Econometrica, Vol. 78, No. 1 (January, 2010), 309–347

RECURSIVE EQUILIBRIUM IN STOCHASTIC OVERLAPPING-GENERATIONS ECONOMIES

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We prove the generic existence of a recursive equilibrium for overlapping-generations economies with uncertainty. "Generic" here means in a residual set of utilities and endowments. The result holds provided there is a sufficient number of potentially different individuals within each cohort.

KEYWORDS: Overlapping generations, Markov equilibrium, recursive equilibrium, transversality theorem.

1. INTRODUCTION

THE OVERLAPPING-GENERATIONS (OLG) model, introduced first by Allais (1947) and Samuelson (1958), is one of the two major workhorses for macroeconomic and financial modeling of open-ended dynamic economies. Following developments in the study of two-period economies, the OLG model has been extended to cover stochastic economies with production and possibly incomplete financial markets. As is the consensus, in dynamic economies the general notion of competitive equilibrium à la Arrow and Debreu, which allows for prices and allocations to depend on histories of arbitrary length, is not always fully satisfactory for a variety of reasons. Among them, it is worth recalling at least the following two. First, from a theoretical viewpoint, when prices have unbounded memory, the notion of rational expectations equilibrium is strained because of the complexity of the forecasts and the expectations coordination involved. Second, the ensuing large dimensionality of the allocation and price sequences strains the ability of approximating solutions with present-day computers through truncations, rendering this general notion of equilibrium not very useful for applied, quantitative work, even in stationary Markovian environments.

Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) gave a general theorem for the existence of stationary Markov equilibria for OLG economies with associated ergodic measure.[2] While these equilibria help bypass the two above-mentioned issues, they are still quite complicated as the state space contains a number of past and current endogenous variables. Therefore, the conceptual issue of whether it is possible to find simpler equilibria—stationary Markov equilibria based on a minimal state space—remains open.

[1]We thank John Geanakoplos, Felix Kubler, David Levine, Michael Magill, Herakles Polemarchakis, Martine Quinzii, and three anonymous referees for useful comments. The paper was written while the first author was visiting the Columbia Business School, whose financial support is gratefully acknowledged.

[2]See, for example, also the earlier work by Spear (1985), Cass, Green, and Spear (1992), and Gottardi (1996).

 DOI: 10.3982/ECTA7219

In parallel developments, and to allow for computational work, the literature in macroeconomics and finance has focused on a simpler notion of time-homogeneous Markov equilibrium, also known as recursive equilibrium.[3] In a recursive equilibrium, the state space is reduced to the exogenous shocks and the initial distribution of wealth for the agents—asset portfolios from the previous period, and capital and storage levels if production is considered. The notion of recursive equilibrium also originates in the long-standing tradition of using recursive methods in economics (see, e.g., Stokey and Lucas (1989) and Ljungqvist and Sargent (2000)), and is the "natural" extension of those methods to stochastic OLG models with heterogeneity. A recursive equilibrium can be thought of as a time-homogeneous Markov equilibrium that is based on a minimal state space.

However, no existence theorem is available for such recursive equilibria. In fact, Kubler and Polemarchakis (2004) provided two examples of nonexistence of recursive equilibrium in OLG exchange economies. The idea that recursive equilibria may not exist is based on the observation that when there are multiple equilibria, the continuation of an equilibrium may depend on past economic variables other than the wealth distribution. That is, the current wealth distribution may not be enough to summarize the information contained in past equilibrium prices and marginal utilities.

While this phenomenon may occur, we prove that it is nongeneric under some qualifying condition. The argument follows from two fundamental observations.

First, we show that competitive equilibria, which are time-homogeneous Markov over a *simple* state space, exist. This is the state space made of the current exogenous state, the current wealth distribution, current commodity prices, and marginal utilities of income for all generations except for the first and the last ("newly born" and "eldest"). This result is similar to that obtained by Duffie et al. (1994, Section 2.5). In fact, we even construct such equilibria that guarantee that the Markov state space contains all the wealth distributions that can be taken to be initial conditions for competitive equilibria of each economy considered.

Second, we show that simple Markov equilibria typically satisfy the condition that prices and multipliers are a function of the current state and of the current wealth distribution, provided that there is a large number of individuals within each generation. We call these equilibria nonconfounding, while we call equilibria that do not satisfy that condition confounding. Clearly, nonconfounding simple time-homogeneous Markov equilibria are recursive.

The trick we use is to find a finite system of equations that must necessarily have a solution so that simple Markov equilibria fail to be nonconfounding.

[3]Examples by now abound; see, for example, Rios Rull (1996), Constantinides, Donaldson, and Mehra (2002), Geanakoplos, Magill, and Quinzii (2004), and Storesletten, Telmer, and Yaron (2004).

This trick allows one to bypass the infinite-dimensional nature of the equilibrium set and the fact that, with overlapping generations, there is an infinite number of individuals and an infinite number of market clearing equations, rendering direct genericity analysis quite problematic.

Specifically, if there exists a confounding equilibrium, there is a pair of Markov states with identical current shock and wealth distribution, but different multipliers or different current prices. However, the joint requirement of equal wealth but different multipliers restricts the future prices following the realization of the two critical Markov states to be such that individuals with equal wealth make different choices. Since individuals have finite lives, the latter is a restriction over finitely many future prices. Equivalently, both the dimension of the relevant price processes and the number of wealth equalities are finite. We show that with a large enough number of potentially different individuals of the same generation (Assumption A1), typically the wealth equalities are not satisfied, thereby establishing the generic existence of recursive equilibria. Since we need to check this for all admissible—and not just equilibrium—pairs of price processes, the number of individuals must be large. It should be noted in passing that this is not at odds with the notion of price-taking behavior, which is assumed in competitive models such as ours.

The notion of genericity we will use will rely also on utility perturbations and, therefore, will only be topological. In fact, due to the infinite dimension of the equilibrium set, we will not be able to establish local uniqueness of competitive equilibria, whether or not it is time-homogeneous. Without this prerequisite, the argument that essentially shows some one-to-one property of prices will have to be made without knowing whether or not such prices are "critical"; in fact, without knowing if they are equilibrium prices. Therefore, we will resort to an argument reminiscent of Mas-Colell and Nachbar (1991), and we will show the existence of recursive equilibria for a *residual* or *nonmeager* subset of parameters, that is, a set of stationary utilities and endowments that is dense and is the countable intersection of open and dense sets. This is a well established notion of genericity for dynamic systems.

Our class of OLG economies has multiple goods, generations, and types within each generation. We present first the simplest model where there are only short-lived real—namely, numéraire—assets in zero net supply. Long-lived assets with nonzero real payoffs and in positive net supply, production (as in Rios Rull (1996)), and economies with individual risk are briefly discussed at the end of the paper. Our result encompasses all these extensions and the argument of the paper goes through unchanged. In fact, in the extensions the Markov state space becomes richer, thereby possibly relaxing Assumption A1.

To simplify the exposition, in this paper we carry out the density part of the argument for complete financial markets. When markets are incomplete, density proofs are much more elaborate. Hence, we leave the density analysis with incomplete markets to a companion paper (Citanna and Siconolfi (2007)).

Our result suggests that the notion of recursive equilibrium is coherent as an exact concept, adding robustness to its interpretation and quantitative use.[4] While the number of agents in each cohort must be large, their characteristics can always be taken to be very similar for the underlying economy to have exact recursive equilibria. We must stress that although we cannot claim that economies with a large number of identical individuals are part of the generic set where recursive equilibrium exists, nothing in the argument indicates otherwise.[5]

The paper is organized as follows. Section 2 introduces the stochastic OLG economy model. Section 3 gives the definition of the relevant notions of equilibrium: competitive, simple Markov, and recursive. Section 4 states the main theorem and outlines the fundamental aspects of the argument. In particular, Section 4.1 shows existence of simple Markov equilibria, while Section 4.2 contains the properties of demand that arise from a distribution of agents in two cohorts. Section 5 discusses extensions to the basic model and related results.

2. THE MODEL

We consider standard stationary OLG economies. Time is discrete, indexed by $t = 0, 1, 2, \ldots$. There are $S > 1$ states of the world that may be realized in each period. At each t, H individuals enter the economy. Each individual $h \in H$ lives $G + 1 \geq 2$ periods, indexed by $a = 0, \ldots, G$, from the youngest ($a = 0$) to the oldest ($a = G$) age. If and when we need to make explicit that a variable or function is of a specific individual of type h (and of age a), we use the superscript h (and ha). At each t, $C \geq 1$ physical commodities are available for consumption. The consumption bundle of an individual of age a is $x_t^a \in \mathbb{R}_{++}^C$. Each individual h has a discounted, time-separable, von Neumann–Morgenstern utility function with time-, state-, and age-invariant Bernoulli utility index u^h at age a. Each $u^h : \mathbb{R}_{++}^C \to \mathbb{R}$ is twice continuously differentiable, differentially strictly increasing, and differentially strictly concave (the Hessian is negative definite), and satisfies the following boundary condition: if $x_{c,t}^{ha} \to 0$, then $\|Du^h(x_t^a)\| \to +\infty$, where Du^h denotes the vector of partial derivatives. The common discount factor is $\delta \in (0, 1]$. At each t, endowments are $e_t^{ha} \in \mathbb{R}_{++}^C$ for $h \in H$ and $0 \leq a \leq G$.

[4]For their interpretation as mere computing devices, see Kubler and Schmedders (2005). When feedback policies are examined in a recursive formulation, their interpretation is of course strengthened by the exact, as opposed to only approximate, nature of recursivity. As we stressed earlier, the focus on recursive equilibria can also be justified by their "simplicity" when they are exact.

[5]On the other hand, if an economy with possibly many identical individuals does not have a recursive equilibrium, independent and arbitrarily small perturbations of their characteristics restore its existence. While Assumption A1 does not necessarily encompass the economies of the examples constructed by Kubler and Polemarchakis (2004), in a previous paper (Citanna and Siconolfi (2008)) we showed that such examples are nonrobust.

At each t, there are competitive spot markets for the exchange of physical commodities. The price vector of the C commodities is $p_t = (\ldots, p_{c,t}, \ldots) \in \mathbb{R}^C_{++}$. Commodity $c = 1$ is dubbed the numéraire commodity and hereafter, unless we say otherwise, we adopt the normalization $p_{1,t} = 1$ for all t. There are also competitive markets for trading S one-period securities in zero net supply, with prices $q_t \in \mathbb{R}^S$. We set $\psi_t \equiv (p_t, q_t)$.

The security payoffs $d_t \in \mathbb{R}^S$ are in units of the numéraire commodity. The portfolio of an individual of age a is $\theta^a_t \in \mathbb{R}^S$. We assume that $\theta^{-1}_t \equiv \theta^G_t \equiv 0$ and, denoting with $w^a_t \equiv d_t \theta^{a-1}_{t-1}$ the financial wealth of an individual of age a at t, that $w^0_t = 0$ for all t. Also, $w^a_0 = d_0 \theta^{a-1}_{-} \in \mathbb{R}$ is the individual initial wealth at $t = 0$ for $a > 0$, while $w_0 \equiv (w^{ha}_0)_{h\in H}$ denotes the initial wealth distribution of the economy.

Finally, we let λ^{ha}_t be the marginal utility of income for all (h, a); that is, $\lambda^{ha}_t = D_1 u^h(x^a_t)$, where D_1 denotes the derivative with respect to the first entry of the vector x^a_t.

To define the various notions of equilibrium, all exogenous and endogenous variables are seen as stochastic processes on a probability space $\langle \mathcal{X}, \mathcal{F}, \Pr\rangle$. First, the sequence of exogenous shocks is seen as a realization of the process $\tilde{s} = (s_t, t = 0, 1, \ldots)$ that is constructed in a standard way through a given $S \times S$ stochastic matrix π, where each $s_t : \mathcal{X} \to S$ is $\mathcal{F}$-measurable, and with initial shock $s_0(\chi) = s_0$ for some $s_0 \in S$ for Pr-a.a. χ. The matrix π is the time-invariant transition of this process and $\pi(s_{t+1}|s_t) = \Pr\{\chi : s_{t+1} = s_{t+1}(\chi)|s_t = s_t(\chi)\}$ defines the probability of shock realization s_{t+1} given s_t, that is, $\tilde{s}$ is a time-homogeneous, first-order Markov chain.

Endowments e^{ha}_t are assumed to be affected only by current realizations of $s \in S$ and are denoted by $e^{ha}_s \in \mathbb{R}^C_{++}$ for $h \in H$, $0 \leq a \leq G$, and $s \in S$. The security payoff d_t is also only affected by the state $s_t \in S$ and is d_{s_t}, and is assumed to give rise to a full-rank, $S \times S$-dimensional matrix $\mathbf{D}$ which is then time and history invariant. Hence, state realizations affect endowments and preferences of all individuals of all ages and security payoffs, while time does not. Thus, the fundamentals of our economies follow a time-homogeneous, first-order Markov process, so that the economy is *stationary*.

In what follows, we keep $(\pi, \mathbf{D})$ fixed and we identify an economy with an endowment and utility profile e^{ha}_s, u^h, for all $h \in H$, $a = 0, \ldots, G$, $s \in S$, and discount factor $\delta \in (0, 1]$. The endowment space E is an open subset of $\mathbb{R}^{H(G+1)SC}_{++}$. The set $\mathcal{U}$ of utilities u^h is the G_δ subset of $\bigtimes_h \mathcal{C}^2(\mathbb{R}^C_{++}, \mathbb{R})$, which satisfies our maintained assumptions and is endowed with the topology of $\mathcal{C}^2$-uniform convergence on compacta. The set $\mathcal{U}$ is a complete metric space, and, by the Baire theorem, countable intersections of open and dense sets are dense in $\mathcal{U}$. Let $\Omega = E \times \mathcal{U} \times (0, 1]$ be the space of economies with the product topology.

3. EQUILIBRIUM

We give a unified view of the different notions of equilibria that we study as Markov processes defined over a common state space. Competitive equilibria of an OLG economy are stochastic processes attached to an initial condition, that is, an initial shock and a wealth distribution. In principle, the entire equilibrium trajectory may depend on the initial condition, and its continuation path may be history dependent. Since we are aiming at a unified treatment, histories and initial conditions must be part of our state space.

For given χ and $\tilde{s}$, a history of length t is an array of shock realizations with identical initial shock, $s^t(\chi) = (s_0, s_1, \dots, s_t)(\chi)$. For a given process $\tilde{s}$, let $S^t_{s_0}$ be the set of all possible histories of length t with initial shock s_0. History s^t precedes history $\bar{s}^{t'}$ and we write $\bar{s}^{t'} \succsim s^t$ if there is an array of $t' - t$ realizations of shocks $s^{t'-t}$ such that $\bar{s}^{t'} = (s^t, s^{t'-t})$. We denote with $\tilde{S}_{s_0} = \bigcup_t S^t_{s_0}$ the set of all possible histories of any length, and denote with initial shock s_0, a tree with nodes s^t and root s_0. When all possible initial conditions need to be considered at once, the union $\tilde{S} = \bigcup_{s_0 \in S} \tilde{S}_{s_0}$ of S trees with distinct roots comes into play.

Let the hyperplane $W \equiv \{w \in \mathbb{R}^{(G+1)H} : w^{h0} = 0$, for all h, and $\sum_{ha} w^{ha} = 0\}$ be the space of wealth distributions. The space of endogenous variables $\Xi = (\mathbb{R}^C_{++} \times \mathbb{R}^S \times \mathbb{R}_{++})_{h,a} \times \mathbb{R}^C_{++} \times \mathbb{R}^S$ has element $\xi = ((x^{ha}, \theta^{ha}, \lambda^{ha})_{h,a}, \psi)$, which comprises the vectors of consumption bundles, of portfolios, of marginal utilities of income for all individuals of all ages, and of prices.

To define trajectories which are history and initial-condition dependent, an array $(s^t, w, \xi) \in \tilde{S} \times W \times \Xi$, specifying a history, current values of the wealth distribution, and endogenous variables, has to be augmented with an initial state $(s_0, w_0, \xi_0) \in Z_0 = S \times W \times \Xi$. The state space is then $Z \subset \tilde{S} \times W \times \Xi \times Z_0$, and we write z to denote a generic element of Z, write ζ for an element of $W \times \Xi$, and write $z_0 = (s_0, w_0, \xi_0)$ for an initial state. We assume that the projection of Z on $\tilde{S}$ is onto, that is, no histories s^t are excluded.

Stochastic processes over Z are defined by specifying a transition. We limit attention to spotless transitions, that is, at each t the process can take as many values as realizations of the exogenous shocks. We can therefore identify a transition with a deterministic map $T : Z \to Z^S$ such that $T(s^t, \zeta, z_0) = ((s^t, s), \zeta_s, z_0)_{s \in S}$. Therefore, $\Pr(z'|z) = \pi(s_{t+1}|s_t)$ for $z = (s^t, \cdot)$ and $z' = ((s^t, s_{t+1}), \cdot)$. We also write $T = (T_s)_{s \in S}$ with $T_s : Z \to Z$ and $z' = ((s^t, s), \zeta, z_0) = T_s(z)$. Notice two aspects of this construction. First, the transition T leaves the initial condition unaltered. Second, the spotless nature of the transition allows us to dispense with any additional restriction on the transition map T such as, for instance, (Borel) measurability with respect to $W \times \Xi$.

The pair (Z, T) generates a family of stochastic processes $\tilde{z} = (\tilde{z}_{z_0}, z_0 \in Z_0)$ or simply a process, each with $z_{z_0,0}(\chi) = (z_0, z_0)$, $z_{z_0,1}(\chi) = T_{s_1(\chi)}(z_{z_0,0}(\chi))$, and, recursively, $z_{z_0,t}(\chi) = T_{s_t(\chi)}[\cdots T_{s_1(\chi))}(z_{z_0,0}(\chi))]$ for Pr-a.a. χ. Naturally a process $\tilde{z}$ generated by (Z, T) induces an endogenous variables process $\tilde{\xi}$ via the natural projection, that is, $\xi_t = \mathrm{proj}_\Xi z_t$, and we say that $\tilde{\xi}$ is generated by (Z, T). Since the stochastic engine of $\tilde{z}_{z_0}$ is the shock process $\tilde{s}$, $\tilde{\xi}_{z_0}$ is adapted to $\tilde{S}$ for all initial conditions z_0.

A process $\tilde{\xi}$ generated by (Z, T) is a family of *competitive equilibrium processes* if, for Pr-a.a. χ and all $t \geq 0$, the following conditions hold:

CONDITION H: For all h,

(1a) $$Du^h(x_t^{ha}) - \lambda_t^{ha} p_t = 0 \quad \text{for all } a,$$

(1b) $$-\lambda_t^{ha} q_t + \delta \mathbb{E}_t\big(\lambda_{t+1}^{h(a+1)} d_{t+1}\big) = 0 \quad \text{for all } a < G,$$

(1c) $$\psi_t[(x_t^{ha} - e_t^{ha}), \theta_t^{ha}] = d_t \theta_{t-1}^{h(a-1)} \quad \text{for all } z' = T(z), \text{ all } a.$$

CONDITION M: $\sum(x_t^{ha} - e_t^{ha}, \theta_t^{ha}) = 0.$

Condition M is the familiar market clearing equation. Condition H is optimality, that is, it is the first-order conditions for the utility maximization problem of any individual h of age a at any t, choosing consumption and asset portfolios facing the competitive sequential budget constraint (1c) for $\tau = t, \dots, G - a$. By the assumptions on u^h and the Markovian structure of the economy, time consistency is satisfied and these conditions apply to the problem faced at any $\tau = t - a, \dots, t$.

A competitive equilibrium is an element $\tilde{\xi}_{\bar{z}}$ from a family of competitive equilibrium processes where $z_0 = \bar{z} = (\bar{s}, \bar{w}, \bar{\xi})$ for Pr-a.a. χ is a given initial condition for the economy. Since the initial condition specifies also the values of the endogenous variables at $\bar{s}$, the same initial pair $(\bar{s}, \bar{w})$ may be associated to multiple continuation paths, one for each distinct vector $\bar{\xi}$. Equivalently, our definition allows for multiple competitive equilibria of an economy ω starting off at $(\bar{s}, \bar{w})$.

As is well known, the existence of a competitive equilibrium $\tilde{\xi}_{\bar{z}}$ for an economy ω restricts the choice of the initial wealth distributions $\bar{w}$ to a bounded subset $W_{\bar{s},\omega}$ of the hyperplane W, with nonempty interior $\mathring{W}_{\bar{s},\omega}$. Given the boundary condition on utilities, for all economies ω and for all initial conditions $(\bar{s}, \bar{w})$, $\bar{w} \in W_{\bar{s},\omega}$, the existence of a competitive equilibrium is established using a standard truncation argument (see Balasko and Shell (1980)).

The equilibrium process $\tilde{\xi}$ is a Markov process, but it is not time-homogeneous. The focus of this paper is on a special kind of stationary, time-homogeneous competitive equilibrium, also known as recursive equilibrium. To study recursive equilibria, our argument will go through and use an intermediate notion of equilibrium that is stationary, time-homogeneous, and Markov, and that we call simple Markov.

A *simple Markov equilibrium* is a family of competitive equilibrium processes $\tilde{\xi}$ generated by a pair (Z, T) that satisfies:

$$T(s^t, \zeta, z_0) = T(\hat{s}^t, \hat{\zeta}, \hat{z}_0) \quad \text{if} \quad (s_t, w^{ha}, \lambda^{ha}, p) = (\hat{s}_t, \hat{w}^{ha}, \hat{\lambda}^{ha}, \hat{p})$$
$$\text{for all } h, a.$$

A *recursive equilibrium* is a simple Markov equilibrium $\tilde{\xi}$ that satisfies:

$$T(s^t, \zeta, z_0) = T(\hat{s}^t, \hat{\zeta}, \hat{z}_0) \quad \text{if} \quad (s_t, w^{ha}) = (\hat{s}_t, \hat{w}^{ha}) \quad \text{for all } h, a.$$

Two aspects are worth noticing. Different restrictions on the transition T deliver different equilibrium notions. The restrictions on the map T translate into restrictions on Z: all elements of Z satisfy the same constraints imposed on T, thereby effectively generating a reduced state space, the subspace of the coordinates making the transition T (potentially) injective. Thus, the reduced state space of a simple Markov equilibrium includes only the current exogenous state s and, as endogenous states, wealth distribution w, commodity prices p, and marginal utilities of income for all generations except for the first and the last, $\lambda \equiv ((\lambda^{ha})_{0<a<G})_{h\in H}$: with some abuse of notation, for a simple Markov, Z is the set of vectors $(s, \zeta) = (s, w, p, \lambda)$. For a recursive equilibrium, the endogenous state is only the wealth distribution w, and Z is the set of vectors $(s, \zeta) = (s, w)$. The transition $T : Z \to Z^S$ maps the current state (s, ζ) into all its S immediate successors (s', ζ'). An endogenous variables function $\xi : Z \to \Xi$ completes the construction. Hence, our definition of recursive equilibrium coincides with the usual formulation found in the literature (see, e.g., Kubler and Polemarchakis (2004) or Rios Rull (1996)). We write Z_ω, T_ω, and ξ_ω when we want to stress their dependence on the economy ω, and write W_s^Z for the s section of the projection of Z on W.[6]

The following notions of initial Markov state and initial state–wealth pair will be important for the subsequent analysis. Consider a simple Markov equilibrium of an economy $\omega \in \Omega$. A state $\bar{z} = (\bar{s}, \bar{\zeta}) \in Z$ is an *initial Markov state* if there does not exist $s \in S$ such that $\bar{z} = T_{\bar{s}}(z)$ for some $z \in Z$. Thus, the economy can just start from an initial Markov state, but it can never reach that state starting from somewhere else—an initial Markov state is an extreme notion of transient state. A pair $(\bar{s}, \bar{w})$ is an *initial state–wealth pair* if $(\bar{s}, \bar{w}, p, \lambda)$ is an initial Markov state for some (p, λ) such that $(\bar{s}, \bar{w}, p, \lambda) \in Z$.

4. GENERIC EXISTENCE OF RECURSIVE EQUILIBRIA

We are going to show that under a qualifying condition on H, recursive equilibria typically exist and are compatible with a large set of initial conditions, that is, of initial wealth distributions.

The qualifying condition on H, which is used below, is the following inequality.

ASSUMPTION A1: $H > 2[(C-1)\sum_{a=0}^{G} S^a + S\sum_{a=0}^{G-1} S^a]$.

[6]That is, $W_s^Z = \{w \in W : (s, w, p, \lambda) \in Z \text{ for some } (p, \lambda)\}$ for a simple Markov and $W_s^Z = \{w \in W : (s, w) \in Z\}$ for a recursive equilibrium.

Our main result is then stated as follows.

THEOREM 1: *Under A1, there exists a residual subset Ω^* of Ω such that every economy ω in Ω^* has a recursive equilibrium. Furthermore, $W^Z_{s,\omega}$ contains an open and full Lebesgue measure subset $W^*_{s,\omega}$ of $\mathring{W}_{s,\omega}$ for all $s \in S$.*

We summarize the logic and the various steps involved in proving Theorem 1. We will first prove that simple Markov equilibria with a large wealth space exist (Proposition 2 in Section 4.1). Then, using a transversality argument, we will show that the wealth distribution is typically a sufficient statistic of the Markov states, that is, that typically a Markov equilibrium is recursive.

More precisely, we want to show that simple Markov equilibria, typically in ω and under A1, have the following injection property: if two Markov states z, $\hat{z}$ are given, with $(s, w) = (\hat{s}, \hat{w})$, then $(p, \lambda) = (\hat{p}, \hat{\lambda})$. We call Markov equilibria that satisfy this property *nonconfounding*. It is immediate that a simple Markov equilibrium is nonconfounding *if and only if* it is a recursive equilibrium. Instead, a pair of Markov states z and $\hat{z}$ violating this property is called *critical* and the corresponding equilibrium is called *confounding*. We also call an exogenous state s critical if $(s, \zeta_1) \in Z$ and $(s, \zeta_2) \in Z$ are critical Markov states for a pair ζ_k, $k = 1, 2$.

If $(\bar{s}, \zeta_1)$ and $(\bar{s}, \zeta_2)$ are a pair of critical Markov states, then

$$w_1 = w_2, \tag{2}$$

and either

$$p_1 \neq p_2 \tag{3}$$

or

$$\lambda_1 \neq \lambda_2. \tag{4}$$

Clearly, (2) and (3) or (4) cannot have a solution if (2) does not have a solution for the H individuals of age $a = 1$ when (3) or (4) holds. The next step is then to establish via perturbation methods that indeed (2) for $a = 1$, and (3) and (4) cannot have a solution.

We can prove that system (2) and conditions (3) and (4) are incompatible at a simple Markov equilibrium if they also are incompatible at prices which are not necessarily equilibrium prices. In fact, we will only use restrictions on prices that arise from equilibrium to put them in a compact set $P(\omega)$. Since (w_1, λ_1) and (w_2, λ_2) are wealth and multiplier values that arise in the finite-dimensional optimization problems of H individuals of various ages, using stationarity and finite lives, the relevant price set $P(\omega)$ will also have finite dimension bounded by the right-hand side of Assumption A1. Hence we reduce the problem of the existence of a recursive equilibrium to the problem of establishing that generically the joint wealth differentials $w_1^{h1} - w_2^{h1}$ that arise in the finite-dimensional optimization problems of H individuals in two cohorts cannot be zero in the price domain $P(\omega)$ that satisfies (3) or (4). This is done in Section 4.2 via Propositions 3 and 4.

4.1. *Existence of Simple Markov Equilibrium*

Markov equilibria have a "large" wealth space when the latter is compatible with a large set of initial wealth distributions of the competitive economy. The next proposition states that simple Markov equilibria exist and their wealth space is large, and also gives two properties of the initial state–wealth pairs that will be important for the generic existence of recursive equilibria.

PROPOSITION 2: *For all $\omega \in \Omega$ and any subsets $O_s \subseteq W_{s,\omega}$, $s \in S$,* (i) *there exists a simple Markov equilibrium with $O_s \subset W^Z_{s,\omega}$ for all $s \in S$,* (ii) *if $(\bar{s}, \bar{w})$ is an initial state–wealth pair, then $\bar{w} \in O_{\bar{s}}$, and* (iii) *for every $\bar{s} \in S$, there is a unique $(\bar{p}, \bar{\lambda})$ such that $(\bar{s}, \bar{w}, \bar{p}, \bar{\lambda})$ is an initial Markov state.*

See the Appendix for the proof.

Since O_s is any subset of $W_{s,\omega}$, simple Markov equilibria can be constructed so that $W^Z_{s,\omega}$ contains all initial conditions of the competitive economy. However, if we take $O_s = \mathring{W}_{s,\omega}$, by Proposition 2(ii) we construct a simple Markov equilibrium with initial wealth distribution contained in an open set, a precondition for later perturbations. Proposition 2(iii) will later allow us to exclude certain configurations of bad states. To get the idea of the proof of Proposition 2(i), pick an economy $\omega \in \Omega$ and consider the family of competitive equilibria $\tilde{\xi}_z$, $z = (s, w, \xi) \in Z_0$, with $w \in O_s$. Competitive equilibria may fail to be simple Markov because at some histories they may generate identical simple Markov states, but different values of some current endogenous variables or different continuation paths. Consider two competitive equilibria $\tilde{\xi}_{\bar{z}}$ and $\tilde{\xi}_z$ (with possibly $\bar{z} = z$) that generate at some histories s^t and $\hat{s}^{t'}$, respectively, the same Markov state. Construct a new equilibrium by grafting $\tilde{\xi}_{\bar{z}}$ onto $\tilde{\xi}_z$: follow the equilibrium process $\tilde{\xi}_z$, but modify it from $\hat{s}^{t'}$ on by using the process $\tilde{\xi}_{\bar{z}}$ as if the history were s^t. Our choice for Z implies that, checking the first-order conditions (1), the grafting technique still defines a competitive equilibrium. Then if, at some histories, multiple competitive equilibria generate identical Markov states, we can select one of them, thereby obtaining unique realizations of the endogenous variables and of the continuation paths.

Duffie et al. (1994) proved the existence of simplified equilibria where, when $G = 1$, the state space can be reduced to the current exogenous shock, the consumption of the young, and their portfolio choices. For comparison, in this same case in our simple Markov equilibria, the state space reduces to the current shock, the current commodity prices, and the wealth distribution of the *current old*. It is this last state component that is the fundamental ingredient of our analysis; instead, it is missing in the construction of their simplified equilibria. We construct a selection directly by applying the grafting technique to the current state, making sure the previous and current equilibrium conditions are satisfied. Duffie et al.'s construction is forward looking and therefore does not make use of the wealth distribution of the old. Furthermore, their use of a measurable selection argument does not allow them to control the position of the initial state–wealth pairs—an important step of our argument (Proposition 2(ii)). Finally, their construction requires further measurability and topological assumptions on the endogenous variable functions and spaces what we avoid.

4.2. *Some Properties of the Wealth Differentials*

As argued above, the existence of recursive equilibria depends on generic properties of the joint wealth differentials of H individuals. Two such properties are of interest, depending on whether the wealth levels w_k^{h1} for some state k are exogenously given as initial wealth levels $\bar{w}_k^{h1}$ or the wealth levels at both states $k = 1, 2$ result from optimization.

Observe that an individual of age a at t faces $N_a = \sum_{a'=0}^{G-a} S^{a'}$ histories $s^{t+a'} \succeq s^t$ before death, $a = 0, \dots, G - a$. Thus, to study the joint wealth differentials, in the first case consider a tree of length G with initial node $s_{01} \in S$ and define over it an arbitrary finite price process $\underline{\psi} \in \mathbb{R}_{++}^{(C-1)N_0} \times \mathbb{R}^{SN_0}$, that is, a vector of prices (p_{s^a}, q_{s^a}) at each node s^a of the tree. As is well known, under our maintained assumptions—namely, stationarity and the boundary condition on u^h—competitive (and, therefore, Markov) equilibrium prices are bounded uniformly in $s^t \in \tilde{S}$ (or $s \in S$) and $(\bar{w}^{ha})_{h\in H, a>0}$, the exogenously given financial wealth of the economy. That is, commodity prices are uniformly bounded above and bounded away from zero, while asset prices are uniformly bounded and bounded away from the boundary of the no-arbitrage region. Hence, if we think of $\underline{\psi}$ as the restriction of a simple Markov equilibrium price process to the finite histories represented by the tree, the equilibrium nature of this price process implies that we can take it in $P(\omega)$, a compact subset of $\mathbb{R}_{++}^{(C-1)N_0} \times \mathbb{R}^{SN_0}$ independent of $\bar{w}$.

We just look at the H individuals born at s_{01}, hence $P(\omega)$ will also satisfy the innocuous additional restriction $q_{s^G} = 0$, for all terminal nodes s^G: individuals born at s_{01} will be old and, in the absence of arbitrage (a necessary condition for the existence of equilibria, embedded in $\underline{\psi}$), they will not trade on the asset market. Individual optimization (regularity of demand) pins down all the endogenous variables of these individuals as (smooth) functions of p, q, and ω. In particular, consider all individuals of age 0 who solve at s_{01} their programming problem when facing the finite process $\underline{\psi} \in P(\omega)$. Let $(\underline{x}^h, \underline{\theta}^h)(\underline{\psi}, \omega)$ be their optimal solution that takes values $(x^{ha}_{s^a}, \theta^{ha}_{s^a})(\underline{\psi}, \omega)$ at s^a and let

$$w^{h1}_{\bar{s}^1}(\underline{\psi}, \omega) = d_{\bar{s}} \theta^{h0}_{s_{01}}(\underline{\psi}, \omega)$$

be the wealth of an individual born at s_{01} and of age $a = 1$ at node $\bar{s}^1 = (s_{01}, \bar{s})$. For given $\bar{w} \in \overset{\circ}{W}_{\bar{s},\omega}$, let

$$f_{\bar{s}^1}(\underline{\psi}, \omega, \bar{w}^1) \equiv (w^{h1}_{\bar{s}^1}(\underline{\psi}, \omega) - \bar{w}^{h1})_{h\in H}.$$

Under Assumption A1, for given $(\underline{\psi}, \bar{w}^1)$, the H equations $f_{\bar{s}^1}(\underline{\psi}, \omega, \bar{w}^1) = 0$ outnumber the unknowns. We use this to show that, for each $\omega \in \Omega$ and $\bar{s} \in S$, and for generic choices of $\bar{w} \in \overset{\circ}{W}_{\bar{s},\omega}$, there is no such finite tree, that is, no $\bar{s}^1$ and no $\underline{\psi} \in P(\omega)$, where $f_{\bar{s}^1}(\underline{\psi}, \omega, \bar{w}^1) = 0$.

PROPOSITION 3: *Let $\omega \in \Omega$ be given. For each $\bar{s} \in S$ there is an open and full Lebesgue measure subset $W^*_{\bar{s},\omega}$ of $\overset{\circ}{W}_{\bar{s},\omega}$ such that $f_{\bar{s}^1}(\underline{\psi}, \omega, \bar{w}^1) = 0$ does not have a solution in $P(\omega)$ for all $\bar{w} \in W^*_{\bar{s},\omega}$ and all $(s_{01}, \bar{s})$.*

For the proof see the Appendix.

The proof of Proposition 3 uses a standard transversality argument through the computation of the derivative of $f_{\bar{s}^1}$ with respect to $\bar{w}^1$.

Continuing the analysis of wealth differentials, we now study the second case. To this end, consider two trees of finite length G each with initial node s_{0k}, $k = 1, 2$. Otherwise identical histories s^a on the two trees may only differ in their initial node. When we want to stress this, we denote with (k, s^a) the history s^a on the k tree. Consider a pair of finite price processes $\underline{\psi}_k \in P(\omega)$ defined over the two trees. With some abuse of notation, the process $\underline{\psi}$ denotes now the pair $(\underline{\psi}_1, \underline{\psi}_2) \in P(\omega) \times P(\omega) = P(\omega)^2$.

Once again, we focus on all the individuals h born at s_{0k}, hence we can further restrict prices in $P(\omega)^2$ to satisfy $q_{k,s^G} = 0$ for all terminal nodes s^G and $k = 1, 2$. The optimizing behavior of the two cohorts (individuals born at s_{0k}; one for each k) is entirely determined by $\underline{\psi} \in P(\omega)^2$ and ω. In particular, for each k, consider all individuals of age 0 solving at s_{0k} their programming problem when facing the finite process $\underline{\psi}_k \in P(\omega)$. Let $(\underline{x}^h_k, \underline{\theta}^h_k)(\underline{\psi}_k, \omega)$ be their optimal solution, taking values $(x^{ha}_{k,s^a}, \theta^{ha}_{k,s^a})(\underline{\psi}_k, \omega)$ at (k, s^a), and let

$$w^{h1}_{k,\bar{s}^1}(\underline{\psi}_k, \omega) = d_{\bar{s}} \theta^{h0}_{s_{0k}}(\underline{\psi}_k, \omega)$$

be the wealth of an individual born at s_{0k} and of age $a = 1$ at node $(k, \bar{s}^1)$. To simplify the notation, hereafter we set $\sigma \equiv (s_{01}, s_{02}, \bar{s})$ and let

$$\hat{f}_\sigma(\underline{\psi}, \omega) \equiv (w^{h1}_{1,\bar{s}^1}(\underline{\psi}_1, \omega) - w^{h1}_{2,\bar{s}^1}(\underline{\psi}_2, \omega))_{h\in H}$$

be the wealth difference between the two age $a = 1$ cohorts at $\bar{s}^1 = (s_{0k}, \bar{s})$, $k = 1, 2$. We further restrict attention to $P(\sigma; \omega) \subset P(\omega)^2$, the set of price processes $\underline{\psi}$ that satisfy inequality (3) or induce (4) at node $\bar{s}^1$ among older cohorts, given that all these cohorts' wealth ($w^{ha}_{k,\bar{s}^1}$ for $a \geq 1$) is k-invariant. We show below that in a residual set of parameters, the system of equations $\hat{f}_\sigma(\underline{\psi}, \omega) = 0$ does not have a solution for any σ and any price process in $P(\sigma; \omega)$.

PROPOSITION 4: *There exists a residual subset Ω^* of Ω such that $\hat{f}_\sigma(\underline{\psi}, \omega) = 0$ does not have a solution in $P(\sigma; \omega)$ for all $\omega \in \Omega^*$ and all σ.*

The proof of Proposition 4 is quite elaborate, but interesting in its own. Since it is central to our technique, reducing the whole issue to a finite-dimensional problem, we devote Section 4.4 to explaining its logic (a perturbation argument), while details and computations are in the Appendix. Although density, in general, is stated in the space of all parameters, the argument will show that when $G = 1$, genericity is only in endowments.

Taking for granted Propositions 2–4, we are now ready to prove Theorem 1.

4.3. *Proof of Theorem 1*

Pick an economy $\omega \in \Omega^*$, the set constructed in Proposition 4. Use Proposition 2(i) and (ii) to construct the state space Z_ω of the simple Markov equilibrium so that if $\bar{w} \in W^Z_{s,\omega}$ and $(\bar{w}, s)$ is an initial state–wealth pair, then $\bar{w} \in W^*_{s,\omega}$ for all $s \in S$, the set defined in Proposition 3. We need to show that such equilibrium is void of critical Markov states. At this junction, two possibilities arise: Case 1, neither $(\bar{s}, \zeta_1)$ nor $(\bar{s}, \zeta_2)$ is initial Markov states; Case 2, there exists k such that $(\bar{s}, \zeta_k)$ is an initial Markov state of the economy. While in Case 1 we just refer to the Markov states as critical, in Case 2 we add the qualification "initial."

CASE 1: Any simple Markov equilibrium of an economy $\omega \in \Omega^*$ is void of critical pairs of Markov states. Suppose not. Pick a pair of critical Markov states $(\bar{s}, \zeta_k)$, $k = 1, 2$. Consider the H individuals born at the node on $\tilde{S}$ predecessor to the one where $(\bar{s}, \zeta_k)$ has realized (s_{0k}) and of age $a = 1$ at the critical state.[7] The (simple Markov) equilibrium price process matters to them only as restricted to the finite histories represented by the pair of trees since they were born, that is, as $\underline{\psi} \in P(\omega)^2$. Thanks to the stationarity of the economy, each pair of such trees is identified only by their initial nodes (s_{01}, s_{02}). The individuals' wealth difference at age $a = 1$ is $\hat{f}_\sigma(\underline{\psi}, \omega)$, defined in Section 4.2. If

[7]The choice of $a = 1$ is dictated to avoid considering combinations of critical states other than the two studied here.

the pair $(\bar{s}, \zeta_k)$, $k = 1, 2$, is critical, then (2), and conditions (3) and (4) correspond to $\hat{f}_\sigma(\underline{\psi}, \omega) = 0$ for some $\underline{\psi} \in P(\sigma; \omega)$ and some (s_{01}, s_{02}), contradicting Proposition 4.

CASE 2: Consider now initial critical pairs of Markov states. By Proposition 2(iii), $(\bar{s}, \zeta_1)$ and $(\bar{s}, \zeta_2)$ cannot both be initial Markov states. Suppose that $(\bar{s}, \zeta_2)$, say, is an initial Markov state of the economy where individuals of age $0 < a \le G$ are endowed with exogenously given financial wealth $(\bar{w}^{ha})_{h\in H}$. Proposition 2(ii) guarantees that for $(\bar{s}, \bar{w})$ we have $\bar{w} \in W^*_{\bar{s},\omega}$. As in Case 1, we look at the H individuals born at the node on $\tilde{S}$ predecessor to the one where $(\bar{s}, \zeta_1)$ has realized, s_{01}. Their wealth when their age is $a = 1$ is their wealth at the critical state, and it is a function of the equilibrium price process over the tree spanning their finite life, that is, of $\underline{\psi} \in P(\omega)$. It is compared with the exogenous wealth $\bar{w}^1$ of individuals of the same age at state $(\bar{s}, \zeta_2)$, and the difference is $f_{(s_{01},\bar{s})}(\underline{\psi}, \omega, \bar{w}^1)$, which is defined in Section 4.2. If $(\bar{s}, \zeta_k)$, $k = 1, 2$, is an initial critical pair, then $f_{(s_{01},\bar{s})}(\underline{\psi}, \omega, \bar{w}^1) = 0$ for some $\underline{\psi} \in P(\omega)$ and some s_{01}. However, this contradicts Proposition 3.

Hence, the Markov equilibrium we constructed is void of both initial critical pairs and critical pairs of Markov states. Thus, the Markov states $(s, \zeta) \in Z_\omega$ are one-to-one in (s, w); equivalently, there exists a function $(p, \lambda)(s, w)$ such that each Markov state z can be written as $(s, w, (p, \lambda)(s, w))$. Equivalently, the transition function of the simple Markov equilibrium can be decomposed as $T_{s'}(z) = [T^r_{s'}(s, w), (p, \lambda)(T^r_{s'}(s, w))]$. Thus, the recursive equilibrium of the economy $\omega \in \Omega^*$ is the recursive state space Z^r_ω and transition and endogenous variables functions T^r, ξ^r defined as

$$Z^r_\omega = \text{proj}_{S\times W} Z_\omega, \quad \text{with} \quad W^{Z^r_\omega}_{s,\omega} \supset W^*_{s,\omega}, \quad \text{for all } s,$$

$$T^r : Z^r_\omega \to (Z^r_\omega)^S,$$

$$\xi^r : Z^r_\omega \to \Xi \quad \text{is} \quad \xi^r(z) = \xi(s, w, (p, \lambda)(s, w)),$$

and it has the desired properties, ending the proof. *Q.E.D.*

Of course, at this stage nothing is said about any regularity property of the transition or the endogenous variables functions, an important topic for future research.

4.4. *Perturbation Analysis*

In this technical subsection we prove Proposition 4. Proposition 4 asserts that while facing different prices, at least one among H individuals will typically have different expenses on a subset of goods—those purchased after a certain date–event. If we look at just one individual, and put no qualification on what "different prices" means, this may not be true. Consider, for example, a Walrasian individual with log-linear utility over three commodities. The individual will spend the same on commodity one, even if he faces different prices for commodities two—and three—provided that prices of commodities two and three are such that the value of the endowment is the same. We could perturb the utility and the endowments of this individual, but still obtain a region of different prices, yielding the same expense on commodity one. It is clear that we need to consider simultaneously many potentially different individuals and that the dimension of potential difference across individuals should be larger than the price dimension—Assumption A1.

We proceed as follows. Since openness and density are local properties, for each ω we need to define a superset of $P(\omega)$ which is locally independent of ω. This will allow perturbations of ω independent of prices. Then, to make the analysis as simple as possible, we transform the price space, the individual programming problems, and the equations $\hat{f}_\sigma(\cdot) = 0$ into a more convenient, but equivalent form. This first transformation suffices to prove Proposition 4 for $G = 1$. However, it will not be powerful enough to carry the result for $G > 1$.

Local Independence of the Price Set From ω: Pick $\omega \in \Omega$, and let $B_\varepsilon(\omega) \subset \Omega$ be an open ball centered at ω for some fixed $\varepsilon > 0$. Under the maintained assumptions, the optimal consumption-portfolio plans $(\underline{x}_k^h, \underline{\theta}_k^h)(\underline{\psi}_k, \omega)$ are continuous and then, by the strict monotonicity of preferences, there is a compact set of prices P such that $P(\omega') \subset P \subset \mathbb{R}_{++}^{(C-1)N_0} \times \mathbb{R}^{SN_0}$ for all $\omega' \in B_\varepsilon(\omega)$. Hence, by the compactness of P, for all h, k, s^a, and $(\underline{\psi}, \omega') \in P \times B_\varepsilon(\omega)$, $x_{k,s^a}^{ha}(\underline{\psi}_k, \omega') \in \bar{X} \subset \mathbb{R}_+^C$—a compact set—and since preferences satisfy the boundary condition, $\bar{X}$ is contained in the interior of the positive cone, that is, $\bar{X} \subset \mathbb{R}_{++}^C$.

What is open and dense in $B_\varepsilon(\omega)$ for any arbitrary such $B_\varepsilon(\omega)$ is open and dense in Ω. Therefore, to keep notation simple, hereafter we identify Ω with the arbitrary $B_\varepsilon(\omega)$.

Transforming the Programming Problems and the Price Space: In our economy, assets pay off in the numéraire commodity and their payoff matrix $\mathbf{D}$ is invertible. Thus, individuals face sequentially complete asset markets. Therefore, the sequence of budget constraints can be compressed into a single one, getting rid of asset prices and portfolios. Hence, we change the price space to

$$P' = \{\underline{p}' \in \mathbb{R}_{++}^{2CN_0} : p'_{1,k,s_0} = 1, k = 1, 2\}.$$

We introduce the operator $\mathbb{E}_{s_t}^\delta$, which applies to any finite process $\underline{L}^a = (L_{t+a'})_{a'=0}^{G-a}$ of N_a histories and is defined as $\mathbb{E}_{s_t}^\delta(\underline{L}^a) = \mathbb{E}_{s_t}\{\sum_{a'=0}^{G-a} \delta^{a'} L_{t+a'}\}$; for simplicity, for $a = 0$ we omit the superscript from $\underline{L}^a$. The programming problems of the individuals born at s_{0k} are

$$\text{(5)} \qquad \max \mathbb{E}_{s_{0k}}^\delta\{u^h(\underline{x})\} \quad \text{s.t.} \quad \mathbb{E}_{s_{0k}}^\delta\{\underline{p}'_k(\underline{x} - \underline{e}_k^h)\} = 0.$$

We need to reformulate the programming problems of individuals of age $a^* = 1, \dots, G$ who have wealth $w^{ha^*}_{k,\bar{s}^1} = d_{\bar{s}} \theta^{h(a^*-1)}_{s_{0k}}$ at $\bar{s}^1 = (s_{0k}, \bar{s})$. Since $p_{k,\bar{s}^1}$ is normalized, while $p'_{k,\bar{s}^1}$ is not, the prices in the budget constraint at $\bar{s}^1$ for these individuals must be divided by $\delta\pi(\bar{s}^1|s_{0k}) p'_{1,k,\bar{s}^1}$, and their programming problem is

$$(6) \qquad \max \mathbb{E}^{\delta}_{\bar{s}^1}\{u^h(\underline{x}^{a^*})\} \quad \text{s.t.} \quad \mathbb{E}^{\delta}_{\bar{s}^1}\left\{\frac{\underline{p}'^{a^*}_k}{p'_{1,k,\bar{s}^1}}(\underline{x}^{a^*} - \underline{e}^{ha^*}_k)\right\} = w^{ha^*}_{k,\bar{s}^1}.$$

Notice that by the definition of the operator $\mathbb{E}^{\delta}_{s_{0k}}$, the coefficient that multiplies $u^h(x_{s^a})$ in the objective function of problem (5) coincides with the coefficient that multiplies $p'_{k,s^a}(x_{s^a} - e^h_{k,s_a})$ in the budget constraint, and the same applies (modulo changing p'_{k,s^a} with $p'_{k,s^a}/p'_{1,k,\bar{s}^1}$) for problem (6).

By Arrow's equivalence theorem, to each $\underline{\psi} \in P^2$ corresponds a unique pair $\underline{p}' \in P'$ such that $\underline{\psi}$ and $\underline{p}'$ are equivalent: the consumption bundles that satisfy the sequential budget constraints (1c) at $\underline{\psi}$ coincide with the consumption bundles that satisfy at $\underline{p}'$ the single budget constraint of (5) (and hence of (6)). We take P' to be the set of processes $\underline{p}'$ equivalent to price pairs $\underline{\psi} \in P^2$, which we refer to as Arrow prices; the two sets have obviously identical dimension $2CN_0 - 2$, the right-hand side of the inequality in Assumption A1. As we took P^2 to be a compact set, P' is also a compact subset of $\mathbb{R}^{2CN_0-2}_{++}$. Also, we take $P'(\sigma; \omega) \subset P'$ to be the set of Arrow prices p' equivalent to the prices $\underline{\psi} \in P(\sigma; \omega)$.

Transforming the Wealth Equations: We reformulate the wealth equation $\hat{f}_\sigma(\cdot) = 0$ without making reference to portfolios. For $\underline{p}' \in P'(\sigma; \omega)$, let $x^{ha}_{k,s^a}(\underline{p}^1_k, \omega)$ be the optimal solutions at (k, s^a) to problems (5). Then, in analogy with the form of the budget constraints (6), we get that the wealth functions are

$$w^{h1}_{k,\bar{s}^1}(\underline{p}'_k, \omega) = \mathbb{E}^{\delta}_{\bar{s}^1}\left(\frac{\underline{p}'^1_k}{p'_{1,k,\bar{s}^1}}(\underline{x}^{h1}_k(\underline{p}^1_k, \omega) - \underline{e}^{h1}_k)\right)$$

for $k = 1, 2$, and we define the functions

$$f^h_\sigma(\tilde{p}'^0, \omega) = w^{h1}_{1,\bar{s}^1}(\underline{p}'_1, \omega) - w^{h1}_{2,\bar{s}^1}(\underline{p}'_2, \omega)$$

and let $f_\sigma = (f^h_\sigma)_{h \in H}$. We are ready to prove Proposition 4 for $G = 1$.

4.4.1. $G = 1$

Remember that for $G = 1$, the endogenous Markov state is reduced to (w, p). If $C = 1$, simple Markov equilibria are already recursive. Therefore, let $C > 1$ and let

$$P'(\sigma;\omega) = P'(\bar{s}) = \left\{ \underline{p}' \in P' : \left\| \frac{p'_{1,\bar{s}^1}}{p'_{1,1,\bar{s}^1}} - \frac{p'_{2,\bar{s}^1}}{p'_{1,2,\bar{s}^2}} \right\| \neq 0 \right\},$$

and for any integer $n > 0$, let

$$P'^n(\bar{s}) = \left\{ \underline{p}' \in P' : \left\| \frac{p'_{1,\bar{s}^1}}{p'_{1,1,\bar{s}^1}} - \frac{p'_{2,\bar{s}^1}}{p'_{1,2,\bar{s}^1}} \right\| \geq \frac{1}{n} \right\}.$$

Obviously, $P'^n(\bar{s}) \subset P'(\bar{s})$, and both $P'^n(\bar{s})$ and $P'(\bar{s})$ are sets that are (locally) independent of ω. Let Ω^n_σ denote the subset of Ω where system $f_\sigma(\underline{p}', \omega) = 0$ does not have a solution in $P'^n(\bar{s})$. If Ω^n_σ is open and dense in Ω, then

$$\Omega^* = \bigcap_{n>0} \bigcap_{\sigma} \Omega^n_\sigma$$

is the intersection of a countable family of open and dense sets; therefore, it is a residual set of Ω (hence, it contains a dense subset), where system $\hat{f}_\sigma(\underline{p}', \omega) = 0$ does not have a solution in $\bar{P}'(\bar{s})$ for all σ. Suppose not. Then there is $\omega^* \in \Omega^*$, σ, and $\underline{p}' \in \bar{P}'(\bar{s})$ such that $f_\sigma(\underline{p}', \omega^*) = 0$. By definition of $P'(\bar{s})$, there must be $\hat{n} > 0$ such that $\underline{p}' \in P'^{\hat{n}}(\bar{s})$. However, the latter implies that $\omega^* \notin \Omega^{\hat{n}}_\sigma$, a contradiction.

To show that Ω^n_σ is open, pick $\omega \in \Omega^n_\sigma$. The compactness of $P'^n(\bar{s})$ implies that $|f_\sigma(\underline{p}', \omega)| \geq \eta$ for some $\eta > 0$ and all $\underline{p}' \in P'^n(\bar{s})$. However, the map f_σ is continuous in all its argument and hence $|f_\sigma(\underline{p}', \omega')| > 0$ for all $\underline{p}' \in P'^n(\bar{s})$ and ω' in an open neighborhood of ω. Thus, the set Ω^n_σ is open.

We now show that Ω^n_σ is dense. It suffices to prove that the Jacobian matrix $D_e f_\sigma(\underline{p}'; e, u, \delta)$ is a surjection for all $(\underline{p}', e, u, \delta) \in P'(\bar{s}) \times \Omega$ (and therefore, in $P'^n(\bar{s}) \times \Omega$). If this is the case, by Assumption A1, $\dim P'^n(\bar{s}) < H$ and there are more equations than unknowns in $f_\sigma(\underline{p}', \omega) = 0$. Hence, by the preimage and the transversality theorems, there is a dense subset E^n_σ of E (so Ω^n_σ of Ω) where $f_\sigma(\underline{p}', \omega) = 0$ has no solution in $P'^n(\bar{s})$.

When $\underline{p}' \in P'^n(\bar{s})$, the vectors $p'_{k,\bar{s}^1}$, $k = 1, 2$, are linearly independent. Therefore, we can find an appropriate perturbation $\Delta e^1_{\bar{s}}$ of $e^1_{\bar{s}}$ such that $p'_{1,\bar{s}^1} \Delta e^1_{\bar{s}} = 1$, while $p'_{2,\bar{s}^1} \Delta e^1_{\bar{s}} = 0$. We show that $D_{\overrightarrow{e^1_{\bar{s}}}} f_\sigma$, the directional derivative of f_σ in the direction identified by the perturbation $\Delta e^1_{\bar{s}}$, is a surjection. First, observe that this perturbation does not affect $w^{h1}_{2,\bar{s}^1}(\underline{p}'_2, \omega)$ and, hence, that $D_{\vec{e}^1_{\bar{s}}} f_\sigma = D_{\vec{e}^1_{\bar{s}}} w^{h1}_{1,\bar{s}^1}$. Second, differentiate the first-order conditions to problem (5) for $k = 1$, drop h and k, and get

(7a) $$H_{s^a} \Delta x_{s^a} - p'^T_{s^a} \Delta\lambda = 0,$$

(7b) $$\sum_a \delta^a \sum_{s^a} \pi(s^a|s_0) p'_{s^a} \Delta x_{s^a} = \delta\pi(\bar{s}^1|s_0),$$

where H_{s^a} is the invertible Hessian at x_{s^a}, a negative definite matrix, and the superscript T stands for transposed. Let $Q_{s^a} = p'_{s^a} H^{-1}_{k,s^a} p'^T_{k,s^a} < 0$ and $Q = \sum_{a,s^a} \delta^a \pi(s^a|s_0) Q_{s^a} < 0$. We get

$$\Delta\lambda = \frac{\delta\pi(\bar{s}^1|s_0)}{Q}.$$

Differentiating the map $w_{1,\bar{s}^1}(\underline{p}'_1, \omega)$, we obtain

$$\Delta w_{\bar{s}^1} = (Q_{\bar{s}^1}\Delta\lambda - 1)\frac{1}{p'_{1,1,\bar{s}^1}} = \left(\frac{\delta\pi(\bar{s}^1|s_0)Q_{\bar{s}^1}}{Q} - 1\right)\frac{1}{p'_{1,1,\bar{s}^1}} < 0.$$

The argument is concluded by observing that $D_{\vec{e}^1_{\bar{s}}} f^{h'}_\sigma = 0$ for all h, h' with $h \neq h'$.

The microeconomics of the result is clear. The normality of all expenditures $p'_{s^a} x_{s^a}$, a by-product of separability of utility, implies that all of them move proportionally to, but less than, a lifetime wealth change. Thus, the change in $w_{\bar{s}^1}$ induced by the perturbation $\Delta e^1_{\bar{s}}$ is negative, since $w_{\bar{s}^1} = p'_{\bar{s}^1}/p'_{1,\bar{s}^1}(x^1_{\bar{s}^1} - e^1_{\bar{s}})$.

4.4.2. $G > 1$

When $G > 1$, multiplier inequalities (4) must be taken into account when defining the set of prices $P'(\sigma; \omega)$. Such inequalities can be generated by price differences across k trees arising at nodes s^a with $a > 1$. Relative to the case when $G = 1$, things are then considerably complicated by stationarity: endowment or utility perturbations at one node reverberate across the trees, possibly rendering the perturbations ineffective, that is, the derivative of the wealth differentials is zero. Fortunately, we show below that when perturbations are ineffective it is because the price differences are only "nominal," that is, only due to labeling, and do not translate to differences in multipliers. To this extent it will be essential to identify equivalence classes of Arrow prices which determine $P'(\sigma, \omega)$, making the effectiveness of the available perturbations immediately apparent and $P'(\sigma, \omega)$ (locally) ω-independent. We then show how to bypass the possibility that the derivative of the wealth differentials is zero by introducing a nesting technique.

Creating Equivalence Classes of Arrow Prices: This section formalizes the necessary condition for $\underline{p}' \in P'(\sigma;\omega)$ by (a) identifying equivalence classes of Arrow prices which determine $\underline{p}' \in P'(\sigma;\omega)$, irrespective of ω, and (b) creating compact sets of prices where the condition holds.

(a) *Identifying equivalence classes of Arrow prices*. We start by making a simple, preliminary observation. Recall that by assumption, all individuals of all ages have identical wealths at $(k, \bar{s}^1)$, that is, $w^{ha}_{1,\bar{s}^1} = w^{ha}_{2,\bar{s}^2} \equiv w^{ha}$ for all h, a. If for all possible specifications of the economy ω' and such wealth w^{ha}, the (optimal) marginal utilities $\lambda^{ha^*}_{1,\bar{s}^1}(\underline{p}'_k, \omega', \bar{w}^{ha^*})$ in problems (6) computed at $\underline{p}'$ are k-invariant for all h and $a^* \geq 1$, and $p'_{1,\bar{s}^1}/p'_{1,1,\bar{s}^1} = p'_{2,\bar{s}^2}/p'_{1,2,\bar{s}^1}$, then $\underline{p}' \notin P'(\sigma, \omega)$. Thus, a necessary condition for $\underline{p}' \in P'(\sigma, \omega)$ is that either $p'_{1,\bar{s}^1}/p'_{1,1,\bar{s}^1} \neq p'_{2,\bar{s}^2}/p'_{1,2,\bar{s}^1}$ or $\lambda^{ha^*}_{1,\bar{s}^1}(\underline{p}'_1, \omega, \bar{w}^{ha^*}) \neq \lambda^{ha^*}_{2,\bar{s}^1}(\underline{p}'_2, \omega, \bar{w}^{ha^*})$ for some $a^* \geq 1$, and some $\omega, \bar{w}^{ha^*}$. In the absence of restrictions on the fundamentals, this necessary condition translates into $p'_{1,s^a}/p'_{1,1,\bar{s}^1} \neq p'_{2,s^a}/p'_{1,2,\bar{s}^1}$ for some $s^a \succeq \bar{s}^1$.

However, in our economies, the cardinality indices u^h are state and age invariant, endowments are stationary, and, by assumption, wealth is k-invariant. The pair of price processes $\underline{p}' = (\underline{p}'_1, \underline{p}'_2)$ can take different values at pairs of identical nodes $s^a \succeq \bar{s}^1$ on the two trees, but they will still generate the same values for $\lambda^{ha}_{k,\bar{s}^1}(\underline{p}'_k, \omega', \bar{w}^{ha})$ for all h, a, and (ω', w^{ha}) if for each period $a \geq 1$, $s^a \succeq \bar{s}^1$, and price realization p, the (discounted) probabilities that $p'_{k,s^a}/p'_{1,k,\bar{s}^1} = p$ are independent of k and the overall wealth of individuals (h, a^*) is k-invariant. The next example makes this point transparent.

EXAMPLE: Consider an economy with $G = 3$ and $S = \{\alpha, \beta\}$, where $\pi(s|s') = \frac{1}{2}$ for all s, s', $s_{0k} = \alpha$ for all k, and $\bar{s} = \alpha$, $\delta = 1$, and wealth is k-invariant. Pick $\underline{p}'$ such that the following statements hold:

- p'_{k,s^a} is k-invariant for $a \leq 2$.
- $p'_{1,(s^2,\alpha)} = p'_{2,(s^2,\beta)} = p^1$ while $p'_{1,(s^2,\beta)} = p'_{2,(s^2,\alpha)} = p^2$ for $s^2 = (\alpha, \alpha, \alpha)$, (α, α, β) and vice versa.
- $p'_{1,(s^2,\alpha)} = p'_{2,(s^2,\beta)} = p^2$ while $p'_{1,(s^2,\beta)} = p'_{2,(s^2,\alpha)} = p^1$ for $s^2 = (\alpha, \beta, \alpha)$, (α, β, β).

If $p^1 \neq p^2$, then $\underline{p}'_1 \neq \underline{p}'_2$. However, this difference is just a matter of relabeling the states and has no real consequences. Indeed, by wealth k-invariance and k-invariance of p'_{k,s^a}, $a \leq 2$, the multipliers $\lambda^{ha^*}_{k,\bar{s}^1}$ associated to problems (6) for individuals of age $a^* \geq 2$ are k-invariant. Furthermore, by the definition of $\underline{p}'$ and the assumptions on endowments and on conditional probabilities, $\mathbb{E}^{\delta}_{s_{0k}}(\underline{p}'_k \underline{e}^h)$ is k-invariant. The latter implies that, for all u^h satisfying the maintained assumptions, the optimal solutions $\underline{x}^h_k$, $k = 1, 2$, to (5) satisfy the following statements:

- x^{ha}_{k,s^a} is k-invariant for $a \leq 2$.

- $x^{h3}_{1,(s^2,\alpha)} = x^{h3}_{2,(s^2,\beta)}$ while $x^{h3}_{1,(s^2,\beta)} = x^{h3}_{2,(s^2,\alpha)}$ for $s^2 = (\alpha, \alpha, \alpha), (\alpha, \alpha, \beta)$.
- $x^{h3}_{1,(s^2,\alpha)} = x^{h3}_{2,(s^2,\beta)}$ while $x^{h3}_{1,(s^2,\beta)} = x^{h3}_{2,(s^2,\alpha)}$ for $s^2 = (\alpha, \beta, \alpha), (\alpha, \beta, \beta)$.

It follows that λ^{h0}_k and w^{h1}_k are k-invariant, so that the wealth invariance condition is satisfied. Finally, $\lambda^{h1}_{k,\bar{s}^1}$ is k-invariant. Thus, $\underline{p}'$ is not an element of $P'(\sigma; \omega)$ for any ω.

Next, we make the observations of this example general. By strict concavity of u^h, if $p'_{1,\hat{s}^a}/p'_{1,1,\bar{s}^1} = p'_{2,\hat{s}^a}/p'_{1,2,\bar{s}^1}$ for two distinct histories $\hat{s}^a \succeq \bar{s}^1$ and $\bar{s}^{a'} \succeq \bar{s}^1$ at ages a and a', then at the optimal solution to (6), $x^{h(a^*+a-1)}_{k,\hat{s}^a} = x^{h(a^*+a'-1)}_{k,\bar{s}^{a'}}$. This allows us to rewrite both problems (6) by expressing prices in terms of their distinct realizations rather than in terms of their realizations at each (k, s^a), $s^a \succeq \bar{s}^1$.

For $\underline{p}' \in P'(\sigma, \omega)$, let

$$\mathbb{P}^1 = \left\{ p \in \mathbb{R}^C_{++} : \frac{p'_{k,s^a}}{p'_{1,k,\bar{s}^1}} = p, \text{ for some } k, s^a \succeq \bar{s}^1 \right\}$$

with cardinality $\mathbb{P}^1 \leq 2\sum_{a=0}^{G-1} S^a$ and generic element $p(\ell)$, where $\mathbb{P}^1$ denotes also the set of price indices ℓ.

For $\ell \in \mathbb{P}^1$ and $a \geq 1$, we define sets of histories of length a and their probability weights as

$$S^a_k(\ell) = \left\{ s^a \succeq \bar{s}^1 : \frac{p'_{k,s^a}}{p'_{1,k,\bar{s}^1}} = p(\ell) \right\}, \quad \Pi[S^a_k(\ell)] = \sum_{s^a \in S^a_k(\ell)} \pi(s^a|\bar{s}^1),$$

where $\Pi[S^a_k(\ell)] = 0$ if $S^a_k(\ell) = \emptyset$. To make the programming problems dependent only on the distinct realizations of the price processes $p'_{k,s^a}/p'_{1,k,\bar{s}^1}$, $s^a \succeq \bar{s}^1$, define

$$\Pi_k(a^*, \ell) = \sum_{a=1}^{G+1-a^*} \delta^{a-1} \Pi[S^a_k(\ell)], \quad a^* \geq 1. \tag{8}$$

To make transparent the overall endowment value (as well as the effectiveness of endowment perturbation) on the two subtrees (k, s^a), $s^a \succeq \bar{s}^1$, define

$$p_{k+}(a, s) = \sum_{s^{a-1}:(s^{a-1},s)\succeq\bar{s}^1} \pi(s^{a-1}, s|\bar{s}^1) \frac{p'_{k,(s^{a-1},s)}}{p'_{k,1,\bar{s}^1}},$$

with $p_{k+}(a, s) = 0$ if $a = 0$ or $a = 1$ and $s \neq \bar{s}$. By definition of $\Pi_k(a^*, \ell)$ and $p_{k+}(a, s)$, and by strict concavity of u^h, problems (6), $1 \leq a^* \leq G$, can be equivalently written as

$$(9)\qquad \max \sum_{\ell\in\mathbb{P}^1} \Pi_k(a^*,\ell)u^h(x(\ell)) \quad \text{s.t.}$$

$$\sum_{\ell\in\mathbb{P}} \Pi_k(a^*,\ell)p(\ell)x(\ell) - \sum_{a=a^*}^{G} \delta^{a-a^*} \sum_s p_{k+}(a,s)e_s^{ha} = w_{k,\bar{s}^1}^{ha^*}.$$

A simple inspection of problems (9) delivers the necessary condition for $\underline{p}' \in P'(\sigma,\omega)$. Consider a pair $\underline{p}'$ such that $p_{k+}(a,s)$ are k-invariant for all (a,s), and $\Pi[S_k^a(\ell)]$ are k-invariant for all ℓ and $a \geq 1$. Then, by (8), also $\Pi_k(a^*,\ell)$ are k-invariant for all ℓ and $a^* \geq 1$, and then the two problems (9) are identical at $\underline{p}'$ and so are their optimal solutions for all (h,a^*). Thus, $\lambda_{k,\bar{s}^1}^{ha^*}(\underline{p}'_k,\omega,w^{ha^*})$ are k-invariant. Hence, the necessary condition for $\underline{p}' \in P'(\sigma,\omega)$ simply is

$$(\text{NC})\qquad \left\| \left(\left(\Pi[S_1^a(\ell)] \right)_{a,\ell}, (p_{1+}(a,s))_{a,s} \right) - \left(\left(\Pi[S_2^a(\ell)] \right)_{a,\ell}, (p_{2+}(a,s))_{a,s} \right) \right\| \neq 0.$$

Since $\pi(s^a|\bar{s}^1) = \pi(s^a|\bar{s})$ for all $s^a \succeq \bar{s}^1$, inequalities (NC) are independent of s_{0k} and ω, and we let $P'(\bar{s})$ denote the set of $\underline{p}'$ satisfying condition (NC).

However, since we need to perturb the map f_σ, we need to make sure that at $\underline{p}' \in P'(\bar{s})$, $\Pi_1(1,\ell) \neq \Pi_2(1,\ell)$; otherwise, individuals of age 0 at s_{0k} may be solving identical programming problems. Thus, consider the set $P'(\bar{s},\delta)$ of prices that satisfy the δ-dependent conditions:

$$(10)\qquad \left\| \left((\Pi_1(1,\ell))_\ell, (p_{1+}(a,s))_{a,s} \right) - \left((\Pi_2(1,\ell))_\ell, (p_{2+}(a,s))_{a,s} \right) \right\| \neq 0.$$

Obviously, $P'(\bar{s},\delta) \subset P'(\bar{s})$. We show below that, generically in δ, if the inequality $\Pi[S_1^a(\ell)] \neq \Pi[S_2^a(\ell)]$ holds true for some ℓ, then $\Pi_{1+}(1,\ell) \neq \Pi_{2+}(1,\ell)$. The latter has two desirable implications: $P'(\bar{s},\delta) = P'(\bar{s})$ and, therefore, $P'(\bar{s},\delta)$ is ω-invariant in the generic set of common discount factors.

LEMMA 5: *There exists an open and dense subset Ω' of Ω such that for all $\omega \in \Omega'$, $P'(\bar{s},\delta) = P'(\bar{s})$.*

See the Appendix for the proof.

(b) *Creating compact sets of prices*. In the analysis for $G = 1$, we proved that Ω^* is residual by defining compact regions of the price domain $P'^n(\bar{s}) \subset P'(\bar{s})$. For $\underline{p}' \in P'^n(\bar{s})$, the effective difference between $\underline{p}'_1$ and $\underline{p}'_2$ is sizeable, a necessary condition for establishing openness of the sets Ω_σ^n. We have to repeat that maneuver. The notion of effective price difference is embedded in the definition of the set $P'(\bar{s})$ and it is precisely defined by (NC) or, equivalently, (10). What does sizeable mean? The details are in the proof of the next lemma, but here is the precise idea. Since we are limiting attention to the set Ω', for $\underline{p}' \in P'(s)$, either $p_{1+}(a, s) \neq p_{2+}(a, s)$ for some a, s or $\Pi_1(1, \ell) \neq \Pi_2(1, \ell)$ for some ℓ. However, the values $\Pi_k(1, \ell)$ depend on the distinct realizations of p'_{k,s^a}, $s^a \succeq \bar{s}^1$, but not on the values $p(\ell)$ of these realizations. Thus, differences in probabilities $\Pi_k(1, \ell)$ may coexist with distinct, but arbitrarily close, values $p(\ell)$ of price realizations. Obviously, if this is the case and if $p_{1+}(a, s) = p_{2+}(a, s)$ for all (a, s), individuals of age $a = 1$ face at $\bar{s}^1$ arbitrarily close price systems on the two trees. Thus, for $\omega \in \Omega'$, we say that in a subset of $P'(\bar{s})$, the difference between $\underline{p}'_1$ and $\underline{p}'_2$ is sizeable if the values of either $\|(p_{1+}(a, s))_{a,s} - (p_{2+}(a, s))_{a,s}\|$ or $\|p(\ell) - p(\ell')\|$ for some pair $\ell \neq \ell'$ are uniformly bounded away from zero by some positive constant.

LEMMA 6: *There exists a countable collection* $\{P'^n(\bar{s})\}_{n=1}^{+\infty}$ *of compact subsets of* $P'(\bar{s})$ *such that* (i) *if* $\underline{p}' \in P'^n$, *the effective difference between* $\underline{p}'_1$ *and* $\underline{p}'_2$ *is sizeable*, (ii) $P'^n(\bar{s}) \subset P'^{n+1}(\bar{s})$, *and* (iii) $\bigcup_n P'^n(\bar{s}) = \text{cl}(P'(\bar{s}))$.

See the Appendix for the proof.

The Nesting Technique: A direct use of the transversality theorem to obtain $f_\sigma \neq 0$ requires the functions $f^h_\sigma(\underline{p}', \omega)$ to have nonzero derivatives with respect to ω^h for all h and for all $\underline{p}' \in P'(\bar{s})$. The analysis would be relatively straightforward if we could find perturbations that disturb the optimal solution on one tree without affecting it on the second. Indeed, this was the essence of the argument for $G = 1$. Unfortunately, for $G \geq 2$ these perturbations are not available in some regions of $P'(\bar{s})$ and in these regions it can be $D_{\omega^h} f^h_\sigma = 0$. However, we are still able to show that $f^h_\sigma \neq 0$ on $P'(\bar{s})$ for a generic set of parameters and for some h. The argument is based on a nesting technique. We first lay out its general mathematical structure and later we will apply it to our problem.

Let Ω'' be an open subset of Ω' and let $\mathfrak{F}$ be a finite family of real-valued maps f^h_j with domain $P'(\bar{s}) \times \Omega''$, with $h \in H$ and $j = 1, \ldots, J$, where J denotes also the set of indices. The maps f^h_j are assumed to be continuous differentiable and to satisfy $D_{\omega^{h'}} f^h_j(\underline{p}', \omega) = 0$ for all j and $h' \neq h$. Let

$$E^h_j = \{(\underline{p}', \omega) \in P'(\bar{s}) \times \Omega'' : D_\omega f^h_j(\underline{p}', \omega) \neq 0 \text{ or } f^h_j(\underline{p}', \omega) \neq 0\}$$

and

$$N^h_j = \{(\underline{p}', \omega) \in E^h_j : f^h_j(\underline{p}', \omega) \neq 0\} \quad \text{for all } j$$

and define the following two conditions on the maps f^h_j:

CONDITION UNIVERSAL: For all h, $P'(\bar{s}) \times \Omega'' \subset E^h_1 \cup E^h_J$.

CONDITION NESTING: For all h, $N^h_J \cap N^h_j \subset \bigcup_{j' \leq j} E^h_{j'-1}$, for all $j > 1$.

We call the first Condition Universal because the family satisfies a nonzero property for all prices and parameters: at any $(\underline{p}', \omega)$ we can extract from the family $\mathfrak{F}$ an auxiliary system of H maps, one for each h, which are nonzero or have nonzero derivative. We call the second Condition Nesting because it allows us to nest the f_1 functions into a cascade of auxiliary systems of maps in $\mathfrak{F}$. The following result suffices for our analysis.

LEMMA 7: *If $\mathfrak{F}$ satisfies Conditions Universal and Nesting, then there exists a dense set $\bar{\Omega} \subset \Omega''$ such that for all $(\underline{p}', \omega) \in P'(\bar{s}) \times \bar{\Omega}$, there exists h with $f^h_1(\underline{p}', \omega) \neq 0$.*

For the proof see the Appendix.

We sketch the reasoning. What is $\bar{\Omega}$? Consider the set of maps g that assign to each individual h a function $f^h_{g(h)}$ from $\mathfrak{F}$. Let $f_g = (f^h_{g(h)})_{h \in H}$,

$$E_g = \bigcap_h E^h_{g(h)}, \quad \text{and} \quad N_g = \bigcup_h N^h_{g(h)}.$$

A straightforward application of the transversality theorem implies that for each g there exists a dense subset Ω_g of Ω'' such that for all $(\underline{p}', \omega) \in P'(\bar{s}) \times \Omega_g$, either $(\underline{p}', \omega) \notin E_g$ or $f_g(\underline{p}', \omega) \neq 0$; equivalently, $(\underline{p}', \omega) \in N_g$. We then let $\bar{\Omega} = \bigcap_g \Omega_g$. Notice that by construction $P'(\bar{s}) \times \bar{\Omega} \cap E_g = N_g$ for all g. If $(\underline{p}', \omega) \in P'(\bar{s}) \times \Omega_g$, $f_g(\underline{p}', \omega) \neq 0$ only if $(\underline{p}', \omega) \in E_g$, but apparently nothing excludes the possibility that $(\underline{p}', \omega) \notin E_g$. So why does $\bar{\Omega}$ work? Here, the nesting technique kicks in. Condition Universal states that $(\underline{p}', \omega) \in E_g$ for some g with $g(h) = 1$ or $g(h) = J$ for all h. Let $H_g = \{h : f^h_{g(h)}(\underline{p}', \omega) \neq 0\}$ and suppose that $g(h) = J$ for all $h \in H_g$; otherwise, the argument is concluded. Condition Nesting now states that $(\underline{p}', \omega) \in E_{g'}$ and, hence, in $N_{g'}$, for some g' with $g'(h) = g(h)$, for $h \in H \backslash H_g$, while $g'(h) < J$ for $h \in H_g$ and $(\underline{p}', \omega) \in N^h_J \cap N^h_{g'(h)}$. Once again Condition Nesting can be applied to move to an auxiliary system $f_{g'}$ with $g' < g$, and iterating finitely many times, eventually we reach the desired conclusion that $(\underline{p}', \omega) \in N_{g^*}$ with $g^*(h) = 1$ for some $h \in H_{g^*}$.

We are now ready to prove Proposition 4 for $G > 1$. First, we exploit the equivalence classes of Arrow prices to prove that the following family of maps $\mathfrak{F}$ satisfies Conditions Universal and Nesting. Define $\mathfrak{F}$ to be the following family of maps (remember that $N_0 = \sum_{a=0}^G S^a$):

- $f^h_1(\underline{p}', \omega) = f^h_\sigma(\underline{p}', \omega)$.
- $f^h_{s^a}(\underline{p}', \omega) = x^{h0}_{s_{01}}(\underline{p}'_1, \omega) - x^{ha}_{s^a}(\underline{p}'_2, \omega)$, $s^a \in \bigcup_{a=0}^G S^a$.

- $f^h_{2+N_0}(\underline{p}', \omega) = \lambda^h_1(\underline{p}'_1, \omega) - \lambda^h_2(\underline{p}'_2, \omega)$.

Put the set of nodes $s^a \in \bigcup_{a=0}^G S^a$ in a one-to-one correspondence with $\{2, \dots, 1 + N_0\}$ and index them by j. All maps are continuously differentiable and obviously satisfy the condition $D_{\omega^{h'}} f^h_j(p', \omega) = 0$ for all j and $h' \neq h$. The following lemma proves that our choice of $\mathfrak{F}$ also satisfies Conditions Universal and Nesting on a dense subset Ω'' of Ω'.

LEMMA 8: *There exists an open and dense set $\Omega'' \subset \Omega'$ such that the family of maps $\mathfrak{F} = (f^h_j)_{j \geq 1}$ satisfies Conditions* Universal *and* Nesting.

For the proof see the Appendix.

To gain intuition on this issue, notice three aspects of our choice. First, the construction of $P'(\bar{s})$ (i.e., of the equivalence classes of Arrow price pairs) implies that on $P'(\bar{s}) \times \Omega'$, either $D_\omega f^h_1 \neq 0$ or $D_\omega f^h_{2+N_0} \neq 0$; that is, it implies Condition Universal. Second, whenever $f^h_{2+N_0} \neq 0$, endowment or utility perturbations yield different changes in optimal consumption bundles across the two trees; that is, if $f^h_{2+N_0} \neq 0$, then either $D_\omega f^h_{s^a} \neq 0$ or $f^h_{s^a} \neq 0$, $s^a \in \bigcup_{a=0}^G S^a$, and Condition Nesting holds true for $j > 1$. Third, whenever $f^h_{s^a} \neq 0$, $s^a \in \bigcup_{a=0}^G S^a$, the optimal bundle $x^h_{s_{01}}(\underline{p}'_1, \omega)$ does not appear on the second tree, thereby allowing for perturbations of utilities that affect optimal solutions on the first, but not on the second tree. These perturbations are powerful enough to show that when $f^h_{s^a} \neq 0$, $s^a \in \bigcup_{a=0}^G S^a$, and $f^h_{2+N_0} \neq 0$, $D_\omega f^h_1 \neq 0$, that is, Condition Nesting holds true also for $j = 1$.

Next, for given n, Ω^n_σ denotes the subset of Ω'' where the system of equations $f_\sigma(\underline{p}', \omega) = 0$ does not have a solution in $P'^n(\bar{s})$, the set introduced in Lemma 6. By the same argument provided for $G = 1$, Ω^n_σ is open. Again, let $\Omega^* = \bigcap_{n>0} \bigcap_\sigma \Omega^n_\sigma$. If the sets Ω^n_σ are also dense, Ω^* is a residual set where $f_\sigma(\underline{p}', \omega) \neq 0$. However, $P'^n(\bar{s}) \times \bar{\Omega} \subset P'(\bar{s}) \times \bar{\Omega}$ and, hence, by Lemma 7, $f_\sigma(\underline{p}', \omega) \neq 0$ for all $(\underline{p}', \omega) \in P'^n(\bar{s}) \times \bar{\Omega}$. Thus, $\bar{\Omega} \subset \Omega^n_\sigma$, thereby concluding the argument.

5. EXTENSIONS

In computational applications, utility functions are parametrically given, most frequently in the constant relative risk aversion (CRRA) class. As already mentioned, when $G = 1$, Proposition 4 can be immediately established by perturbing only the endowments, therefore covering this class. However, when our density result is based on local utility perturbations, it does not immediately cover those economies, as perturbations now have to be parametric and cannot alter the utility functional form.

On the other hand, many of our simplifying assumptions on the economic environments can be dropped either without altering the results or by actually sharpening them. For example, the simple demographic structure can be generalized going from a constant population process to any exogenous time-homogeneous finite Markov chain only with an increased notational burden. Utilities can be assumed to be state or age dependent, actually facilitating our proofs. Beliefs can differ across agents. We also have assumed that financial assets are short-lived and in zero net supply, that there is no production, and that all risk is aggregate. Our result clearly survives all such extensions.

Long-Lived Assets: We can add long-lived assets in positive supply (i.e., Lucas' trees), storage, and even production to our model as done in Rios Rull (1996) or Kubler and Polemarchakis (2004) without altering any of our results. Obviously now beginning-of-period financial wealth w has to be defined to include the value of long-lived assets held by the individuals and the value of stored commodities. The latter implies that the wealth of the individuals depends on all current prices, their portfolios of long-lived assets, and the stored amount of commodities. The state spaces of Markov and recursive equilibria have to be expanded, since they must now include portfolios of long-lived assets and amounts of stored commodities. If there is production, the capital distribution across firms also needs to be included as an endogenous state. Now, at a critical Markov pair, not only wealths, but also portfolios of long-lived assets, stored commodities, and capital must be invariant. Hence, if anything, by adding equations, these extensions can potentially weaken the degree of heterogeneity needed to rule out the existence of critical pairs and critical initial pairs of Markov states.

Idiosyncratic Risk: Recall that the issue here is not whether we can include idiosyncratic risk in our model, rather whether this inclusion can be used to substantially reduce the ex ante heterogeneity in Assumption A1. The results will depend on how one models idiosyncratic shocks. We just give a hint of how to carry out the analysis for what is the hardest case for our approach.

For each type h, there is a large number of individuals subject to individual shocks that affect endowments (such as unemployment, accidental loss risk, and so on). In each period the set of states of uncertainty is $S \times \Sigma$, with Σ denoting the set of individual states, and individual risks are independent and identically distributed. Ex ante identical individuals of age 0 enter the economy under different uninsured realizations of individual risk: this is the key feature that can be exploited to weaken Assumption A1. The maneuver comes, however, at a cost. To perturb independently the various functions of identical individuals born at different personal states, we need a richer set of perturbations: utilities have to be age and (aggregate) state dependent.

Competitive—and, therefore, Markov and recursive—equilibrium prices are affected by the realizations of the aggregate, but not of the individual states. Thus, the only exogenous variable entering the definition of a Markov state is $s \in S$. A Markov (or recursive) state must now specify, for each type and age, a distribution of wealth. The state space includes vectors of the form $w^{ha}_{\sigma^a}$, $\sigma^a \in \Sigma^a$. The definitions of confounding and nonconfounding equilibria, and critical and initial critical pairs are identical. However, the density argument in

Proposition 4 changes considerably and is more demanding. The essence is to perturb independently individuals indexed by the same h, but by different σ at birth. Since these individuals were endowed at birth with different endowments, their overall wealth and, hence, their consumption plans will typically be different. With state and age dependent utility perturbations, this is sufficient to establish that generically the system of $H\Sigma$ equations $f_{(s_{01},s_{02},\bar{s})} = 0$ cannot have a solution. In other words, Assumption A1 can be weakened:

ASSUMPTION A1′: $H\Sigma > 2[C\sum_{a=0}^{G} S^a - 1]$.

Therefore, even $H = 1$ is compatible with the existence of a recursive equilibrium if Σ is large enough, as we wanted to show.

APPENDIX

PROOF OF PROPOSITION 2: (i) Pick an economy $\omega \in \Omega$ and any family of subsets $O_s \subset W_{s,\omega}$, $s \in S$. Consider the family of competitive equilibrium processes $\tilde{\xi}$. They are parametrized by initial states (s_0, w_0, ξ_0). Hereafter, let Z_0 denote the projection of the initial states of the family of competitive equilibrium processes onto the set of variables (s, w, p, λ), that is, onto the (smaller) state space of the simple Markov equilibrium. For all $\bar{z} = (\bar{s}, \bar{w}, \bar{p}, \bar{\lambda}) \in Z_0$ with $\bar{w} \in O_{\bar{s}}$, select a unique competitive equilibrium, thereby describing a family of competitive equilibria $\tilde{\xi}_{\bar{z}}$ parametrized by such elements $\bar{z} \in Z_0$. We call such $\bar{z}$ an initial condition. The construction of the simple Markov equilibrium is based on an observation that we state under the form of a separate claim. For each history s^t, $t \geq 1$, the competitive equilibrium realization at s^t, $\xi_{\bar{z},s^t} = [x, \theta, \lambda, \psi]_{\bar{z},s^t}$, and the corresponding financial wealth distribution, $w_{\bar{z},s^t} = d_{s_t}\theta_{\bar{z},s^{t-1}}$, define a Markov state $[s_t, (w, p, \lambda)_{\bar{z},s^t}]$. We call two competitive equilibria realizations $\xi_{z,\hat{s}^t}$ and $\xi_{\bar{z},\bar{s}^{t'}}$ *Markov equivalent* if they generate the same Markov states (at $\hat{s}^t$ and $\bar{s}^t$). Given two equilibrium processes $\tilde{\xi}_{\bar{z}}$ and $\tilde{\xi}_z$, and a pair of histories $\bar{s}^{t'}, \hat{s}^t \in \tilde{S}$, $t' > 0$, such that $\xi_{\bar{z},\bar{s}^{t'}}$ and $\xi_{z,\hat{s}^t}$ are Markov equivalent, we define a binary operation $[_{\hat{s}^t} \curlywedge_{\bar{s}^{t'}}] : \Xi^{\tilde{S}} \times \Xi^{\tilde{S}} \to \Xi^{\tilde{S}}$ called *grafting* and denote its result by $\tilde{\xi}_z[_{\hat{s}^t} \curlywedge_{\bar{s}^{t'}}]\tilde{\xi}_{\bar{z}}$. It is the process defined as

$$\{\xi_z[_{\hat{s}^t} \curlywedge_{\bar{s}^{t'}}]\xi_{\bar{z}}\}_{s^{t^*}} = \begin{cases} \xi_{z,s^{t^*}} & \text{for } s^{t^*} \not\succeq \hat{s}^t, \\ \xi_{\bar{z},\bar{s}^{t'+\tau}} & \text{for } s^{t^*} = \hat{s}^{t+\tau}, \tau \geq 0. \end{cases}$$

Notice that the grafting operation $[_{\hat{s}^t} \curlywedge_{\bar{s}^{t'}}]$ can be applied to the same competitive equilibrium $\tilde{\xi}_z$ at two distinct Markov equivalent histories.

CLAIM 9: *For each pair of* (*not necessarily distinct*) *competitive equilibria* $\tilde{\xi}_z$ *and* $\tilde{\xi}_{\bar{z}}$ *with Markov equivalent realizations at* $\hat{s}^t$ *and* $\bar{s}^{t'}$, *the grafted process* $\tilde{\xi}_z[_{\hat{s}^t} \curlywedge_{\bar{s}^{t'}}]\tilde{\xi}_{\bar{z}}$ *is a competitive equilibrium process that starts from the initial condition* $z \in Z_0$.

The proof of Claim 9 is put off to the end of this argument. Claim 9 implies that we can apply the operator $[_{\hat{s}^t} \curlywedge_{\bar{s}^{t'}}]$ countably many times and still obtain a competitive equilibrium. This is what we do to construct a simple Markov equilibrium. We build the functions T and $\xi : Z \to \Xi$, and define the state space Z recursively as follows.

Start from $t = 0$. Drop the subscript ω, consider the set Z_0 and the selection of equilibrium processes $\xi_{\bar{z}}$, $\bar{z} \in Z_0$, $\bar{w} \in O_{\bar{s}}$, and start defining the endogenous map ξ:

(i) Set $\xi(\bar{z}) \equiv \xi_{\bar{z}}$ for all $\bar{z} \in Z_0$.

Moving to $t = 1$, the competitive equilibrium $\tilde{\xi}_{\bar{z}}$ determines $\xi_{\bar{z},s^1}$ and $w_{\bar{z},s^1}$, $s^1 = (\bar{s}, s)$, for all $\bar{z} \in Z^0$. Therefore, it describes uniquely on Z_0 the continuation of $\bar{z}$, that is, the S Markov states $z' = (s, \zeta_{\bar{z},s^1})$, $s \in S$, and $\zeta = (w, p, \lambda)$, immediately following $\bar{z}$.

(ii) Set $T_s(\bar{z}) = (s, \zeta_{\bar{z},s^1})$ for $\bar{z} \in Z_0$ and $Z_1 = \bigcup_{s \in S}\{T_s(Z_0)\}$.

A predecessor of $z' = (s', \zeta') \in Z_1$ is $z \in Z_0$ such that $z' = T_{s'}(z)$; $z'_- \subset Z_0$ denotes the set of predecessors of z'. Predecessors may not be unique, since the same (endogenous) state z' can be generated by the equilibrium processes of different initial conditions $\bar{z}$. Partition the sets Z_1 into (Z_{1a}, Z_{1b}, Z_{1c}), three disjoint and exhaustive subsets defined as follows:

(a) $Z_{1a} = Z_0 \cap Z_1$ is the set of states that are both initial and successors to the initial conditions; (b) $Z_{1,b} = \{z \in Z_1 \backslash Z_0 : \# z_- = 1\}$ is the set of states with a unique predecessor, but that are not initial conditions; and (c) $Z_{1c} = \{z \in Z_1 \backslash Z_0 : \# z_- > 1\}$ is the set of states with multiple predecessors, but that are not initial conditions.

If (a), then the state $\bar{z}$ is both initial condition—and $\tilde{\xi}_{\bar{z}}$ is the equilibrium associated to it—as well as a successor of an initial condition z, and ξ_z is the competitive equilibrium associated to it. Now, with $s^0 = \bar{s}$ and $s^1 = (s, \bar{s})$, ξ_{z,s^1} and $\xi_{\bar{z},s^0}$ are Markov equivalent states. Apply $[_{\hat{s}^t} \curlywedge_{\bar{s}^t}]$ as $\tilde{\xi}_z[_{s^1} \curlywedge_{s^0}]\tilde{\xi}_{\bar{z}} = \tilde{\xi}^*_z$. From Claim 9, $\tilde{\xi}^*_z$ is a new competitive equilibrium starting at z and, by construction, $\tilde{\xi}^*_z$ has the same continuation path at s^1 of $\tilde{\xi}_{\bar{z}}$ at $s^0 = \bar{s}$.

(iii) Thus set $T_s(\bar{z}) = (s, \zeta^*_{\bar{z},(\bar{s},s)})$ and $\xi(z) = \xi^*_{\bar{z},(\bar{s},s)}$ for $z \in Z_{1a}$.

If (b), $\tilde{\xi}_{\bar{z}}$ will be the unique competitive equilibrium that has generated $z' \in Z_{1b}$ at $s^1 = (\bar{s}, s')$.

(iv) Then set $T_s(z') = (s', \zeta_{\bar{z},(s^1,s)})$ and $\xi(z') = \xi_{\bar{z},s^1}$ for $\zeta \in Z_{1b}$.

If (c), there may be multiple competitive values for the current endogenous variables at $z = (s, \zeta) \in Z_{1c}$ as well as multiple continuations $(s, \zeta_{z,(s^1,s')})$, $s' \in S$, depending on which competitive process $\tilde{\xi}_{\bar{z}}$, $\bar{z} \in z_-$, we follow. In such a case, we first select arbitrarily one predecessor, $\phi(z) = (s^*, \zeta^*) \in z_-$. Then, for all $z' \in z_-$, we define the new competitive equilibrium $\tilde{\xi}_{z'}[_{s',s} \curlywedge_{s^*,s}]\tilde{\xi}_{\phi(z)}$. Again by Claim 9, this grafting operation uniquely defines the continuation of the Markov state z as $(\hat{s}, \zeta_{\phi(z),(s^{*1},\hat{s})})$, $\hat{s} \in S$ and $s^{*1} = (s^*, s)$, determined by the competitive equilibrium $\tilde{\xi}_{\phi(z)}$.

(v) Thus, we set $T_{\hat{s}}(z) = \zeta_{\phi(z),(s^{*1},\hat{s})}$ and $\xi(z) = \xi_{\phi(z),s^{*1}}$ for $z \in Z_{1c}$.

By construction, the maps ξ and T are well defined functions on $Z_0 \cup Z_1$: each Markov state $z \in Z_0 \cup Z_1$ selects a unique competitive equilibrium and, hence, a unique continuation.

Let $Z_2 = \bigcup_{s \in S} T_s(Z_1)$ be the sets of endogenous Markov states obtained by using operations (iii)–(v). Since each Markov state $z \in Z_0 \cup Z_1$ selects a unique competitive equilibrium and a unique continuation, moving to $t = 3$ we can treat $Z_0 \cup Z_1$ as Z_0, and Z_2 as Z_1, and repeat operations (iii)–(v) thereby creating sets Z_3 and extending through the same operations the transition T and the maps ξ to $Z_2 \backslash (Z_0 \cup Z_1)$.

Apply recursively this procedure, thereby creating sets Z_t, $t = 0, 1, \dots,$ and defining the maps T and ξ over $\bigcup_{t \geq 0} Z_t$. This describes a Markov equilibrium of $\omega \in \Omega$. Notice that the space of Markov states is $Z = \bigcup_{t \geq 0} Z_t$. Obviously, $Z_0 \subset Z$, and hence $O_s \subset W^Z_{s,\omega}$, $s \in S$, and we have proven Proposition 2(i). Furthermore, by construction, $\bar{z} = (\bar{s}, \bar{w}, \bar{p}, \bar{\lambda})$ is an initial Markov state only if $\bar{z} \in Z_0$ and $\bar{z} \notin Z_t$ for all $t > 1$. Then $\bar{w} \in O_s$ (Proposition 2(ii)) and $(\bar{s}, \bar{w}, \bar{p}, \bar{\lambda}) \in Z_0$ for a unique choice of $(\bar{p}, \bar{\lambda})$, the competitive equilibrium value taken at the initial state $\bar{z}$, that is, Proposition 2(iii).

PROOF OF CLAIM 9: The argument is simple, so we just sketch it. By construction, $\tilde{\xi}^*_z \equiv \tilde{\xi}_z[_{\hat{s}^t} \lambda_{\bar{s}^{t\prime}}]\tilde{\xi}_{\bar{z}}$ satisfies the market clearing conditions. Thus, it suffices to check that $\tilde{\xi}^*_z$ satisfies the first-order conditions of the individual programming problems. However, since $\tilde{\xi}^*_z$ coincides at s^t with either $\tilde{\xi}_z$ or with $\tilde{\xi}_{\bar{z}}$, it satisfies the optimality conditions $Du^h(x^{ha}) - \lambda^{ha} p = 0$ and the budget constraints for all s^t, h, a. For the same reason, $\tilde{\xi}^*_z$ satisfies as well the no arbitrage equations

$$(11) \qquad -\lambda^{ha}_{s^t} q_{s^t} + \delta \sum_s \delta \pi(s|s_t) \lambda^{h(a+1)}_{(s^t,s)} d_s = 0, \quad a < G$$

for $s^t \neq \hat{s}^t$ and $s^t \neq \hat{s}^{t-1}$, the immediate predecessor of $\hat{s}^t$, since then only variables defined by $\tilde{\xi}_z$ or $\tilde{\xi}_{\bar{z}}$ appear. Hence, it only remains to be checked that (11) is satisfied at $\hat{s}^t$ and $\hat{s}^{t-1}$. The definition of Markov equivalent realization implies that $(\bar{s}_{t'}, \lambda_{\bar{z},\bar{s}^{t'}}) = (\hat{s}_t, \lambda_{z,\hat{s}^t})$, while the construction of $\tilde{\xi}^*_z$ implies that $\lambda^*_{z,\hat{s}^t} = \lambda_{\bar{z},\bar{s}^{t'}} = \lambda_{z,\hat{s}^t}$. By direct inspection, the latter implies that both equations are satisfied, thereby concluding the argument. *Q.E.D.*

PROOF OF PROPOSITION 3: For given ω and $\bar{s} \in S$, let

$$\bar{W}^1_{\bar{s},\omega} = \{\bar{w}^1 = (\bar{w}^{h1})_{h \in H} : (\bar{w}^1, (\bar{w}^a)_{a>1}) \in \mathring{W}_{\bar{s},\omega} \text{ for some } (\bar{w}^{ha})_{h,a>1}\}.$$

Given the properties of $\mathring{W}_{\bar{s},\omega}$, $\bar{W}^1_{\bar{s},\omega}$ is an open and bounded subset of $\mathbb{R}^H$ if $G > 1$ and it coincides with $\mathring{W}_{\bar{s},\omega}$ otherwise. Let $\bar{s}^1 = (s_{01}, \bar{s})$ (i.e., pick $s_{01} \in S$

and the corresponding finite tree). For $\bar{w}^1 \in \bar{W}^1_{\bar{s},\omega}$, $f_{(s_{01},\bar{s}),\bar{w}^1} : P(\omega) \to \mathbb{R}^H$, where $\dim P(\omega) < H$ by Assumption A1, and there are fewer equations than unknowns. Furthermore, the map $f_{(s_{01},\bar{s})}$ is independent of $\bar{w}^{ha}$, $a > 1$. If $G > 1$, the Jacobian matrix of the map $f_{(s_{01},\bar{s})}(\cdot)$ with respect to $\bar{w}^1 \equiv (\bar{w}^{h1})_{h\in H}$ is equal to the identity matrix. Thus, for given $\omega \in \Omega$, the transversality and preimage theorems imply that for $\bar{w} \in W^*_{(s_{01},\bar{s}),\omega}$, a full Lebesgue measure subset of $\bar{W}_{\bar{s},\omega}$, $f_{(s_{01},\bar{s})}(p, q, \omega, \bar{w}^{h1}) = 0$ does not have a solution in $P(\omega)$. Furthermore, since $P(\omega)$ is compact, the natural projection onto $\mathring{W}_{\bar{s},\omega}$ restricted to $P(\omega)$ is proper, and $W^*_{(s_{01},\bar{s}),\omega}$ is also open.

If $G = 1$, then $\sum_h \bar{w}^{h1} = 0$. Assumption A1 now reads $H > 2[C(1+S) - 1]$, which implies $H - 1 > C(S+1) - 1 = \dim P(\omega)$. Thus, drop the function $w^{h1}_{(s_{01},\bar{s})}(\cdot) - \bar{w}^{h1}$ from the map $f_{(s_{01},\bar{s})}$ for $h = 1$ and call $f^{\setminus}_{(s_{01},\bar{s})}$ the map so obtained. The latter is independent of $\bar{w}^{11}$, while full-rank perturbations of it can be obtained by perturbing independently $\bar{w}^{h1}$, $h > 1$. Hence, by the same argument used above, for $\bar{w} \in W^*_{(s_{01},\bar{s}),\omega}$, an open and full Lebesgue measure subset of $\bar{W}_{\bar{s},\omega}$, $f^{\setminus}_{(s_{01},\bar{s})}(\cdot) = 0$ (and therefore $f_{(s_{01},\bar{s})}(\cdot) = 0$) does not have a solution in $P(\omega)$.

Now the set $W^*_{\bar{s},\omega} = \bigcap_{s_{01}} W^*_{(s_{01},\bar{s}),\omega}$ is an open and full Lebesgue measure subset of $\mathring{W}_{\bar{s},\omega}$ that satisfies the required property. *Q.E.D.*

PROOF OF LEMMA 5: We want to show that there exists an open and dense set of parameters Ω' such that for $(e, u, \delta) \in \Omega'$, if $\underline{p}' \in P'(\bar{s})$, then $\underline{p}' \in P'(\bar{s}, \delta)$. Pick $\underline{p}' \in P'(\bar{s})$ and, to avoid trivialities, assume that

$$(12) \qquad p_{1+}(a, s) = p_{2+}(a, s) \quad \text{for all } a, s.$$

Then there exists a and ℓ' such that

$$(13) \qquad \Pi[S^a_1(\ell')] \neq \Pi[S^a_2(\ell')].$$

Write expand the conditions $\Pi_1(1, \ell) = \Pi_2(1, \ell)$. They are a polynomial in δ, namely,

$$(14) \qquad \sum_{a\geq 1} \delta^{a-1} \phi(a, \ell) = 0,$$

where

$$\phi(a, \ell) = \left[\sum_{s^a \in S_+(a,\ell)} \pi(s^a|\bar{s}^1) - \sum_{s^a \in S_{2+}(a,\ell)} \pi(s^a|\bar{s}^1) \right] = \Pi[S^a_1(\ell)] - \Pi[S^a_2(\ell)].$$

The coefficients $\phi = (\phi(a, \ell))_{a,\ell}$ are uniquely determined by the Arrow price pairs $\underline{p}'$ through the determination of the sets $S^a_k(\ell)$. Thus, the function and the notation $\phi_{\underline{p}'}$ are well defined. Condition (13) can be written as $\phi_{\underline{p}'}(a, \ell') \neq 0$. Since the trees are finite, there are only finitely many sets $S_k^a(\ell)$ or, equivalently, there are only finitely many coefficient values ϕ. Let Φ be their set. By definition of $p_{k+}(a, s)$, (12) implies that $p'_{k,\bar{s}^1}/p'_{1,k,\bar{s}^1}$ and $p'_{k,s^2}/p'_{1,k,\bar{s}^1}$ for all $s^2 > \bar{s}^1$ are k-invariant. Thus, $\phi(1, \ell) = \phi(2, \ell) = 0$ for all ℓ and $\phi \in \Phi$. Now, for $\bar{a} \geq 3$, let

$$\Phi(\bar{a}) = \{\phi \in \Phi : \phi(a, \ell) = 0, \text{ for all } \ell \text{ and } a < \bar{a}, \phi(\bar{a}, \ell') \neq 0, \text{ for some } \ell'\}.$$

If $\underline{p}' \in P'(\bar{s})$ and (12) holds, then $\phi_{\underline{p}'} \in \Phi(\bar{a})$, for some $\bar{a} \geq 3$. The equation $\Pi_1(1, \ell') = \Pi_2(1, \ell')$ now reads

$$\sum_{a \geq \bar{a}}^{G} \delta^{a-\bar{a}} \phi(a, \ell') = 0.$$

Since the coefficient for the degree-zero term is $\phi(\bar{a}, \ell') \neq 0$, the latter is a nonzero polynomial and by Theorem 14, in Zariski and Samuel (1960, Chap. I, p. 38) the set of zeros of the polynomial is closed and has measure zero in $(0, 1]$. Let Ω_ϕ be the complement of this set and let $\Omega(\bar{a}) = \bigcap_{\phi \in \Phi(\bar{\alpha})} \Omega_\phi$. By construction, Ω_ϕ and, therefore, $\Omega(\bar{a})$ are open and dense. Finally set $\Omega' = \bigcap_{\bar{a} \geq 3} \Omega(\bar{a})$ as an open and dense subset set of Ω. Most importantly, if $\underline{p}' \in P'(\bar{s})$, then $\phi_{\underline{p}'} \in \Phi(\bar{a})$ for some $\bar{a}$; then if $\omega = (e, u, \delta) \in \Omega'$, $\Pi_{1+}(\ell') \neq \Pi_{2+}(\ell')$ for some ℓ' or $\underline{p}' \in P'(\bar{s}, \delta)$. *Q.E.D.*

PROOF OF LEMMA 6: Each $\underline{p}' \in P'(\bar{s})$ uniquely determines the set of indices $\mathbb{P}^1$, the sets of histories of length a, $S_k^a(\ell)$ for all k, $\ell \in \mathbb{P}^1$ and $a \geq 1$, and the values $p_{k+}(a, s)$ for all a, s. First, the possible configurations of the index set $\mathbb{P}^1$ and of histories $S_k^a(\ell)$ are finite. Second, if two pairs of Arrow prices determine the same $\mathbb{P}^1$, $S_k^a(\ell)$ for all k, $\ell \in \mathbb{P}^1$, and $a \geq 1$, they deliver identical values for $\Pi[S_k^a(\ell)]$. Then partition $P'(\bar{s})$ into a collection of J disjoint and exhaustive subsets $P'_j(\bar{s})$, where J is the cardinality of the sets $(\mathbb{P}^1, (S_k^a(\ell))_{k,\ell \in \mathbb{P}^1, a \geq 1})$ generated by the elements of $P'(\bar{s})$ and j is their indices. Two distinct price pairs in $P'_j(\bar{s})$ determine the same sets of price indices $\mathbb{P}^1$ and of states $S_k^a(\ell)$, but distinct values $p(\ell)$. Therefore, without ambiguity denote with $\mathbb{P}_j^1$ the set of price indices generated by $\underline{p}' \in P'_j(\bar{s})$. Define $P'^n_j(\bar{s})$ as the subset of $P'_j(\bar{s})$ that satisfies

$$\|p(\ell) - p(\ell')\| \geq \frac{1}{n} \quad \text{for all } \ell \text{ and } \ell' \in \mathbb{P}_j^1, j = 1, \ldots, J.$$

Then define $P'^n_0(\bar{s})$ as

$$\left\|(p_{1+}(a, s) - p_{2+}(a, s))_{a,s}\right\| \geq \frac{1}{n}.$$

Finally, let

$$P'^n(\bar{s}) = \bigcup_{j=0}^{J} P_j'^n(\bar{s}).$$

The sets $P_j'^n(\bar{s})$ are compact for all j and, therefore, so is $P'^n(\bar{s})$. Lemma 5 guarantees that for $\omega \in \Omega'$ and $\underline{p}' \in P'^n(\bar{s}) \subset P'(\bar{s})$, inequalities (10) are satisfied. Obviously $P'^n(\bar{s}) \subset P'^{n+1}(\bar{s})$ and $\bigcup_n P'^n(\bar{s}) = \mathrm{cl}(P'(\bar{s}))$ by construction. *Q.E.D.*

Utility Perturbations: For the proofs that follow, we use utilities to perturb the equations $f_j^h = 0$. To do so, we use a locally finite, linear parametrization of the utility functions. Pick $N > 1$ distinct consumption bundles $x_j \in \mathbb{R}_{++}^C$, $j = 1, \ldots, N$. We perturb the gradient of $u^h(\cdot)$ around the bundles x_j, $j = 1, \ldots, N$. For each j, pick a pair of open balls $B_{\varepsilon_i}(x_j)$, $i = 1, 2$, centered around x_j and such that (i) $\varepsilon_2 > \varepsilon_1$ and (ii) $\bigcap_j \mathrm{cl}(B_{\varepsilon_2}(x_j)) = \emptyset$. Then pick smooth "bump" functions Φ_j such that $\Phi_j(x; x_j) = 1$ for $x \in B_{\varepsilon_1}(x_j)$, and $\Phi_j(x; x_j) = 0$ for $x \notin \mathrm{cl}(B_{\varepsilon_2}(x_j))$. For given arbitrary vectors $\Delta u = (\Delta u_j)_{j=1}^N \in \mathbb{R}^{CN}$ and scalar η, define the utility function

$$u_\eta^h(x, \Delta u) = u^h(x) + \eta \sum_{j=1}^{N} \Phi_j(x; x_j) \sum_c \Delta u_{j,c} x_c.$$

For any given Δu, we can pick η so close to zero that $u_\eta^h(\cdot)$ is arbitrarily close to $u^h(\cdot)$ in the $\mathcal{C}^2$-uniform convergence topology and it satisfies, therefore, all the maintained assumptions. We identify a utility perturbation with the (Gateaux) derivative of D (the derivative as a linear map of functions) at u^h in the direction $\sum_j \Phi_j(x; x_j)\Delta u_j$, that is, with a vector Δu_j. Which finite set of points x_j is used depends on $\underline{p}'$, so the derivative of u^h, hence of f, is not finitely parametrized.

PROOF OF LEMMA 7: As a preliminary step, to apply the infinite-dimensional version of transversality, we need the space of utilities to be Banach. This is done as follows. Let $\bar{X} \subset \mathbb{R}_{++}^C$ be the compact set defined in Section 4.4. Recall that, thereafter, we have identified Ω with $B_\varepsilon(\omega)$ for a given ω. By construction, when $(\psi, \omega') \in P \times B_\varepsilon(\omega)$, optimal consumptions are contained in $\bar{X}$. This is going to stay true when we move to P' and, in particular, to $P'(\bar{s})$. Since density is a local property, in what follows we can identify $\mathcal{U}$ with the utilities u^h restricted to the compact domain $\bar{X}$, a complete subspace of $\mathcal{C}^2(\bar{X}, \mathbb{R})$ when endowed with the uniform topology, and a Banach space. If a set is dense in Ω in this topology, it is also dense when Ω is endowed with the original, coarser topology. For simplicity, we keep the notation $\mathcal{U}$, Ω, Ω'', and f_j^h unchanged.

Let $\mathcal{G} = \{g : H \to \{1, \dots, 2 + \sum_a S^a\}\}$ be the set of maps that assign for each h a map $f^h_{g(h)} \in \mathfrak{F}$. Then, for all $g \in \mathcal{G}$, $f_g = (f^h_{g(h)})_{h \in H} : P'(\bar{s}) \times \Omega'' \to \mathbb{R}^H$ are continuously differentiable and E_g are (relatively) open sets of $P'(\bar{s}) \times \Omega''$. Let $\phi_g : E_g \to \mathbb{R}^H$ be the restriction of f_g to E_g and let $\Omega(g) = \text{proj}_\Omega(E_g)$. By construction, either $\phi_g \neq 0$ or $D\phi_g$ is onto $\mathbb{R}^H$, and since $\ker D\phi_g$ is finite codimensional in $\mathbb{R}^{\dim P'(\bar{s})} \times \Omega(g)$, ϕ_g is transversal to zero. Then, by the parametric transversality theorem (see Abraham, Marsden, and Ratiu (1988, Theorem 3.6.22)), $\phi_{g,\omega}$ is also transversal to zero for all $\omega \in \Omega^*(g)$, a dense subset of $\Omega(g)$. However, since by Assumption A1, $\dim P < H$, by the preimage theorem, $\phi^{-1}_{g,\omega}(0) = \emptyset$ for $\omega \in \Omega^*(g)$. Then $\Omega_g = \Omega^*(g) \cup \Omega'' \backslash \Omega(g)$ is a dense subset of Ω'' and so is $\overline{\Omega} = \bigcap_g \Omega_g$. Bear in mind that by construction $E_g \cap P'(\bar{s}) \times \overline{\Omega} = N_g$ for all g. We now show that for all $(\underline{p}', \omega) \in P'(\bar{s}) \times \overline{\Omega}$, there exists h such that $f^h_1(\underline{p}', \omega) \neq 0$. Pick any such $(\underline{p}', \omega)$. By Condition Universal, $E^h_1 \cup E^h_J = P'(\bar{s}) \times \Omega''$ for all h, and hence $(\underline{p}', \omega) \in E_g$, and, therefore, $(\underline{p}', \omega) \in N_g$, for some $g \in \mathcal{G}$ with $g(h) = 1$ or J for all h. Let $H_g = \{h : (\underline{p}', \omega) \in N^h_{g(h)}\}$. By construction of N_g and E_g, H_g is nonempty. If $g(h) = 1$ for some $h \in H_1$, then $f^h_1(\underline{p}', \omega) \neq 0$, a contradiction that concludes the argument. Otherwise, $g(h) = J$ for all $h \in H_g$. Since $(\underline{p}', \omega) \in N^h_J$, by Condition Nesting, $(\underline{p}', \omega) \in E^h_{j(h)}$ with $j(h) < J$ for all $h \in H_1$. Let $g^1 \in \mathcal{G}$ be defined as $g^1(h) = g(h)$ for $h \in H \backslash H_g$, while $g^1(h) = j(h)$ for $h \in H_g$. Since $\omega \in \overline{\Omega}$, $(\underline{p}', \omega) \in N_{g^1}$. By construction of H_g and g^1, $f^h_{g^1(h)}(\underline{p}', \omega) = 0$ for all $h \in H \backslash H_g$. Since $(\underline{p}', \omega) \in N_{g^1}$, it must be that $H_{g^1} = \{h : f^h_{g^1(h)}(\underline{p}', \omega) \neq 0\} \subset H_g$ is nonempty. For $h \in H_{g^1}$, it is $(\underline{p}', \omega) \in N^h_{g^1(h)} \cap N^h_J$ and, therefore, Condition Nesting implies that $(\underline{p}', \omega) \in E^h_{j'(h)}$ for some $j'(h) < g^1(h)$. Iterating finitely many times, the argument concludes that $(\underline{p}', \omega) \in N_{g^*}$ for g^* with $g^*(h) = 1$ for some $h \in H_{g*}$. *Q.E.D.*

PROOF OF LEMMA 8:

Preliminary Computations. In analogy with the notation in the text, for $\underline{p}' \in P'$, let

$$\mathbb{P} = \{\hat{p} \in \mathbb{R}^C_{++} : p'_{k,s^a} = \hat{p}, \text{ for some } k, s^a\}$$

with cardinality $\mathbb{P} \leq 2\sum_{a=0}^{G-1} S^a$, generic element $\hat{p}(\ell)$, and $\mathbb{P}$ denoting also the set of price indices ℓ. For $\ell \in \mathbb{P}$, we define sets of histories and probability weights

$$S_k(\ell) = \{s^a : p'_{k,s^a} = \hat{p}(\ell)\}, \quad \Pi_k(\ell) = \sum_{s^a \in S_k(\ell)} \delta^a \pi(s^a | s_{0k}),$$

where $\Pi_k(\ell) = 0$ if $S_k(\ell) = \emptyset$ and

$$p_k(a,s) = \delta^{a-1} \sum_{s^{a-1}:(s^{a-1},s)} \pi(s^{a-1}, s|\bar{s}^1) p'_{k,(s^{a-1},s)}.$$

Given when $p'_{1,1,\bar{s}^1} \neq p'_{1,1,\bar{s}^1}$, $\underline{p}' \in P'(\bar{s})$, $\hat{p}(\ell) \in \mathbb{P}$ generates two distinct values in the set $\mathbb{P}_1$: $p(\ell^1) = (\hat{p}(\ell))/p'_{1,1,\bar{s}^1}$ and $p(\ell^2) = (\hat{p}(\ell))/p'_{1,2,\bar{s}^1}$. To keep the two index sets consistent, use the convention $\ell \in \mathbb{P} \cap \mathbb{P}^1$ if and only if $p(\ell) = (\hat{p}(\ell))/p'_{1,k,\bar{s}^1}$ for some $k = 1, 2$.

Pick $\underline{p}' \in P'(\bar{s})$ and, with this notation at hand, rewrite the individual programming problem (5) as

$$\text{(15)} \qquad \max \sum_{\ell \in \mathbb{P}} \Pi_k(\ell) u^h(x_k(\ell)) \quad \text{s.t.}$$

$$\sum_{\ell \in \mathbb{P}} \Pi_k(\ell) \hat{p}(\ell) x_k(\ell) = \sum_{a,s} p_k(a,s) e_s^{ha}.$$

The first-order conditions associated to problem (15) are, for $\ell \in \mathbb{P}$,

$$Du^h(x_k(\ell)) - \lambda_k^h \hat{p}(\ell) = 0,$$

$$\sum_{\ell \in \mathbb{P}} \Pi_k(\ell) \hat{p}(\ell) x_k(\ell) - \sum_{a,s} p_k(a,s) e_s^{ha} = 0.$$

Drop h. We perturb the utility function around $x_k(\ell)$ without disturbing it around $x_k(\ell')$, $\ell' \neq \ell$. We denote such a perturbation by $\Delta u(\ell)$ and denote the endowment perturbation by Δe_s^a, while $(\Delta x, \Delta\lambda, \Delta w)$ is their effect on the variables (x, λ, w).

Differentiating the first-order conditions, we get

$$\text{(l)} \qquad H_k(\ell)\Delta x_k(\ell) - \hat{p}^T(\ell)\Delta\lambda_k - \Delta u(\ell) = 0,$$

$$\text{(bc)} \qquad \sum_{\ell \in \mathbb{P}} \Pi_k(\ell) \hat{p}(\ell) \Delta x_k(\ell) - \sum_{a,s} p_k(a,s) \Delta e_s^a = 0,$$

where $H_k(\ell)$ is the invertible Hessian at $x_k(\ell)$ and the superscript T stands for transpose. While differentiating the map $w_{k,\bar{s}^1}$, recalling that $p(\ell) = (\hat{p}(\ell))/p'_{1,k,\bar{s}^1}$, we get

$$\text{(W)} \qquad \Delta w_{k,\bar{s}^1} = \sum_{\ell \in \mathbb{P}} \Pi_k(1,\ell) p(\ell) \Delta x_k(\ell) - \sum_{a>0,s} p_{k+}(a,s) \Delta e_s^a.$$

Define

$$Q_k(\ell) \equiv \hat{p}(\ell) H_k^{-1}(\ell) \hat{p}^T(\ell).$$

Since $H_k^{-1}(\ell)$ is a negative definite matrix, the terms $Q_k(\cdot)$ are negative, and so is $Q_k = \sum_{\ell\in\mathbb{P}} \Pi_k(\ell)Q_k(\ell)$. From equation (1), by performing elementary computations, we get

$$\Delta\lambda_k = \left\{\sum_{a,s} \frac{p_k(a,s)\Delta e_s^a}{Q_k} - \sum_{\ell\in\mathbb{P}} \frac{\Pi_k(\ell)\hat{p}(\ell)H_k^{-1}\Delta u(\ell)}{Q_k}\right\},$$

and taking into account that $p(\ell) = (\hat{p}(\ell))/p_{1,k,\bar{s}^1}$, we get

$$\Delta w_{k,\bar{s}^1} = Q_{k+}\Delta\lambda_k + \sum_{\ell\in\mathbb{P}^1} \Pi_k(1,\ell)p(\ell)H_k^{-1}(\ell)\Delta u(\ell) - \sum_{a>0,s} p_{k+}(a,s)\Delta e_s^a,$$

where $Q_{k+} = \sum_{\ell\in\mathbb{P}^1} \Pi_{k+}(\ell)(Q_k(\ell))/p'_{1,k,s^1} < 0$.

Denote by λ_k the value $\lambda_k(\underline{p}'_k, \omega)$. By the first-order conditions of the individual problems, if $\Pi_k(\ell) > 0$ for all k and if $\lambda_1 = \lambda_2$, then $x_1(\ell) = x_2(\ell)$ at the optimal solutions of the two programming problems. However, if $\lambda_1 \neq \lambda_2$, for any pair ℓ and ℓ_ℓ such that $\Pi_1(\ell) > 0$ and $\Pi_2(\ell_\ell) > 0$, $x_1(\ell) = x_2(\ell_\ell)$ if and only if $\hat{p}(\ell) = (\lambda_1/\lambda_2)\hat{p}(\ell_\ell)$. Hereafter, for each ℓ (with $\Pi_1(\ell) > 0$), we denote with ℓ_ℓ the index associated to $\hat{p}(\ell) = (\lambda_1/\lambda_2)\hat{p}(\ell_\ell)$—and if $\lambda_1 = \lambda_2$, $\ell_\ell = \ell$. Also bear in mind that $H_1^{-1}(\ell) = H_2^{-1}(\ell_\ell)$. We denote by $\ell = 1$ the price equivalence class identified by $p'_{1,s_{01}}$ and denote by ℓ_1 the price equivalence class associated to $p'_{2,s^a} = (\lambda_1/\lambda_2)p'_{1,s_{01}}$. Define $f_{\ell_1} = x_1(1) - x_2(\ell_1)$ and observe that f_{ℓ_1} is smooth in ω if $\Pi_2(\ell_1) > 0$ or, equivalently, if $\lambda_1(\underline{p}'_1, \omega) \neq \lambda_2(\underline{p}'_2, \omega)$, while $f_{\ell_1}(\underline{p}', \omega) \neq 0$ otherwise. Notice that $f_{\ell_1} \neq 0$ if and only if $f_j \neq 0$, $j = 2, \ldots, \sum_{a=0}^G S^a$, and that $D_\omega f_j = D_\omega f_{\ell_1}$ for all $j = 2, \ldots, 1 + \sum_{a=0}^G S^a$ such that $f_j = 0$. Therefore, the set E_2^h is just the set

$$E_2^h = \{(\underline{p}', \omega) : [f_{\ell_1}^h = 0 \text{ and } D_\omega f_{\ell_1} \neq 0] \text{ or } f_{\ell_1} \neq 0\}.$$

By taking into account that $\hat{p}(\ell) = (\lambda_1/\lambda_2)\hat{p}(\ell_\ell)$, the derivatives of the map f_{ℓ_1} are

$$D_{u(\ell)}f_{\ell_1} = H_1^{-1}(1)\hat{p}^T(1)\hat{p}(\ell)H_1^{-1}(\ell) \times\left[\left(\frac{\lambda_1}{\lambda_2}\right)^2 \frac{\Pi_2(\ell_\ell)\Delta u(\ell)}{Q_2} - \frac{\Pi_1(\ell)\Delta u(\ell)}{Q_1}\right] \tag{16}$$

and

$$D_{e_s^a}f_{\ell_1} = H^{-1}(1)\hat{p}^T(1)\left[\left(\frac{\lambda_1}{\lambda_2}\right)\frac{p_2(a,s)}{Q_2} - \frac{p_1(a,s)}{Q_1}\right]. \tag{17}$$

Hereafter, to simplify notation, we set $J = 2 + N_0$. We are now ready to establish Condition Universal. If, for $(\underline{p}', \omega) \in P'(\bar{s}) \times \Omega'$, either $f_J^h(\underline{p}', \omega) \neq 0$ or $f_1^h(\underline{p}', \omega) \neq 0$, Condition Universal holds true. Hence, in the next claim we are concerned only with the complementary case.

CLAIM 10: *Suppose that* $(\underline{p}', \omega) \in P'(\bar{s}) \times \Omega'$ *and* $f_1^h(\underline{p}', \omega) = 0$. *If* $(s_{0k}, p'_{k,s_{0k}})$ *is one-to-one in* k, *then* $D_{\omega^h} f_1^h(\underline{p}', \omega) \neq 0$; *otherwise, and if* $f_J^h(\underline{p}', \omega) = 0$, *then either* $D_{\omega^h} f_1^h(\underline{p}', \omega) \neq 0$ *or* $D_{\omega^h} f_J^h(\underline{p}', \omega) \neq 0$.

PROOF: Drop h. We argue by contradiction, assuming that $D_\omega f_1 = 0$. If $D_{\Delta e_s^a} f_1 = 0$ for all (a, s), then

$$\frac{Q_{1+}}{Q_1} p_1(a, s) - p_{1+}(a, s) = \frac{Q_{2+}}{Q_2} p_2(a, s) - p_{2+}(a, s) \quad \text{for all } (a, s). \tag{18}$$

Equation (18) computed at $(0, s_{01})$ implies that $s_{01} = s_{02}$. Moreover, since by the adopted normalization $p_{1,k,s_0} = 1$, it also implies that p'_{k,s_0} and Q_{k+}/Q_k are k-invariant, a contradiction. Assume, therefore, that $(s_{0k}, p'_{k,s_{0k}})$ is k-invariant and that also, by contradiction, $D_{\omega^h} f_1 = D_{\omega^h} f_J = 0$. If $D_{\Delta e_s^a} f_J = 0$ for all (a, s), then $(p_1(a, s))/Q_1 = (p_2(a, s))/Q_2$ for all (a, s). Computed at $(0, s_0)$, the latter implies $Q_k = Q$ for $k = 1, 2$; computed at any other (a, s) implies that $p_k(a, s)$ is k-invariant and we set $p_k(a, s) = p(a, s)$. Thus, since $p_k(1, \bar{s}) = \delta\pi(\bar{s}^1|s_0) p'_{k,\bar{s}^1}$, it is $p'_{1,1,\bar{s}^1} = p'_{1,2,\bar{s}^1} \equiv p'_{1,\bar{s}^1}$. Then $D_{e_s^a} f_1 = 0$ or, equivalently, equation (18) reads

$$\frac{Q_{1+}}{Q} p(a, s) - p_{1+}(a, s) = \frac{Q_{2+}}{Q} p(a, s) - p_{2+}(a, s).$$

By taking into account that the first entry of $p_{k+}(1, \bar{s})$ is equal to 1, the last equation computed at $(1, \bar{s})$ implies that $Q_{k+} = Q_+$ and then that

$$p_{1+}(a, s) = p_{2+}(a, s) \quad \text{for all } a, s. \tag{19}$$

We turn next to utility perturbations. For each ℓ, pick the perturbation $\Delta u(\ell)$ and bear in mind that since λ_k is k-invariant, so is $x_k(\ell)$ k-invariant. By direct computation, if $D_{\Delta u(\ell)} f_1 = 0$ (and recalling that $p(\ell) H_k^{-1}(\ell) \Delta u(\ell) = (c_k(\ell))/p_{1,\bar{s}^{1'}} = 1/p'_{1,\bar{s}^1}$), then

$$\frac{\Pi_1(1, \ell)}{p'_{1,\bar{s}^1}} - \frac{\Pi_1(\ell) Q_+}{Q} = \frac{\Pi_2(1, \ell)}{p'_{1,\bar{s}^1}} - \frac{\Pi_2(\ell) Q_+}{Q} \quad \text{for all } \ell. \tag{20}$$

Similarly, if $D_{\Delta u(\ell)} f_J = 0$ for all ℓ, then $(\Pi_1(\ell))/Q = (\Pi_2(\ell))/Q$ for all ℓ. These equations and equation (20) immediately imply that

$$\Pi_1(\ell) = \Pi_2(\ell) \quad \text{and} \quad \Pi_{1+}(\ell) = \Pi_{2+}(\ell) \quad \text{for all } \ell.$$

The latter together with (19) contradict the assumption $\underline{p}' \in P'(\bar{s})$. *Q.E.D.*

The next lemma shows that if $\lambda_1^h \neq \lambda_2^h$, then either $f_j^h \neq 0$ or $D_{\omega^h} f_j^h \neq 0$ for all $j = 2, \ldots, J-1$. In other words, it shows that $N_J^h \subset \bigcap_{2 \le j \le J-1} E_j^h$. Since, both $N_J \cap N_j \subset N_J$ and $\bigcap_{2 \le j \le J-1} E_{j'}^h \subset \bigcup_{j' < j} E_{j'}^h$ for all $j > 1$, the claim shows that Condition Nesting holds true for all $j > 1$.

CLAIM 11: *Suppose that* $(\underline{p}', \omega) \in P'(\bar{s}) \times \Omega'$ *is such that* (i) $f_1^h(\underline{p}', \omega) = 0$ *and* (ii) $f_J^h(\underline{p}', \omega) \neq 0$. *If* $f_j^h(\underline{p}', \omega) = 0$, *then* $D_{\omega^h} f_j^h(\underline{p}', \omega) \neq 0$ *for all* $1 < j < J$.

PROOF: Drop h. Arguing by contradiction, assume that both $D_\omega f_j = 0$ and $f_j(\underline{p}', \omega) = 0$ for some $1 < j < J$. Then, by equation (17), $D_e f_{\ell_1} = 0$ reads

$$\left(\frac{\lambda_1}{\lambda_2}\right) \frac{p_2(a, s)}{Q_2} = \frac{p_1(a, s)}{Q_1} \quad \text{for all } (a, s).$$

Computing the latter at $(0, s_0)$ (and since $p_{1,k,s_0} = 1$), we get that

$$\left(\frac{\lambda_1}{\lambda_2}\right) \frac{1}{Q_2} = \frac{1}{Q_1}.$$

Next we move to utility perturbations. Then by equation (16), $D_{\Delta u(\ell')} f_{\ell_1} = 0$ reads

$$\left(\frac{\lambda_1}{\lambda_2}\right) \Pi_2(\ell_\ell) = \Pi_1(\ell) \quad \text{for all } \ell.$$

Summing across ℓ and noticing the k-invariance of $\sum_\ell \Pi_k(\ell)$, we get $\lambda_1 = \lambda_2$, an immediate contradiction with Claim 11(ii). *Q.E.D.*

The final claim establishes Condition Nesting for $j = 1$, thereby concluding the argument.

CLAIM 12: *There exists an open and dense subset* Ω'' *of* Ω' *such that if* $(\underline{p}', \omega) \in P'(\bar{s}) \times \Omega''$ *and* $f_j^h(\underline{p}', \omega) \neq 0$ *for all* $j \geq 2$, *then* $D_{\omega^h} f_1^h(\underline{p}', \omega) \neq 0$.

PROOF: Drop h and, assume that $(s_{0k}, p'_{k,s_{0k}}) \equiv (s_0, p_{s_0})$ is k-invariant; otherwise Claim 10 implies the thesis. Since $f_j^h(\underline{p}', \omega) \neq 0$ for all $j \geq 2$, it is $\Pi_2(\ell_1) = 0$. Furthermore, since $p'_{k,s_{0k}} = p_{s_0}$ for all k, if $\Pi_1(\ell) = 0$ for all $\ell \neq 1$, there must be at least two distinct indexes $\ell \neq \ell_1$ and $\ell' \neq \ell_1$ with both $\Pi_2(\ell') > 0$ and $\Pi_2(\ell) > 0$; otherwise, $p'_{k,s^a} = p'_{1,s_0}$ for all k and s^a, contradicting $\underline{p}' \in P'(\bar{s})$. Second, given s_{0k} is k-invariant and given $\underline{p}' = (\underline{p}'_1, \underline{p}'_2)$, then

$D_\omega f_1((\underline{p}'_1, \underline{p}'_2), \omega) = -D_\omega f_1((\underline{p}'_2, \underline{p}'_1), \omega)$. Therefore, up to a relabeling of the two trees, there is no loss of generality in assuming that $p'_{1,s^a} \neq p'_{1,s_0}$ for some s^a.

We now argue by contradiction, that is, we assume that $D_\omega f_1 = 0$. If $D_{\Delta e^a_s} f_1 = 0$ for all a, s, then equation (18) holds true and $p'_{1,k,\bar{s}^1}$ is k-invariant, thereby implying as already argued the k-invariance of Q_{k+}/Q_k.

Thus, if $D_{\Delta u(\ell)} f_1 = 0$, then

$$\left[\frac{\Pi_1(1,\ell)}{p'_{1,\bar{s}^1}} - \frac{Q_{1+}}{Q_1}\Pi_1(\ell)\right] = \left(\frac{\lambda_1}{\lambda_2}\right)\left[\frac{\Pi_2(1,\ell_\ell)}{p'_{1,\bar{s}^1}} - \frac{Q_{1+}}{Q_1}\Pi_2(\ell_\ell)\right],$$

and summing across ℓ and observing that $\sum \Pi_k(\ell) \equiv \Pi$ and $\sum \Pi_k(1,\ell) \equiv \Pi_+$ are k-invariant, it must be that $\Pi_+/\Pi = p'_{1,\bar{s}^1} Q_{k+}/Q_k$ since $\lambda_1 \neq \lambda_2$. Therefore,

$$\left[\frac{\Pi_1(1,\ell)}{\Pi_+} - \frac{\Pi_1(\ell)}{\Pi}\right] = \left(\frac{\lambda_1}{\lambda_2}\right)\left[\frac{\Pi_2(1,\ell_\ell)}{\Pi_+} - \frac{\Pi_2(\ell_\ell)}{\Pi}\right] \quad \text{for all } \ell.$$

However, since $\Pi_2(\ell_1) = \Pi_2(1,\ell_1) = 0$, it is

$$\Pi\Pi_1(1,1) - \Pi_1(1)\Pi_+ = 0. \tag{21}$$

Since

$$\Pi = \sum_{j=0}^{G} \delta^j = \frac{1-\delta^{G+1}}{1-\delta},$$

while

$$\Pi_+ = \sum_{j=0}^{G-1} \delta^j = \frac{1-\delta^G}{1-\delta}$$

(21) is a polynomial equation of degree $2G$ in δ of the form

$$(1-\delta^{G+1})(a_0 + a_1\delta + \cdots + a_G\delta^{G-1}) - (1-\delta^G)(b_0 + b_1\delta + \cdots + b_G\delta^G) \equiv \sum_{j=0}^{2G} t_j\delta^j = 0,$$

where

$$a_n = \sum_{s^{n+1} \in S_1(n+1,1)} \pi(s^{n+1}|\bar{s}^1)$$

and

$$b_n = \sum_{s^n \in \bar{S}_1(n,1)} \pi(s^n|s_0)$$

for

$$\bar{S}_1(n,1) = \{s^a : p_{1,s^a} = p_{1,s_{01}},\ \text{for } a = n\}.$$

We want to show that this polynomial is nonzero, that is, that $t_j \neq 0$ for some j. By trivial computations,

$$t_j = a_j - b_j \quad \text{for} \quad j = 0, \dots, G-1, t_G = b_0 - b_G,$$

while

$$t_{j+(G+1)} = b_{j+1} - a_j \quad \text{for} \quad j = 0, \dots, G-1.$$

Thus, if $t_j = 0$, for all j, then $b_j = b_{j+1}$, $j \geq 1$. Since, by construction, $b_0 = 1$ and $t_G = b_0 - b_G$, then $t_j = 0$ for all j if and only if $b_j = a_j = 1$ for all j. Obviously, the latter contradicts the assumption $p'_{1,s^a} \neq p'_{1,s_0}$ for some s^a. Since the polynomial is nonzero, by Theorem 14 in Zariski and Samuel (1960, Chap. I, p. 38), the set of zeros of the polynomial is closed and has measure zero in $[0,1]$. Most importantly, there are only finitely many pairs of nonempty and exhaustive subsets of the tree. Each of these pairs uniquely determines the vectors of coefficients t of the polynomial above and we only consider pairs that deliver a vector t with $t_j \neq 0$ for some j. The union of the zeros of such polynomials intersected with $(0,1]$ is the finite union of closed and measure zero sets. Thus, its complement Δ is open and of full measure in $(0,1]$. Let $\Omega'' = \Omega' \cap [E \times \mathcal{U} \times \Delta]$. By construction, for all $(\underline{p}', \omega) \in P'(\bar{s}) \times \Omega''$, no equation (21) is satisfied or, equivalently, $D_\omega f_1 \neq 0$, concluding the proof. *Q.E.D.*

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Manuscript received June, 2007; final revision received June, 2009.