Creating Sales with Stock-outs*

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Stock-outs convey information about the propensity of other consumers to purchase a product and this can increase the willingness of marginally interested consumers to buy. But in order to leverage stock-outs, firms must be able to capture the extra demand. We show how asymmetric inventory allocations to ex ante identical retailers may increase the expected satisfied demand compared to symmetric inventory allocations; when one retailer stocks out, the other retailer faces increased demand, not only due to overflow demand, but also due to an increase in the residual demand triggered by the stock-out information. In short, stock-outs can trigger herding behavior. Taking consumer reactions to stock-outs into account may lead to higher inventory investment (to capture the 'herd') and asymmetric inventory allocation (one retailer is 'sacrificed' to trigger the herd) for high margin products with a low prior on the quality (i.e. 'brand perception'). In other cases, accounting for consumer reactions to stock-outs can lead to lower investment in inventory.

Key words: Strategic consumer behavior, inventory management

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1. Introduction and Motivation

In 1994, Mighty Morphin Power Rangers were hard to find. This created a frenzied search by parents, many of whom even camped outside stores in order to buy Power Rangers as soon as they came in (Collins, New York Times, Dec. 5, 1994). Other toys and innovative products experienced similar phenomena: Cabbage Patch Kids in 1983, Beanie Babies in the 1990s, Tickle me Elmo in 1998, Pokeman in 1999, Play station 2 in 2000, Nike Airforce1 in 2002, iPod mini and Nitendo DS in 2004, iPod nano, in 2005 (Wingfield and Guth, Wall Street Journal, Dec 2, 2005). Why do we observe so many stock-outs of these products?

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Classical newsvendor logic provides one explanation: Demand for innovative products is difficult to forecast and production processes may be inflexible, suppliers and subcontractors may have to be lined up in advance, lead times can be long, etc. Hence, firms may have to commit to a production decision long before observing demand. If production exceeds demand, the firm incurs costs of overstocking, otherwise, it incurs lost sales or costs of understocking. The optimal production quantity will trade off these two costs. Sometimes firms lose the "bet" on the upside and demand is greater than supply, which can account for the sorts of availability problems reported above. This no doubt accounts for some observed shortages. Also firms may try spreading out demand when production costs decrease because of learning. This can lead to more stock-outs for new products (see e.g. Holloway and Lee, 2006). And limited inventory may increase customer store visits and lead to sales of other products while the customer is in the store. Still, it is surprising to find such extreme unavailability repeatedly over many generations and types of products given the often high margins that are lost. And such high profile shortages often raise suspicion about company motives in the popular business press.[1] Could it be that something more subtle than an unlucky production decision is at work?

We think so. A common characteristic of the products cited above is that they are new, innovative, difficult to evaluate and/or have little (or no) market history. This creates a great deal of uncertainty among potential customers about the utility (quality) of the product. As a result, customers may try to acquire information about product quality through other channels. In the absence of readily available historical information, customers may consult "expert" opinions, product reviews, the advice of friends and colleagues, etc. But with new or highly experiential products (e.g. a new video game) reviews can at best convey only a general sense of product quality. So another important source of information is the purchasing decisions of fellow consumers. And this information - the fact that droves of other consumers are "voting" their approval with their wallets

[1] See for example Business Week, Nov. 21, 2005, "Moore Addresses Xbox 360 Shortage 'Conspiracy'" in which a Microsoft executive addressed criticism that the company created an artificial scarcity of its popular game console to whip up holiday hype.

- is in many ways the truest indication of whether a product is good or bad. Given this fact, why shouldn't a firm try to encourage such signaling in the market?

There are many different channels through which consumers learn about the purchasing decisions of others. Web-enabled technologies facilitate consumer-to-consumer interactions. Shopping web sites rank products by popularity, scores and narrative reviews by customers are posted, etc. But despite these advances, simple availability (or lack thereof) remains a strong signal of which products are most popular. Just as the prospect of a sell-out concert or sporting event creates buzz and stimulates interest among potential fans, backlogs and stockouts create a sense that a product is "hot" and widely in demand. And such information can confirm positive (but uncertain) believes about a product and create a sense of affirmation that stimulates new customers to buy.

In this paper, we study when stock-outs can be leveraged by a firm to signal high product quality. We refer to 'herding behavior' as an increased willingness to purchase the product after a stock-out occurs. Inducing herding behavior leads to an interesting paradox from an Operations Management point of view: Can it ever be optimal for a firm to *limit* its inventory investment in order to trigger stock-outs and create more sales by inducing a herd? Other questions emerge in this context: When potential consumers gain information from observing stock-outs, how does this information influence total sales? How does the firm's inventory investment and allocation to retailers impact this herding behavior? And how should a firm take this strategic consumer behavior into account when allocating and investing in inventory?

To answer these questions, we study herding behavior in a 'newsvendor' context. During a season, consumers are rational Bayesian agents and consider purchasing a product from one of two retailers. Before making a purchasing decision, some agents observe how many retailers are out of stock and take that information into account (strategic agents), while other agents make a purchasing decision ignoring the stock-out information (myopic agents). As long as retailers are not out of stock, a sale can be made to the consumer if s/he decides to purchase the product. Otherwise, any potential sale is lost. Before the start of the season, the firm decides how much inventory to invest and how to allocate its inventory to the retailers. During the season, there is no possibility to replenish the inventory.

We study how observed stock-outs impact consumer purchasing behavior and total realized sales and how the initial inventory level impacts the firm's expected profits. While stock-outs may signal that the product quality is high, increasing the willingness to buy, stock-outs also make it more difficult for consumers to obtain the product. Hence, increasing profits through stock-outs is tricky. We find that (1) when agents observe one retailers out of stock, the purchasing probability increases, (2) when taking the strategic consumer behavior into account, asymmetric allocation of inventory to otherwise identical retailers may be more profitable than symmetric allocation, and (3) the total inventory investment may be higher or lower than the inventory the firm would have invested assuming that all agents are myopic. Our model provides insights into how manufacturers can increase sales through well-managed shortages.

The remainder of this paper is organized as follows: in the next section, we review the related literature. In the sections following, we set up a model with a single retailers and analyze it. Next, we extend the single retailer model to a two-retailer model and analyze it. In each of these sections, we derive the consumer equilibrium for a given inventory strategy and then, we determine the optimal inventory strategy. Finally, we discuss the results and conclude the paper.

2. Related Literature

The link between product availability and product quality has been explored in different research streams. In one behavioral experiment (Verhallen 1982), subjects were shown three recipe books that differed in availability (available, unavailable and unavailable that changed to available). When the market reasons for unavailability were given, subjects rated the unavailable books higher; i.e. agents inferred from the limited availability that the product must have high demand and therefore be of high quality. The author explains this reaction using commodity theory, a theory rooted in psychology predicting that scarcity enhances the value (or desirability) of anything that can be possessed, is useful to its possessor, and is transferable from one person to another (Lynn, 1991).

In the economics literature, consumer inference from other agents' actions has been studied in a recent stream of research. Banerjee (1992) and Bikchandani, Hirschleifer and Welch (1992) analyze the equilibrium outcome when a sequence of individuals makes decisions with incomplete information about the value of an asset. The asset can either be of negative or of positive value. Each individual has private but inaccurate information about the asset value and observes the outcome of the decisions (to buy the asset or not) of his predecessors. Agents do not observe the predecessor's private information. The authors demonstrate that the influence of the observed decisions of the predecessors could be so strong that individuals ignore completely their own information and follow their predecessor's decision. This is called "herding". Herding can be socially inefficient as agents can make the wrong decision; i.e. buying an asset with negative value or not buying a high value asset. In a retail context, it is common that agents interpret stock-outs as a proxy of the previous' agent's purchasing decisions[2]. In the psychology and herding literature, typically, the focus is on explaining the decisions of individual agents or subjects. How a firm can influence these decisions is not studied.

Our work has some connections to prior operations literature as well. Traditionally, availability levels are considered to be a consequence of exogenous consumer demand and the firm's inventory policy (Van Ryzin and Mahajan, 1999, Lippman and McCardle, 1997). The examples above suggest that consumer purchasing behavior and availability may be determined simultaneously; that they may be endogenously determined in an equilibrium (Gaur and Park, 2007, Cachon and Kok, 2007). This is especially true when agents do not have accurate information about a product but observe public product availability information. Consumers may then complement their own private information with availability information when they make a purchasing decision. One literature stream is concerned with management of a category of products that are distinguished by some attribute (van Ryzin and Mahajan, 1999, Gans, 2001 and Gaur and Park, 2007). van Ryzin and Mahajan study how to optimally select which variants need to be offered in the category and how

[2] Websites may also list e.g. rankings of recent sales of books or CDs (the New York Times) or may announce publicly when a product has reached a certain threshold sales (e.g. a CD has earned gold or platinum).

much inventory of each should be stocked, taking consumer characteristics and the cost of supply into account. They model a trend following population as an exogenously given probability that all demand for the category will be for one particular variant (as in herding). Gans (2001) study customer search behavior. Gans studies customer loyalty to a certain vendor when the quality experience is noisy. Customers may sample different vendors and accumulate their experiences before settling with one supplier. During each visit, customers update their prior about the quality of the vendor. Stock and Balachander (2005) analyze when 'scracity strategies' signal product quality to uninformed consumers and may yield higher profits for the seller. A stream of papers discusses how inventory levels impact demand. Balakrishnan et al. (2004) analyze optimal lot-sizing when stocking large quantities stimulates demand. A recent stream of papers incorporates strategic consumer behavior in newsvendor models. The research in this stream mainly focuses on the problem of how consumers' possible waiting behavior affects a seller's performance. Liu and van Ryzin (2005) find that the resulting threat of shortages creates an incentive for customers to purchase early at higher current prices. Several papers look at how mechanisms to alleviate the impact the strategic consumer behavior: Su and Zhang (2005) study quantity and price commitment, Lai et al. (2007) study posterior price matching policies and Cachon and Swinney (2007) study quick, in-season replenishment. These authors find that these mechanisms can increase the seller's profit. Finally, Debo et al. study strategic queue joining behavior when the value of the service for which the queue is generated is unknown. They show that some consumers may not join the queue in equilibrium unless it is long enough. Veeraraghavan and Debo (2007a) and Veeraraghavan and Debo (2007b) study the selection of a queue when the relative value of the services is unknown. They show that, some consumers may join the longer queue in equilibrium, depending on the waiting costs, the queue buffer size and the heterogeneity with respect to prior service value information.

None of the papers above provides an answer rooted in inventory management theory as how a firm can possibly benefit from inducing herd behavior. Therefore, we develop and analyze in the following sections a simple newsvendor model in which we allow agents to react strategically to stock-outs.

3. Preview of the Models and Insights

In this section, we give a preview of the insights that will be obtained in the next sections. We first model and analyze a single retailer model (in §4 and §5) and then a two retailer model (in §6 and §7). We show in §7 that the analysis for multiple *symmetric* retailers follows the same pattern as for a single retailer. With a single retailer, the only purchasing decision that is relevant is when there is *no stock-out*, because otherwise, when there is a stock-out, any potential sale is lost. We find that the lack of a stock-out has negative implications on the sales due to strategic consumer behavior. This effect is severe when the initial inventory is low. Then, strategic consumers attribute not observing a stock-out to low product quality, and they are more reluctant to purchase the product. For larger initial inventory levels, this negative effect is reduced and eliminated entirely when the initial inventory is high enough that the stock-out probability is zero. Then, strategic consumers ignore the absence of a stock-out (as it is expected) when determining whether to purchase the product or not. Hence, when the product margins are low and investing in large inventory is expensive, the strategic reaction of rational consumers to the absence of stock-outs depresses the optimal investment in inventory.

Another implication of selling via a single retailer is that the stock-out signal is irrelevant for the firm when no replenishment is possible after a stock-out. While the stock-out signal increases consumers' willingness to buy, the firm cannot cash in on this effect. Therefore, we introduce two ex ante identical retailers (in §6 and §7). We find that when the smallest retailer stocks out, this signal also increases the strategic consumers' willingness to purchase. With two retailers, the consumers that observe the stock-out can be satisfied from the remaining inventory at the larger retailer.

In order to determine the optimal allocation of inventory to two ex ante symmetric retailers, the trade-off is the following: on one hand, the larger the inventory at the smallest retailer, the stronger the impact of a stock-out on the strategic consumers' willingness to purchase. The intuition is that it is more likely that a high quality product creates such a stock-out. On the other hand, the absence of a stock-out makes strategic consumers more reluctant to buy. So with a larger smallest retailer, more consumers will be reluctant to buy as more will observe no stock-outs. As a consequence, the small retailer should not be too large. We will show that there may be an interior non-symmetric solution to the inventory allocation problem.

Moreover, when the margins are high, the total inventory invested will be larger than the inventory invested when all agents are myopic. This is also intuitive as the stock-outs (triggered by the small retailer) increases the pool of interested consumers (post stock-out) for whom inventory must be available at the large retailer. In this case, stock-outs are not created by an aggregate shortage of inventory, but rather by a deliberate asymmetry in the allocation of inventory to retailers.

Finally, we will show that a stock-out signal together with an appropriate inventory investment strategy leads to significantly more profits when the prior about the product quality (or brand perception) is low and the private signal containing quality-related information is noisy. In these cases, without stock-out information, the expected sales are low. Hence, the product information triggered by stock-outs is highly valuable for such a firm.

4. The Single Retailer Model

The model has two stages. In stage one, the firm determines the inventory investment, after which Nature determines the product quality. In stage two, which represents the selling season, a continuum of agents (consumers) arrives in a random sequence and make purchasing decisions based on their observed information. All agents observe privately an individual signal that is correlated with the product quality. Some agents observe whether the retailer is out of stock or not. The firm's profits are determined by the realized sales and inventory investment. The agents' utility is determined by the product quality and their purchasing decision. We elaborate each of these processes further below:

The firm's problem: At the beginning of the season, the firm decides the inventory investment level, Δ. Each sale results in r revenue and each unit of inventory costs c $(< r)$. Leftover inventory at the end of the season has no salvage value. The potential market size is $\lambda > 0$. The sales are determined by the agent's willingness to buy (as explained in the next paragraph) and the product availability.

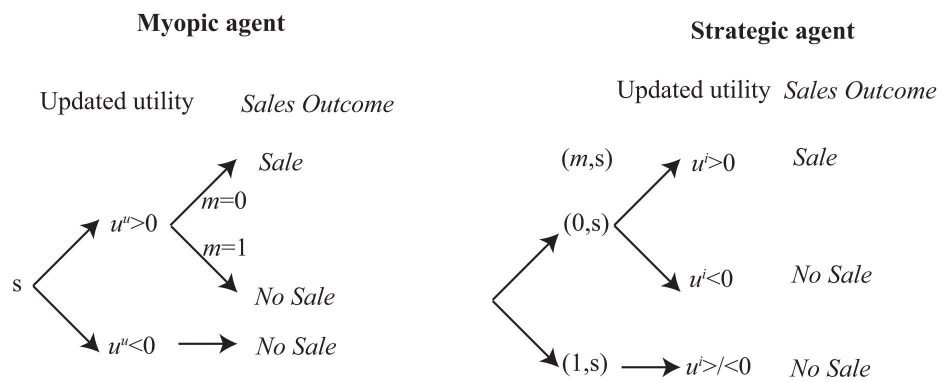

Figure 1 Sequence of events for myopic and strategic agents when there is are $m=1$ or $m=0$ retailers out of stock. u^u and u^i are, respectively, the updated utilities of the uninformed and informed agents based on their private signal s.

The agent's problem: We assume that agents observe the initial inventory level, Δ, before making a purchasing decision, but, they do not observe the actual inventory level when they make a purchase. The product quality is a random variable, $\omega \in \{\ell, h\}$ and the agent's net value of purchasing the product is a function of the product quality, v_ω, where $v_h = -v_\ell = v$[3]. The realization of ω is unobservable. The common prior about the quality is $p_0 = \Pr(\omega = h)$. Every agent observes privately a signal s that depends on the product quality. The signal density when the product quality is ω is $g_\omega(s)$. $g_\omega(s)$ is continuous and positive over $[\underline{s}, \overline{s}]$, with $g_h(\underline{s}) = 0$, $g_\ell(\overline{s}) = 0$ and $g_h(s)/g_\ell(s)$ strictly increasing over $[\underline{s}, \overline{s}]$. A fraction α of the market only observes their private signal when making a purchasing decision. The remaining fraction $1-\alpha$ of the market observes in addition whether the retailer is out of stock ($m=1$) or not ($m=0$) before deciding whether to buy the product (or not). If an agent decides to buy the product, but, the retailer is out of stock, the potential sale is lost. Agents make only one purchasing decision, after which they disappear from the system. We refer to agents that observe the number of retailers that are out of stock as the 'informed' or 'strategic' agents and to the other agents as the 'uninformed' or 'myopic' agents. Figure 1 illustrates the sequence of events and sales outcomes for the myopic and strategic agents.

The equilibrium conditions: Without loss of generality, we can restrict the action space of the

[3] The model can easily be generalized to $v_h \neq v_\ell$

informed agents to the 'purchasing thresholds', $s^i_m(\Delta) \in [\underline{s}, \overline{s}]$ for $m \in \{0,1\}$, where the informed agent buys the product if the realization of his private signal is higher than the threshold. The informed agent's purchasing threshold is only relevant for the case in which there is no-stock-out (i.e. $m = 0$[4]) and is a function of the retailer's initial inventory Δ. Therefore, we only study $s_0(\Delta)$. Similarly, let $s^u(\Delta) \in [\underline{s}, \overline{s}]$ be the purchasing threshold of the uninformed agents. The combined strategy is denoted as $\boldsymbol{s}(\Delta) = (s^i_0(\Delta), s^u(\Delta))$. We denote the equilibrium purchasing strategy for a given inventory investment, Δ, as $\boldsymbol{s}^*(\Delta)$. As the uninformed agents do only observe their private signal, their utility depends on their private signal only. Let $u^u(s, \Delta)$ be the updated product utility after signal s is observed. For a given initial inventory, Δ, the equilibrium purchasing threshold of the uninformed agents, s^{u*}, satisfies:

$$u^u(s^{u*}, \Delta) = 0. \tag{1}$$

Let $u^i(m, s, \boldsymbol{s}, \Delta)$ be the updated product utility after an informed focal agent observes $m \in \{0,1\}$ and signal s. As the product availability information, m, depends on the relative demand versus the inventory, the informed focal agent's utility also depends on the strategy of all agents $\boldsymbol{s}$ and the retailer's initial inventory Δ. For a given initial inventory, the equilibrium purchasing threshold of the uninformed agents, $s^{i*}_0(\Delta)$, when there is no stock-out, satisfies:

$$u^i\left(0, s^{i*}_0, \boldsymbol{s}^*(\Delta), \Delta\right) = 0, \tag{2}$$

i.e. when the focal agent's expected updated utility with private signal s^{i*}_0 is zero when all other agents play $\boldsymbol{s}^*(\Delta)$, an equilibrium is reached. Let $\Pi(\boldsymbol{s}, \Delta)$ be the firm's expected profit for a given inventory, Δ, and customer purchasing behavior $\boldsymbol{s}$, then, the firm's equilibrium inventory investment, Δ^*, is:

$$\Delta^* \in \arg\max_{\Delta \geq 0} \, \Pi(\boldsymbol{s}^*(\Delta), \Delta). \tag{3}$$

The model parameters: We parameterize the density of the private signal, $g_\omega(s)$ by means of one parameter, $\kappa \in (0, \infty)$ such that $g_h(s) \sim (1+s)^\kappa$ and $g_\ell(s) \sim (1-s)^\kappa$ for $s \in [-1, 1]$. Higher

[4] When $m = 1$, no inventory is left over, hence, the purchasing decision becomes irrelevant.

values of κ imply that the private signal is more informative. Without loss of generality, we can normalize λ, v and r to 1. The independent parameters are: the prior on the product quality, p_0, the informativeness of the signal, κ, product cost (or margin), c (or $r-c$) and the fraction of myopic agents in the market α.

The quantity p_0 is the common prior or brand perception, i.e. the uniform assessment of product quality before customers observe any other information. One can interpret this prior as being due to brand reputation or a history of successful introductions of new products. For example, consumers arguably had a high prior on Apply iPhones due to Apple's success with iPods; 'iPod killers' from other manufacturers (e.g. Sirius or Microsoft) arguably started with lower priors. Fisher Price toys may have a stronger prior than Bandai toys; the latter, is less well known but had an enormous unanticipated success with the Mighty Morphin Power rangers in the mid-nineties. Game boxes are other examples that fit well; the prior of a new product may be determined through the success of previous product launches. Industry "hype" about a new product introduction may also affect the market prior.

The signal strength, κ, is a measure of how easy it is for customers to independently assess a product's quality. If they can perfectly assess its quality, then $\kappa=\infty$; if they cannot assess quality at all, then $\kappa=0$ (the signal is pure noise). The signal strength is related to two primary factors: 1) the inherent difficulty in evaluating a product without experiencing it, and 2) the information available about product quality. This second factor is partially controllable by the firm (e.g. by advertising, providing specifications, encouraging reviews, etc.), while the first factor is an intrinsic feature of the product and largely uncontrollable. For products with subjective features of quality, the signal strength is inherently lower even if firms try to provide detailed information. Books and music CDs are examples. They have many subjective dimensions and are therefore fundamentally difficult to evaluate without actually consuming them (e.g. "You can't judge a book by it's cover"). Privately observed information like a review or a friend's opinion can help, but even this sort of information can be misleading (With whom does one agree 100% on movies and music?) and may not change one's prior (brand perception). Toys are another example of products that are inherently difficult to evaluate because they are experiential products. Moreover, the purchasers (parents) are not the ultimate users (children), and may have difficulty assessing their appeal without outside information about whether the toy will be enjoyable and of enduring value to their child.

Other products are more tangible and easier to describe and evaluate without experiencing them. A digital camera or laptop computer with objective performance measures (e.g. screen size, processor speed, megapixels, gigabytes of memory, etc.) is easier to evaluate using a good review or by simply reading product specifications. These product categories would inherently have higher values of the signal strength κ.

The product margin is $r - c$. For new and innovative products that are our focus, margins are typically relatively high, see e.g. Fisher (1997); firms try creating a new market and hence can command high prices. However, for products like books and music CDs, even though there is considerable uncertainty about their quality, margins are often lower as the competition in these markets is fierce.

Finally, the fraction of agents that do not take the product availability into account is α. These could be simply myopic agents that purchase a product based on their private information only, or, these could be agents that do not observe the product availability at different stores. As often shortages of 'hot' products are mentioned in the popular press, increased access to media reduces the fraction of uninformed agents. For products that are sold on-line, the fraction of uninformed agents may be lower than for products sold in brick-and-mortar stores. Hence, α is a measure of the relative importance of public (stock-out) information.

5. Analysis of a Single Retailer

We first derive the equilibrium conditions and characterize the possible equilibria.

5.1. Customer purchasing behavior of given inventory investment

Without loss of generality, we can restrict the analysis to $\Delta \in [0, \lambda]$, otherwise, there is enough inventory to satisfy all potential demand and without any loss of revenues, the inventory can be decreased, leading to savings in purchasing costs .

The uninformed agents' strategy: First, we characterize the uninformed agent's strategy. When only considering the private signal, the utility, updated after observing signal s, using Bayes' rule, is:

$$u^u(s,\Delta) = p'(s)\,v + (1 - p'(s))(-v) \tag{4}$$

where

$$p'(s) = \frac{g_h(s)\,p_0}{g_h(s)\,p_0 + (1-p_0)\,g_\ell(s)}. \tag{5}$$

Observe that the uninformed agent's utility, u^u, is independent of Δ. Let $l(s) = \frac{g_h(s)}{g_\ell(s)}$, $\theta = \frac{p_0}{1-p_0}$. It is obvious that when only observing a private signal, the equilibrium action is characterized by means of a threshold, $\hat{s}$, defined implicitly as $u^u(\hat{s},\Delta) = 0$, or:

$$\theta l(\hat{s}) = 1. \tag{6}$$

In Equation (6), notice two factors: $\theta \times l = 1$; the first factor of the left hand side captures the prior about the product quality, the second factor of the left hand side captures the information in the private signal. The right hand side is equal to 1 because the utility gain from buying a high quality product is the same as the utility loss from buying a low quality product (i.e. $v_h = -v_\ell = v$). The updated utility of Equation (4) is strictly positive if and only if $s > \hat{s}$. Hence, the unique solution of the equilibrium condition of Equation (1) is $s^{*u}(\Delta) = \hat{s}$. $\hat{s}$ is the 'myopic threshold,' i.e. ignoring the product availability information, an agent purchases when his private signal is larger than $\hat{s}$. As $l(s)$ is unbounded and strictly positive and $\theta > 0$, there always exist a threshold $\hat{s}$.

The informed agents' strategy: Now, we characterize the informed agent's strategy. Assume that a focal informed agent observes signal s. Assume that $m \in \{0,1\}$ and all other agents follow a threshold strategy s_0, then, the focal agent wants to buy the product if

$$p''(m,s,s_0,\Delta)\,v + (1 - p''(m,s,s_0,\Delta))(-v) > 0, \tag{7}$$

where $p''(m,s,s_0,\Delta)$ is the updated probability that the product quality is high. With Bayes' rule, we express $p''(m,s,s_0,\Delta)$ as follows:

$$p''(m,s,s_0,\Delta) = \frac{g_h(s)\,p_0 p^h(m,s_0,\Delta)}{p_0 g_h(s)\,p^h(m,s_0,\Delta) + (1-p_0)\,g_\ell(s)\,p^\ell(m,s_0,\Delta)}. \tag{8}$$

p'' is similar to p' in Equation (5), except that now the stock-out information plays a role via $p^{\omega}(m, s_0, \Delta)$. $p^{\omega}(m, s_0, \Delta)$ is the probability that $m \in \{0,1\}$ is observed when the product quality is ω and all agents play strategy s_0 and the inventory investment is Δ. Hence, we first calculate $p^{\omega}(m, s_0, \Delta)$. To that end, we define:

$$P^{\omega}(s) = \alpha \overline{G}_{\omega}(\hat{s}) + (1-\alpha)\overline{G}_{\omega}(s).$$

When the retailer is not out of stock, the purchasing probability is $P^{\omega}(s_0)$; with probability α, the customer is uninformed and wants to purchase the product if the private signal realization is higher than $\hat{s}$. With probability $1-\alpha$, the customer is informed and wants to buy if the private signal realization is higher than s_0. Now, we define:

$$\overline{\lambda}^{\omega}(s_0, \Delta) = \frac{\Delta}{P^{\omega}(s_0)}. \tag{9}$$

As $P^{\omega}\overline{\lambda}^{\omega} = \Delta$, a potential population of the size $\overline{\lambda}^{\omega}$ will lead to Δ sales when the product quality is ω. Depending on the size of the potential population, λ, we can now determine the probability that a random agent observes that no or one retailer is out of stock, $p^{\omega}(m, s_0, \Delta)$: It is the volume of agents that observes no retailer out of stock divided by the total volume of agents. When the market potential, λ, is less than $\overline{\lambda}^{\omega}$, the retailer will not stock out, hence, $p^{\omega}(0) = 1$ and $p^{\omega}(1) = 0$. When λ, is in larger than $\overline{\lambda}^{\omega}$, with probability $\overline{\lambda}^{\omega}/\lambda$, no stock-out will be observed. We have obtained:

Lemma 1. *Given s_0, the probability of observing no stock-out when the product quality is ω is given by:*

$$p^{\omega}(0, s_0, \Delta) = \min\left(1, \frac{\overline{\lambda}^{\omega}(s_0, \Delta)}{\lambda}\right).$$

(And $p^{\omega}(1, s_0, \Delta) = 1 - p^{\omega}(0, s_0, \Delta)$.)

The focal informed agent uses $p^{\omega}(m, s_0, \Delta)$ to make a purchasing decision with Equations (7) and (8). If his decision threshold, s_0, is the same as the conjectured threshold, s_0, an equilibrium is obtained. The equilibrium condition of Equation (2) can be written in terms of s_0 as follows:

$$s_0^* : \theta l(s_0) = \frac{p^{\ell}(0, s_0, \Delta)}{p^{h}(0, s_0, \Delta)}. \tag{10}$$

Compare Equation (10) with Equation (6). Notice three factors in: $\theta \times l_0 \times \frac{p_0^h}{p_0^\ell} = 1$; the first factor captures the prior about the product quality, the second factor captures the information in the private signal, the third factor capture the information in the public signal (i.e. the retailer is not out of stock). The latter factor will determine how the absence of a stock-out influences the strategic agents' purchasing behavior.

Now, define:

$$\tilde{s} : \theta l(s) = P^\ell(s) / P^h(s), \tag{11}$$

then, when the small retailer stocks out for both quality levels, the equilibrium threshold when observing no stock-out is determined by $\tilde{s}$, (Equation (10) reduces to Equation (11)). It is interesting that $\tilde{s}$ is independent of the inventory investment and market size; it only depends on the prior quality (θ), the fraction of uninformed agents (α) and the signal distribution ($\overline{G}_\omega(s)$). Recall that $\hat{s}$ is the myopic threshold, i.e. when there is never a stock-out. It is independent of the inventory investment. Similarly, $\tilde{s}$, which is the threshold when there is always a stock-out is independent of the inventory investment. The following Lemma provides properties of $\hat{s}$ and $\tilde{s}$:

LEMMA 2. *(i) $\hat{s}$ and $\tilde{s}$ decrease in θ.*
(ii) There exist a unique $\tilde{s} \in (\underline{s}, \overline{s})$ for $\alpha \in (0,1]$.
(iii) $\tilde{s} \geq \hat{s}$ and decreases for $\alpha \in (0,1]$.

The myopic purchasing threshold has some intuitive properties: as the prior about the quality (brand perception), θ, increases, the purchasing threshold decreases (i.e. the agents become less 'picky'). $\tilde{s}$ always exist in $(\underline{s}, \overline{s})$ and increases in the prior about the quality (brand perception). Finally, $\tilde{s}$ is always higher than or equal to $\hat{s}$.

The Impact of Inventory Investment on Strategic Customer Purchasing Behavior: For any retailer inventory investment Δ, we now characterize the agent equilibrium s_0^*. In the next subsection, we will determine the optimal inventory investment, taking the dependency of s_0^* on Δ into account.

Case	Large Retailer (l, h)	Δ	$s_0^*(\Delta)$
i	(Stock-out, Stock-out)	$0 \leq \Delta \leq \tilde{\lambda}^{\ell}$	$\tilde{s}$
ii	(Leftover, Stock-out)	$\tilde{\lambda}^{\ell} \leq \Delta \leq \hat{\lambda}^{h}$	$s_A(\Delta, \lambda)$
iii	(Leftover, Leftover)	$\hat{\lambda}^{h} \leq \Delta \leq \lambda$	$\hat{s}$

Table 1 **Equilibrium customer purchasing strategy as a function of initial inventory.**

The following Proposition specializes the equilibrium conditions of Equation (10) for different values of Δ.

PROPOSITION 1. *(i) The equilibrium purchasing threshold is given in Table 1, where:* $\tilde{\lambda}^{\ell} \doteq \lambda P^{\ell}(\tilde{s})$, $\hat{\lambda}^{h} \doteq \lambda P^{h}(\hat{s})$ *and*

$$s_A(\Delta, \lambda) : \theta l\,(s_A) = \frac{P^h\,(s_A)}{\Delta}\lambda.$$

(ii) Larger inventories decrease the purchasing threshold of the strategic agents: $\frac{\partial s_0^*}{\partial \Delta} \leq 0$.

The results are intuitive. When the inventory is large, no stock-out will ever occur, hence, agents ignore inventory information and the myopic threshold, $\hat{s}$ is the equilibrium threshold (case iii). When the inventory is very small, a stock-out will occur irrespective of the product quality, and the equilibrium threshold is $\tilde{s}$ (case i), which is again independent of the inventory.

For intermediate inventory levels, the purchasing threshold is a function of the inventory level (case ii). When the inventory increases, agents become less 'picky' (i.e. their purchasing threshold decreases). This result is expected from Lemma 2(iv), since for low inventory levels the equilibrium purchasing level is $\tilde{s}$, which is higher than the equilibrium purchasing level at high inventory levels, which is $\hat{s}$. When the strategic agents observe no stock-outs, they are less 'surprised' when the inventory is larger because the stock-out probability depends less on the product quality. The intuition is that large inventories make it more difficult to assess the difference between high and low quality when there are no stock-outs. This effect is similar to as Balakrishnan, et al., (2004), who argue that large inventories may stimulate demand. In our model, this same phenomenon emerges endogenously as (a subgame) equilibrium.

5.2. Profit maximizing inventory investment

In this subsection, we characterize the optimal inventory investment.

The Expected Satisfied Demand: We can write expected satisfied demand as:

$$S(\Delta, s_0) = \mathbb{E}_\omega \left[\min\left(\Delta, P^\omega\left(s_0\right)\lambda\right)\right].$$

In order to obtain the optimal inventory investment and allocation, it is helpful to decompose the marginal revenues with respect to Δ into direct and the strategic agent terms:

$$\underbrace{\frac{\partial}{\partial \Delta} S(\Delta, s_0^*)}_{\text{direct effect } \geq 0} + \underbrace{\frac{\partial}{\partial s_0} S(\Delta, s_0^*) \frac{\partial s_0^*}{\partial \Delta}}_{\text{strategic agent effect } \geq 0}.$$

It is intuitive that the strategic agent effect (keeping the inventory constant) is positive: increasing inventory makes agents less picky, and less picky agents lead to more sales. It is also intuitive that the direct effect is positive; a larger inventory has a non-negative impact on the expected satisfied demand (keeping the agent purchasing behavior constant).

The myopic inventory investment: It is useful to consider the optimal inventory investment strategy assuming that all agents are myopic (i.e. $\max_{\Delta \in [0,\lambda]} \; rS(\Delta, \hat{s}) - c\Delta$). Then, the demand is bi-valued (depending on the quality realization) and, hence, there are only two possible inventory investment levels (assuming that $\frac{c}{r} < 1$): $\hat{\lambda}^h$ and $\hat{\lambda}^\ell = P^\ell\left(\hat{s}\right)\lambda$. Such a bi-valued demand model has been used in the literature to model demand for fashion products (Lippman and McCardle, 1997, van Ryzin and Mahajan, 1999). According to the newsvendor logic, the large inventory investment, $\hat{\lambda}^h$ is optimal when $p_0 > \frac{c}{r}$. Otherwise, the low inventory inventory investment, $\hat{\lambda}^\ell$ is optimal when $p_0 < \frac{c}{r}$. The break-even point is thus $p_0 = \frac{c}{r}$. The profits (normalized by λr) for the large inventory investment are $\hat{\pi}^h = \left(p_0 - \frac{c}{r}\right) P^h\left(\hat{s}\right) + (1 - p_0) P^\ell\left(\hat{s}\right)$ and for the small inventory investment $\hat{\pi}^\ell = \left(p_0 - \frac{c}{r}\right) P^\ell(\hat{s}) + (1 - p_0) P^\ell\left(\hat{s}\right)$. We refer to products with $p_0 > \frac{c}{r}$ ($p_0 < \frac{c}{r}$) as high (low) margin products for the myopic newsvendor.

The optimal inventory investment: The firm's profit, as a function of the agent's purchasing strategy is:

$$\Pi(\Delta, s_0) = rS(\Delta, s_0) - c\Delta,$$

and the optimal inventory investment is determined by: $\max_{\Delta \in [0,\lambda]} \Pi(\Delta, s_0^*(\Delta))$. In the next Proposition, we rewrite the firm's decision problem. For notational convenience, the profit π is normalized with respect to $r\lambda$.

PROPOSITION 2. *The inventory optimization problem of Equation (3) for a single retailer is solved by:* $\Delta^* = \frac{P^h(s^*)}{\theta l(s^*)}\lambda$, *where* s^* *solves*

$$\max_{\hat{s} \le s \le \tilde{s}} \ \pi^o(s) \doteq (1-p_0)\left(P^\ell(s) + \left(1 - \frac{c}{p_0 r}\right)\frac{P^h(s)}{l(s)}\right).$$

The optimal profit is $\pi^o(s^*)\lambda r$.

In general, $\pi(s)$ is not convex over $[\hat{s}, \tilde{s}]$. Hence, it is possible that the optimization problem of Proposition 1 has an interior solution. For example, for $c = 0.175$, $p = 0.15$, $r = 1$, $v_h = 1$, $v_\ell = -1$, $\alpha = 0.25$, $\kappa = 2$, $\lambda = 10$, the solution s^* satisfies: $\hat{s} < s^* < \tilde{s}$. However, for slight perturbations of the parameters, the solution moves to one of the corners of the interval. For most parameter values we tested, $\pi(s)$ is a convex function over $[\underline{s}, \overline{s}]$. In that case, only the two extreme cases $s = \hat{s}$ and $s = \tilde{s}$ will be optimal. These correspond respectively with inventory investment $\hat{\lambda}^h$, covering the demand when the product quality is high and inventory investment $\tilde{\lambda}^\ell$, only covering the demand when the product quality is low.

We explain Proposition 1 for the parameter values for which $\pi(s)$ is convex and hence only the two boundary points, $\hat{s}$ and $\tilde{s}$ are candidate optimizers. From the definitions (Equations (6) and (11)), it follows that Δ^* is either $\hat{\lambda}^h$ or $\tilde{\lambda}^\ell$. It is easy to see that when the critical ratio c/r is low enough, the optimal inventory is $\hat{\lambda}^h$, which is the optimal inventory when all agents are myopic. In other words, strategic agent behavior does not impact the single retailer optimal inventory for low cost or high margin products. When critical ratio c/r is high, the optimal inventory with strategic agents, $\tilde{\lambda}^\ell$ is *lower* than the optimal inventory without strategic agents. Hence, it is the absence of a stock-out that makes the strategic agents more reluctant to buy the product and depresses the optimal inventory investment for low margin/high cost products.

Before concluding the single retailer discussion, it is interesting to assess the updated utility of the agents that observe a stock-out in cases where $\Delta^* = \tilde{\lambda}^\ell$ (when investing $\hat{\lambda}^h$, no stock-out ever occurs). It is easy to find using Equation (8) that the updated probability satisfies: $p''(1, \tilde{s}, \tilde{s}, \Delta^*) > p''(0, \tilde{s}, \tilde{s}, \Delta^*)$, i.e. strategic agents that observe a stock-out at the single retailer have a higher expected utility than those that do not observe a stock-out and are therefore more willing to buy. Unfortunately, with a single retailer, this increased willingness to buy cannot be exploited because it only occurs when there is not stock left in the system. To capitalize on the positive effect of stock-outs, a firm needs need to provide agents with other outlets for purchasing. This leads us to consider next a two-retailer model.

6. The Two Retailer Model

The model set-up is the same as for the single retailer case, except that the firm determines the inventory investment *and* allocations of inventory to two retailers. In the second stage, some agents observe in addition to their private quality signal how many retailers are out of stock (zero, one or two). We assume there are no customer switching costs between retailers.

The firm's problem: Without loss of generality, assume that retailer 1 has inventory $Q_1 = Q$ and retailer 2 has inventory $Q_1 = Q + \Delta$, where $Q \geq 0$ and $\Delta \geq 0$. We denote (Q, Δ) by $\boldsymbol{I} \in \mathbb{R}_+^2$. At the beginning of the season, the firm decides the inventory investment and allocation, $\boldsymbol{I}$. The revenues and costs are the same as for the single retailer case.

The agent's problem: We assume that agents observe $\boldsymbol{I}$, at the moment that they make a purchasing decision.

As for the single retailer case, a fraction α of the market only observes their private signal before making a purchasing decision and picks a retailer at random (i.e. each retailer is selected with probability 1/2) if they want to buy (based on their private signal). If the selected retailer is out of stock, the consumer tries the other retailer. If both retailers are out of stock, the sale is lost. The remaining fraction $1 - \alpha$ of the market observes the number of retailers out of stock ($m \in \{0, 1, 2\}$) before making a purchasing decision. If no retailers are out of stock and they decide to buy, they pick a retailer at random. If one retailer is out of stock and they decide to buy, they go to the retailer with stock. If both retailers are out of stock, they do not buy. Figure 2 illustrates the sequence of events and sales outcomes for the myopic and strategic agents.

The equilibrium conditions: Let $\boldsymbol{s}^i(\boldsymbol{I}) = (s_0^i(\boldsymbol{I}), s_1^i(, \boldsymbol{I}))$ be the purchasing strategy of the informed agents and let $s^u(\boldsymbol{I})$ be the purchasing strategy of the uninformed agents. The combined

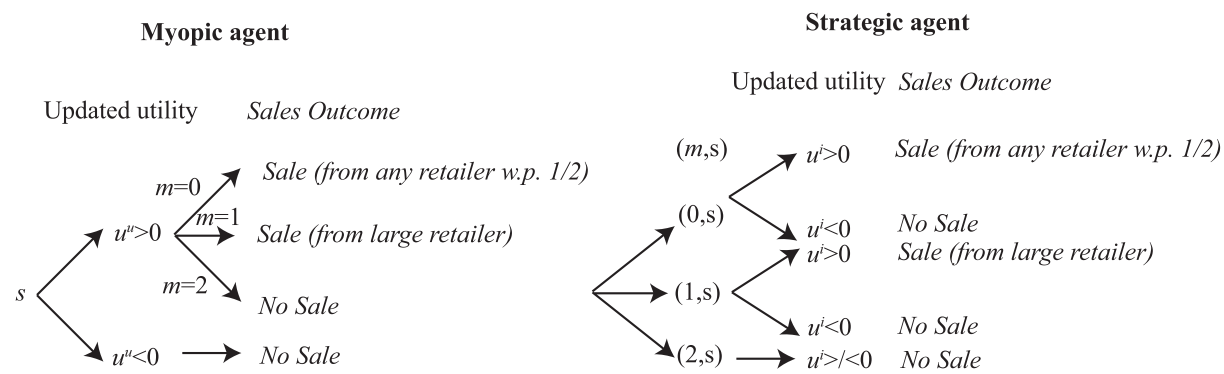

Figure 2 Sequence of events for myopic and strategic agents when there are two retailers that are both out of stock ($m=2$), only the large retailer is out of stock ($m=1$), or none are out of stock ($m=0$). u^u and u^i are, respectively, the updated utilities of the uninformed and informed agents based on their private signal s.

strategy is denoted as $\boldsymbol{s}(\boldsymbol{I}) = (\boldsymbol{s}^i(\boldsymbol{I}), s^u(\boldsymbol{I}))$. We denote the equilibrium purchasing strategy for a given inventory investment and allocation, $\boldsymbol{I}$, as $\boldsymbol{s}^*(\boldsymbol{I})$. As the uninformed agents do only observe their private signal, their utility depends on their private signal only: $u^u(s, \boldsymbol{I})$ which is the updated product utility after signal s is observed. Hence, the equilibrium threshold for $s^{*u}(\boldsymbol{I})$ satisfies:

$$u^u(s^{*u}, \boldsymbol{I}) = 0. \tag{12}$$

For the informed agents, let $u^i(m, s, \boldsymbol{s}, \boldsymbol{I})$ be the updated product utility after a focal agent observes $m \in \{0,1\}$ retailers out of stock and signal s and the strategy of all agents is $\boldsymbol{s}$. Then, $\boldsymbol{s}^{i*}(\boldsymbol{I})$ is an equilibrium when:

$$u^i\left(m, s_m^{i*}(\boldsymbol{I}), \boldsymbol{s}^*(\boldsymbol{I}), \boldsymbol{I}\right) = 0 \text{ for } m \in \{0,1\}. \tag{13}$$

We denote the equilibrium purchasing strategy for a given inventory allocation, $\boldsymbol{I}$, as $\boldsymbol{s}^*(\boldsymbol{I})$. Let $\Pi(\boldsymbol{s}, \boldsymbol{I})$ be the firm's expected profit for a given inventory allocation, $\boldsymbol{I}$, and customer purchasing behavior $\boldsymbol{s}(\boldsymbol{I})$, then, the firm's equilibrium inventory investment, $\boldsymbol{I}^*$, is:

$$\boldsymbol{I}^* \in \arg\max_{\boldsymbol{I} \in \mathbb{R}_+^2} \Pi(\boldsymbol{s}^*(\boldsymbol{I}), \boldsymbol{I}). \tag{14}$$

Similarly as for the single retailer case, we first derive the equilibrium conditions and characterize the possible equilibria when the firm distributes the product via two retailers.

7. Analysis of Two Retailers

As for the single retailer case, we first derive the equilibrium conditions and characterize the possible equilibria.

7.1. Customer purchasing behavior for a given inventory investment and allocation

As for the single retailer, without loss of generality, we can restrict the analysis to $Q \in [0, \lambda/2]$, $\Delta \in [0, \lambda]$ and $2Q + \Delta \leq \lambda$.

The uninformed agents' strategy: As for the single retailer case, the unique solution of the equilibrium condition of Equation (12) is $s^{*u}(\boldsymbol{I}) = \hat{s}$ and is independent of $\boldsymbol{I}$.

The informed agents' strategy: Assume that a focal informed agent observes signal s, there are $m \in \{0, 1, 2\}$ retailers out of stock and all other agents follow a threshold strategy $\boldsymbol{s}$, then, the focal agent wants to purchase if

$$p''(m, s, \boldsymbol{s}, \boldsymbol{I})\, v + (1 - p''(m, s, \boldsymbol{s}, \boldsymbol{I}))(-v) > 0, \tag{15}$$

where $p''(m, s, \boldsymbol{s}, \boldsymbol{I})$ is the updated probability that the product quality is high. With Bayes' rule, we express $p''(m, s, \boldsymbol{s}, \boldsymbol{I})$ as follows:

$$p''(m, s, \boldsymbol{s}, \boldsymbol{I}) = \frac{g_h(s)\, p_0 p^h(m, \boldsymbol{s}, \boldsymbol{I})}{p_0 g_h(s)\, p^h(m, \boldsymbol{s}, \boldsymbol{I}) + (1 - p_0)\, g_\ell(s)\, p^\ell(m, \boldsymbol{s}, \boldsymbol{I})}. \tag{16}$$

$p^\omega(m, \boldsymbol{s}, \boldsymbol{I})$ is the probability that m stockouts are observed when the product quality is ω and all agents play strategy $\boldsymbol{s}$ and the inventory investment and allocation is $\boldsymbol{I}$. Hence, we calculate $p^\omega(m, \boldsymbol{s}, \boldsymbol{I})$.

Figure 3 illustrates graphically the inventory depletion path of both retailers, for given quality ω and agent strategy (s_0, s_1). As long as no retailers are out of stock, the depletion rate at both retailers is the same, $P_\omega(s_0)/2$. Both retailers split the market equally. When the small retailer is out of stock, the remaining retailer captures the whole market *and* the depletion rate changes to $P_\omega(s_1)$. Now, we define:

$$\underline{\lambda}^\omega(\boldsymbol{s}, \boldsymbol{I}) = \frac{Q}{\frac{1}{2} P^\omega(s_0)} \text{ and } \overline{\lambda}^\omega(\boldsymbol{s}, \boldsymbol{I}) = \frac{Q}{\frac{1}{2} P^\omega(s_0)} + \frac{\Delta}{P^\omega(s_1)}. \tag{17}$$

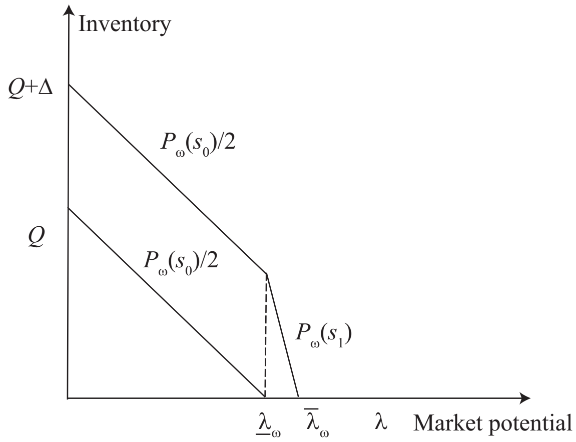

Figure 3 **Depletion path of both retailer's inventory for product quality ω. One retailer has inventory Q and the other one has inventory $Q+\Delta$. Slopes of lines are indicated.**

A potential population of $\underline{\lambda}^{\omega}$ will lead to Q sales at each retailer, assuming these occur when there are no retailers out of stock when the product quality is ω: When there are no stock-outs, the purchasing probability of a retailer is $\frac{1}{2}P^{\omega}$. As $\frac{1}{2}P^{\omega}\underline{\lambda}^{\omega}=Q$, a potential population of $\underline{\lambda}^{\omega}$ will lead to exactly Q sales when the product quality is ω. Similarly, a potential population of $\overline{\lambda}^{\omega}-\underline{\lambda}^{\omega}$ leads to Δ sales, assuming these occur when there is one retailer out of stock and the product quality is ω as $P^{\omega}(\overline{\lambda}^{\omega}-\underline{\lambda}^{\omega})=\Delta$. We can now use these expressions to compute $p^{\omega}(m,\boldsymbol{s},\boldsymbol{I})$:

Lemma 3. *Given* $\boldsymbol{s}$*, the probability of observing* $m\in\{0,1,2\}$ *retailers out of stock when the product quality is* ω *is given by:*

$$p^{\omega}(0,\boldsymbol{s},\boldsymbol{I})=\min\left(1,\frac{\underline{\lambda}^{\omega}(\boldsymbol{s},\boldsymbol{I})}{\lambda}\right) \text{ and } p^{\omega}(1,\boldsymbol{s},\boldsymbol{I})=\min\left(1,\frac{\overline{\lambda}^{\omega}(\boldsymbol{s},\boldsymbol{I})}{\lambda}\right)-\frac{\underline{\lambda}^{\omega}(s_0,\boldsymbol{I})}{\lambda} \text{ for } 1\geq\frac{\underline{\lambda}^{\omega}(s_0,\boldsymbol{I})}{\lambda}$$

(and 0 *otherwise).* $p^{\omega}(2,\boldsymbol{s},\boldsymbol{I})=1-p^{\omega}(0,\boldsymbol{s},\boldsymbol{I})-p^{\omega}(1,\boldsymbol{s},\boldsymbol{I})$.

Proof of Lemma 3: The probability that a random agent observes that zero or one retailer is out of stock, conditional on the agent's strategy and the product quality, $p^{\omega}(m,\boldsymbol{s},\boldsymbol{I})$ is the volume of agents that observes zero or one retailer out of stock divided by the total volume of agents. When the market potential, λ, is less than $\underline{\lambda}^{\omega}$, no retailer will stock out, hence, $p^{\omega}(0)=1$ and $p^{\omega}(1)=0$. When λ, is in between $\underline{\lambda}^{\omega}$ and $\overline{\lambda}^{\omega}$, with probability $\underline{\lambda}^{\omega}/\lambda$, no stock-out will be observed and with probability $(\lambda-\underline{\lambda}^{\omega})/\lambda$, one stock-out will be observed. Finally, when λ, is in larger than

Case	Small Retailer (ℓ, h)	Large Retailer (ℓ, h)
I	(Stock-out, Stock-out)	(Stock-out, Stock-out)
II	(Stock-out, Stock-out)	(Leftover, Stock-out)
III	(Stock-out, Stock-out)	(Leftover, Leftover)
IV	(Leftover, Stock-out)	(Leftover, Stock-out)
V	(Leftover, Stock-out)	(Leftover, Leftover)
VI	(Leftover, Leftover)	(Leftover, Leftover)

Table 2 Overview of the possible end-of-season inventory realizations (low quality, high quality), for both small and large retailers with inventory Q and $Q+\Delta$ respectively.

$\overline{\lambda}^{\omega}$, with probability $\underline{\lambda}^{\omega}/\lambda$, no stock-out will be observed and with probability $(\overline{\lambda}^{\omega}-\underline{\lambda}^{\omega})/\lambda$, one stock-out will be observed. ■

The focal informed agent now uses $p^{\omega}(m, \boldsymbol{s}, \boldsymbol{I})$ to make a purchasing decision according to Equations (15) and (16). If his decision threshold, s_m, is the same as the conjectured threshold, $\boldsymbol{s}$, an equilibrium is obtained. The equilibrium condition of Equation (13) can be written in terms of $\boldsymbol{s}^*$ as follows:

$$\boldsymbol{s}^* : \theta l\left(s_m^*\right) = \frac{p^{\ell}\left(m, \boldsymbol{s}^*, \boldsymbol{I}\right)}{p^{h}\left(m, \boldsymbol{s}^*, \boldsymbol{I}\right)},\ m \in \{0,1\}. \tag{18}$$

Note the similarity between Equation (18) for two retailers, which is of the form: $\theta \times l_m \times \frac{p_m^h}{p_m^{\ell}} = 1$, and Equation (10) for a single retailer. Now, for $m=1$, the factor $l_1 \times p_1^h/p_1^{\ell}$ captures how a stock-out (at the small retailer) influences the purchasing behavior of the strategic agents. It is intuitive that the fraction of the population that observes no retailers out of stock only depends on the purchasing threshold when there are no stock-outs; i.e. $p^{\omega}(0, \boldsymbol{s}, \boldsymbol{I})$ is only a function of s_0. On the other hand, the fraction of the population that observes one retailer out of stock may depend on both purchasing thresholds s_0 and s_1. An implication is that the equilibrium threshold s_0 can be determined first using a single equilibrium condition (Equation (18) for $m=0$). The obtained s_0^* is subsequently used in the equilibrium condition for s_1 (Equation (18) for $m=1$).

The Impact of Inventory Investment and Allocation on Strategic Purchasing Behavior: For any retailer inventory strategy $\boldsymbol{I}$, we now characterize the agent equilibrium $\boldsymbol{s}^*$. In the next subsection, we will determine the optimal inventory allocation, taking the dependency of $\boldsymbol{s}^*$ on $\boldsymbol{I}$ into account. Recall that there are two possible quality realizations and two retailers that can

Region	$s_0^*(\boldsymbol{I})$	$s_1^*(\boldsymbol{I})$
$\Omega_I \quad = \{\boldsymbol{I} \geq \mathbf{0} : 0 \leq Q < \tilde{\lambda}^\ell/2, 0 \leq \Delta < \tilde{\lambda}^\ell - 2Q\}$	$\tilde{s}$	$\tilde{s}$
$\Omega_{II} \quad = \{\boldsymbol{I} \geq \mathbf{0} : 0 \leq Q < \tilde{\lambda}^\ell/2, \tilde{\lambda}^\ell - 2Q \leq \Delta < \Delta_A(Q)\}$	$\tilde{s}$	$s_A(\Delta, \lambda - Q/(\frac{1}{2}P^\ell(\tilde{s})))$
$\Omega_{III} \quad = \{\boldsymbol{I} \geq \mathbf{0} : \tilde{\lambda}^\ell/2 \leq Q < \hat{\lambda}^h/2, \Delta_A(Q) \leq \Delta \leq \lambda - 2Q\}$	$\tilde{s}$	$s_B(Q, \lambda)$
$\Omega_{IV} \quad = \{\boldsymbol{I} \geq \mathbf{0} : \tilde{\lambda}^\ell/2 \leq Q < \hat{\lambda}^h/2, 0 \leq \Delta < \Delta_B(Q, \lambda)\}$	$s_A(2Q, \lambda)$	$\underline{s}$
$\Omega_V \quad = \{\boldsymbol{I} \geq \mathbf{0} : \tilde{\lambda}^\ell/2 \leq Q \leq \hat{\lambda}^h/2, \Delta_B(Q, \lambda) \leq \Delta \leq \lambda - 2Q\}$	$s_A(2Q, \lambda)$	$\underline{s}$
$\Omega_{VI} \quad = \{\boldsymbol{I} \geq \mathbf{0} : \hat{\lambda}^h/2 \leq Q \leq \lambda/2, \Delta \leq \lambda - 2Q\}$	$\hat{s}$	N.A.

Table 3 Overview of the customer purchasing equilibrium for all cases identified in Table 3.

either stock-out or have leftover inventory at the end of the season. The different possibilities at the small and large retailer for each quality realization are illustrated in Table 2. The following Proposition specializes the equilibrium conditions of Equation (18) for different values of $\boldsymbol{I}$. To that end, we define:

$$\Delta_A(Q) = \left(\lambda - \frac{Q}{\frac{1}{2}P^h(\tilde{s})}\right) P^h(s_B(Q, \lambda)) \text{ for } 0 \leq Q \leq \tilde{\lambda}^\ell/2$$
$$\text{and } \Delta_B(Q) = \left(\lambda - \frac{2Q}{P^h(s_A(2Q, \lambda))}\right) P^h(\underline{s}) \text{ for } \tilde{\lambda}^\ell/2 \leq Q \leq \hat{\lambda}^h/2.$$

Now, we can characterize the properties of the purchasing equilibrium as a function of the inventory strategy (**I**):

Proposition 3. *(i) The (Q, Δ)-space can be divided into six regions defined in Table 3 that lead to each of the possible end–of-season realizations of Table 2. The equilibrium purchasing threshold in each region is also given in Table 3, with:*

$$s_B(Q, \lambda) : \theta l(s_B) = \frac{\lambda - \frac{Q}{\frac{1}{2}P^\ell(\tilde{s})}}{\lambda - \frac{Q}{\frac{1}{2}P^h(\tilde{s})}}.$$

(ii) A stock-out does not increase the strategic agents' purchasing threshold: $s_0^* \geq s_1^*$ (whenever s_1^* is defined).

(iii) Larger inventories decrease the purchasing thresholds of the strategic agents: $\frac{\partial s_0^*}{\partial Q} \leq 0$ and $\frac{\partial s_1^*}{\partial \Delta} \leq 0$ (whenever s_1^* is defined).

Proposition 3(i) is illustrated in Figure 4. The solid lines indicate the boundaries between the different regions. The dotted lines indicate $\hat{\lambda}^\omega = 2Q + \Delta$ for $\omega \in \{h, \ell\}$, i.e. the two possible inventory investment levels when all consumers are myopic.

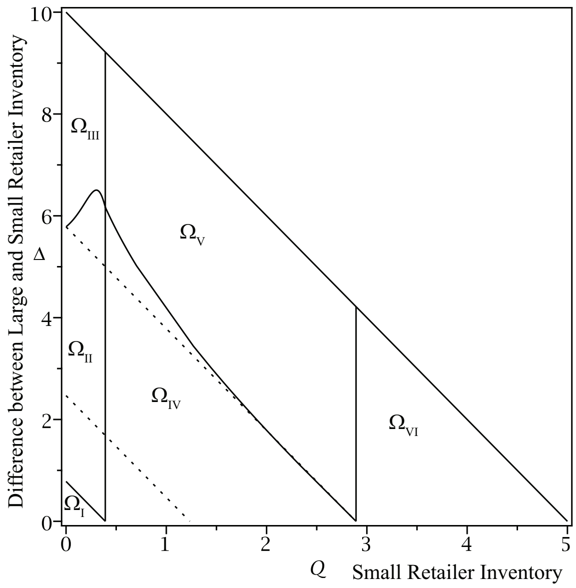

Figure 4 **Regions in the Q-Δ space in which different equilibria occur. Parameters are:** $\lambda = 10$, $p = 0.45$, $v_h = 1$, $v_\ell = -1$ **and** $\kappa = 0.6$. **The cases of Table 2 are indicated in the Figure.**

With Proposition 3(i), as long as no retailer is out of stock, the equilibrium purchasing behavior, $s_0^*(\boldsymbol{I})$, is similar to the single retailer case except that the total inventory is $2Q$ (instead of Δ). The small retailer inventory Q plays a major role in determining the equilibrium purchasing strategy of the informed agents. Proposition 3(i) defines a 'small small retailer,' $(0 < Q < \hat{\lambda}^\ell/2)$ a 'medium small retailer' $(\hat{\lambda}^\ell/2 < Q < \hat{\lambda}^h/2)$ and a 'large small retailer,' $(\hat{\lambda}^h/2 < Q < \lambda/2)$ depending on the value of small retailer inventory, Q. When the small retailer has a lot of inventory (i.e. more or less the same inventory allocated as the large retailer; region Ω_{VI}), the myopic purchasing strategy is an equilibrium as no retailer ever stocks out[5]. This is why for the large small retailers, the purchasing equilibrium is $(\hat{s}, \bullet)$. For medium small retailers (regions Ω_{IV} and Ω_V), the stock-out signal becomes *perfectly* informative of high product quality; stock-outs occur only when the product quality is high. Hence, upon observing a stock-out at the medium small retailer, all strategic agents know that the product quality is high and they 'rush' to the large retailer to obtain the product. This is why for the medium small retailers, the purchasing equilibrium is $(\bullet, \underline{s})$. Note that the absence of stock-outs do not imply low quality for sure. This is due to the fundamental single-sidedness of inventory: as long as the inventory is strictly positive, there exists for both quality realizations a

[5] Off the equilibrium path, an inventory signal is very informative, but, this happens with probability zero.

strictly positive probability that a agent observes no stock-out. Hence, a no-stock-out situation can never perfectly reveal low quality. For small small retailers (regions Ω_I, Ω_{II} and Ω_{III}), the stock-out signal is still informative, but, less than for medium small retailers. The reason is that the small retailer stocks out for any product quality. Nevertheless, more frequently for high quality products. Hence, upon observing a stock-out, some strategic agents will not rush to the large retailer to obtain the product. For the small small retailers, the purchasing equilibrium is $(\tilde{s}, \bullet)$.

The difference between the small and large retailer, Δ, plays an intuitive role in the inventory realization at both retailers at the end of the season. When Δ is large (regions Ω_{III}, Ω_V and Ω_{VI}), the large retailer has excess inventory, irrespective of the product quality. When this difference is medium (regions Ω_{II} and Ω_{IV}), the large retailer stocks out for high quality products only. Finally, when the difference is low (region Ω_I), the large retailer stocks out for any product quality. At $\Delta = \Delta_A(Q)$ and $\Delta = \Delta_B(Q)$, the large retailer breaks even when the product quality is high.

Furthermore, Proposition 3(ii) shows that stock-out information may reduce the agent's purchasing threshold. This confirms the intuition that stock-outs are positively related to product quality when the quality is uncertain. This result provides an affirmative answer to the question whether stock-outs may positively influence consumer purchasing behavior.

Finally, with Proposition 3(iii), we find that keeping the difference between the small and large retailer inventory fixed, when the small retailer inventory increases, agents become less 'picky' (i.e. their purchasing threshold decreases). Keeping the small retailer inventory fixed, when the difference between the small and large retailer inventory increases, agents become less picky. The intuition is the same as for the single retailer case of Proposition 1.

We found in Proposition 3(i), for medium small retailers in regions Ω_{IV} and Ω_V that through appropriate inventory investment and allocation, a stock-out can become perfectly informative about high quality. The next question is whether perfectly communicating high quality is ex ante optimal for the firm. In order to answer this question, we need to analyze how the expected satisfied demand changes a as function of the inventory investment and allocation.

7.2. Profit maximizing inventory investment and allocation

We next characterize the optimal inventory investment and allocation policy. We can write expected satisfied demand as:

$$S(\boldsymbol{I},\boldsymbol{s}) = \mathbb{E}_\omega\left[P^\omega(s_0)\min\left(\underline{\lambda}^\omega(\boldsymbol{s},\boldsymbol{I}),\lambda\right) + \min\left(\Delta, P^\omega(s_1)\left(\lambda - \underline{\lambda}^\omega(\boldsymbol{s},\boldsymbol{I})\right)^+\right)\right].$$

The expected satisfied demand depends on the quality realization. Recall that one retailer has inventory Q, while the other retailer has inventory $Q+\Delta$. When the potential demand is small, $\lambda < \underline{\lambda}^\omega$, the small retailer will not stock out, hence, the large retailer will not stock out either and the sales are simply $P^\omega(s_0)\lambda$. When the potential demand is high enough, $\lambda > \underline{\lambda}^\omega$, the small retailer will stock out and have $Q = \frac{1}{2}P^\omega\underline{\lambda}^\omega$ sales. The large retailer will also have Q sales at least. Hence, the sales are at least $2Q = P^\omega\underline{\lambda}^\omega$. In addition, the remaining potential market (with size: $\lambda - \underline{\lambda}^\omega$) now purchases a product at rate $P^\omega(s_1)$. If the remaining inventory at the large retailer, Δ, is high enough, the sales are $P^\omega(s_1)(\lambda - \underline{\lambda}^\omega)$, otherwise, the remaining sales are Δ (and the large retailer stocks out too).

Lemma 4. *Allocating inventory $I \in [0,\lambda]$ to one retailer and none to the other retailer yields the same expected satisfied demand as allocating $I/2$ to both retailers:* $S((0,I),\boldsymbol{s}^*(0,I)) = S((I/2,0),\boldsymbol{s}^*(I/2,0))$.

With Lemma 4, allocating all inventory to a single retailer yields exactly the same expected sales as distributing equally all inventory over the two retailers. Even though the purchasing behavior is slightly different, the outcome from the firm's perspective is identical. With all inventory allocated to one retailer, the other retailer will go out of stock for sure. As a result, the mass of agents that moves when no retailer is out of stock is zero. The bulk of the market purchases at the large retailer, after having observed one stock-out. However, as the stock-out is predictable, the purchasing strategy s_1^*, is determined by $P^h(s)\lambda/\Delta$ (see Proposition 3(i), region Ω_{II}). When both retailers have identical initial inventory allocated, it will never be the case that one retailer is out of stock, while the other retailer is not. The purchasing strategy s_0^*, is determined by $\frac{1}{2}P^h(s)\lambda/Q$

(see Proposition 3(i), region Ω_{IV} and Ω_V). As a result, when $\frac{1}{2}\Delta = Q$, completely asymmetric or completely symmetric inventory allocation results in the same strategic consumer purchasing behavior and thus equal expected satisfied sales.

In order to obtain the optimal inventory investment and asymmetric allocation, it is insightful to decompose the marginal revenues for $\boldsymbol{I}$ into a direct and an indirect term caused by strategic agent behavior:

$$\underbrace{\frac{\partial}{\partial I} S(\boldsymbol{I}, \boldsymbol{s}^*(\boldsymbol{I}))}_{\text{direct effect} \leqslant 0} + \underbrace{\frac{\partial}{\partial s_0} S(\boldsymbol{I}, \boldsymbol{s}^*(\boldsymbol{I})) \frac{\partial s_0^*}{\partial I} + \frac{\partial}{\partial s_1} S(\boldsymbol{I}, \boldsymbol{s}^*(\boldsymbol{I})) \frac{\partial s_1^*}{\partial I}}_{\text{strategic agent behavior effect} \geq 0} \text{ for } I = Q \text{ or } \Delta.$$

Strategic agent behavior effect: A lower purchasing threshold when the small retailer is out of stock always increases the expected satisfied demand. When $s_1 < s_0$ (which is the case in equilibrium, see Proposition 3-ii), a lower purchasing threshold when no retailers are out of stock leads to higher sales even if the small retailer stocks out. When s_0 decreases (but stays above s_1), the small retailer inventory depletes 'faster' (i.e. there is less potential demand required to deplete Q) and hence increases the pool of potential agents after the stock-out event occurs ($\lambda - \underline{\lambda}^{\omega}$). We have thus obtained that a lower purchasing threshold increases the expected satisfied demand (i.e. $\frac{\partial S}{\partial s_m} \leq 0$ for $m = 0, 1$), keeping all else equal. As with Proposition 3-iii, $\frac{\partial s_m^*}{\partial I} \leq 0$, the impact of an inventory increase on the expected satisfied demand is non-negative. This is intuitive.

Direct effect: Keeping the small retailer's inventory (and the agent purchasing strategy) fixed, a larger large retailer (i.e. increasing Δ) only increases the expected satisfied demand. This is intuitive. Increasing Q (keeping Δ fixed) is more subtle: Increasing Q increases the expected satisfied demand at the small retailer, but, if the depletion rate after a stock-out occurs is large than before (i.e. when $s_0 > s_1$), then, increasing Q *reduces* the total expected satisfied demand. This is somewhat surprising, but, intuitive when noticing that the potential demand is transformed with higher probability into a real demand after the stock-out occurred (if $s_0 < s_1$). In other words, as potential sales are converted more effectively *after* the stock-out occurs, the expected demand is higher when the stock-out occurs early, i.e. only using a low volume of agents that observe no stock-out.

The optimal inventory allocation: Combining the two different effects, the optimal inventory strategy is not obvious. Increasing Q on the one hand reduces the purchasing threshold (Proposition 3-iii), which is, due to the strategic agent behavior effect beneficial for the expected satisfied demand. On the other hand, due to the direct effect, increasing Q leads to fewer sales as strategic agents are pickier when not observing a stock-out. Hence, it may be the case that the total expected sales decrease as the small retailer inventory increases.

In general[6], the firm's profit is: $\Pi(\boldsymbol{I},\boldsymbol{s}) = rS(\boldsymbol{I},\boldsymbol{s}) - c(2Q+\Delta)$ and the firm's optimization problem of Equation (14) becomes: $\max_{\mathbf{I}\geq\mathbf{0}}\{rS(\boldsymbol{I},\boldsymbol{s}) - c(2Q+\Delta)\}$. In the next Proposition, we rewrite the firm's optimization problem by eliminating inventory strategies that are dominated. For notational convenience, the profit π is normalized with respect to $r\lambda$. Then, we can state:

PROPOSITION 4. *No (Q,Δ) interior to any region defined in Proposition 3(i) (defined in Table 3) can be optimal. The only candidates optimal inventory strategies are at the boundary of regions Ω_{II} and Ω_{IV}.*

The inventory optimization problem of Equation (14) can be written as follows:

$$\pi^* = \max\begin{cases} \max_{\underline{s}\leq s\leq\hat{s}} \ \pi_{II}(s) \doteq \pi^o(s)\theta l(s) + \left(1 - \frac{c}{r} - \frac{\theta l(s)}{P^h(\tilde{s})}\pi^o(s)\right)\frac{1-\theta l(s)}{\frac{1}{P^\ell(\tilde{s})} - \frac{\theta l(s)}{P^h(\tilde{s})}} \\ \max_{\hat{s}\leq s\leq\tilde{s}} \ \pi_{IV}(s) \doteq \pi^o(s) + \left(p_0 - \frac{c}{r}\right)^+ P^h(\underline{s})\frac{\theta l(s)-1}{\theta l(s)}. \end{cases}$$

where $\pi^o(s)$ is defined in Proposition 1.

(i) If $s^*\in[\underline{s},\hat{s}]$, then, $\boldsymbol{s}^* = (\tilde{s}, s^*)$ and $Q^*\in(0,\tilde{\lambda}^\ell/2)$ and $\Delta^* = \Delta_A(Q^*)$.

(ii) If $s^*\in[\hat{s},\tilde{s}]$ and $p_0 > \frac{c}{r}$, then, $\boldsymbol{s}^* = (s^*,\underline{s})$ and $Q^*\in(\tilde{\lambda}^\ell/2, \hat{\lambda}^h/2)$ and $\Delta^* = \Delta_B(Q^*)$.

(iii) If $s^*\in[\hat{s},\tilde{s}]$ and $p_0 < \frac{c}{r}$, then, $\boldsymbol{s}^* = (s^*,\underline{s})$ and $Q^*\in(\tilde{\lambda}^\ell/2,\hat{\lambda}^h/2)$ and $\Delta^* = 0$, or $\boldsymbol{s}^* = (\tilde{s}, s^*)$ and $Q^* = 0$ and $\Delta^*\in(\tilde{\lambda}^\ell,\hat{\lambda}^h)$.

The optimal profit is $\pi^*\lambda r$.

Proposition 4 is intuitive: In regions Ω_{III}, Ω_V and Ω_{VI}, for a fixed Q, the agent equilibrium does not depend on Δ (see Proposition 3). As in these regions, the large retailer always has leftovers,

[6] Note that when all agents are myopic, i.e. $s_0 = s_1 = \hat{s}$, then the expected satisfied demand reduces to: $\mathbb{E}_\omega[\min(P^\omega(\hat{s})\lambda, \Delta+2Q)]$, the classical satisfied demand of a (bi-valued) single newsvendor problem when the total inventory is $\Delta+2Q$.

the expected satisfied demand does not depend on Δ. As a result, increasing Δ, which is expensive, does not increase the expected satisfied sales. Thus, it is never optimal for the firm to select $\boldsymbol{I}$ inside regions Ω_{III}, Ω_V and Ω_{VI}. In region Ω_I, both retailers always stock out. The expected satisfied demand is thus $2Q+\Delta$. As $r>c$ by assumption, it is always optimal to increase the total inventory in region Ω_I. In Proposition 4, it is proven that no interior $\boldsymbol{I}$ is ever optimal in regions Ω_{II} and Ω_{IV}. As a result, the only candidate inventory strategies are: (1) When the large retailer exactly breaks even for the high quality products i.e. when $(Q,\Delta)=(Q,\Delta_A(Q))$ (when the small retailer is small; $\underline{s}\leq s^*\leq \hat{s}$, with profits $\pi_{II}(s)$) or $(Q,\Delta_B(Q))$ (when the small retailer is medium (when $\hat{s}\leq s^*\leq \tilde{s}$, with profits $\pi_{IV}(s)$). (2) When there is exactly one retailer (i.e. either $Q=0$ and $\tilde{\lambda}^\ell\leq\Delta\leq\hat{\lambda}^h$), which, with Lemma 4 is equivalent to the two retailer case with symmetric inventory allocation (i.e. the case $\Delta=0$ and $\tilde{\lambda}^\ell/2\leq Q\leq\hat{\lambda}^h/2$).

Proposition 4 implies thus that in Figure 4, the optimal inventory investment and allocation strategy is one of the points that are on the boundary between regions Ω_I-Ω_{II}, Ω_{II}-Ω_{III} or Ω_{IV}-Ω_V, or at the boundary of region Ω_{II} with $Q=0$ or at the boundary of region Ω_{IV} with $\Delta=0$.

Numerical Experiments: With Proposition 4, we can now numerically determine the optimal inventory strategy. In Figure 5, we plot the total inventory investment, allocation and profits as a function of the critical ratio c/r for a representative set of parameter values of the parameters (κ,p,α).

Note from the dashed lines in the first column in Figure 5 that with only myopic agents (i.e. when $\boldsymbol{s}=(\hat{s},\hat{s})$), the optimal inventory investment decreases from $\hat{\lambda}^h$ to $\hat{\lambda}^\ell$ as the critical ratio c/r increases (i.e. crosses p_0). With strategic agents, note from the solid lines in the first column that in all cases for high margin products (i.e. low c/r ratio) the optimal total inventory investment is higher than the myopic inventory with an imbalanced retailer network. This is interesting and solves the paradox of creating more sales with stock-outs: At first sight, one may think that stock-outs need to be generated with *less* inventory and hence, that less inventory leads to more sales. Our model shows that this is not necessarily the case: it is optimal to invest in *more* inventory in total, but, allocate the inventory asymmetrically over the two retailers. A stock-out may then be

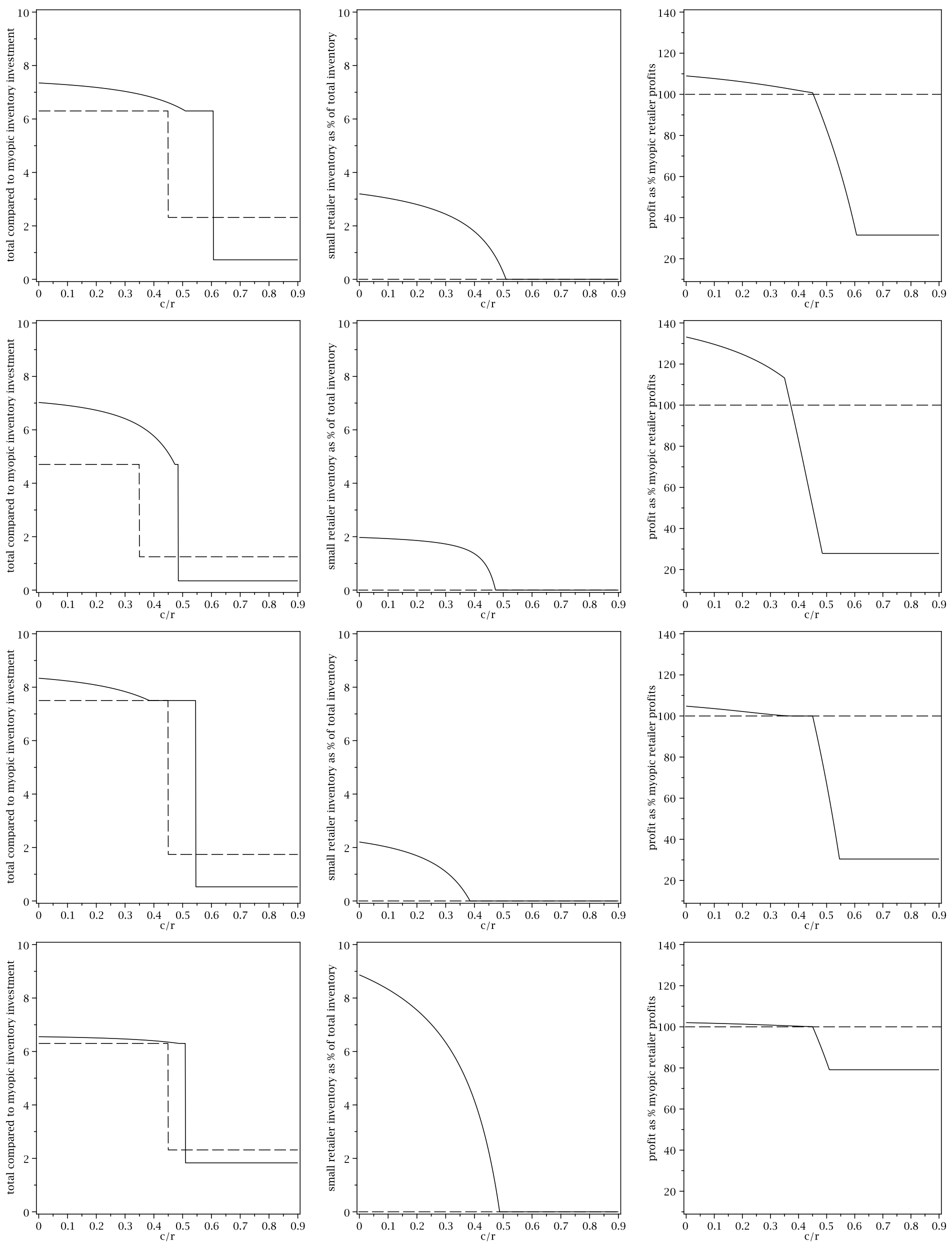

Figure 5 First column: the optimal (solid line) and the myopic (dashed line) total inventory investment as a function of c/r. Second column: the small retailer inventory as a percentage of the total inventory as a function of c/r. Third column: the firm's profits as a percentage of the optimal profits of the myopic retailer as a function of c/r. The parameters are: $(\kappa, p, \alpha) = (0.75, 0.45, 0.25)$ in the first line (base case). In the second, third and fourth line, only one parameter changes from the base case: $p = 0.35$, $\kappa = 1.25$ and $\alpha = 0.75$ respectively.

generated at the smallest of the two retailers, which then leads to more sales that are satisfied by the largest of the two retailers. Furthermore, note from the solid lines in the first column that for low margin products (i.e. high c/r ratio) the optimal total inventory investment is lower than the myopic inventory. This effect is similar to the single retailer case (see Proposition 2). The numerical experiments reveal thus that the strategic agent behavior introduces extra volatility to the realized demand, compared to the situation in which all agents are myopic: the low quality realization leads to lower sales because more strategic agents observe no stock-outs, which makes them 'pickier'. Conversely, the high quality realization leads to more sales, triggered by the small retailers and satisfied by the large retailer. Due to this increased volatility, when margins are high, the optimal inventory investment is higher and when margins are low, the optimal inventory investment is lower than the optimal myopic inventory investment.

From the second column of Figure 5, note that the small retailer's inventory is a few percent of the total inventory. Recall that two forces determine the size of the small retailer: on the one hand, the small retailer should be large enough that the quality communication to the agent after a stock-out is significant. (For medium small retailer, the stock-out perfectly reveals high quality.) On the other hand, the larger the small retailer, the more consumers are 'wasted' to generate the stock-out. Hence, an early stock-out is preferred. From our numerical experiments, we found that the incentive to stock-out early is mostly strong enough that some signal informativeness is sacrificed for a larger post-stock-out market. When there are more myopic agents (in the fourth line of Figure 5, the fraction of myopic agents is higher than in the first line of Figure 5), the small retailer's inventory represents a larger fraction of the total inventory. This is because with fewer strategic agents the information content of a stock-out is weaker and hence a larger small retailer is required. At the same time, less strategic agents are wasted to generate a stock-out.

From the third column of Figure 5, note that the profit increase with respect to the myopic profits for high margin products can be substantial (more than 30% in some cases), especially for high margin products with a low prior or brand perception (in the second line of Figure 5, the prior is lower than in the first line of Figure 5) and noisy private signals (in the fist line of Figure 5, the noise is higher than in the third line of Figure 5). As the higher sales are generated through higher investments in inventory, it is logical that high margin products are better candidates for improved profits. Furthermore, without additional (public) stock-out information, the expected sales of a firm with a low prior (brand perception) and noisy information are very low. When the private signal does not contain any information and the prior (brand perception) is low, the sales will be zero. Hence, the public stock-out signal is most useful in these circumstances. For products with a high prior about the product quality (strong brands), the profit increase is much lower. Also, from the third column of Figure 5, note that the profit decrease with respect to the myopic profits for low margin products can also be substantial.

8. Conclusions

While there has been speculation in the business and popular press about the reasons for frequent stock-outs of new, innovative products, to the best of our knowledge, no theory provides a rationale for the how stock-out influence the agent purchasing behavior and the optimal inventory investment and allocation of the firm. We believe that our model sheds a new light into this issue. We explained how firms can generate sales through stock-outs, which enable strategic customers to infer that the product is worthwhile because many other customers are buying it. We demonstrate how and when a firm can leverage this effect through inventory investment and allocation to retailers.

In order to leverage the stock-out effect, we find that it is optimal for a retailer to asymmetrically allocate inventory to retailers. One retailer is sacrificed to generate the stock-out signal when the product quality is high. This triggers a herd of consumers whose demand for the product is satisfied by the other (large) retailer. The total inventory investment in such cases is larger than the inventory investment that a retailer would make assuming that all agents are myopic. Thus, stock-outs in our theory are due to an aggregate shortage of supply, but rather the result of a deliberate imbalance of inventory allocations to retailers. The total expected satisfied demand (i.e. realized sales) is also larger than the demand when all consumers are myopic.

The nature of the asymmetrical inventory allocation is also interesting. Even though for a slightly imbalanced allocation a stock-out can perfectly communicate to the market that the product is of high quality, it is mostly optimal to have strongly imbalanced inventory allocations (i.e. a 'small' small retailer), which dilutes the information contained in the stock-out (now, a stock-out is more likely), but allows a stock-out signal to be generated early when the remaining potential market is large. Yet asymmetric inventory allocations are optimal only when the prior about the product quality (brand perception) is relatively low and margins are high. In that case, stock-out signals are valuable complementary public signals for the firm and if the product margin is low, it is optimal to invest in *less* total inventory at both retailers than the inventory a retailer would have invested assuming that all agents are myopic.

Do these findings map to stylized facts? The story of Apple's shortage of iPods could be interpreted as one example. Apple distributed iPods through BestBuy and their own Apple stores but allocated less inventory to BestBuy (Wingfield, 2004, Slattery, 2008). This created obvious tension with BestBuy, who incurred stock-outs for the new iPods. The fact that BestBuy was stocked out of iPods arguably lead to Apple stores gaining spill over demand from BestBuy and moreover, according to our theory, a potentially increased overall demand due to the stock-out information that created an increased willingness to buy among consumers.

From our numerical experiments, we observed that taking strategic consumer behavior into account is important for products with a low prior (brand perception) and with weak signals. Hence, taking herding reactions to stock-outs into account is important for a weak brand (think of a less branded MP3 player) or when previous product launches failed or in a crowded market with easy entry in which many (dubious) alternatives are available (think on-line music). Markets for books and music fall in the latter category. Furthermore, taking herding reactions to stock-outs into account is especially relevant for products with that are difficult to evaluate (e.g. communicating the taste of a food in a restaurant, or the quality of a new book).

Of course, our model is stylized and makes a number of assumptions that would be desirable to relax or extend. In order to keep the analysis tractable and insightful, we assume that the potential market size is deterministic and common knowledge. We only introduced one source of uncertainty: the quality of the product. An interesting extension would be to allow for stochastic demand. Then, strategic agents would not know the exact market size, but, only the distribution. We conjecture that our insights obtained for deterministic demand still hold, as long as the extra uncertainty about the potential demand is not too large. Also, we kept the retailers ex ante symmetric. In reality, a firm may distribute the product via different retailer channels which have different margins and different customer segments.

Furthermore, we assumed that strategic consumers observed at no cost the product availability at both retailers and, in case one retailer was out of stock, they purchase from the other retailer with no switching cost. In the physical world, consumers may live closer to one retailer and incur switching costs if they need to go to the other retailer because of a stock-out. In the extreme case, when the switching costs are very high, there is less to be gained from the herding effect triggered by a stock-out.

Our model is also a single period model in which agents do not observe any information about the sequence in which they move and/or the time in the season when they move. They only observe how many retailers are out of stock. In case that agents would move sequentially and in case they would have multiple purchasing opportunities, the model would become significantly more complex. In a two-period model, for example, the stock-out outcome of the first period may be leveraged in the second period (even with a single retailer). In addition, the firm's cost may decrease over time due to learning. Considering multi-period extensions of our newsvendor model would be a challenging and interesting avenue for further research.

In our model, we kept the retailer passive. We found that the manufacturer may have strong incentives to favor otherwise identically retailers. Allowing the retailer to react strategically (e.g. by changing its effort to promote the manufacturer's product) would be an interesting avenue of further research.

We assumed that potential customers are only differentiated through their private signal realization and by their reaction to a stock-out (strategic vs. non-strategic) and customers value the product in exactly the same way if they knew the quality. In reality, even if customers knew the quality perfectly, it may be that some customers are willing to pay more than other customers. Hence, introducing such product differentiation is another interesting avenue for future research.

We considered purchasing decisions of strategic consumers to be rational. The rationality assumption may seem restrictive. Psychological effects will also impact purchasing decisions; consumers may not be perfect Bayesian agents. As in the traditional herding literature, our model is a benchmark, from which possible deviations may need to be established when observing real consumer choices. Our focus is on understanding how consumers may take into account operational factors, like observed stock-outs, and how a firm needs to take such behavior into account when determining optimal operational strategies (like inventory investment and allocation). We believe that our modeling framework can be extended in many interesting directions, taking strategic consumer behavior into account and hope that this will be a fruitful area for further research.

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10. Appendix: Statements in the Main Body

In the single retailer case discussion of Section 5, we claimed that there exist a critical cutoff inventory investment cost that is higher than the inventory investment cost, above which it is optimal to invest in low inventory. It is optimal to invest in low inventory when:

$$\begin{aligned}
\pi^o(\hat{s}) \ > \ \pi^o(\tilde{s}) &\Leftrightarrow \frac{\left(p_0 - \frac{c}{r}\right) P^h(\hat{s}) + p_0 l(\hat{s}) P^\ell(\hat{s})}{\theta l(\hat{s})} > \frac{\left(p_0 - \frac{c}{r}\right) P^h(\tilde{s}) + p_0 l(\tilde{s}) P^\ell(\tilde{s})}{\theta l(\tilde{s})} \\
&\Leftrightarrow \left(p_0 - \frac{c}{r}\right) P^h(\hat{s}) + (1-p_0) P^\ell(\hat{s}) > \frac{\left(p_0 - \frac{c}{r}\right) P^h(\tilde{s}) + p_0 l(\tilde{s}) P^\ell(\tilde{s})}{P^h(\tilde{s}) / P^\ell(\tilde{s})} \\
&\Leftrightarrow \frac{c}{r} \underbrace{\left(P^\ell(\tilde{s}) - P^h(\hat{s})\right)}_{<0} > p_0 \left(P^\ell(\tilde{s}) - P^h(\hat{s})\right) + (1-p_0)\left(P^\ell(\tilde{s}) - P^\ell(\hat{s})\right) \\
&\Leftrightarrow \frac{c}{r} < \frac{p_0 \left(P^h(\hat{s}) - P^\ell(\tilde{s})\right) + (1-p_0)\left(P^\ell(\hat{s}) - P^\ell(\tilde{s})\right)}{P^h(\hat{s}) - P^\ell(\tilde{s})} \\
&\Leftrightarrow \frac{c}{r} < p_0 + (1-p_0) \frac{P^\ell(\hat{s}) - P^\ell(\tilde{s})}{P^h(\hat{s}) - P^\ell(\tilde{s})}
\end{aligned}$$

The latter determines the cutoff cost $\hat{p}_0$ which is strictly larger than p_0.

In the single retailer case discussion of Section 5, we claimed that when $m = 1$, the posterior after observing a stock-out is higher than before observing a stock-out when the inventory investment is low (and hence a stock-out occurs for both quality realizations). Formally, this can be written as: $p''(1, \tilde{s}, \tilde{s}, \Delta^*) > p''(0, \tilde{s}, \tilde{s}, \Delta^*)$, which is true if:

$$\frac{1}{1 + \frac{1-p_0}{p_0} \frac{g_\ell(\tilde{s})}{g_h(\tilde{s})} \frac{p^\ell(1,\tilde{s},\Delta^*)}{p^h(1,\tilde{s},\Delta^*)}} > \frac{1}{1 + \frac{1-p_0}{p_0} \frac{g_\ell(\tilde{s})}{g_h(\tilde{s})} \frac{p^\ell(0,\tilde{s},\Delta^*)}{p^h(0,\tilde{s},\Delta^*)}} \Leftrightarrow \frac{p^\ell(1, \tilde{s}, \Delta^*)}{p^h(1, \tilde{s}, \Delta^*)} < \frac{p^\ell(0, \tilde{s}, \Delta^*)}{p^h(0, \tilde{s}, \Delta^*)}$$

and as

$$\begin{aligned}
p^h(1, \tilde{s}, \Delta^*) &= 1 - \frac{\overline{\lambda}^h(\tilde{s}, \Delta^*)}{\lambda}, \ p^\ell(1, \tilde{s}, \Delta^*) = 1 - \frac{\overline{\lambda}^\ell(\tilde{s}, \Delta^*)}{\lambda} \text{ and} \\
p^h(0, \tilde{s}, \Delta^*) &= \frac{\overline{\lambda}^h(\tilde{s}, \Delta^*)}{\lambda}, \ p^\ell(0, \tilde{s}, \Delta^*) = \frac{\overline{\lambda}^\ell(\tilde{s}, \Delta^*)}{\lambda},
\end{aligned}$$

we obtain that the above condition is satisfied iff

$$\frac{1 - \frac{\overline{\lambda}^\ell(\tilde{s},\Delta^*)}{\lambda}}{1 - \frac{\overline{\lambda}^h(\tilde{s},\Delta^*)}{\lambda}} < \frac{\frac{\overline{\lambda}^\ell(\tilde{s},\Delta^*)}{\lambda}}{\frac{\overline{\lambda}^h(\tilde{s},\Delta^*)}{\lambda}}.$$

The latter is true because $\frac{\overline{\lambda}^\ell(\tilde{s},\Delta^*)}{\lambda} < \frac{\overline{\lambda}^h(\tilde{s},\Delta^*)}{\lambda}$, is always satisfied (recall that by definition $\overline{\lambda}^\omega(s_0, \Delta) = \Delta / P^\omega(s_0)$ and hence $\Delta^* / P^\ell(\tilde{s}) < \Delta^* / P^h(\tilde{s})$).

11. Appendix: Proofs of Lemmas and Propositions

Proof of Lemma 1: Has been argued in the text.

Proof of Lemma 2:

Part (i): Is trivial, as θ increases, the left hand sides of Equations (6) and (11) increase, as the right side is independent of s, the intersection decreases when θ increases.

Part (ii): It is easy to see that there exists always a $\tilde{s} \in (\underline{s}, \overline{s})$ because

$$\begin{aligned} \frac{P^h(\underline{s})}{P^\ell(\underline{s})} &= \frac{\alpha\overline{G}_h(\hat{s}) + (1-\alpha)}{\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)} > \theta l(\underline{s}) = 0 \text{ and} \\ \frac{P^h(\overline{s})}{P^\ell(\overline{s})} &= \frac{\overline{G}_h(\hat{s})}{\overline{G}_\ell(\hat{s})} < \theta l(\overline{s}) = +\infty \end{aligned}$$

and both $\theta l(s)$ and $\frac{\alpha\overline{G}_h(\hat{s})+(1-\alpha)\overline{G}_h(s)}{\alpha\overline{G}_\ell(\hat{s})+(1-\alpha)\overline{G}_\ell(s)}$ are continuous functions in s over $(\underline{s}, \overline{s})$. Due to Rolle's theorem, there must exist at least one intersection point of these two functions.

Part (iii): Now, we take such an intersection point, $\tilde{s}$ and prove that it decreases for $\alpha \in (0,1]$ and is always higher than $\hat{s}$. For $\alpha = 1$, it is easy to establish that $\tilde{s} > \hat{s}$. Proving that $\tilde{s}$ decreases in α then completes the proof that $\tilde{s}$ is always higher than $\hat{s}$ for any $\alpha \in (0,1]$. When $\alpha = 1$:

$$\theta l(\tilde{s}) = \frac{\overline{G}_h(\hat{s})}{\overline{G}_\ell(\hat{s})} > 1$$

as $\theta l(\hat{s}) = 1$ and $l(s)$ is increasing in s, it follows that $\tilde{s} \geq \hat{s}$.

It is easy to see that $\tilde{s}$ decreases in α:

$$\theta l'(\tilde{s})\frac{d\tilde{s}}{d\alpha} = \frac{d}{ds}\left(\frac{P^h(s)}{P^\ell(s)}\right)\bigg|_{s=\tilde{s}}\frac{d\tilde{s}}{d\alpha} + \frac{d}{d\alpha}\left(\frac{\alpha\overline{G}_h(\hat{s}) + (1-\alpha)\overline{G}_h(\tilde{s})}{\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(\tilde{s})}\right)$$

and $$\frac{d}{ds}\left(\frac{P^h(s)}{P^\ell(s)}\right)\bigg|_{s=\tilde{s}} = \frac{-(1-\alpha)g_h(\tilde{s})\left(\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(\tilde{s})\right) + \left(\alpha\overline{G}_h(\hat{s}) + (1-\alpha)\overline{G}_h(\tilde{s})\right)(1-\alpha)g_\ell(\tilde{s})}{\left(\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(\tilde{s})\right)^2}.$$

Note that by definition of $\tilde{s}$ (Equation (11)):

$$\frac{d}{ds}\left(\frac{P^h(s)}{P^\ell(s)}\right)\bigg|_{s=\tilde{s}} = -(1-\alpha)g_\ell(\tilde{s})\left(\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(\tilde{s})\right)\left(\frac{g_h(\tilde{s})}{g_\ell(\tilde{s})} - \frac{\alpha\overline{G}_h(\hat{s}) + (1-\alpha)\overline{G}_h(\tilde{s})}{\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(\tilde{s})}\right) = 0,$$

hence

$$\frac{d\tilde{s}}{d\alpha} = \frac{-\overline{G}_h(\tilde{s})\overline{G}_\ell(\hat{s}) + \overline{G}_\ell(\tilde{s})\overline{G}_h(\hat{s})}{\theta l'(\tilde{s})\left(\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(\tilde{s})\right)^2}.$$

Note that as $l' > 0$ we obtain that $\frac{d\tilde{s}}{d\alpha} < 0 \Leftrightarrow \frac{\overline{G}_h(\tilde{s})}{\overline{G}_\ell(\tilde{s})} > \frac{\overline{G}_h(\hat{s})}{\overline{G}_\ell(\hat{s})}$. As $\frac{\overline{G}_h(\tilde{s})}{\overline{G}_\ell(\tilde{s})}$ is increasing, $\frac{d\tilde{s}}{d\alpha} < 0$ when $\tilde{s} > \hat{s}$. As for $\alpha = 1$, $\tilde{s} > \hat{s}$, it follows that $\tilde{s}$ is decreasing at $\alpha = 1$ and hence $\tilde{s} > \hat{s}$ for all $\alpha \in (0,1]$.

Notice that at $\tilde{s}$, the second derivative of $\frac{P^h(s)}{P^\ell(s)}$ is negative:

$$\begin{aligned}
\frac{d^2}{ds^2}\left(\frac{P^h(s)}{P^\ell(s)}\right)\Bigg|_{s=\tilde{s}} &= -\left(\frac{g_h(s)}{g_\ell(s)} - \frac{\alpha\overline{G}_h(\hat{s}) + (1-\alpha)\overline{G}_h(s)}{\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(s)}\right)\frac{d}{ds}\left((1-\alpha)g_\ell(s)\left(\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(s)\right)\right)\Bigg|_{s=\tilde{s}} \\
&\quad - \left((1-\alpha)g_\ell(s)\left(\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(s)\right)\right)\frac{d}{ds}\left(\frac{g_h(s)}{g_\ell(s)} - \frac{\alpha\overline{G}_h(\hat{s}) + (1-\alpha)\overline{G}_h(s)}{\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(s)}\right)\Bigg|_{s=\tilde{s}} \\
&= -\underbrace{\left(\frac{g_h(\tilde{s})}{g_\ell(\tilde{s})} - \frac{\alpha\overline{G}_h(\hat{s}) + (1-\alpha)\overline{G}_h(\tilde{s})}{\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(\tilde{s})}\right)}_{=0}\frac{d}{ds}\left((1-\alpha)g_\ell(s)\left(\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(s)\right)\right)\Bigg|_{s=\tilde{s}} \\
&\quad -(1-\alpha)g_\ell(\tilde{s})\left(\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(\tilde{s})\right)\left(l'(\tilde{s}) - \underbrace{\frac{d}{ds}\frac{\alpha\overline{G}_h(\hat{s}) + (1-\alpha)\overline{G}_h(s)}{\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(s)}\Bigg|_{s=\tilde{s}}}_{=0}\right) \\
&= -(1-\alpha)g_\ell(\tilde{s})\left(\alpha\overline{G}_\ell(\hat{s}) + (1-\alpha)\overline{G}_\ell(\tilde{s})\right)l'(\tilde{s}) < 0.
\end{aligned}$$

Hence, at $s = \tilde{s}$, $\frac{P^h(s)}{P^\ell(s)}$ achieves a maximum, which is determined by $\frac{d}{ds}\left(\frac{P^h(s)}{P^\ell(s)}\right)\Big|_{s=\tilde{s}} = 0 \Leftrightarrow \theta l(\tilde{s}) = \frac{P^h(\tilde{s})}{P^\ell(\tilde{s})}$. Assume that there are two solutions to $\theta l(\tilde{s}) = \frac{P^h(\tilde{s})}{P^\ell(\tilde{s})}$. Then, $\frac{P^h(s)}{P^\ell(s)}$ has two maxima. As $\frac{P^h(s)}{P^\ell(s)}$ is continuous in s, it must also have a minimum, also determined by $\theta l(\tilde{s}) = \frac{P^h(\tilde{s})}{P^\ell(\tilde{s})}$. This is a contraction with $\frac{d^2}{ds^2}\left(\frac{P^h(s)}{P^\ell(s)}\right)\Big|_{s=\tilde{s}} < 0$. As a result, there can only be one maximum of $\frac{P^h(s)}{P^\ell(s)}$ which is $\tilde{s}$ and it is unique.

Proof of Proposition 1: Part (i): Recall the the equilibrium condition is:

$$s_0^* : \theta l(s_0) = \frac{p^\ell(0, s_0, \Delta)}{p^h(0, s_0, \Delta)}.$$

There are three cases possible:

(case i) When: $p^\ell(0, s_0, \Delta) = p^h(0, s_0, \Delta) = 1$, or $\overline{\lambda}^\ell(s_0, \Delta) > \lambda$ (as then also $\overline{\lambda}^h(s_0, \Delta) > \lambda$), or $\Delta > \lambda P^\ell(s_0)$ and $\Delta > \lambda P^h(s_0)$, then: $p^\ell(0, s_0, \Delta)/p^h(0, s_0, \Delta) = 1$. Notice that the solution is $s_0 = \hat{s}$, hence, the condition for this case to be an equilibrium is: $\Delta > \lambda P^h(\hat{s})$.

(case ii) When: $p^\ell(0, s_0, \Delta) = 1$ and $p^h(0, s_0, \Delta) < 1$, or $\Delta > \lambda P^\ell(s_0)$ and $\Delta < \lambda P^h(s_0)$, then: $p^\ell(0, s_0, \Delta)/p^h(0, s_0, \Delta) = \lambda/\underline{\lambda}^h(s_0, \Delta) = \lambda P^h(s_0)/\Delta$. By definition, $s_A(\Delta, \lambda)$ is the solution of $\theta l(s_0) = \lambda P^h(s_0)/\Delta$, hence the condition for this case to be and equilibrium is: $\lambda P^h(s_A(\Delta, \lambda)) >$

$\Delta > \lambda P^{\ell}(s_A(\Delta, \lambda))$.

(case iii) When: $p^{\ell}(0, s_0, \Delta) < 1$ and $p^{h}(0, s_0, \Delta) < 1$, then: $p^{\ell}(0, s_0, \Delta)/p^{h}(0, s_0, \Delta) = P^{h}(s_0)/P^{\ell}(s_0)$. Notice that the solution is $s_0 = \tilde{s}$, hence, the condition for this case to be an equilibrium is: $\Delta > \lambda P^{\ell}(\tilde{s})$.

Part (ii): Obviously, in case (i), $\frac{d\hat{s}}{d\Delta} = 0$. In case (ii), it follows from deriving: $s_A(\Delta, \lambda)$ with respect to Δ:

$$\begin{aligned}
\frac{d}{d\Delta}[\theta l(s_A)] &= \frac{d}{d\Delta}[\lambda P^{h}(s_A)/\Delta] \\
\theta l'(s_A)\frac{ds_A}{d\Delta} &= \lambda\frac{\frac{dP^{h}(s_A)}{ds}\frac{ds_A}{d\Delta}\Delta - P^{h}(s_A)}{\Delta^2} \\
\frac{ds_A}{d\Delta} &= -\frac{\lambda\frac{P^{h}(s_A)}{\Delta^2}}{\theta l'(s_A) - \lambda\frac{\frac{dP^{h}(s_A)}{ds}}{\Delta}}
\end{aligned}$$

as $\frac{dP^{h}(s_A)}{ds} < 0$, it follows that $\frac{ds_A}{d\Delta} < 0$. Obviously, in case (iii), $\frac{d\tilde{s}}{d\Delta} = 0$.

Proof of Proposition 2: It is obvious that $\Delta \in [0, P^{\ell}(\tilde{s})\lambda)$ and $\Delta \in (P^{h}(\hat{s})\lambda, \lambda]$ can never be an equilibrium as in the first case, the firm always sells Δ (with a positive margin) and in the second case, the firm always sells $P^{h}(\hat{s})\lambda$, which does not depend on Δ.

For $\Delta \in [P^{\ell}(\tilde{s})\lambda, P^{h}(\hat{s}_0)\lambda]$, we have that: $\theta l(s_0^*) = \frac{P^{h}(s_0^*)\lambda}{\Delta}$ and hence, the expected satisfied sales are:

$$S(\Delta, s_0^*) = p_0\Delta + (1-p_0)P^{\ell}(s_0^*)\lambda$$

as $\Delta = \frac{P^{h}(s_0^*)\lambda}{\theta l(s_0^*)}$, we obtain that

$$\begin{aligned}
\Pi(\Delta, s_0^*) &= rS(\Delta, s_0^*) - c\Delta = \left(\left(p_0 - \frac{c}{r}\right)\frac{P^{h}(s_0^*)}{\theta l(s_0^*)} + (1-p_0)P^{\ell}(s_0^*)\right)\lambda r \\
&= \frac{\left(p_0 - \frac{c}{r}\right)P^{h}(s_0^*) + p_0 l(s_0^*)P^{\ell}(s_0^*)}{\theta l(s_0^*)}\lambda r
\end{aligned}$$

hence, with $\pi^{o}(s) \doteq \left(p_0 - \frac{c}{r}\right)\frac{P^{h}(s)}{\theta l(s)} + (1-p_0)P^{\ell}(s)$, the equilibrium profits are written as $\Pi(\Delta, s_0^*) = \pi(s_0^*)r\lambda$. As $\Delta \in [P^{\ell}(\tilde{s})\lambda, P^{h}(\hat{s}_0)\lambda]$, it follows that $s_0^* \in [\hat{s}, \tilde{s}]$. We can thus write the inventory optimization problem as

$$\max_{\hat{s} \le s_0 \le \tilde{s}} \pi^{o}(s_0).$$

Proof of Lemma 3: Has been argued in the text.

Proof of Proposition 3: For convenience of notation, we drop $\boldsymbol{I}$ from the arguments and use one argument for $\underline{\lambda}^{\omega}(s_0)$ (and two for $\overline{\lambda}^{\omega}(s_0, s_1)$).

Part (i) is proven as follows. There can only be six possible end-of-season inventory realizations, given by Table 2. We partition the (Q, Δ) space in six regions:

$$\begin{aligned}
\Omega_I &= \left\{\boldsymbol{I} : 0 < Q \le \tfrac{\lambda}{2} P^{\ell}(\tilde{s}), 0 < \Delta \le \lambda P^{\ell}(\tilde{s}) - 2Q\right\} \\
\Omega_{II} &= \left\{\boldsymbol{I} : 0 < Q \le \tfrac{\lambda}{2} P^{\ell}(\tilde{s}), \lambda P^{\ell}(\tilde{s}) - 2Q < \Delta \le \Delta_A(Q)\right\} \\
\Omega_{III} &= \left\{\boldsymbol{I} : 0 < Q \le \tfrac{\lambda}{2} P^{\ell}(\tilde{s}), \Delta_A(Q) < \Delta \le \lambda - 2Q\right\} \\
\Omega_{IV} &= \left\{\boldsymbol{I} : \tfrac{\lambda}{2} P^{\ell}(\tilde{s}) < Q \le \tfrac{\lambda}{2} P^{h}(\hat{s}), 0 < \Delta \le \Delta_B(Q)\right\} \\
\Omega_{V} &= \left\{\boldsymbol{I} : \tfrac{\lambda}{2} P^{\ell}(\tilde{s}) < Q \le \tfrac{\lambda}{2} P^{h}(\hat{s}), \Delta_B(Q) < \Delta \le \lambda - 2Q\right\} \\
\Omega_{VI} &= \left\{\boldsymbol{I} : \tfrac{\lambda}{2} P^{h}(\hat{s}) < Q \le \tfrac{\lambda}{2}, 0 < \Delta \le \lambda - 2Q\right\}
\end{aligned}$$

where $\Delta_A(Q) = \left(\lambda - \frac{Q}{\frac{1}{2}P^h(\tilde{s})}\right) P^h(s_B(Q, \lambda))$ and $\Delta_B(Q) = (\lambda - \frac{2Q}{P^h(s_B(Q,\lambda))}) P^h(\underline{s})$ are the values of Δ in regions Ω_{II} and Ω_{IV} respectively that make $s_A(\Delta, \lambda - \frac{Q}{\frac{1}{2}P^{\ell}(\tilde{s})}) = s_B(Q, \lambda)$ and $s_A(2Q, \lambda) = \underline{s}$ respectively for a given Q. We conjecture that in the above regions, the equilibrium thresholds will lead to the end-of-season inventory realization. We for each of the possible end-of-season inventory realizations, the following thresholds are equilibria:

$$\boldsymbol{s}^* = \begin{cases}
(\tilde{s}, \tilde{s}), & \boldsymbol{I} \in \Omega_I, \\
(\tilde{s}, s_A(\Delta, \lambda - Q/(\frac{1}{2}P^{\ell}(\tilde{s})))), & \boldsymbol{I} \in \Omega_{II}, \\
(\tilde{s}, s_B(Q, \lambda)), & \boldsymbol{I} \in \Omega_{III}, \\
(s_A(2Q, \lambda), \underline{s}), & \boldsymbol{I} \in \Omega_{IV}, \\
(s_A(2Q, \lambda), \underline{s}), & \boldsymbol{I} \in \Omega_{V}, \\
(\hat{s}, s_1) \text{ for any } s_1 \in [\underline{s}, \overline{s}], & \boldsymbol{I} \in \Omega_{VI}.
\end{cases}$$

It is trivial to verify that these equilibria satisfy Equation (13), each time assuming that the stock-out event of Table 2 in the particular region holds. For example, in region Ω_{II}, the conjectured end-of-season inventory is that the small retailer always stocks out while the large retailer stocks out only when the quality is high. With Equation (13), we obtain in that case:

$$p^{\omega}(0, \boldsymbol{s}, \boldsymbol{I}) = \frac{\underline{\lambda}^{\omega}(\boldsymbol{s}, \boldsymbol{I})}{\lambda} \text{ for } \omega \in \{\ell, h\} \text{ and}$$
$$p^{\ell}(1, \boldsymbol{s}, \boldsymbol{I}) = 1 - \frac{\underline{\lambda}^{\ell}(s_0, \boldsymbol{I})}{\lambda} \text{ and } p^{h}(0, \boldsymbol{s}, \boldsymbol{I}) = \frac{\overline{\lambda}^{h}(\boldsymbol{s}, \boldsymbol{I})}{\lambda} - \frac{\underline{\lambda}^{h}(s_0, \boldsymbol{I})}{\lambda}$$

$$\theta l(s_0^*) = \frac{p^{\ell}(0, \boldsymbol{s}^*, \boldsymbol{I})}{p^{h}(0, \boldsymbol{s}^*, \boldsymbol{I})} = \frac{\frac{\underline{\lambda}^{\ell}(\boldsymbol{s}^*, \boldsymbol{I})}{\lambda}}{\frac{\underline{\lambda}^{h}(\boldsymbol{s}^*, \boldsymbol{I})}{\lambda}} = \frac{P^{\ell}(s_0^*)}{P^{h}(s_0^*)}$$

$$\theta l\left(s_1^*\right) = \frac{p^\ell\left(1, \boldsymbol{s}^*, \boldsymbol{I}\right)}{p^h\left(1, \boldsymbol{s}^*, \boldsymbol{I}\right)} = \frac{1 - \frac{\underline{\lambda}^\ell\left(s_0^*, \boldsymbol{I}\right)}{\lambda}}{\frac{\overline{\lambda}^h\left(\boldsymbol{s}^*, \boldsymbol{I}\right)}{\lambda} - \frac{\underline{\lambda}^h\left(s_0^*, \boldsymbol{I}\right)}{\lambda}}$$

$$= \frac{\lambda - \underline{\lambda}^\ell\left(s_0^*, \boldsymbol{I}\right)}{\overline{\lambda}^h\left(\boldsymbol{s}^*, \boldsymbol{I}\right) - \underline{\lambda}^h\left(s_0^*, \boldsymbol{I}\right)} = \frac{P^h\left(s_1^*\right)}{\Delta}\left(\lambda - \frac{Q}{\frac{1}{2}P^h\left(s_0^*\right)}\right)$$

From the first equation, with Equation (11), $s_0^* = \tilde{s}$. Plugging $s_0^* = \tilde{s}$ in the second equation, with the definition of $s_A(\Delta, \lambda)$ in Proposition 1, we obtain $s_1^* = s_A(\Delta, \lambda - Q/(\frac{1}{2}P^\ell(\tilde{s})))$. The same can be done for all other regions.

Now, it remains to be proven that with the proposed equilibria, the end-of-season realizations are consistent with the regions in which they are. Before proving this, we first introduce inequalities

$$\underline{\lambda}^h(s_0) < \underline{\lambda}^\ell(s_0) \text{ and } \overline{\lambda}^h(s_0, s_1) < \overline{\lambda}^\ell(s_0, s_1) \text{ and} \tag{19}$$

$$\underline{\lambda}^\ell(s_0) < \overline{\lambda}^\ell(s_0, s_1) \text{ and } \underline{\lambda}^h(s_0) < \overline{\lambda}^h(s_0, s_1) \tag{20}$$

that follow immediately from the definitions (Equation (17)). With these equilibria events in Cases I-VI must satisfy the following four conditions (one condition per quality realization for each retailer):

	Small Retailer	Large Retailer
I	$(\underline{\lambda}^\ell(\tilde{s}) \leq \lambda,\ \underline{\lambda}^h(\tilde{s}) \leq \lambda)$	$(\overline{\lambda}^\ell(\tilde{s}, \tilde{s}) \leq \lambda,\ \overline{\lambda}^h(\tilde{s}, \tilde{s}) \leq \lambda)$
II	$(\underline{\lambda}^\ell(\tilde{s}) \leq \lambda,\ \underline{\lambda}^h(\tilde{s}) \leq \lambda)$	$(\overline{\lambda}^\ell(\tilde{s}, s_A(\Delta, \lambda - \frac{Q}{\frac{1}{2}P^\ell(\tilde{s})})) \geq \lambda,\ \overline{\lambda}^h(\tilde{s}, s_A(\Delta, \lambda - \frac{Q}{\frac{1}{2}P^\ell(\tilde{s})})) \leq \lambda)$
III	$(\underline{\lambda}^\ell(\tilde{s}) \leq \lambda,\ \underline{\lambda}^h(\tilde{s}) \leq \lambda)$	$(\overline{\lambda}^\ell(\tilde{s}, s_B(Q, \lambda)) \geq \lambda,\ \overline{\lambda}^h(\tilde{s}, s_B(Q, \lambda)) \geq \lambda)$
IV	$(\underline{\lambda}^\ell(s_A(2Q, \lambda)) \geq \lambda,\ \underline{\lambda}^h(s_A(2Q, \lambda)) \leq \lambda)$	$(\overline{\lambda}^\ell(s_A(2Q, \lambda), \underline{s}) \geq \lambda,\ \overline{\lambda}^h(s_A(2Q, \lambda), \underline{s}) \leq \lambda)$
V	$(\underline{\lambda}^\ell(s_A(2Q, \lambda)) \geq \lambda,\ \underline{\lambda}^h(s_A(2Q, \lambda)) \leq \lambda)$	$(\overline{\lambda}^\ell(s_A(2Q, \lambda), \underline{s}) \geq \lambda,\ \overline{\lambda}^h(s_A(2Q, \lambda), \underline{s}) \geq \lambda)$
VI	$(\underline{\lambda}^\ell(\hat{s}) \geq \lambda,\ \underline{\lambda}^h(\hat{s}) \geq \lambda)$	$(\overline{\lambda}^\ell(\hat{s}, \underline{s}) \geq \lambda,\ \overline{\lambda}^h(\hat{s}, \underline{s}) \geq \lambda)$

Table 4 Conditions for the Small and Large retailer, when the quality realization is (low, high), that lead to possible stock-out realization (I-VI).

We prove that the conditions in Table 4 are indeed satisfied with the conjectured equilibrium in Ω_j, where $j \in \{\text{I, II, III, IV, V, VI}\}$. First, we rewrite the description of the Ω-regions in terms of $\overline{\lambda}^\omega$ and $\underline{\lambda}^\omega$, $\omega \in \{h, \ell\}$. Then, we verify the four conditions on the end-of-season inventory realization:

1. **Case I:** By definition of Ω_I, $0 < \underline{\lambda}^\ell(\tilde{s}) \leq \lambda$ and $\overline{\lambda}^\ell(\tilde{s}, \tilde{s}) \leq \lambda$.

Small Retailer: As $\underline{\lambda}^h(\tilde{s}) < \underline{\lambda}^\ell(\tilde{s}) \leq \lambda$ (Equation (19)), the small retailer's conditions are satisfied.

Large Retailer: As $\overline{\lambda}^h(\tilde{s},\tilde{s}) < \overline{\lambda}^\ell(\tilde{s},\tilde{s}) \leq \lambda$ (Equation (19)), the large retailer's conditions for Case I is also satisfied.

2. **Case II:** By definition of Ω_{II}, $0 < \underline{\lambda}^\ell(\tilde{s}) \leq \lambda$ and $\lambda \leq \overline{\lambda}^\ell(\tilde{s},\tilde{s})$ and $\overline{\lambda}^h(\tilde{s}, s_A(\Delta, \lambda - Q/(\frac{1}{2}P^\ell(\tilde{s})))) \leq \lambda$.

Small Retailer: The small retailer's conditions are satisfied for the same reasons as in case I.

Large Retailer: As $\overline{\lambda}^h(\tilde{s}, s_A(\Delta, \lambda - Q/(\frac{1}{2}P^\ell(\tilde{s})))) \leq \lambda$, we only need to establish the large retailer's condition: $\lambda \leq \overline{\lambda}^\ell(\tilde{s}, s_A(\Delta, \lambda - \frac{Q}{\frac{1}{2}P^\ell(\tilde{s})}))$. As $\lambda \leq \overline{\lambda}^\ell(\tilde{s},\tilde{s})$, it follows that $\tilde{s} \geq s_A(\Delta, \lambda - \frac{Q}{\frac{1}{2}P^\ell(\tilde{s})})$ (in Part (ii) of this proof, we demonstrate that s_A is decreasing in Δ for a given Q, and when $\lambda = \overline{\lambda}^\ell(\tilde{s},\tilde{s})$, $s_A = \tilde{s}$). Also, it is easy to see $\overline{\lambda}^\ell(\tilde{s}, s_1)$ increases in s_1. Therefore: $\overline{\lambda}^\ell(\tilde{s},\tilde{s}) \leq \overline{\lambda}^\ell(\tilde{s}, s_A(\Delta, \lambda - \frac{Q}{\frac{1}{2}P^\ell(\tilde{s})}))$ and as $\lambda \leq \overline{\lambda}^\ell(\tilde{s},\tilde{s})$, the second condition for the large retailer is also satisfied.

3. **Case III:** By definition of Ω_{III}, $0 < \underline{\lambda}^\ell(\tilde{s}) \leq \lambda$ and $\lambda \leq \overline{\lambda}^h(\tilde{s}, s_B(Q,\lambda))$ and $\Delta + 2Q \leq \lambda$.

Small Retailer: The small retailer's conditions are satisfied for the same reasons as in case I.

Large Retailer: The large retailer's conditions are also satisfied as $\lambda \leq \overline{\lambda}^h(\tilde{s}, s_B(Q,\lambda)) < \overline{\lambda}^\ell(\tilde{s}, s_B(Q,\lambda))$ (Equation (19)).

4. **Case IV:** By definition of Ω_{IV}, $\lambda < \underline{\lambda}^\ell(\tilde{s})$ and $\underline{\lambda}^h(\hat{s}) \leq \lambda$ and $\overline{\lambda}^h(s_A(2Q,\lambda), \underline{s}) \leq \lambda$.

Small Retailer: As $\underline{\lambda}^h(\hat{s}) \leq \lambda$, it follows that $s_A(2Q,\lambda) < \tilde{s}$ (in Part (ii) of this proof, we demonstrate that s_A is decreasing in Q for a given Δ, and when $\underline{\lambda}^h(\hat{s}) = \lambda$, $s_A = \hat{s}$, see also discussion for Proposition 1), from which it follows that $\frac{P^h(s_A(2Q,\lambda))}{P^\ell(s_A(2Q,\lambda))} > \theta l\,(s_A(2Q,\lambda))$. Note that $2Q = P^h\,(s_A(2Q,\lambda))\,/\theta l\,(s_A(2Q,\lambda))$, then, we rewrite $\underline{\lambda}^\ell(s_A(2Q,\lambda)) = \frac{2Q}{P^\ell(s_A(2Q,\lambda))}$ as

$$\underline{\lambda}^\ell(s_A(2Q,\lambda)) = \frac{P^h\,(s_A(2Q,\lambda))}{P^\ell(s_A(2Q,\lambda))}\frac{1}{\theta l\,(s_A(2Q,\lambda))}\lambda > \lambda$$

which is the first condition for the small retailer. As $\underline{\lambda}^h(\hat{s}) \leq \lambda$, it follows that $\hat{s} < s_A(2Q,\lambda)$ (see discussion for Proposition 1) and $\theta l\,(s_A(2Q,\lambda)) > 1$. Now, we rewrite $\underline{\lambda}^h(s_A(2Q,\lambda)) = \frac{2Q}{P^h(s_A(2Q,\lambda))}$ as:

$$\underline{\lambda}^h(s_A(2Q,\lambda)) = \frac{P^h\,(s_A(2Q,\lambda))}{P^h(s_A(2Q,\lambda))}\frac{1}{\theta l\,(s_A(2Q,\lambda))}\lambda = \frac{1}{\theta l\,(s_A(2Q,\lambda))}\lambda < \lambda,$$

which is the second condition for the small retailer.

Large Retailer: As $\overline{\lambda}^h(s_A(2Q,\lambda),\underline{s}) \leq \lambda$ and $\overline{\lambda}^\ell(s_A(2Q,\lambda),\underline{s}) > \underline{\lambda}^\ell(s_A(2Q,\lambda))$ (Equation (20)) and $\underline{\lambda}^\ell(s_A(2Q,\lambda)) \geq \underline{\lambda}^\ell(\tilde{s})$, the large retailer conditions are also satisfied.

5. **Case V:** By definition of Ω_V, $\lambda < \underline{\lambda}^\ell(\tilde{s})$ and $\underline{\lambda}^h(\hat{s}) \leq \lambda$ and $\lambda \leq \overline{\lambda}^h(s_A(2Q,\lambda),\underline{s})$ and $\Delta + 2Q \leq \lambda$.

Small Retailer: The small retailer's conditions are satisfied for the same reasons as in case IV.

Large Retailer: As $\lambda \leq \overline{\lambda}^h(s_A(2Q,\lambda),\underline{s}) \leq \overline{\lambda}^\ell(s_A(2Q,\lambda),\underline{s})$ (Equation (19)), the large retailer conditions are also satisfied.

6. **Case VI:** By definition of Ω_{VI}, $2Q \leq \lambda$ and $\lambda \leq \underline{\lambda}^h(\hat{s})$ and $\Delta + 2Q \leq \lambda$.

Small Retailer: As $\lambda \leq \underline{\lambda}^h(\hat{s}) < \underline{\lambda}^\ell(\hat{s})$ (Equation (19)), the small retailer's conditions are satisfied.

Large Retailer: As $\overline{\lambda}^h(\hat{s},\underline{s}) > \underline{\lambda}^h(\hat{s})$ (Equation 20) and $\overline{\lambda}^\ell(\hat{s},\underline{s}) \geq \overline{\lambda}^h(\hat{s},\underline{s})$ (Equation 19), the large retailer conditions are also satisfied.

Hence, we have proven that the equilibrium purchasing thresholds are determined by Table 3.

Part (ii) is proven based as follows: For $\boldsymbol{I} \in \Omega_I \cup \Omega_{IV} \cup \Omega_V \cup \Omega_{VI}$, the statement that $s_1^* \leq s_0^*$ when both s_1^* and s_0^* exist is trivial. Remains to be proven:

$$\begin{cases} \boldsymbol{I} \in \Omega_{II}: & \tilde{s} \geq s_A(\Delta, \lambda - Q/(\frac{1}{2}P^\ell(\tilde{s}))) \\ \boldsymbol{I} \in \Omega_{III}: & \tilde{s} \geq s_B(Q,\lambda) \end{cases}$$

For $\boldsymbol{I} \in \Omega_{II}$ notice that $s_A(\Delta,\lambda)$ with Proposition 1 $s_A(\Delta,\lambda) \leq \tilde{s}$ and in **Part (iii)** below, it will be proven that $s_A(\Delta, \lambda - Q/(\frac{1}{2}P^\ell(\tilde{s})))$ decreases in Q, hence, for any $\boldsymbol{I} \in \Omega_{II}$, $\tilde{s} \geq s_A(\Delta, \lambda - Q/(\frac{1}{2}P^\ell(\tilde{s})))$.

For $\boldsymbol{I} \in \Omega_{III}$, notice that $s_B(0,\lambda) = \hat{s}$, which is less than $\tilde{s}$ (Lemma 2) and in **Part (iii)** below, it will be proven that $s_B(Q,\lambda)$ decreases in Q, hence, for any $\boldsymbol{I}$ in Ω_{III}, $\tilde{s} \geq s_B(Q,\lambda)$.

Part (iii) is proven by implicitly deriving the defining equilibrium equations with respect to Q and Δ:

- $s_A(\Delta, \lambda - Q/(\frac{1}{2}P^\ell(\tilde{s})))$: Deriving $\theta l\left(s^*\right) = \frac{P^h\left(s^*\right)}{\Delta}(\lambda - \frac{Q}{\frac{1}{2}P^\ell(\tilde{s})})$ with respect to Δ, we obtain:

$$\theta l'\left(s^*\right) \frac{\partial s^*}{\partial \Delta} = \frac{\Delta \frac{dP^h\left(s^*\right)}{ds} \frac{\partial s^*}{\partial \Delta} - P^h\left(s^*\right)}{\Delta^2} \left(\lambda - \frac{Q}{\frac{1}{2}P^\ell\left(\tilde{s}\right)}\right)$$

$$
\begin{aligned}
\frac{\partial s^*}{\partial \Delta} &= -\frac{\frac{P^h\left(s^*\right)}{\Delta^2}\left(\lambda - \frac{Q}{\frac{1}{2}P^\ell(\tilde{s})}\right)}{\theta l'\left(s^*\right) - \frac{\frac{dP^h(s^*)}{ds}}{\Delta}\left(\lambda - \frac{Q}{\frac{1}{2}P^\ell(\tilde{s})}\right)} \text{ (isolating } \frac{\partial s^*}{\partial \Delta}\text{)} \\
\frac{\partial s^*}{\partial \Delta} &= -\frac{\frac{1}{\Delta}\theta l\left(s^*\right)}{\theta l'\left(s^*\right) - \frac{\frac{dP^h(s^*)}{ds}}{P^h(s^*)}\theta l\left(s^*\right)} \text{ (by definition of } s^*\text{)} \\
\frac{\partial s^*}{\partial \Delta} &= -\frac{1}{\Delta}\frac{1}{\frac{l'(s^*)}{l(s^*)} - \frac{\frac{dP^h(s^*)}{ds}}{P^h(s^*)}} \text{ (as } \frac{dP^h\left(s^*\right)}{ds} < 0\text{)} \\
&< 0
\end{aligned}
$$

and with respect to Q, we obtain:

$$
\begin{aligned}
\theta l\left(s^*\right)\frac{\partial s^*}{\partial Q} &= \frac{\frac{dP^h\left(s^*\right)}{ds}\frac{\partial s^*}{\partial Q}}{\Delta}\left(\lambda - \frac{Q}{\frac{1}{2}P^\ell\left(\tilde{s}\right)}\right) - \frac{1}{\Delta}\frac{P^h\left(s^*\right)}{\frac{1}{2}P^\ell\left(\tilde{s}\right)} \\
\frac{\partial s^*}{\partial Q} &= -\frac{\frac{1}{\Delta}\frac{P^h\left(s^*\right)}{\frac{1}{2}P^\ell(\tilde{s})}}{\theta l\left(s^*\right) - \frac{\frac{dP^h(s^*)}{ds}}{\Delta}\left(\lambda - \frac{Q}{\frac{1}{2}P^\ell(\tilde{s})}\right)} \text{ (isolating } \frac{\partial s^*}{\partial Q}\text{)} \\
\frac{\partial s^*}{\partial Q} &= -\frac{\frac{1}{\Delta}\frac{P^h\left(s^*\right)}{\frac{1}{2}P^\ell(\tilde{s})}}{\theta l\left(s^*\right) - \frac{\frac{dP^h(s^*)}{ds}}{P^h(s^*)}\theta l\left(s^*\right)} \text{ (by definition of } s^*\text{)} \\
\frac{\partial s^*}{\partial Q} &= -\frac{1}{\theta\Delta}\frac{\frac{P^h\left(s^*\right)}{\frac{1}{2}P^\ell(\tilde{s})l(s^*)}}{1 - \frac{\frac{dP^h(s^*)}{ds}}{P^h(s^*)}} \text{ (as } \frac{dP^h\left(s^*\right)}{ds} < 0\text{)} \\
&< 0
\end{aligned}
$$

Hence, for a given Q, $s_A(\Delta, \lambda - Q/(\frac{1}{2}P^\ell(\tilde{s})))$ decreases from $\tilde{s}$ (when $\lambda P^\ell\left(\tilde{s}\right) - \Delta = 2Q$) to $s_B(Q,\lambda)$ as Δ increases to $\Delta_A(Q)$. For a given Δ, $s_A(\Delta, \lambda - Q/(\frac{1}{2}P^\ell(\tilde{s})))$ decreases from $s_{A-ii}(0,\Delta)$ to $\underline{s}$ as Q increases from 0 to $\frac{\lambda}{2}P^h\left(\hat{s}\right)$.

- $s_B(Q,\lambda)$: Deriving $\theta l\left(s^*\right) = \frac{\lambda - \frac{2Q}{P^\ell(\tilde{s})}}{\lambda - \frac{2Q}{P^h(\tilde{s})}}$ with respect to Q, we obtain:

$$
\begin{aligned}
\theta l'\left(s^*\right)\frac{\partial s^*}{\partial Q} &= \frac{-\frac{2}{P^\ell(\tilde{s})}\left(\lambda - \frac{2Q}{P^h(\tilde{s})}\right) + \left(\lambda - \frac{2Q}{P^\ell(\tilde{s})}\right)\frac{2}{P^h(\tilde{s})}}{\left(\lambda - \frac{2Q}{P^h(\tilde{s})}\right)^2} \\
\frac{\partial s^*}{\partial Q} &= 2\frac{\frac{1}{P^h(\tilde{s})} - \frac{1}{P^\ell(\tilde{s})}}{\left(\lambda - \frac{2Q}{P^\ell(\tilde{s})}\right)\left(\lambda - \frac{2Q}{P^h(\tilde{s})}\right)}\frac{l\left(s^*\right)}{l'\left(s^*\right)}\lambda \text{ (isolating } \frac{\partial s^*}{\partial Q}\text{)} \\
&< 0 \text{ (as } l'\left(s^*\right) > 0 \text{ and } \lambda > \frac{2Q}{P^\ell\left(\tilde{s}\right)} > \frac{2Q}{P^h\left(\tilde{s}\right)}\text{)} \\
\frac{\partial s^*}{\partial \Delta} &= 0
\end{aligned}
$$

Hence, for a given Q, $s_B(Q,\lambda)$ decreases from $\hat{s}$ to $\underline{s}$ as Q increases from 0 to $\frac{\lambda}{2}P^{\ell}(\tilde{s})$.

- $s_A(2Q,\lambda)$: Deriving $\theta l(s^*) = \frac{\frac{1}{2}P^h(s^*)\lambda}{Q}$ with respect to Q, we obtain:

$$\begin{aligned}
\theta l'(s^*)\frac{\partial s^*}{\partial Q} &= \frac{\frac{dP^h(s^*)}{ds}Q\frac{\partial s^*}{\partial Q} - P^h(s^*)}{Q^2}\frac{1}{2}\lambda \\
\frac{\partial s^*}{\partial Q} &= -\frac{\frac{P^h(s^*)}{Q^2}\frac{1}{2}\lambda}{\theta l'(s^*) - \frac{\frac{dP^h(s^*)}{ds}}{Q}\frac{1}{2}\lambda} \text{ (isolating } \frac{\partial s^*}{\partial Q}) \\
\frac{\partial s^*}{\partial Q} &= -\frac{\theta l(s^*)}{Q\theta l'(s^*) - \frac{1}{2}\frac{dP^h(s^*)}{ds}\lambda} \text{ (by definition of } s^*) \\
&< 0 \text{ (as } \frac{dP^h(s^*)}{ds} < 0) \\
\frac{\partial s^*}{\partial \Delta} &= 0
\end{aligned}$$

Hence, for a given Q, $s_A(2Q,\lambda)$ decreases from $\tilde{s}$ to $\hat{s}$ as Q increases from $\frac{\lambda}{2}P^{\ell}(\tilde{s})$ to $\frac{\lambda}{2}P^h(\hat{s})$.

Proof of Proposition 4: In Proposition 3, we obtained an equilibrium, $\boldsymbol{s}^*(\boldsymbol{I})$, for each $\boldsymbol{I} \in \Omega \doteq \Omega_I \cup \Omega_{II} \cup \Omega_{III} \cup \Omega_V \cup \Omega_{IV} \cup \Omega_{VI}$. In this Proposition, we characterize the solution of

$$\max_{\boldsymbol{I}\in\Omega} rS(\boldsymbol{s}^*(\boldsymbol{I}),\boldsymbol{I}) - c(2Q+\Delta).$$

We split this optimization problem over Ω into three parts: First, we analyze the optimization over $\Omega_I \cup \Omega_{III} \cup \Omega_V \cup \Omega_{VI}$, then, over Ω_{II} and finally over Ω_{IV}. We show that for none of these regions, an optimal allocation, $\boldsymbol{I}$, can be inside the region. We identify which boundaries contain the optimal allocation and rewrite the optimization problem over these boundaries.

Solution of Ω_I, Ω_{III}, Ω_V and Ω_{VI}: The expected satisfied demand is determined in a similar fashion as when the prior was weak, only, when the low inventory retailer stocks out, the 'consumer conversion rate' is not equal to 1. As when $\lambda\overline{G}_{\ell}(\hat{s}_0) > 2Q$, the low inventory retailer always stocks out, the sales are $2Q$ for sure. Now, we can write the demand as follows:

$$S(\boldsymbol{s}^*(\boldsymbol{I}),\boldsymbol{I}) = \begin{cases} (Q,\Delta)\in\Omega_I : & 2Q+\Delta \\ (Q,\Delta)\in\Omega_{III} : & 2Q + p_0P^h(s_1^*)\left(\lambda - \frac{Q}{\frac{1}{2}P^h(\tilde{s})}\right) + (1-p_0)P^{\ell}(s_1^*)\left(\lambda - \frac{Q}{\frac{1}{2}P^{\ell}(\tilde{s})}\right) \\ (Q,\Delta)\in\Omega_V : & p_0\left(2Q+\lambda - \frac{Q}{\frac{1}{2}P^h(s_0^*)\lambda}\right) + (1-p_0)P^{\ell}(s_0^*))\lambda \\ (Q,\Delta)\in\Omega_{VI} : & (p_0P^h(s_0^*) + (1-p_0)P^{\ell}(s_0^*))\lambda \end{cases}$$

The profits are $rS(Q,\Delta) - c(2Q+\Delta)$. It is obvious that for any $r > c > 0$, there will never be an optimal (Q,Δ) in $\Omega_I \cup \Omega_{III} \cup \Omega_V \cup \Omega_{VI}$. In $\Omega_{III} \cup \Omega_V \cup \Omega_{VI}$, the total inventory investment can be reduced without losing sales. In Ω_I (as long as $r > c$), increasing the total inventory investment increases profits.

Solution over Ω_{II}: For $(Q,\Delta) \in \Omega_{II}$, we have that: $\theta l(s_1^*) = \frac{P^h(s_1^*)}{\Delta}\left(\lambda - \frac{Q}{\frac{1}{2}P^\ell(\tilde{s})}\right)$ and hence:

$$S(s_1,Q,\Delta) = 2Q + p_0\Delta + (1-p_0)P^\ell(s_1)\left(\lambda - \frac{Q}{\frac{1}{2}P^\ell(\tilde{s})}\right)$$

we obtained the following optimization problem:

$$\begin{aligned}\Pi_{II} = & \max_{\Delta\ge 0, Q\ge 0, \underline{s}\le s_1\le \bar{s}} rS(s_1,Q,\Delta) - c(2Q+\Delta) \\ & s.t. \begin{cases} \theta l(s_1) = \frac{P^h(s_1)}{\Delta}\left(\lambda - \frac{2Q}{P^\ell(\tilde{s})}\right), \\ \lambda P^\ell(\tilde{s}) - 2Q \le \Delta \le \left(\lambda - \frac{2Q}{P^h(\tilde{s})}\right)P^h(s_1), \\ 0 \le Q \le \lambda P^\ell(\tilde{s})/2. \end{cases}\end{aligned}$$

The latter can be rewritten as:

$$\begin{aligned}\frac{\Pi_{II}}{\lambda r} = & \max_{\underline{s}\le s_1\le\tilde{s}} \max_{\Delta\ge 0, Q\ge 0} \left\{\left(1-\frac{c}{r}\right)P^\ell(\tilde{s}) + \left(p_0\left(1+\frac{P^\ell(s_1)}{P^h(s_1)}l(s_1)\right) - \frac{c}{r} - \left(1-\frac{c}{r}\right)P^\ell(\tilde{s})\frac{\theta l(s_1)}{P^h(s_1)}\right)\frac{\Delta}{\lambda}\right\} \\ & s.t. \begin{cases} \frac{2Q}{\lambda} = \left(1 - \frac{\theta l(s_1)}{P^h(s_1)}\frac{\Delta}{\lambda}\right)P^\ell(\tilde{s}), \\ P^\ell(\tilde{s}) - \frac{2Q}{\lambda} \le \frac{\Delta}{\lambda} \le \left(1 - \frac{2Q}{\lambda P^h(\tilde{s})}\right)P^h(s_1), \\ 0 \le \frac{2Q}{\lambda} \le P^\ell(\tilde{s}). \end{cases}\end{aligned}$$

Note that when we substitute $\frac{2Q}{\lambda}$ from the first constraint into the second constraint, we obtain that:

$$P^\ell(\tilde{s}) - \left(1 - \frac{\theta l(s_1)}{P^h(s_1)}\frac{\Delta}{\lambda}\right)P^\ell(\tilde{s}) \le \frac{\Delta}{\lambda} \le \left(1 - \left(1 - \frac{\theta l(s_1)}{P^h(s_1)}\frac{\Delta}{\lambda}\right)\frac{P^\ell(\tilde{s})}{P^h(\tilde{s})}\right)P^h(s_1) \Leftrightarrow$$

$$0 \le \left(1 - \frac{P^\ell(\tilde{s})}{P^h(s_1)}\theta l(s_1)\right)\frac{\Delta}{\lambda} \text{ and } \left(\frac{P^h(\tilde{s})}{P^h(s_1)} - \theta l(s_1)\frac{P^\ell(\tilde{s})}{P^h(s_1)}\right)\frac{\Delta}{\lambda} \le P^h(\tilde{s}) - P^\ell(\tilde{s}).$$

As $\Delta \ge 0$, we must have that $1 - P^\ell(\tilde{s})\frac{\theta l(s_1)}{P^h(s_1)} \ge 0 \Leftrightarrow \frac{P^h(s_1)}{P^\ell(\tilde{s})} \ge \theta l(s_1)$, or: $s_1 \le \tilde{s}$. Otherwise, the feasible region is empty. We can thus restrain the domain $\underline{s} \le s_1 \le \bar{s}$ to: $\underline{s} \le s_1 \le \tilde{s}$ and rewrite:

$$\frac{\Pi_{II}}{\lambda r} = \max_{\underline{s}\le s_1\le\tilde{s}} \pi_{II}(s_1),$$

where:

$$\pi_{II}(s_1) = \left(1-\frac{c}{r}\right)P^\ell(\tilde{s}) + \max_{\Delta\ge 0, Q\ge 0}\left(p_0\left(1+\frac{P^\ell(s_1)}{P^h(s_1)}l(s_1)\right) - \frac{c}{r} - \left(1-\frac{c}{r}\right)\frac{P^\ell(\tilde{s})}{P^h(s_1)}\theta l(s_1)\right)\frac{\Delta}{\lambda}$$

$$s.t.\begin{cases} \frac{2Q}{\lambda} = P^{\ell}(\tilde{s})\left(1-\frac{\Delta}{\lambda}\frac{\theta l(s_1)}{P^h(s_1)}\right), \\ \left(\frac{P^h(\tilde{s})}{P^h(s_1)} - \theta l(s_1)\frac{P^{\ell}(\tilde{s})}{P^h(s_1)}\right)\frac{\Delta}{\lambda} \leq P^h(\tilde{s}) - P^{\ell}(\tilde{s}), \\ 0 \leq \frac{2Q}{\lambda} \leq P^{\ell}(\tilde{s}). \end{cases}$$

Now, we consider two cases:

1. The case that the coefficient of Δ is positive: $p_0\left(1+\frac{P^{\ell}(s_1)}{P^h(s_1)}l(s_1)\right) > \frac{c}{r} + \left(1-\frac{c}{r}\right)\frac{P^{\ell}(\tilde{s})}{P^h(s_1)}\theta l(s_1)$ (and $s_1 \leq \tilde{s}$), then, Δ must be as high as possible. Note that $P^h(\tilde{s}) - P^{\ell}(\tilde{s}) \geq 0$ (more sales are made when the quality is high for a given threshold) and, by definition of $\tilde{s}$ (see Equation (11)):

$$\frac{P^h(\tilde{s})}{P^h(s_1)} - \theta l(s_1)\frac{P^{\ell}(\tilde{s})}{P^h(s_1)} > 0 \Leftrightarrow \frac{P^h(\tilde{s})}{P^{\ell}(\tilde{s})} > \theta l(s_1) \Leftrightarrow s_1 \leq \tilde{s}$$

as $s_1 \leq \tilde{s}$, we can rewrite the second constraint in $\pi_{II}(s_1)$. Furthermore, as Q does not appear in the objective function, we can drop $Q \geq 0$ from the optimization problem by imposing that $1 \geq \frac{\Delta}{\lambda}\frac{\theta l(s_1)}{P^h(s_1)}$. We obtain

$$\pi_{II}(s_1) = \left(1-\frac{c}{r}\right)P^{\ell}(\tilde{s}) + \max_{\Delta \geq 0}\left(p_0\left(1+\frac{P^{\ell}(s_1)}{P^h(s_1)}l(s_1)\right) - \frac{c}{r} - \left(1-\frac{c}{r}\right)\frac{P^{\ell}(\tilde{s})}{P^h(s_1)}\theta l(s_1)\right)\frac{\Delta}{\lambda}$$

$$s.t.\begin{cases} \frac{\Delta}{\lambda} \leq \frac{P^h(\tilde{s}) - P^{\ell}(\tilde{s})}{\frac{P^h(\tilde{s})}{P^h(s_1)} - \theta l(s_1)\frac{P^{\ell}(\tilde{s})}{P^h(s_1)}}, \\ \frac{\Delta}{\lambda} \leq \frac{P^h(s_1)}{\theta l(s_1)}. \end{cases}$$

Which one of the two constraints on $\frac{\Delta}{\lambda}$ will be the tightest one depends on s_1:

$$\frac{P^h(s_1)}{\theta l(s_1)} > \frac{P^h(\tilde{s}) - P^{\ell}(\tilde{s})}{\frac{P^h(\tilde{s})}{P^h(s_1)} - \theta l(s_1)\frac{P^{\ell}(\tilde{s})}{P^h(s_1)}} \Leftrightarrow \frac{1}{\theta l(s_1)} > \frac{P^h(\tilde{s}) - P^{\ell}(\tilde{s})}{\underbrace{P^h(\tilde{s}) - \theta l(s_1)P^{\ell}(\tilde{s})}_{>0 \Leftrightarrow s_1 < \tilde{s}}} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{\theta l(s_1)}P^h(\tilde{s}) - P^{\ell}(\tilde{s}) > P^h(\tilde{s}) - P^{\ell}(\tilde{s}) \Leftrightarrow 1 > \theta l(s_1) \Leftrightarrow s_1 \leq \hat{s},$$

(where we used the definition of $\hat{s}$, Equation (6) and $\tilde{s}$, Equation (11)) hence, the largest Δ is:

$$\begin{cases} \frac{\Delta}{\lambda} = \frac{P^h(\tilde{s}) - P^{\ell}(\tilde{s})}{\frac{P^h(\tilde{s})}{P^h(s_1)} - \theta l(s_1)\frac{P^{\ell}(\tilde{s})}{P^h(s_1)}}, & \underline{s} \leq s_1 \leq \hat{s}, \\ \frac{\Delta}{\lambda} = \frac{P^h(s_1)}{\theta l(s_1)}, & \hat{s} \leq s_1 \leq \tilde{s}. \end{cases}$$

Plugging these values of Δ into the objective function, we obtain:

$$\pi_{II}(s_1) = \left(1-\frac{c}{r}\right)P^{\ell}(\tilde{s}) + \begin{cases} \left(p_0\left(1+\frac{P^{\ell}(s_1)}{P^h(s_1)}l(s_1)\right) - \frac{c}{r} - \left(1-\frac{c}{r}\right)\frac{P^{\ell}(\tilde{s})}{P^h(s_1)}\theta l(s_1)\right)\frac{P^h(\tilde{s}) - P^{\ell}(\tilde{s})}{\frac{P^h(\tilde{s})}{P^h(s_1)} - \theta l(s_1)\frac{P^{\ell}(\tilde{s})}{P^h(s_1)}}, & \underline{s} \leq s_1 \leq \hat{s}, \\ \left(p_0\left(1+\frac{P^{\ell}(s_1)}{P^h(s_1)}l(s_1)\right) - \frac{c}{r} - \left(1-\frac{c}{r}\right)\frac{P^{\ell}(\tilde{s})}{P^h(s_1)}\theta l(s_1)\right)\frac{P^h(s_1)}{\theta l(s_1)}, & \hat{s} \leq s_1 \leq \tilde{s}, \end{cases}$$

or with $\pi^o(s) = \frac{p_0 P^\ell(s) l(s) + \left(p_0 - \frac{c}{r}\right) P^h(s)}{\theta l(s)}$, the latter expression can be rewritten as:

$$\pi_{II}(s_1) = \begin{cases} \left(1-\frac{c}{r}\right) P^\ell(\tilde{s}) + \left(\theta l(s_1)\pi^o(s_1) - \left(1-\frac{c}{r}\right) P^\ell(\tilde{s})\theta l(s_1)\right) \frac{\frac{1}{P^\ell(\tilde{s})} - \frac{1}{P^h(\tilde{s})}}{\frac{1}{P^\ell(\tilde{s})} - \frac{\theta l(s_1)}{P^h(\tilde{s})}}, & \underline{s} \le s_1 \le \hat{s}, \\ \pi^o(s_1), & \hat{s} \le s_1 \le \tilde{s}, \end{cases}$$
$$= \begin{cases} \theta l(s_1)\pi^o(s_1) + \left(1 - \frac{c}{r} - \frac{\theta l(s_1)\pi^o(s_1)}{P^h(\tilde{s})}\right) \frac{1-\theta l(s_1)}{\frac{1}{P^\ell(\tilde{s})} - \frac{\theta l(s_1)}{P^h(\tilde{s})}}, & \underline{s} \le s_1 \le \hat{s}, \\ \pi^o(s_1), & \hat{s} \le s_1 \le \tilde{s}. \end{cases}$$

Note that

$$\pi^o(\tilde{s}) = \frac{p_0 P^\ell(\tilde{s}) l(\tilde{s}) + \left(p_0 - \frac{c}{r}\right) P^h(\tilde{s})}{\theta l(\tilde{s})} = (1-p_0) P^\ell(\tilde{s}) + \left(p_0 - \frac{c}{r}\right) P^\ell(\tilde{s}) = \left(1 - \frac{c}{r}\right) P^\ell(\tilde{s}).$$

2. The case that the coefficient of Δ is negative: $p_0\left(1 + \frac{P^\ell(s_1)}{P^h(s_1)} l(s_1)\right) < \frac{c}{r} + \left(1-\frac{c}{r}\right)\frac{P^\ell(\tilde{s})}{P^h(s_1)}\theta l(s_1)$ (and $s_1 \le \tilde{s}$), then, Δ must be as low as possible: We can set $\Delta = 0$

$$\pi_{II}(s_1) = \left(1 - \frac{c}{r}\right) P^\ell(\tilde{s})$$
$$s.t. \begin{cases} \frac{2Q}{\lambda} = P^\ell(\tilde{s}), \\ 0 \le P^h(\tilde{s}) - P^\ell(\tilde{s}), \\ 0 \le \frac{2Q}{\lambda} \le P^\ell(\tilde{s}). \end{cases}$$

which is equal to $\pi^o(\tilde{s})$.

3. **Summary:** We obtained that:

$$\pi_{II}(s)$$
$$= \max \begin{cases} \pi^o(s)\theta l(s) + \left(1 - \frac{c}{r} - \frac{\pi^o(s)\theta l(s)}{P^h(\tilde{s})}\right) \frac{1-\theta l(s)}{\frac{1}{P^\ell(\tilde{s})} - \frac{\theta l(s)}{P^h(\tilde{s})}}, & \underline{s} \le s \le \hat{s},\ p_0\left(1 + \frac{P^\ell(s)}{P^h(s)} l(s)\right) > \frac{c}{r} + \left(1-\frac{c}{r}\right)\frac{P^\ell(\tilde{s})}{P^h(s_1)}\theta l(s), \\ \pi^o(s), & \hat{s} \le s \le \tilde{s},\ p_0\left(1 + \frac{P^\ell(s)}{P^h(s)} l(s)\right) > \frac{c}{r} + \left(1-\frac{c}{r}\right)\frac{P^\ell(\tilde{s})}{P^h(s_1)}\theta l(s), \\ \pi^o(\tilde{s}), & \underline{s} \le s \le \tilde{s},\ p_0\left(1 + \frac{P^\ell(s)}{P^h(s)} l(s)\right) < \frac{c}{r} + \left(1-\frac{c}{r}\right)\frac{P^\ell(\tilde{s})}{P^h(s)}\theta l(s). \end{cases}$$

It is impossible that when $p_0\left(1 + \frac{P^\ell(s)}{P^h(s)} l(s)\right) < \frac{c}{r} + \left(1-\frac{c}{r}\right)\frac{P^\ell(\tilde{s})}{P^h(s_1)}\theta l(s)$, the maximum of the first two cases is higher than $\pi^o(\tilde{s})$. Hence, $\pi_{II}(s)$ reduces to:

$$\pi_{II}(s) = \max \begin{cases} \pi^o(s)\theta l(s) + \left(1 - \frac{c}{r} - \frac{\pi^o(s)\theta l(s)}{P^h(\tilde{s})}\right) \frac{1-\theta l(s)}{\frac{1}{P^\ell(\tilde{s})} - \frac{\theta l(s)}{P^h(\tilde{s})}}, & \underline{s} \le s \le \hat{s}, \\ \pi^o(s), & \hat{s} \le s \le \tilde{s}. \end{cases}$$

Solution over Ω_{IV}: For $(Q,\Delta) \in \Omega_{IV}$, we have that: $\theta l(s_0^*) = \frac{\frac{1}{2} P^h(s_0^*)\lambda}{Q}$ and hence:

$$S(s_1, Q, \Delta) = p_0(2Q + \Delta) + (1-p_0) P^\ell(s_0)\lambda$$

we obtained the following optimization problem:

$$\Pi_{IV} = \max_{\Delta\geq 0, Q\geq 0, \underline{s}\leq s_0\leq \bar{s}} rS(s_1, Q, \Delta) - c(2Q+\Delta)$$
$$s.t. \begin{cases} \theta l\left(s_0\right) = \frac{P^h(s_0)\lambda}{2Q}, \\ 0 \leq \Delta \leq \left(\lambda - \frac{2Q}{P^h(s_0)}\right) P^h\left(\underline{s}\right), \\ \lambda P^\ell\left(\tilde{s}\right)/2 \leq Q \leq \lambda P^h\left(\hat{s}\right)/2. \end{cases}$$

This problem can be rewritten as

$$\frac{\Pi_{IV}}{\lambda r} = \max_{\Delta\geq 0, Q\geq 0, \underline{s}\leq s_0\leq \bar{s}} \left(p_0 - \frac{c}{r}\right)\frac{P^h\left(s_0\right)}{\theta l\left(s_0\right)} + \left(1-p_0\right)P^\ell\left(s_0\right) + \left(p_0 - \frac{c}{r}\right)\frac{\Delta}{\lambda}$$
$$s.t. \begin{cases} \frac{2Q}{\lambda} = \frac{P^h(s_0)}{\theta l(s_0)}, \\ 0 \leq \frac{\Delta}{\lambda} \leq \left(1 - \frac{1}{\theta l(s_0)}\right) P^h\left(\underline{s}\right), \\ \lambda P^\ell\left(\tilde{s}\right)/2 \leq Q \leq \lambda P^h\left(\hat{s}\right)/2. \end{cases}$$

Note that, by definition of $\hat{s}$, Equation (6) and $\tilde{s}$, Equation (11), we must restraint the domain of s_0:

$$P^\ell\left(\tilde{s}\right) \leq \frac{P^h\left(s_0\right)}{\theta l\left(s_0\right)} \leq P^h\left(\hat{s}\right) \Leftrightarrow \hat{s} \leq s_0 \leq \tilde{s}$$

otherwise, there is no feasible solution of Q. With this restriction, as Q does not appear in the objective function, we obtain:

$$\frac{\Pi_{IV}}{\lambda r} = \max_{\hat{s}\leq s_0\leq \tilde{s}} \left\{ \left(p_0 - \frac{c}{r}\right)\frac{P^h\left(s_0\right)}{\theta l\left(s_0\right)} + \left(1-p_0\right)P^\ell\left(s_0\right) + \max_{\Delta\geq 0}\left\{\left(p_0 - \frac{c}{r}\right)\frac{\Delta}{\lambda}\right\}\right\}$$
$$s.t.\ 0 \leq \frac{\Delta}{\lambda} \leq \left(1 - \frac{1}{\theta l\left(s_0\right)}\right) P^h\left(\underline{s}\right).$$

Now, we consider two cases:

1. $p_0 > \frac{c}{r}$, then Δ must be as high as possible. As there is only one constraint, we obtain: $\Pi_{IV} = \max_{\hat{s}\leq s_0\leq \tilde{s}} \pi_{IV}(s_0)\lambda r$, where

$$\pi_{IV}(s_0) = \left(p_0 - \frac{c}{r}\right)\frac{P^h\left(s_0\right)}{\theta l\left(s_0\right)} + \left(1-p_0\right)P^\ell\left(s_0\right) + $$
$$+ \left(p_0 - \frac{c}{r}\right)\left(1 - \frac{1}{\theta l\left(s_0\right)}\right) P^h\left(\underline{s}\right)$$

and recalling that $\pi^o\left(s\right) = \frac{p_0 P^\ell(s) l(s) + \left(p_0 - \frac{c}{r}\right) P^h(s)}{\theta l(s)}$, we obtain:

$$\pi_{IV}(s_0) = \pi^o\left(s_0\right) + \left(p_0 - \frac{c}{r}\right)\left(1 - \frac{1}{\theta l\left(s_0\right)}\right) P^h\left(\underline{s}\right).$$

2. $p_0 < \frac{c}{r}$, then Δ must be as low as possible, i.e. $\Delta = 0$. We obtain: $\Pi_{IV} = \max_{\hat{s} \leq s_0 \leq \tilde{s}} \pi_{IV}(s_0)\lambda r$ and

$$\pi_{IV}(s_0) = \pi^o(s_0)$$

3. **Summary:** We have obtained that:

$$\pi_{IV}(s) = \begin{cases} \pi^o(s) + \left(p_0 - \frac{c}{r}\right)\left(1 - \frac{1}{\theta l(s)}\right) P^h(\underline{s}), & p_0 > \frac{c}{r}, \hat{s} \leq s_0 \leq \tilde{s}, \\ \pi^o(s), & p_0 < \frac{c}{r}, \hat{s} \leq s_0 \leq \tilde{s}. \end{cases}$$

Solution over $\Omega_{II} \cup \Omega_{IV}$: Now, we combine the two optimization problems:

$$\begin{aligned} \pi_{II}(s) &= \begin{cases} \theta l(s)\pi^o(s) + \left(\left(1 - \frac{c}{r}\right) - \frac{\theta l(s)\pi^o(s)}{P^h(\tilde{s})}\right) \frac{1 - \theta l(s)}{\frac{1}{P^\ell(\tilde{s})} - \frac{\theta l(s)}{P^h(\tilde{s})}}, & \underline{s} \leq s \leq \hat{s}, \\ \pi^o(s), & \hat{s} \leq s \leq \tilde{s}, \end{cases} \\ \text{and } \pi_{IV}(s) &= \begin{cases} \pi^o(s) + \left(p_0 - \frac{c}{r}\right)\left(1 - \frac{1}{\theta l(s)}\right) P^h(\underline{s}), & p_0 > \frac{c}{r}, \hat{s} \leq s_0 \leq \tilde{s}, \\ \pi^o(s), & p_0 < \frac{c}{r}, \hat{s} \leq s_0 \leq \tilde{s}. \end{cases} \end{aligned}$$

The optimization problems reduce to:

$$\begin{aligned} \pi_{II}(s) &= \max_{\underline{s} \leq s \leq \hat{s}} \left\{ \theta l(s)\pi^o(s) + \left(\left(1 - \frac{c}{r}\right) - \frac{\pi^o(s)\theta l(s)}{P^h(\tilde{s})}\right) \frac{1 - \theta l(s)}{\frac{1}{P^\ell(\tilde{s})} - \frac{\theta l(s)}{P^h(\tilde{s})}} \right\} \\ \text{and } \pi_{IV}(s) &= \max_{\hat{s} \leq s \leq \tilde{s}} \left\{ \pi^o(s) + \left(p_0 - \frac{c}{r}\right)^+ \left(1 - \frac{1}{\theta l(s)}\right) P^h(\underline{s}) \right\}. \end{aligned}$$

When maximizing $\pi_{II}(s)$ over $\underline{s} \leq s \leq \hat{s}$ and $\pi_{IV}(s)$ over $\hat{s} \leq s \leq \tilde{s}$, we can recover $\boldsymbol{s}^*$ and the optimal values of Q^* and Δ^*: It is easy to see that:

(i) If $s^* \in [\underline{s}, \hat{s}]$, by definition in Ω_{II}: then, $\boldsymbol{s}^* = (\tilde{s}, s^*)$ and $Q^* \in (0, \tilde{\lambda}^\ell/2)$ and $\Delta^* = \Delta_A(Q^*)$.

(ii) If $s^* \in [\hat{s}, \tilde{s}]$ and $p_0 > \frac{c}{r}$, then, by definition in Ω_{IV}: $\boldsymbol{s}^* = (s^*, \underline{s})$ and $Q^* \in (P^\ell(\tilde{s})\lambda^\ell/2, P^h(\hat{s})\lambda^h/2)$ and $\Delta^* = \Delta_B(Q^*)$.

(iii) If $s^* \in [\hat{s}, \tilde{s}]$ and $p_0 < \frac{c}{r}$, then, by definition in Ω_{IV}: $\boldsymbol{s}^* = (s^*, \underline{s})$ and $Q^* \in (P^\ell(\tilde{s})\lambda^\ell/2, P^h(\hat{s})\lambda^h/2)$ and $\Delta^* = 0$, or, by definition in Ω_{II}: $\boldsymbol{s}^* = (\tilde{s}, s^*)$ and $Q^* = 0$ and $\Delta^* \in (P^\ell(\tilde{s})\lambda^\ell/2, P^h(\hat{s})\lambda^h/2)$.