \partial Q}.$$

Then the relevant first-order conditions are:

$$\frac{\partial u}{\partial c_1} - J_w = 0, \quad \text{(A2)}$$

$$\frac{\partial u}{\partial c_2} - PJ_w = 0, \quad \text{(A3)}$$

$$\left[\frac{\partial \mu_1}{\partial q_1} - r\right] J_w + \left[\frac{\partial G_1}{\partial q_1}\right][\sigma_w' J_w + S' J_{wx} + G' J_{wQ}] = 0, \quad \text{(A4)}$$

$$\left[P\frac{\partial \mu_2}{\partial q_2} + \beta_P + P\sigma_P G_2' - rP\right] J_w + \left[P\frac{\partial G_2}{\partial q_2} + \sigma_P\right][\sigma_w' J_w + S' J_{wx} + G' J_{wQ}] = 0, \quad \text{(A5)}$$

$$\left[\frac{\partial \mu_1}{\partial k_1} - y\right] J_w = 0, \tag{A6}$$

$$\left[P\frac{\partial \mu_2}{\partial k_2} - y\right] J_w = 0, \tag{A7}$$

and

$$[\beta_F - rF]J_w + \sigma_F \sigma_w' J_{ww} + \sigma_F S' J_{wx} + \sigma_F G' J_{wQ} = 0. \tag{A8}$$

IV. Proof of Theorem 1

The proof is by actual substitution. First we impose the restrictions on $\mu_i(\cdot)$, $G_i(\cdot)$, and $Z(t)$ from Section II. These are substituted in (A1)–(A8). It may be verified that the following controls and prices satisfy (A1)–(A8).

$J = \frac{1}{\rho} \ln w + \psi(Q)$ is the indirect value function. The consumption function is $c = \rho w$. The consumption good investment is $q = w$. The capital good investment is $k = \frac{Kw}{Q}$. The number of bonds held is $m = 0$. The equilibrium interest rate is $r = \alpha\mu\left(\frac{Q}{K}\right)^{\alpha-1}$ and the equilibrium rental rate is $y = (1-\alpha)\mu\left(\frac{Q}{K}\right)^{\alpha}$. Q.E.D.

V. Existence of a Nontrivial Steady State Distribution

The proof follows directly from Merton [9] and is provided only for completeness. In Equation (12), note that $h(\vartheta)$ is continuous. Furthermore

$$\lim_{\vartheta \to +\infty} h(\vartheta) = -\left(\rho + \frac{\sigma^2}{2}\right) < 0$$

$$\lim_{\vartheta \to -\infty} h(\vartheta) = \infty$$

It then follows that there exists an $\underline{x} > -\infty$ such that for all $\vartheta \in [-\infty, \underline{x}]$, there exists a $\delta_1 > 0$ such that

$$h(\vartheta) \geqq h(\underline{x}) \geqq \delta_1$$

In a similar manner, there exists an $\bar{x} < \infty$ such that for all $\vartheta \in [\bar{x}, \infty]$, there exists a $\delta_2 < 0$ such that

$$h(\vartheta) \leqq h(\bar{x}) \leqq \delta_2$$

Consider a Wiener process $W_1(t)$ with drift δ_1 and variance σ^2 defined on the interval $[-\infty, \underline{x}]$ where $\underline{x}$ is a reflecting barrier. Cox and Miller [7] have demonstrated such a process has a nondegenerate steady state and hence $-\infty$ is an inaccessible boundary. Since $h(\vartheta) \geqq \delta_1$ for the process, followed by ϑ as well, the boundary is not accessible.

In a similar manner, Cox and Miller [7] show that for the process $W_2(t)$ with drift δ_2 and variance σ^2 defined on the interval $[\bar{x}, \infty]$ where $\bar{x}$ is a reflecting barrier, ∞ is inaccessible so long as $\sigma_2 < 0$. Since $h(\vartheta) \leqq \delta_2$ in this interval $-\infty$ is inaccessible for ϑ.

The proof is complete once it is noted that $\vartheta \equiv \ln Q$. Intuitively when ϑ takes very high values, the drift term $h(\vartheta)$ becomes negative. Similarly when ϑ takes large negative values $h(\vartheta)$ becomes positive. This accounts for the existence of a nontrivial distribution. Q.E.D.

VI. Proof of Theorem 2

The proof is by actual substitution. First we substitute the restrictions on $\mu_i(\cdot)$, $G_i(\cdot)$, and $Z(t)$ from Section III into (A1)–(A8). It may be verified that the following controls and prices satisfy (A1)–(A8).

The indirect value function is

$$J = \frac{\lambda_1 + \lambda_2}{\rho} \ln w + \psi(Q_1, Q_2).$$

The optimal consumption of goods is

$$c_1 = \frac{\lambda_1 \rho}{\lambda_1 + \lambda_2} w, \qquad c_2 = \frac{\lambda_1 \rho}{\lambda_1 + \lambda_2} \frac{Q_2}{Q_1} w.$$

The optimal consumption good investments are given by

$$q_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2} w, \qquad q_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2} w.$$

The optimal capital good investments are

$$k_1 = \frac{\mu_1^{(\alpha^{-1})} K q_1}{Q_1 \mu_1^{(\alpha^{-1})} + Q_2 (P\mu_2)^{(\alpha^{-1})}},$$

$$k_2 = \frac{(P\mu_2)^{(\alpha^{-1})} K q_2}{Q_1 \mu_1^{(\alpha^{-1})} + Q_2 (P\mu_2)^{(\alpha^{-1})}}.$$

The number of bonds held is $m = 0$. The relative spot price is given by

$$P = \frac{\lambda_2}{\lambda_1} \frac{Q_1}{Q_2}.$$

The equilibrium rental rate is

$$y = (1 - \alpha) K^{-\alpha} [Q_1 \mu_1^{1/\alpha} + Q_2 (P\mu_2)^{1/\alpha}]^{\alpha}.$$

and the equilibrium rate of interest is given by (15). Q.E.D.

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