Coordinating Vertical Partnerships for Horizontally Differentiated Products

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We develop a theoretical model to investigate the incentives for coordinating the positioning and pricing of horizontally differentiated products in the context of a vertical trading relationship between a retailer and multiple suppliers. We also use the model to analyze various trading relationships, such as category management, and characterize the channel efficiency of each, leading to interesting insights about when such practices may be most effective. In particular, we show that when competing suppliers sell through a common retail intermediary, the classical neo-Hotelling incentives to differentiate products is eliminated (or significantly reduced) in certain channel added value regimes, resulting in a loss in channel profit. The incentives to differentiate are recovered, however, if all products are controlled by a single supplier as in category management. Depending on the channel added value regime, the gain in efficiency from a monopolist supplier's incentive to differentiate can offset the loss in efficiency due to positioning distortion and cost difference distortion. These results provide a theoretical explanation for why category management may lead to gains in supply chain efficiency. Lastly, we show numerically how category management together with a simple profit target for the retailer can, in many cases, achieve both full supply chain coordination and a Pareto improving allocation of profits to both the supplier and retailer.

Key words: horizontal differentiation, neo-Hotelling model, category management, channel coordination, channel efficiency

1. Introduction

Category management, a common retail practice, is a vertical partnership between a leading supplier (the *category captain*) and a retailer that aims to increase profit of an entire product category by coordinating assortment, promotion and pricing decisions (see Federal Trade Commission 2001, Harris and McPartland 1993, McLaughlin 1994, Nielsen Marketing Research 1992, Verra 1998). While putting category decisions in the hands of a category captain can improve coordination, the market power it creates may lead to channel inefficiency because the category captain has incentives to bias decisions in favor of its own brands (see Steiner 2001, Basuroy et al. 2001, Kurtuluş and Toktay 2004, Gruen and Shah 2000, Zenor 1994). How best to balance these two effects is an important problem in retailing practice.

We develop a theoretical model to investigate the incentives for coordinating product positioning and pricing of horizontally differentiated products[1] in the context of a vertical trading relationship between a monopolist retailer and multiple suppliers. The model, together with its variations, allows us to analyze a number of trading relationships and characterize the channel efficiency of each one, leading to interesting insights about when such practices may be most effective.

In particular, we first consider a two echelon decentralized system with two competing suppliers and a monopolist retailer. The suppliers each offers a horizontally differentiated product defined by a price-attribute bundle. The retailer sets a retail price which directly affects the demand. Horizontal differentiation of product attributes is described through the neo-Hotelling model introduced by Salop (1979). Readers are referred to Lancaster (1990) and Anderson et al. (1992, Ch. 8) for discussions on locational competition and discriminatory pricing models.

Due to the presence of a downstream retailer, we find that suppliers do not always have the incentive to maximally differentiate, in contrast to the direct selling model of Salop (1979). Instead, there are cost regimes where suppliers have partial or complete indifference to differentiating horizontally.

We subsequently study the fully centralized system and the monopolist supplier model based on an appropriate modification of the decentralized system. When comparing channel profits of the three models, we first observe that supply chain inefficiency may be attributable to the following causes:

[1] Products are horizontally differentiated when they are different in features or attributes that cannot be ordered, such as color, style or taste. In contrast, vertically differentiated products can be ordered through a customer's perceived difference in quality.

- *Double Marginalization*: This is the distortion introduced as a result of the marginal cost to the retailer being greater than the marginal cost to the channel. See Spengler (1950) for a definition;
- *Positioning Distortion*: A novel addition from our model, where the retailer's profit is maximized when the two products maximally differentiate but the suppliers in equilibrium are indifferent to product positioning;
- *Cost Difference Distortion*: Another novel addition from our model, where the production cost differences between competing products further exacerbate losses in channel efficiency.

Taking all these various factors into account, we show that under certain channel added value regimes, concentrating positioning and pricing power into the hands of a single supplier improves channel efficiency, because this arrangement eliminates positioning distortion and cost difference distortion. These results, in a stylized sense, suggest where category management provides benefits to a channel.

Finally, we consider a practical modification. In a typical category management partnership, the category captain often has some form of profit sharing agreement with the retailer in exchange for gaining market power over the category. In particular, we consider the case where the retailer imposes a profit target on the category, which moderates some of the market power given to the monopolist supplier. We provide insight via a numerical study, and show in many cases that this simple modification achieves both full channel coordination, which also eliminates the loss due to double marginalization, and a Pareto improving allocation of channel profits.

2. Decentralized System

We first consider a two echelon decentralized system involving two competing suppliers that supply a monopolist retailer. Each supplier offers a single product. The suppliers first simultaneously set product positions and then wholesale prices. The retailer takes the product positions and wholesale prices as given and sets a retail price for each product. Each player maximizes his own profit. Our objective is to characterize an equilibrium for the optimal (product position, wholesale price, retail price) triple.

One can think of this as a stylized model of category management: selecting a product position can be thought of as choosing a specific product from a firm's broader product line to match the consumer preferences and competitive offerings at a given retailer. The constraint that each firm only offers one product mimics the fact that a manufacturer can only place a limited number of its products on store shelves (in this case only one). Suppliers in the category compete against each other in terms of both their product offering and price, and retailers set prices for the category based on demand functions and wholesale prices.

We use the neo-Hotelling model to describe the market, which is typically represented with a circle of unit circumference. Consumers are assumed to be uniformly distributed on that circle. In addition, we use the term *product domain* to describe the demand of a product sold in this neo-Hotelling market. The product domain is realized wherever the purchase utility, a function of both retail price and distance to the product position, is positive. An illustration of the neo-Hotelling model is provided in Figure 1. For a market that contains only one product, the product domain is symmetric around the location of the product.

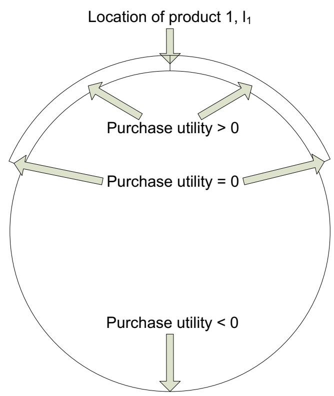

Figure 1 Product Domain of a Neo-Hotelling Model

The reader will note that our results would likely be different if we used the finite line topology (classic Hotelling model) instead of the circle, or infinite-line topology (neo-Hotelling model). This is based on the fact that the equilibrium for the two-competitor finite-line model is achieved with both competitors relocating to the mid-point, whereas the equilibrium for the two-competitor infinite-line model, described in Salop (1979), is achieved with both competitors maximally differentiating their positions. However, we work only with the neo-Hotelling model because it has become the dominant model of horizontal differentiation in the literature. Replicating our analyses under the classic Hotelling model is a subject for future investigation.

For product i, let l_i be its position (see Figure 2). In a multi-product Hotelling setup, the distance between the two products, expressed as $|l_1 - l_2|$, is used to characterize the differentiation of product features. Let w_i be its wholesale price, c_i be its production cost, p_i be its retail price, and D_i be its product domain. In addition, let v be the consumer valuation[2], and θ be the linear position cost to the consumer. A consumer buys product i if and only if its net utility is positive and larger than the competing product; that is

$$v - \theta|l_i - l| - p_i > 0$$

[2] Horizontal differentiation implies that v stays the same for both products. We assume v is fixed.

and

$$v - \theta|l_i - l| - p_i > v - \theta|l_j - l| - p_j, \quad j \neq i.$$

If we assume full rationality, the decentralized system can be solved via backward induction. First we analyze the retailer's problem by solving for (p_1^*, p_2^*). Then we determine (w_1^*, w_2^*) in the suppliers' game. And finally we find the optimal differentiation in product positions $|l_1 - l_2|$.

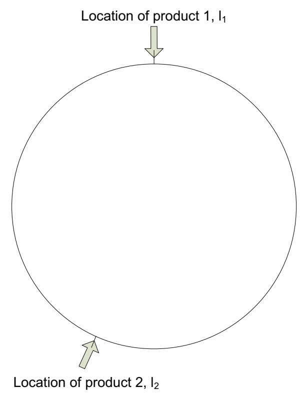

Figure 2 **Neo-Hotelling Setup with Two Products**

2.1. Retailer's Problem

Taking product positions and wholesale prices as given, the retailer solves a joint profit maximization problem to obtain the optimal retail prices. In order to determine the product domains in the objective function, we divide our analysis between symmetric and asymmetric positioning of the two products.

2.1.1. Products in Symmetric Positions. When the two products are positioned symmetrically ($|l_1 - l_2| = \frac{1}{2}$), it is possible for the product domains to be (S1) separable, (S2) barely touching[3], or (S3) overlapping on both ends. An illustration is provided in Figure 3. Let l be the position of a consumer and the corresponding net utility of purchasing product i be $v - \theta|l_i - l| - p_i$.

When the product domains are separable, a consumer makes a purchase decision of product i independent of all other products. Thus he buys product i if

$$v - \theta|l_i - l| - p_i \geq 0.$$

Let l^* be the borderline consumer position between buying and not buying. Then,

$$|l_i - l^*| = \frac{v - p_i}{\theta}.$$

[3] Symmetrically positioned product domains *barely touch* when they extend the entire market without overlapping.

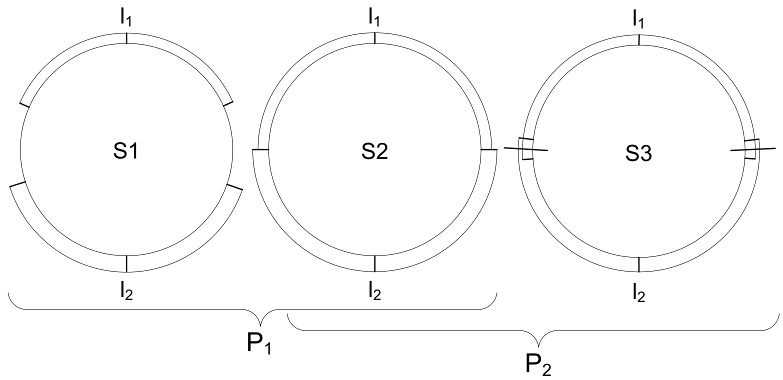

Figure 3 **Domain Scenarios for Products in Symmetric Positions**

We consider scenarios (S1) and (S2) together, i.e.

$$|l_1 - l^*| + |l_2 - l^*| = \frac{v - p_1}{\theta} + \frac{v - p_2}{\theta} \leq \frac{1}{2},$$

where strict inequality describes (S1) and equality describes (S2). By symmetry of the circle, product domains under (S1) or (S2) can be written as

$$D_i = 2|l_i - l^*| = \frac{2(v - p_i)}{\theta}.$$

The retailer then solves the following problem:

$$\begin{aligned} \text{(P1)} \quad \max_{p_1, p_2} \quad & (p_1 - w_1)\frac{2(v - p_1)}{\theta} + (p_2 - w_2)\frac{2(v - p_2)}{\theta} \\ \text{s.t.} \quad & \frac{v - p_1}{\theta} + \frac{v - p_2}{\theta} \leq \frac{1}{2}. \end{aligned}$$

Now we consider scenarios (S2) and (S3) together, i.e.

$$\frac{v - p_1}{\theta} + \frac{v - p_2}{\theta} \geq \frac{1}{2},$$

where strict inequality describes (S3) and equality again describes (S2). When the product domains overlap, a consumer has to decide between purchasing the two products. He will buy product i, if, for $j \neq i$,

$$v - \theta|l_i - l| - p_i \geq v - \theta|l_j - l| - p_j.$$

Let l^* denote the position where a consumer is indifferent between the two products (represented with a straight line that cuts across the overlapped domains of scenario (S3) in Figure 3). We can solve the following system to obtain the domains under (S2) or (S3):

$$\begin{cases} v - \theta|l_1 - l^*| - p_1 = v - \theta|l_2 - l^*| - p_2 \\ |l_1 - l^*| + |l_2 - l^*| = \frac{1}{2} \end{cases}$$

$$\Rightarrow \quad D_1 = 2|l_1 - l^*| = \frac{1}{2} + \frac{p_2 - p_1}{\theta};\ D_2 = 2|l_2 - l^*| = \frac{1}{2} + \frac{p_1 - p_2}{\theta}.$$

The retailer then solves the following problem:

$$\text{(P2)} \quad \max_{p_1, p_2} \quad (p_1 - w_1)\left(\frac{1}{2} + \frac{p_2 - p_1}{\theta}\right) + (p_2 - w_2)\left(\frac{1}{2} + \frac{p_1 - p_2}{\theta}\right)$$
$$\text{s.t.} \quad \frac{v - p_1}{\theta} + \frac{v - p_2}{\theta} \geq \frac{1}{2}.$$

2.1.2. Products in Asymmetric Positions. Assume $|l_1 - l_2|$ refers to the shorter arc between l_1 and l_2, and $1 - |l_1 - l_2|$ refers to the longer arc. When the products are located asymmetrically ($0 < |l_1 - l_2| < \frac{1}{2}$)[4], it is possible for product domains to be (AS1) separable, (AS2) barely touching on one end but not touching on the other end, (AS3) overlapping on one end but not touching on the other end, (AS4) overlapping on one end but barely touching on the other end, or (AS5) overlapping on both ends. An illustration is provided in Figure 4.

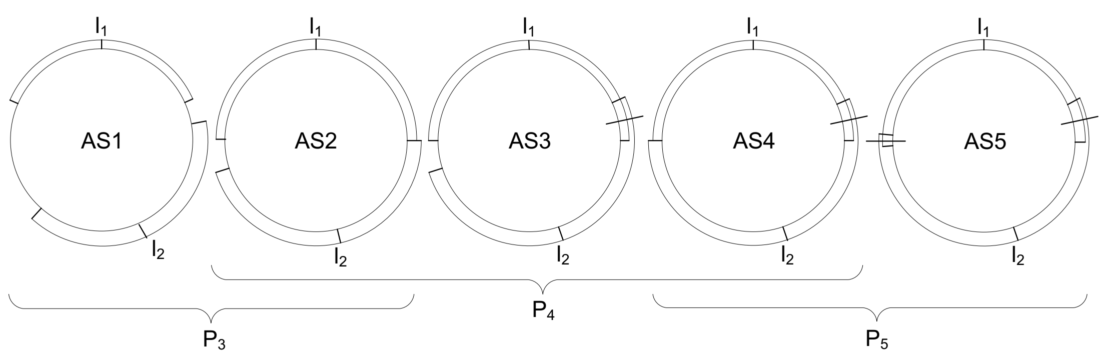

Figure 4 Domain Scenarios for Products in Asymmetric Positions

We consider (AS1) and (AS2) together, i.e.

$$\frac{v - p_1}{\theta} + \frac{v - p_2}{\theta} \leq |l_1 - l_2|,$$

where inequality describes (AS1) and equality describes (AS2). The retailer solves the following problem, which has the same objective function as in (P1):

$$\text{(P3)} \quad \max_{p_1, p_2} \quad (p_1 - w_1)\frac{2(v - p_1)}{\theta} + (p_2 - w_2)\frac{2(v - p_2)}{\theta}$$
$$\text{s.t.} \quad \frac{v - p_1}{\theta} + \frac{v - p_2}{\theta} \leq |l_1 - l_2|.$$

[4] $|l_1 - l_2| = 0$ corresponds to cannibalization of one product by another. We will discuss later in the suppliers' game that this scenario in general does not lead to an equilibrium solution.

(AS2), (AS3) and (AS4), considered together, can be described by the following:

$$|l_1 - l_2| \leq \frac{v - p_1}{\theta} + \frac{v - p_2}{\theta} \leq 1 - |l_1 - l_2|,$$

where an active lower bound describes (AS2), an active upper bound describes (AS4), and all the interior points describe (AS3). The domain for product i on the longer arc between l_1 and l_2 is separable from that of product j ($j \neq i$), while on the shorter arc it overlaps with that of product j. Let l^* denote the position where a consumer is indifferent between the two products on the shorter arc between l_1 and l_2 (represented with a straight line that cuts across the overlapped domains of scenarios (AS3), (AS4) and (AS5) in Figure 4). In order to find the domain on the overlapping side, we solve the following system:

$$\begin{cases} v - \theta|l_1 - l^*| - p_1 = v - \theta|l_2 - l^*| - p_2 \\ |l_1 - l^*| + |l_2 - l^*| = |l_1 - l_2| \end{cases}$$
$$\Rightarrow \quad |l_1 - l^*| = \frac{1}{2}|l_1 - l_2| + \frac{p_2 - p_1}{2\theta}; \; |l_2 - l^*| = \frac{1}{2}|l_1 - l_2| + \frac{p_1 - p_2}{2\theta}.$$

The domain for product i under (AS2), (AS3) or (AS4) consists of two pieces. The first piece measures from the product position l_i to the point where a consumer is indifferent between buying and not buying. The second piece measures from l_i to the point where a consumer is indifferent between buying product i and product j. Therefore,

$$D_1 = \frac{v - p_1}{\theta} + \left(\frac{1}{2}|l_1 - l_2| + \frac{p_2 - p_1}{2\theta}\right) = \frac{1}{2}|l_1 - l_2| + \frac{p_2 - 3p_1}{2\theta} + \frac{v}{\theta};$$
$$D_2 = \frac{v - p_2}{\theta} + \left(\frac{1}{2}|l_1 - l_2| + \frac{p_1 - p_2}{2\theta}\right) = \frac{1}{2}|l_1 - l_2| + \frac{p_1 - 3p_2}{2\theta} + \frac{v}{\theta}.$$

The retailer solves the following problem:

$$\text{(P4)} \quad \max_{p_1, p_2} \quad (p_1 - w_1)\left(\frac{1}{2}|l_1 - l_2| + \frac{p_2 - 3p_1}{2\theta} + \frac{v}{\theta}\right) + (p_2 - w_2)\left(\frac{1}{2}|l_1 - l_2| + \frac{p_1 - 3p_2}{2\theta} + \frac{v}{\theta}\right)$$
$$\text{s.t.} \quad |l_1 - l_2| \leq \frac{v - p_1}{\theta} + \frac{v - p_2}{\theta} \leq 1 - |l_1 - l_2|.$$

Now we consider (AS4) and (AS5) together, i.e.

$$\frac{v - p_1}{\theta} + \frac{v - p_2}{\theta} \geq 1 - |l_1 - l_2|,$$

where strict inequality describes (AS5) and equality describes (AS4). Let l_1^* be the position on the shorter arc between l_1 and l_2 where a customer is indifferent between buying either of the two products, and let l_2^* be similarly defined as the product indifference point on the longer arc between l_1 and l_2. We can solve the following systems to obtain the product domains:

$$l_1^* \begin{cases} v - \theta|l_1 - l_1^*| - p_1 = v - \theta|l_2 - l_1^*| - p_2 \\ |l_1 - l_1^*| + |l_2 - l_1^*| = |l_1 - l_2| \end{cases}$$

$$\Rightarrow |l_1 - l_1^*| = \frac{1}{2}|l_1 - l_2| + \frac{p_2 - p_1}{2\theta};\ |l_2 - l_1^*| = \frac{1}{2}|l_1 - l_2| + \frac{p_1 - p_2}{2\theta};$$

$$l_2^* \begin{cases} v - \theta|l_1 - l_2^*| - p_1 = v - \theta|l_2 - l_2^*| - p_2 \\ |l_1 - l_2^*| + |l_2 - l_2^*| = 1 - |l_1 - l_2| \end{cases}$$

$$\Rightarrow |l_1 - l_2^*| = \frac{1}{2}(1 - |l_1 - l_2|) + \frac{p_2 - p_1}{2\theta};\ |l_2 - l_2^*| = \frac{1}{2}(1 - |l_1 - l_2|) + \frac{p_1 - p_2}{2\theta}.$$

The domain for each product under (AS4) or (AS5) can be written as:

$$D_1 = |l_1 - l_1^*| + |l_1 - l_2^*| = \frac{1}{2} + \frac{p_2 - p_1}{\theta};\ D_2 = |l_2 - l_1^*| + |l_2 - l_2^*| = \frac{1}{2} + \frac{p_1 - p_2}{\theta}.$$

The retailer solves the following problem:

$$\text{(P5)} \quad \max_{p_1, p_2} \quad (p_1 - w_1)\left(\frac{1}{2} + \frac{p_2 - p_1}{\theta}\right) + (p_2 - w_2)\left(\frac{1}{2} + \frac{p_1 - p_2}{\theta}\right)$$
$$\text{s.t.} \quad \frac{v - p_1}{\theta} + \frac{v - p_2}{\theta} \geq 1 - |l_1 - l_2|.$$

2.1.3. Characterization of Retailer's Optimal Response. The solutions to (P1)–(P5) (see Appendix A.1) are used to construct the retailer's optimal response to a given set of wholesale prices (w_1, w_2) and relative product position $|l_1 - l_2|$. We first define the quantity $v - \frac{w_1 + w_2}{2}$ as the *retail added value*. The results, as shown in the following proposition, are fully specified by comparing this quantity against a function of the linear position cost θ.

Proposition 1. *When product positions are symmetric, the retailer's optimal solution can be segmented as follows:*

- *Region 1 ($v - \frac{w_1 + w_2}{2} < \frac{\theta}{2}$) - Separable solution is optimal;*
- *Region 2 ($v - \frac{w_1 + w_2}{2} > \frac{\theta}{2}$) - Barely touching solution is optimal.*

When product positions are asymmetric, the retailer's optimal solution can be segmented as the following:

- *Region 3 ($v - \frac{w_1 + w_2}{2} < \theta|l_1 - l_2|$) - Separable solution is optimal;*
- *Region 4 ($\theta|l_1 - l_2| < v - \frac{w_1 + w_2}{2} < \frac{3}{2}\theta|l_1 - l_2|$) - The solution where one end barely touches and the other end does not touch is optimal;*
- *Region 5 ($\frac{3}{2}\theta|l_1 - l_2| < v - \frac{w_1 + w_2}{2} < \theta - \frac{1}{2}\theta|l_1 - l_2|$) - The solution where one end overlaps and the other end does not touch is optimal;*
- *Region 6 ($v - \frac{w_1 + w_2}{2} > \theta - \frac{1}{2}\theta|l_1 - l_2|$) - The solution where one end overlaps and the other end barely touches is optimal.*

Proof: See Appendix A.1. Q.E.D.

We use two figures to illustrate the two product position scenarios discussed in Proposition 1. Figure 5 corresponds to the symmetric case (regions 1 and 2), whereas Figure 6 corresponds to the asymmetric case (regions 3 through 6). In both figures, the retail added value $v - \frac{w_1+w_2}{2}$ is the axis and characterizes the segments. The retailer's optimal solution in each of those segments is illustrated with the corresponding neo-Hotelling product domain. Table 1 summarizes the solution to each of the 6 regions discussed in Proposition 1.

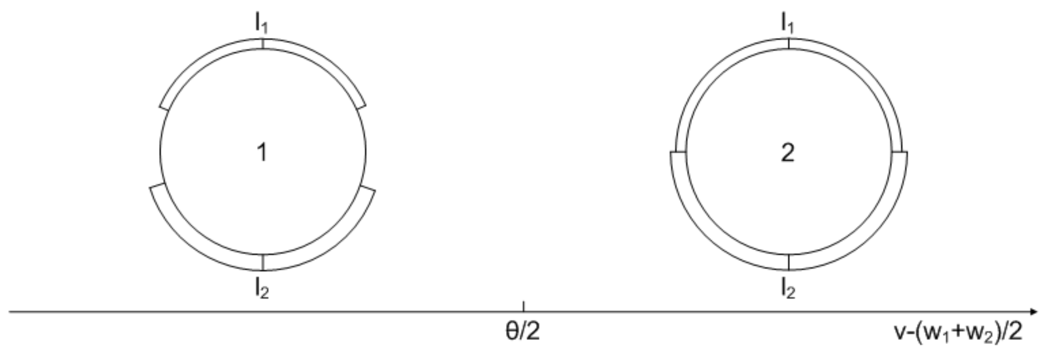

Figure 5 Retailer's Optimal Solution under Symmetric Product Positions

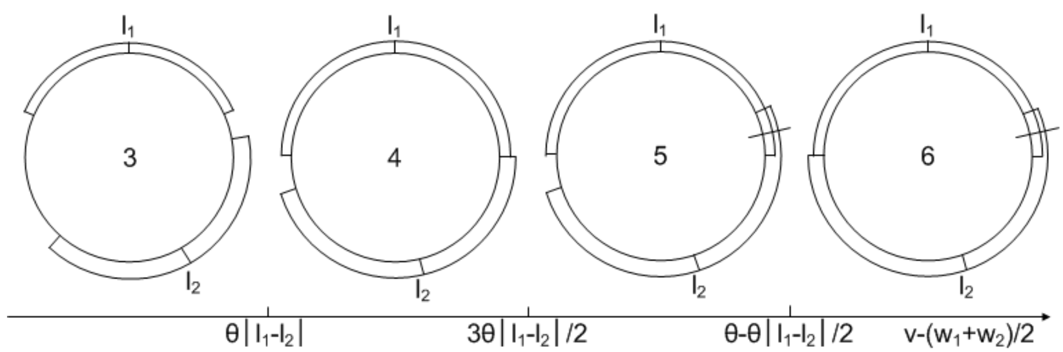

Figure 6 Retailer's Optimal Solution under Asymmetric Product Positions

Regions	p_1^*	p_2^*	π^*
1, 3	$\frac{v+w_1}{2}$	$\frac{v+w_2}{2}$	$\frac{(v-w_1)^2}{2\theta} + \frac{(v-w_2)^2}{2\theta}$
2	$v + \frac{w_1}{4} - \frac{w_2}{4} - \frac{\theta}{4}$	$v + \frac{w_2}{4} - \frac{w_1}{4} - \frac{\theta}{4}$	$v + \frac{(w_1-w_2)^2}{4\theta} - \frac{w_1+w_2}{2} - \frac{\theta}{4}$
4	$v + \frac{w_1-w_2}{4} - \frac{\theta}{2}\lvert l_1-l_2\rvert$	$v - \frac{w_1-w_2}{4} - \frac{\theta}{2}\lvert l_1-l_2\rvert$	$(2v-w_1-w_2)\lvert l_1-l_2\rvert - \theta\lvert l_1-l_2\rvert^2$ $+\frac{(w_1-w_2)^2}{4\theta}$
5	$\frac{v+w_1}{2} + \frac{\theta}{4}\lvert l_1-l_2\rvert$	$\frac{v+w_2}{2} + \frac{\theta}{4}\lvert l_1-l_2\rvert$	$\frac{1}{4\theta}\left(v-w_1+\frac{\theta}{2}\lvert l_1-l_2\rvert\right)^2$ $+\frac{1}{4\theta}\left(v-w_2+\frac{\theta}{2}\lvert l_1-l_2\rvert\right)^2 + \frac{1}{8\theta}(w_1-w_2)^2$
6	$\frac{w_1-w_2}{4} + v + \frac{\theta}{2}\lvert l_1-l_2\rvert - \frac{\theta}{2}$	$\frac{w_2-w_1}{4} + v + \frac{\theta}{2}\lvert l_1-l_2\rvert - \frac{\theta}{2}$	$v - \frac{\theta}{2} + \frac{\theta}{2}\lvert l_1-l_2\rvert - \frac{w_1+w_2}{2} + \frac{(w_1-w_2)^2}{4\theta}$

Table 1 Summary of Results for the Retailer's Problem in Decentralized System

When the two products are symmetrically located (as in Figure 5) and when the retail added value is below the critical value of $\frac{\theta}{2}$ (region 1), it is optimal for the retailer, by setting the appropriate retail prices, to induce product domains that only partially cover the entire market. When the retail added value is above $\frac{\theta}{2}$ (region 2), however, the optimal retail prices induce domains that completely cover the market—hence, barely touching. In other words, the optimal retail prices are constant (if we assume v and w_i to be constant) within each of the respective regions (regions 1 and 2), but the pricing levels step down as retail added value increases past the critical value of $\frac{\theta}{2}$.

When the two products are asymmetrically located (as in Figure 6) and when the retail added value is below $\theta|l_1 - l_2|$ (region 3), it is again optimal for the retailer to induce product domains that only partially cover the entire market. As the retail added value increases past $\theta|l_1 - l_2|$ (region 4), it becomes optimal to drop the retail prices so that the product domains barely touches on one end (due to the asymmetric positioning), while holding them constant until the retail added value reaches $\frac{3}{2}\theta|l_1 - l_2|$. Similarly, the transition from region 4 to region 5 (overlapping on one end but not touching on the other end) and then to region 6 (overlapping on the one end but not touching on the other end), as the retail added value continues to increase, implies two more stepwise drops in the optimal retail prices.

In both the symmetric and asymmetric cases, there is no further drop in optimal retail prices when the product domains cover the entire market (as in regions 2 and 6).

2.2. Suppliers' Game and Equilibrium Analysis

We next consider the "position-then-price" game for the suppliers[5]. To start, note that cannibalization – in which product i takes the same position as product j with a lower wholesale price, which induces the retailer to drop product j from the assortment – cannot be an equilibrium solution as long as it is not the only option, because the cannibalized firm can strictly increase its profit by moving its product position away from its competitor.

We next construct a two-stage game (product position then wholesale price) and find the equilibrium for the suppliers. Via backward induction and the assumption of total rationality, the suppliers know that the monopolist retailer will behave optimally in a manner prescribed by the 6 regions in Figure 6. Furthermore, domain functions for the suppliers are symmetric in each region[6]. This implies that we can first solve for the suppliers' equilibrium wholesale prices in each region, and then find the best overall region, which is subsequently used to determine the optimal product positions.

[5] Simultaneous selection of position and price does not lead to an equilibrium (see Anderson et al. 1992, Section 8.3).

[6] In other words, interchanging subscripts i and j in product i's domain gives product j's domain.

Let $\bar{w}_j, j \neq i$ be the wholesale price of the competing product j as given to supplier i. Suppliers' domain functions are found by substituting results from Table 1. We formulate a set of unconstrained profit maximization problems for the suppliers, and use the derived results to characterize equilibrium outcomes.

Regions 1 and 3 (refer to Figures 5 and 6) have the same domain functions, since both correspond to separable domain solutions.

$$D_1 = \frac{2(v - p_1^*)}{\theta} = \frac{v - w_1}{\theta};\ D_2 = \frac{2(v - p_2^*)}{\theta} = \frac{v - w_2}{\theta}.$$

Each supplier maximizes its own profit, i.e.

$$\text{(SP13)} \quad \max_{w_1} \quad (w_1 - c_1)\frac{v - w_1}{\theta}; \quad \max_{w_2} \quad (w_2 - c_2)\frac{v - w_2}{\theta}.$$

The domain functions for region 2 are

$$D_1 = \frac{2(v - p_1^*)}{\theta} = \frac{1}{2} - \frac{w_1}{2\theta} + \frac{w_2}{2\theta};\ D_2 = \frac{2(v - p_2^*)}{\theta} = \frac{1}{2} - \frac{w_2}{2\theta} + \frac{w_1}{2\theta}.$$

The domain functions for region 6 are

$$D_1 = \frac{1}{2}|l_1 - l_2| + \frac{p_2^* - 3p_1^*}{2\theta} + \frac{v}{\theta} = \frac{1}{2} - \frac{w_1}{2\theta} + \frac{w_2}{2\theta};\ D_2 = \frac{1}{2}|l_1 - l_2| + \frac{p_1^* - 3p_2^*}{2\theta} + \frac{v}{\theta} = \frac{1}{2} - \frac{w_2}{2\theta} + \frac{w_1}{2\theta}.$$

Observe that regions 2 and 6 (refer to Figures 5 and 6) share the same domain functions. Each supplier maximizes its own profit, i.e.

$$\text{(SP26)} \quad \max_{w_1} \quad (w_1 - c_1)\left(\frac{1}{2} - \frac{w_1}{2\theta} + \frac{\bar{w}_2}{2\theta}\right); \quad \max_{w_2} \quad (w_2 - c_2)\left(\frac{1}{2} - \frac{w_2}{2\theta} + \frac{\bar{w}_1}{2\theta}\right).$$

The domain functions for region 4 (refer to Figure 6) are

$$\begin{aligned} D_1 &= \frac{1}{2}|l_1 - l_2| + \frac{p_2^* - 3p_1^*}{2\theta} + \frac{v}{\theta} = |l_1 - l_2| - \frac{w_1}{2\theta} + \frac{w_2}{2\theta}; \\ D_2 &= \frac{1}{2}|l_1 - l_2| + \frac{p_1^* - 3p_2^*}{2\theta} + \frac{v}{\theta} = |l_1 - l_2| - \frac{w_2}{2\theta} + \frac{w_1}{2\theta}. \end{aligned}$$

Each supplier maximizes its own profit, i.e.

$$\text{(SP4)} \quad \max_{w_1} \quad (w_1 - c_1)\left(|l_1 - l_2| - \frac{w_1}{2\theta} + \frac{\bar{w}_2}{2\theta}\right); \quad \max_{w_2} \quad (w_2 - c_2)\left(|l_1 - l_2| - \frac{w_2}{2\theta} + \frac{\bar{w}_1}{2\theta}\right).$$

The domain functions for region 5 (refer to Figure 6) are

$$\begin{aligned} D_1 &= \frac{1}{2}|l_1 - l_2| + \frac{p_2^* - 3p_1^*}{2\theta} + \frac{v}{\theta} = \frac{1}{4}|l_1 - l_2| + \frac{v}{2\theta} + \frac{w_2}{4\theta} - \frac{3w_1}{4\theta}; \\ D_2 &= \frac{1}{2}|l_1 - l_2| + \frac{p_1^* - 3p_2^*}{2\theta} + \frac{v}{\theta} = \frac{1}{4}|l_1 - l_2| + \frac{v}{2\theta} + \frac{w_1}{4\theta} - \frac{3w_2}{4\theta}. \end{aligned}$$

Each supplier maximizes its own profit, i.e.

$$\text{(SP5)} \quad \max_{w_1} \quad (w_1 - c_1)\left(\frac{1}{4}|l_1 - l_2| + \frac{v}{2\theta} + \frac{\bar{w}_2}{4\theta} - \frac{3w_1}{4\theta}\right);$$
$$\max_{w_2} \quad (w_2 - c_2)\left(\frac{1}{4}|l_1 - l_2| + \frac{v}{2\theta} + \frac{\bar{w}_1}{4\theta} - \frac{3w_2}{4\theta}\right).$$

The solutions to the unconstrained maximization problems (detailed in Appendix A.2) have some immediate properties that are summarized in the following lemmas.

LEMMA 1. *The transition of each supplier's product domain is continuous between (i) regions 1 and 2 in the symmetric positioning case, and (ii) regions 3, 4, 5, and 6 in the asymmetric positioning case.*

Proof: See Appendix A.2. Q.E.D.

LEMMA 2. *Both suppliers have incentives to move toward maximum product differentiation for the optimal solutions of regions 4 and 5.*

Proof: See Appendix A.2. Q.E.D.

The following theorem characterizes the suppliers' optimal responses to the downstream retailer. Similar to the retail added value $v - \frac{w_1+w_2}{2}$, we define the quantity $v - \frac{c_1+c_2}{2}$ as the *channel added value*.

THEOREM 1. *When the channel added value $v - \frac{c_1+c_2}{2} \geq \frac{3}{2}\theta$, the following equilibria exist:*

1. $v - \frac{c_1+c_2}{2} \geq 2\theta$*: The suppliers are completely indifferent to the locations of their products;*
2. $\frac{7}{4}\theta \leq v - \frac{c_1+c_2}{2} < 2\theta$*: The suppliers switch from being indifferent to locations to having incentives to differentiate as $|l_1 - l_2|$ decreases below the threshold $\frac{c_1+c_2-2v+4\theta}{\theta}$;*
3. $\frac{3}{2}\theta \leq v - \frac{c_1+c_2}{2} < \frac{7}{4}\theta$*: The suppliers have incentive to maximally differentiate their products (i.e. $|l_1 - l_2| = \frac{1}{2}$).*

In all cases, the retailer sets prices such that product domains extend over the entire market.

When the channel added value $v - \frac{c_1+c_2}{2} \leq \theta$, the suppliers have sufficient incentives to differentiate in order to maintain separable product domains. In particular, they switch from having sufficient incentives to differentiate to having maximal incentives to differentiate as $|l_1 - l_2|$ decreases past the threshold $\frac{2v-c_1-c_2}{4\theta}$. The retailer, in this case, sets prices to maintain the separable domains.

When $\theta < v - \frac{c_1+c_2}{2} < \frac{3}{2}\theta$, no equilibrium exists.

Proof: See Appendix A.2. Q.E.D.

According to Theorem 1, it is optimal for the suppliers to only sufficiently differentiate their products when the channel added value is below the critical threshold of θ. No equilibrium exists when the channel added value is between θ and $\frac{3}{2}\theta$. As the channel added value increases from

$\frac{3}{2}\theta$ to $\frac{7}{4}\theta$, the suppliers have incentives to maximally differentiate their products. The incentives to differentiate the product positions will diminish once again, as the channel added value increases past $\frac{7}{4}\theta$. The suppliers become completely indifferent[7] to the product positions, when the channel added value reaches beyond the critical threshold of 2θ.

The retailer, in response, sets the optimal retail prices that result in separable domains when the channel added value is below θ. When the channel added value is above $\frac{3}{2}\theta$, it is optimal for the retailer to let product domains extend over the entire market—this means the product domains either barely touch on both ends, in the case of maximally differentiated product positions, or barely touch on one end but overlap on the other end, in the case of partially differentiated product positions.

3. Benchmark Systems

In this section, we compare several benchmark systems to the decentralized system in the previous section. Specifically, we consider the fully centralized system in Section 3.1, which allows us to assess channel profit loss. We also consider a monopolist supplier system in Section 3.2, in which the two products are supplied by a single vendor. Comparing the monopolist supplier system to the decentralized system gives additional insights into the relationship between between the upstream and downstream players in a supply chain.

3.1. Fully Centralized System

We consider a fully centralized system where the retailer both selects positions (by picking $|l_1 - l_2|$) and then prices (by determining (p_1, p_2)) the two products, over the cost of acquiring product i, which is simply the production cost c_i. The fully centralized system can again be solved via backward induction. First we analyze the retailer's problem by solving for (p_1^*, p_2^*). We observe that the retailer's problem shares the same formulations as (P1) through (P5) in §2.1 for the decentralized system with w_i replaced by c_i. Next we find the optimal differentiation in product positions $|l_1 - l_2|$, which is summarized in Theorem 2.

Theorem 2. *When the channel added value $v - \frac{c_1+c_2}{2} > \frac{\theta}{2}$, the retailer positions the two products symmetrically and sets prices such that the two domains cover the entire market without overlapping (barely touching). When the channel added value $v - \frac{c_1+c_2}{2} < \frac{\theta}{2}$, the retailer has sufficient incentives to differentiate the products in order to maintain domain separability.*

[7] Complete indifference excludes cannibalization, where both products are found at the same location. We mentioned previously that cannibalization cannot be an equilibrium solution as long as it is not the only option, because the cannibalized firm can strictly increase its profit by moving away.

Proof: See Appendix B. Q.E.D.

In a fully centralized the system, the retailer determines both the retail price and the product positions. According to Theorem 2, it is again optimal to experience separable domains when the channel added value is low, albeit at a lower threshold of $\frac{\theta}{2}$ than the threshold of θ in Theorem 1. In this case, the retailer only has sufficient incentives to differentiate the two products in order to maintain domain separability. As the channel added value increases past $\frac{\theta}{2}$, on the other hand, the retailer has maximal incentives to differentiate while ensuring that product domains cover the entire market.

When all market powers are concentrated within the hands of the retailer, it is not optimal to disregard the positions, in the case of product domains that barely touch on one end but overlap on the other end, of those products with high channel added value (greater than $\frac{\theta}{2}$).

3.2. Monopolist Supplier

In this model, the two products are supplied by a single firm. The supplier first selects the product differentiation $|l_1 - l_2|$ and then picks the wholesale prices (w_1, w_2). The retailer's problem is the same as in §2.1 for the decentralized system.

We will now present the set of problems faced by the monopolist supplier. The constraints in each problem, defined on the aggregate wholesale price $w_1 + w_2$, are algebraically equivalent to the retail added value thresholds $v - \frac{w_1+w_2}{2}$ that we used to define the six solution regions in Figures 5 and 6.

The product domains in regions 1 and 3 are separable. The monopolist supplier solves the following problem:

$$\text{(SP13}'\text{)} \quad \max_{w_1, w_2} \quad (w_1 - c_1)\frac{v - w_1}{\theta} + (w_2 - c_2)\frac{v - w_2}{\theta}$$
$$\text{s.t.} \quad w_1 + w_2 \geq \alpha,$$

where $\alpha = 2v - \theta$ for region 1 and $\alpha = 2v - 2\theta|l_1 - l_2|$ for region 3.

Regions 2 and 6 have the same domain functions due to location indifference. The monopolist supplier solves the following problem:

$$\text{(SP26}'\text{)} \quad \max_{w_1, w_2} \quad (w_1 - c_1)\left(\frac{1}{2} - \frac{w_1}{2\theta} + \frac{w_2}{2\theta}\right) + (w_2 - c_2)\left(\frac{1}{2} - \frac{w_2}{2\theta} + \frac{w_1}{2\theta}\right)$$
$$\text{s.t.} \quad w_1 + w_2 \leq \beta,$$

where $\beta = 2v - \theta$ for region 2 and $\beta = 2v - 2\theta + \theta|l_1 - l_2|$ for region 6.

The monopolist supplier solves the following problem in region 4:

$$\text{(SP4}'\text{)} \quad \max_{w_1, w_2} \quad (w_1 - c_1)\left(|l_1 - l_2| - \frac{w_1}{2\theta} + \frac{w_2}{2\theta}\right) + (w_2 - c_2)\left(|l_1 - l_2| - \frac{w_2}{2\theta} + \frac{w_1}{2\theta}\right)$$

$$\text{s.t.} \quad 2v - 3\theta|l_1 - l_2| \le w_1 + w_2 \le 2v - 2\theta|l_1 - l_2|.$$

The monopolist supplier solves the following problem in region 5:

$$\begin{aligned}
(\text{SP5}') \quad \max_{w_1, w_2} \quad & (w_1 - c_1)\left(\frac{1}{4}|l_1 - l_2| + \frac{v}{2\theta} + \frac{w_2}{4\theta} - \frac{3w_1}{4\theta}\right) \\
& + (w_2 - c_2)\left(\frac{1}{4}|l_1 - l_2| + \frac{v}{2\theta} + \frac{w_1}{4\theta} - \frac{3w_2}{4\theta}\right) \\
\text{s.t.} \quad & 2v - 2\theta + \theta|l_1 - l_2| \le w_1 + w_2 \le 2v - 3\theta|l_1 - l_2|.
\end{aligned}$$

The optimal solution is summarized in the following theorem.

Theorem 3. *When the channel added value $v - \frac{c_1+c_2}{2} > \theta$, the supplier positions the two products symmetrically, and the retailer sets prices that let product domains cover the entire market. When the channel added value $v - \frac{c_1+c_2}{2} < \theta$, the retailer and the supplier have sufficient incentives to differentiate the products in order to maintain domain separability.*

Proof: See Appendix B. Q.E.D.

Compared to the decentralized system, there is only one supplier at the upstream. In terms of market power concentration, the monopolist supplier system ranks between the decentralized system and the fully centralized system. Interestingly, this intermediate concentration in market power is sufficient to eliminate incentives to partially differentiate the product positions when the channel added value is above the critical threshold of θ. We note that in this case the retailer sets prices that allow product domains to cover the entire market, so any partial incentives to differentiate, implying product domains that barely touch on one end but overlap on the other end, are suboptimal. There is still sufficient incentives to differentiate, in order to maintain domain separability, when the channel added value is below θ.

The intermediate concentration in market power (in the monopolist supplier system) also appears to contribute to a higher critical threshold of θ than the full concentration in market power (in the fully centralized system), which has a critical threshold of $\frac{\theta}{2}$ for channel added value. In other words, the region for maximal differentiation, in terms of channel added values, is bigger (by exactly $\frac{\theta}{2}$) in a centralized system than that in a monopolist supplier system.

3.2.1. Monopolist Supplier with Retailer Profit Target. While the monopolist supplier has an incentive to optimally differentiate its products, giving monopoly power to a single manufacturer without constraints may result in the manufacturer appropriating significant profits from the downstream retailer. To remedy that situation, we analyze a slight modification on the monopolist supplier system by introducing a guaranteed profit target for the retailer. While closed-form solutions in this case become analytically intractable, we present a numerical analysis in the next section.

4. Model Comparison

In terms of channel profit, we know that the fully centralized system always dominates either the monopolist supplier system or the decentralized system, because all market powers are concentrated in the hands of the monopolist player. Comparing the decentralized system with the monopolist system, however, leads to the following conclusion:

Theorem 4. • *When the channel added value $v - \frac{c_1+c_2}{2} \geq \frac{3}{2}\theta$, channel profit of the monopolist supplier system dominates that of the decentralized system;*

• *When the channel added value $v - \frac{c_1+c_2}{2} \leq \theta$, the monopolist supplier system achieves the same channel profit as that of the decentralized system.*

Proof: See Appendix C. Q.E.D.

The reader may recall from Theorem 1 that no equilibrium exists for the decentralized system when the channel added value is between θ and $\frac{3}{2}\theta$, hence the omission from Theorem 4.

Looking at Theorems 1, 2, and 3 collectively, we are able to summarize the optimal response for each of the three trading relationships under four channel added value regimes.

Observation 1. *Theorems 1, 2, and 3 collectively imply the following:*

• $0 \leq v - \frac{c_1+c_2}{2} < \frac{\theta}{2}$*: The players in all three relationships have sufficient incentives to differentiate, in order to maintain domain separability;*

• $\frac{\theta}{2} \leq v - \frac{c_1+c_2}{2} \leq \theta$*: The players in the decentralized system and the monopolist supplier system have sufficient incentives to differentiate in order to maintain domain separability; the monopolist player in the fully centralized system positions the two products symmetrically and sets prices such that the two domains cover the entire market;*

• $\theta < v - \frac{c_1+c_2}{2} < \frac{3}{2}\theta$*: no equilibrium exists for the decentralized system, but the players in the centralized and monopolist supplier systems have maximal incentives to differentiate, while the retail prices are set such that the two product domains cover the entire market;*

• $\frac{3}{2}\theta \leq v - \frac{c_1+c_2}{2} < \frac{7}{4}\theta$*: The players in all three systems have maximal incentives to differentiate, while the retail prices are set such that the two product domains cover the entire market;*

• $v - \frac{c_1+c_2}{2} \geq \frac{7}{4}\theta$*: The suppliers in the decentralized system are either partially or completely indifferent to differentiate the two products, while the players in the fully centralized system and the monopolist supplier system still have maximal incentives to differentiate. In all cases, the retail products are set such that the two product domains cover the entire market.*

Channel inefficiency arises when the channel profit of the fully centralized monopolist system is strictly greater than a candidate system. There are three potential sources of channel inefficiency, namely 1) double marginalization, 2) positioning distortion, and 3) cost difference.

Losses due to *double marginalization* (Spengler 1950) occur when an upstream player in a two-echelon vertical trading partnership imposes a wholesale price that is strictly greater than its production cost. This is a familiar supply chain inefficiency, and is the main source of channel inefficiency within trading relationships where there are only sufficient incentives to differentiate the products in order to maintain domain separability.

Positioning distortion occurs when the suppliers in an equilibrium that involves a decentralized system are indifferent to the asymmetrically positioned products (i.e., $|l_1 - l_2| \neq \frac{1}{2}$), and yet the optimal retailer profit, and hence channel profit, suffers as a result of this asymmetric positioning. This is a more novel inefficiency that occurs in our model. In certain channel added value regimes (when $v - \frac{c_1+c_2}{2} \geq \frac{7}{4}\theta$), the suppliers simply do not gain or lose by differentiating from their competition, even though the retailer is directly affected by the product positions. We can quantify the loss in channel profit due to positioning distortion as $\frac{\theta}{4} - \frac{\theta}{2}|l_1 - l_2|$, which is derived in Appendix C. Clearly, this quantity drops to zero when the product positions are maximally differentiated, i.e., when $|l_1 - l_2| = \frac{1}{2}$.

Cost difference distortion is also novel and is an inefficiency due to the difference in production costs, i.e., when $c_1 \neq c_2$. This distortion occurs in channel added value regimes where the product domains cover the entire market (when $v - \frac{c_1+c_2}{2} > \theta$). We only consider cost difference distortion under the situation where product positions are maximally differentiated, and attribute the additional loss in efficiency to positioning distortion when the products are asymmetrically positioned. We demonstrate algebraically in Appendix C that cost difference distortion is eliminated when $c_1 = c_2$ and $v - \frac{c_1+c_2}{2} > \theta$.

Table 2 gives a summary of Theorem 4, Observation 1, as well as the main sources for channel inefficiency:

$v - \frac{c_1+c_2}{2} \in$	$[0, \frac{\theta}{2})$	$[\frac{\theta}{2}, \theta]$	$[\frac{3}{2}\theta, \frac{7}{4}\theta)$	$[\frac{7}{4}\theta, \infty)$
Channel profit	C≥MS=D	C≥MS=D	C≥MS≥D	C≥MS≥D
Cause	DM	DM	CD	PD, CD

Table 2 **Comparison of Channel Profits (C - centralized system, D - decentralized system, MS - monopolist supplier model, CD - cost difference, PD - positioning distortion, DM - double marginalization)**

4.1. Numerical Results

We use numerical experiments to compare the channel profits of the centralized, decentralized and monopolist supplier systems. We assume $v = 1$ in all experiments, and consider the following cases for c_1 and c_2, where $c_1 \leq c_2$ without loss of generality:

- $c_1 = 0.4$, $c_2 = 0.6$;
- $c_1 = c_2 = 0.5$.

The set-up of the experiment implies that the channel added value is held constant, i.e., $v - \frac{c_1+c_2}{2} = 0.5$. Lastly, we vary $\theta \geq 0$ to examine all solution regimes.

Table 3 summarizes the result of one numerical experiment ($v = 1, c_1 = 0.4, c_2 = 0.6$). The product differentiation cost parameter θ is varied to reflect the 4 cases discussed in Table 2, in the same order that was first presented. Except for the centralized system, where channel profit equals retailer profit, we present a pair of profit numbers in the parentheses for each system, with the first number corresponding to channel profit and second number corresponding to retailer profit. We also characterize the optimal regions (refer to Figures 5 and 6 for an illustration) that result in an equilibrium for all parties involved in each system.

θ	2.00	0.60	0.30	0.26
$v - \frac{c_1+c_2}{2} \in$	$[0, \frac{\theta}{2})$	$[\frac{\theta}{2}, \theta]$	$[\frac{3}{2}\theta, \frac{7}{4}\theta)$	$[\frac{7}{4}\theta, \infty)$
Centralized profit Optimal region	0.130 1 or 3	0.367 2	0.458 2	0.473 2
Decentralized profit: best As % of centralized Optimal region	(0.098,0.033) (75.4%,25.4%) 1 or 3	(0.325,0.108) (88.6%,29.4%) 1 or 3	(0.444,0.129) (96.9%,28.2%) 2	(0.456,0.179) (96.4%,37.8%) 2
Decentralized profit: worst As % of centralized Optimal region	n/a	n/a	n/a	(0.412,0.135) (87.1%,28.5%) 6
Monopolist supplier profit As % of centralized Optimal region	(0.098,0.033) (75.4%,25.4%) 1 or 3	(0.325,0.108) (88.6%,29.4%) 1 or 3	(0.45,0.083) (98.3%,18.1%) 2	(0.464,0.075) (98.1%,15.9%) 2
Monopolist supplier with retailer target As % of centralized Optimal region	(0.128,0.100) (98.5%,76.9%) 1 or 3	(0.367,0.250) (100.0%,68.1%) 2	(0.458,0.250) (100.0%,54.6%) 2	(0.473,0.150) (100.0%,31.7%) 2

Table 3 Numerical Results ($v = 1$, $c_1 = 0.4$, $c_2 = 0.6$, (#,#)=(channel, retailer) except for the centralized system)

The reader may recall that in a decentralized system, it is optimal to have partial incentives to differentiate the product positions when the channel added value is greater than $\frac{7}{4}\theta$. In this case, the product domains barely touch on one end and overlap on the other end. It is clear from the numerical examples marked "best" and "worst" in Table 3 that suppliers make a total profit of $0.456 - 0.179 = 0.412 - 0.135 = 0.277$ regardless of product positions[8], but the retailer profit, and hence channel profit, suffers as the product distance $|l_1 - l_2|$ declines from a maximum of $\frac{1}{2}$. This is a clear demonstration of the positioning distortion phenomenon, which is mitigated by moving from the decentralized system to the monopolist supplier system.

[8] In fact, the individual supplier profits are also shown to be constant, otherwise we would not have an equilibrium solution for the suppliers.

The cost difference distortion is evident when we compare two sets of production costs (c_1, c_2) under two different channel added value regimes in Table 4. Here the channel inefficiency for the decentralized and monopolist supplier systems is eliminated completely, as the difference between c_1 and c_2 drops to zero. Furthermore, in the cases where production costs are not equal, the cost difference distortion is mitigated when we move again from the decentralized system to the monopolist supplier system.

θ	0.30		0.26	
$v - \frac{c_1+c_2}{2} \in$	$[\frac{3}{2}\theta, \frac{7}{4}\theta)$		$[\frac{7}{4}\theta, \infty)$	
Production costs	$c_1 = 0.4, c_2 = 0.6$	$c_1 = c_2 = 0.5$	$c_1 = 0.4, c_2 = 0.6$	$c_1 = c_2 = 0.5$
Centralized profit Optimal region	0.458 2	0.425 2	0.473 2	0.435 2
Decentralized profit: best As % of centralized Optimal region	(0.444,0.129) (96.9%,28.2%) 2	(0.425,0.125) (100.0%,29.4%) 2	(0.456,0.179) (96.4%,37.8%) 2	(0.435,0.175) (100.0%,40.2%) 2
Monopolist supplier profit As % of centralized Optimal region	(0.450,0.083) (98.3%,18.1%) 2	(0.425,0.075) (100.0%,17.6%) 2	(0.464,0.075) (98.1%,15.9%) 2	(0.435,0.065) (100.0%,14.9%) 2

Table 4 Numerical Results for Cost Difference Distortion ($v = 1$, (#,#)=(channel, retailer))

Comparing retailer profits in the decentralized system with that of the monopolist supplier model, we see that the retailer is less profitable under the latter than the former, especially when the channel added value is high ($v - \frac{c_1+c_2}{2} \geq \frac{3}{2}\theta$). In other words, the retailer suffers when market power is concentrated into the hands of the monopolist supplier. However, the use of a retailer profit target[9] (as seen from the last row of Table 3) limits the market power of the monopolist supplier, thereby increasing channel efficiency. In this particular example, the channel achieves zero loss in all cases except when $\theta = 2$. In addition, while the monopolist supplier system is not able to reduce the loss in efficiency due to double marginalization (see the cases corresponding to $\theta = 2$ and 0.6, where the monopolist supplier system achieve the same channel profit as the decentralized system due to separable product domains), imposing a retailer profit target does significantly reduce, if not eliminate completely, this classic source of channel inefficiency.

In summary, depending on the channel added value regime, when turning over category power to a single supplier, the gain in efficiency from the supplier's incentives to differentiate can offset the loss in efficiency due to positioning distortion and cost difference distortion. These results provide a theoretical explanation for why category management may lead to gains in supply chain efficiency. Moreover, imposing a simple profit target for the retailer can, in many cases, achieve both full supply chain coordination, eliminating all three sources of channel inefficiency including

[9] In the numerical experiments, we set retailer target to 0.1 for $\theta = 2$, 0.25 for $\theta = 0.6$ and 0.3, 0.15 for $\theta = 0.26$.

double marginalization, and a Pareto improving allocation of profits to both the supplier and retailer.

5. Conclusion

We developed a theoretical model to investigate the incentives for coordinating the positioning and pricing of horizontally differentiated products in the context of a vertical trading relationship between a retailer and multiple suppliers. We also compared it to the fully centralized system and the monopolist supplier system, the latter being akin to a common supply chain arrangement called category management.

The models allow us to characterize the channel efficiency of each trading relationship, leading to interesting insights about when such practices may be most effective. In particular, we are able to show that when competing suppliers sold through a common retail intermediary in certain channel added value regimes, it is possible to eliminate, if not significantly reduce, the classical neo-Hotelling incentives to differentiate. The resulting loss in channel profits is called positioning distortion. The incentives to differentiate are recovered, however, if all products are controlled by a single supplier as in category management. The monopolist supplier arrangement is also able to mitigate, but not fully eliminate, cost difference distortion—another source of channel inefficiency that arises when the production costs are not equal. These results provide a theoretical explanation for why category management may lead to gains in supply chain efficiency.

Lastly, we demonstrate via numerical experiments that category management, together with a simple profit target for the retailer, can reduce or eliminate double marginalization, the classic source of channel inefficiency, and in many cases achieve both full supply chain coordination and a Pareto improving allocation of profits to both the supplier and retailer in this vertical trading relationship.

For future research, we hope to extend the model to accommodate more than 2 suppliers and more than 1 retailer, and investigate whether the conclusions we have reached here can be generalized to a more competitive market. We are also looking to investigate a more accurate portrayal of the category captain, which in reality controls the price of its own product but the positioning of all products in the category—in comparison, the monopolist supplier, as we have introduced in this paper, controls the pricing and positioning of all products in its category.

Appendix A: Technical Details of Decentralized System

A.1. Retailer's Problem

Solution to (P1): First observe that the objective function is concave, and the constraint is linear. Therefore, Karush-Kuhn-Tucker (KKT) conditions are both sufficient and necessary. The Lagrangian is

$$\mathcal{L}(\mathbf{p}, \lambda) = (p_1 - w_1)\frac{2(v-p_1)}{\theta} + (p_2 - w_2)\frac{2(v-p_2)}{\theta} + \lambda\left(\frac{1}{2} - \frac{v-p_1}{\theta} - \frac{v-p_2}{\theta}\right),$$

where λ is the Lagrange multiplier. The first order conditions $\nabla_{\mathbf{p}}\mathcal{L}(\mathbf{p},\lambda)=0$ yield the following:

$$p_1 = \frac{v+w_1}{2} + \frac{\lambda}{4};\ p_2 = \frac{v+w_2}{2} + \frac{\lambda}{4}. \tag{1}$$

Assuming that strict complementarity holds, we have the following 2 cases:

1. $\lambda = 0 \Leftrightarrow$ The separable domain solution (S1) is optimal and dominates the barely touching solution (S2).

$$\text{KKT} \Rightarrow \quad \frac{1}{2} - \frac{v-p_1}{\theta} - \frac{v-p_2}{\theta} > 0; \tag{2}$$

$$(1) \Rightarrow \quad p_1^* = \frac{v+w_1}{2};\ p_2^* = \frac{v+w_2}{2}. \tag{3}$$

The corresponding retail profit is

$$\pi^* = \frac{(v-w_1)^2}{2\theta} + \frac{(v-w_2)^2}{2\theta}. \tag{4}$$

Plugging (3) into (2), we obtain

$$v - \frac{w_1+w_2}{2} < \frac{\theta}{2},$$

which represents the corresponding retail added value condition for this case to hold.

2. $\lambda > 0 \Leftrightarrow$ The barely touching solution (S2) is optimal and dominates the separable domain solution (S1).

$$\text{KKT} \Rightarrow \quad \frac{1}{2} - \frac{v-p_1}{\theta} - \frac{v-p_2}{\theta} = 0; \tag{5}$$

$$(1),(5) \Rightarrow \quad \lambda = 2v - \theta - (w_1+w_2) > 0 \ \Leftrightarrow$$

$$v - \frac{w_1+w_2}{2} > \frac{\theta}{2} \text{ (the retail added value condition for this case to hold);}$$

$$p_1^* = v + \frac{w_1}{4} - \frac{w_2}{4} - \frac{\theta}{4};\ p_2^* = v + \frac{w_2}{4} - \frac{w_1}{4} - \frac{\theta}{4}. \tag{6}$$

The corresponding retail profit is

$$\pi^* = v + \frac{(w_1-w_2)^2}{4\theta} - \frac{w_1+w_2}{2} - \frac{\theta}{4}. \tag{7}$$

Weak complementarity ($\lambda = 0$, $\frac{1}{2} - \frac{v-p_1}{\theta} - \frac{v-p_2}{\theta} = 0$) holds at the borderline of the two cases. The retail added value condition in this case is $v - \frac{w_1+w_2}{2} = \frac{\theta}{2}$.

Solution to (P2): It can be verified that the objective function is concave, and the constraint is linear. Therefore, the KKT conditions are both sufficient and necessary. The Lagrangian is

$$\mathcal{L}(\mathbf{p},\lambda) = (p_1-w_1)\left(\frac{1}{2} + \frac{p_2-p_1}{\theta}\right) + (p_2-w_2)\left(\frac{1}{2} + \frac{p_1-p_2}{\theta}\right) + \lambda\left(\frac{v-p_1}{\theta} + \frac{v-p_2}{\theta} - \frac{1}{2}\right).$$

The first order conditions $\nabla_{\mathbf{p}}\mathcal{L}(\mathbf{p},\lambda)=0$ yield

$$p_1 - p_2 = \frac{\theta}{4} + \frac{w_1-w_2}{2} - \frac{\lambda}{2};\ p_1 - p_2 = -\frac{\theta}{4} + \frac{w_1-w_2}{2} + \frac{\lambda}{2}.$$

Clearly, we have that

$$\frac{\theta}{4} - \frac{\lambda}{2} = -\frac{\theta}{4} + \frac{\lambda}{2} \ \Leftrightarrow\ \lambda = \frac{\theta}{2} > 0.$$

Since $\lambda > 0$, the inequality constraint in (P2) is always active by complementarity, i.e. the barely touching solution (S2) will always dominate the overlapping domain solution (S3).

Solution to (P3): First observe that the objective function is the same as in (P1) and is thus concave, and the constraint is linear. Therefore, the KKT conditions are both sufficient and necessary. The Lagrangian is:

$$\mathcal{L}(\mathbf{p},\lambda) = (p_1 - w_1)\frac{2(v-p_1)}{\theta} + (p_2 - w_2)\frac{2(v-p_2)}{\theta} + \lambda\left(|l_1 - l_2| - \frac{v-p_1}{\theta} - \frac{v-p_2}{\theta}\right).$$

The first order conditions $\nabla_{\mathbf{p}}\mathcal{L}(\mathbf{p},\lambda) = 0$ yield the following, just as in (P1):

$$p_1 = \frac{v+w_1}{2} + \frac{\lambda}{4};\ p_2 = \frac{v+w_2}{2} + \frac{\lambda}{4}. \tag{8}$$

Assuming that strict complementarity holds, we have the following 2 cases:

1. $\lambda = 0 \Leftrightarrow$ The separable domain solution (AS1) is optimal and dominates the solution where one end barely touches and the other end does not touch (AS2).

$$\text{KKT} \Rightarrow \quad |l_1 - l_2| - \frac{v-p_1}{\theta} - \frac{v-p_2}{\theta} > 0 \tag{9}$$

$$(8) \Rightarrow \quad p_1^* = \frac{v+w_1}{2};\ p_2^* = \frac{v+w_2}{2}. \tag{10}$$

The corresponding retail profit is

$$\pi^* = \frac{(v-w_1)^2}{2\theta} + \frac{(v-w_2)^2}{2\theta}.$$

Plugging (10) into (9), we obtain

$$v - \frac{w_1+w_2}{2} < \theta|l_1 - l_2|, \tag{11}$$

which represents the corresponding retail added value condition for this case to hold.

2. $\lambda > 0 \Leftrightarrow$ The solution where one end barely touches and the other end does not touch (AS2) is optimal and dominates the separable domain solution (AS1).

$$\text{KKT} \Rightarrow \quad |l_1 - l_2| - \frac{v-p_1}{\theta} - \frac{v-p_2}{\theta} = 0 \tag{12}$$

$$(8), (12) \Rightarrow \quad \lambda = 2v - 2\theta|l_1 - l_2| - (w_1 + w_2) > 0 \ \Leftrightarrow$$

$$v - \frac{w_1+w_2}{2} > \theta|l_1 - l_2| \text{ (the retail added value condition for this case to hold);} \tag{13}$$

$$p_1^* = v + \frac{w_1}{4} - \frac{w_2}{4} - \frac{\theta}{2}|l_1 - l_2|;\ p_2^* = v + \frac{w_2}{4} - \frac{w_1}{4} - \frac{\theta}{2}|l_1 - l_2|. \tag{14}$$

The corresponding retail profit is:

$$\pi^* = 2v|l_1 - l_2| - (w_1 + w_2)|l_1 - l_2| - \theta|l_1 - l_2|^2 + \frac{(w_1 - w_2)^2}{4\theta}.$$

Weak complementarity ($\lambda = 0$, $|l_1 - l_2| - \frac{v-p_1}{\theta} - \frac{v-p_2}{\theta} = 0$) holds at the borderline of the two cases. The retail added value condition in this case is $v - \frac{w_1+w_2}{2} = \theta|l_1 - l_2|$.

Solution to (P4): It is easy to verify that the objective function is concave, and the constraints are linear. As a result, KKT conditions are both sufficient and necessary. Let μ_1 be the lagrange multiplier associated with the lower bound and μ_2 be lagrange multiplier associated with the upper bound. The Lagrangian is:

$$\begin{aligned}\mathcal{L}(\mathbf{p},\mu_1,\mu_2) =& (p_1 - w_1)\left(\frac{1}{2}|l_1 - l_2| + \frac{p_2 - 3p_1}{2\theta} + \frac{v}{\theta}\right) + (p_2 - w_2)\left(\frac{1}{2}|l_1 - l_2| + \frac{p_1 - 3p_2}{2\theta} + \frac{v}{\theta}\right)\\ &+ \mu_1\left(\frac{v-p_1}{\theta} + \frac{v-p_2}{\theta} - |l_1 - l_2|\right) + \mu_2\left(1 - |l_1 - l_2| - \frac{v-p_1}{\theta} - \frac{v-p_2}{\theta}\right).\end{aligned}$$

The first order conditions $\nabla_{\mathbf{p}}\mathcal{L}(\mathbf{p},\mu_1,\mu_2)=0$

$$\Rightarrow \quad \begin{cases}\theta|l_1-l_2|+2p_2-6p_1+2v+3w_1-w_2-2\mu_1+2\mu_2=0,\\ \theta|l_1-l_2|+2p_1-6p_2+2v+3w_2-w_1-2\mu_1+2\mu_2=0;\end{cases} \tag{15}$$

$$\Rightarrow \quad p_1-p_2=\frac{w_1-w_2}{2}. \tag{16}$$

Assuming that strict complementarity holds, we have the following cases:

$$\mu_1>0 \Rightarrow \frac{v-p_1}{\theta}+\frac{v-p_2}{\theta}-|l_1-l_2|=0; \tag{17}$$

$$\mu_1=0 \Rightarrow \frac{v-p_1}{\theta}+\frac{v-p_2}{\theta}-|l_1-l_2|>0; \tag{18}$$

$$\mu_2>0 \Rightarrow 1-|l_1-l_2|-\frac{v-p_1}{\theta}-\frac{v-p_2}{\theta}=0; \tag{19}$$

$$\mu_2=0 \Rightarrow 1-|l_1-l_2|-\frac{v-p_1}{\theta}-\frac{v-p_2}{\theta}>0. \tag{20}$$

Note first that the upper and lower bounds cannot both be simultaneously active when product positions are asymmetric. This implies that μ_1 and μ_2 cannot both be >0. We have the following 3 cases:

1. $\mu_1>0$, $\mu_2=0 \Leftrightarrow$ The solution where one end of the product domains barely touches while the other end does not touch (AS2) is optimal and dominates (AS3) and (AS4).

$$(16),(17) \Rightarrow \quad p_1^*=v+\frac{w_1-w_2}{4}-\frac{\theta}{2}|l_1-l_2|;\ p_2^*=v-\frac{w_1-w_2}{4}-\frac{\theta}{2}|l_1-l_2|. \tag{21}$$

The corresponding retail profit is

$$\pi^*=(2v-w_1-w_2)|l_1-l_2|-\theta|l_1-l_2|^2+\frac{(w_1-w_2)^2}{4\theta}.$$

As a sanity check, observe that the results obtained above are identical to (14) when solving (P3). We will check that (20) still holds by plugging in (21), and indeed by symmetry,

$$1-|l_1-l_2|-\frac{v-p_1^*}{\theta}-\frac{v-p_2^*}{\theta}=1-2|l_1-l_2|>0.$$

Plugging (21) into either equation in (15), we have that

$$2\mu_1=3\theta|l_1-l_2|+w_1+w_2-2v>0 \Leftrightarrow v-\frac{w_1+w_2}{2}<\frac{3}{2}\theta|l_1-l_2|,$$

which we then intersect with (13) to obtain the retail added value condition for this case to hold, i.e.,

$$\theta|l_1-l_2|<v-\frac{w_1+w_2}{2}<\frac{3}{2}\theta|l_1-l_2|.$$

2. $\mu_1=0$, $\mu_2=0 \Leftrightarrow$ The solution where one end overlaps while the other end does not touch (AS3) is optimal and dominates (AS2) and (AS4).

$$(15) \Rightarrow \quad p_1^*=\frac{v+w_1}{2}+\frac{\theta}{4}|l_1-l_2|;\ p_2^*=\frac{v+w_2}{2}+\frac{\theta}{4}|l_1-l_2|. \tag{22}$$

The corresponding optimal retail profit is

$$\pi^*=\frac{1}{4\theta}\left(v-w_1+\frac{\theta}{2}|l_1-l_2|\right)^2+\frac{1}{4\theta}\left(v-w_2+\frac{\theta}{2}|l_1-l_2|\right)^2+\frac{1}{8\theta}(w_1-w_2)^2.$$

Plugging (22) into (18) and (20) gives

$$\frac{3}{2}\theta|l_1-l_2|<v-\frac{w_1+w_2}{2}<\theta-\frac{1}{2}\theta|l_1-l_2|,$$

which represents the retail added value condition for this case. Observe that (13) is not violated.

3. $\mu_1 = 0$, $\mu_2 > 0 \Leftrightarrow$ The solution where one end overlaps while the other end barely touches (AS4) is optimal and dominates (AS2) and (AS3).

$$(16),(19) \Rightarrow \quad p_1^* = \frac{w_1 - w_2}{4} + v + \frac{\theta}{2}|l_1 - l_2| - \frac{\theta}{2};\ p_2^* = \frac{w_2 - w_1}{4} + v + \frac{\theta}{2}|l_1 - l_2| - \frac{\theta}{2}. \tag{23}$$

The corresponding retail profit is

$$\pi^* = v - \frac{\theta}{2} + \frac{\theta}{2}|l_1 - l_2| - \frac{w_1 + w_2}{2} + \frac{(w_1 - w_2)^2}{4\theta}. \tag{24}$$

Plugging (23) into either equation in (15) gives the retail added value condition for this case, i.e.,

$$2\mu_2 = 2v - 2\theta + \theta|l_1 - l_2| - w_1 - w_2 > 0 \ \Leftrightarrow\ v - \frac{w_1 + w_2}{2} > \theta - \frac{1}{2}\theta|l_1 - l_2|.$$

Observe that (13) is not violated.

Weak complementarity holds at the borderline of (AS2) and (AS3), i.e. $v - \frac{w_1+w_2}{2} = \frac{3}{2}\theta|l_1 - l_2|$, and that of (AS3) and (AS4), i.e. $v - \frac{w_1+w_2}{2} = \theta - \frac{1}{2}\theta|l_1 - l_2|$.

Solution to (P5): First observe that the objective function is the same as in (P2) and is thus concave, and the constraint is linear. Therefore, the KKT conditions are both sufficient and necessary. The Lagrangian is

$$\mathcal{L}(\mathbf{p}, \lambda) = (p_1 - w_1)\left(\frac{1}{2} + \frac{p_2 - p_1}{\theta}\right) + (p_2 - w_2)\left(\frac{1}{2} + \frac{p_1 - p_2}{\theta}\right) + \lambda\left(\frac{v - p_1}{\theta} + \frac{v - p_2}{\theta} - 1 + |l_1 - l_2|\right).$$

The first order conditions $\nabla_{\mathbf{p}}\mathcal{L}(\mathbf{p}, \mu_1, \mu_2) = 0$ yield

$$p_1 - p_2 = \frac{\theta}{4} + \frac{w_1 - w_2}{2} - \frac{\lambda}{2};\ p_1 - p_2 = -\frac{\theta}{4} + \frac{w_1 - w_2}{2} + \frac{\lambda}{2}.$$

Clearly, we have that

$$\frac{\theta}{4} - \frac{\lambda}{2} = -\frac{\theta}{4} + \frac{\lambda}{2} \ \Leftrightarrow\ \lambda = \frac{\theta}{2} > 0.$$

Since $\lambda > 0$, the inequality constraint in (P5) is always active by complementarity, i.e. the solution where one end overlaps while the other end barely touches (AS4) will always dominate solution where both ends overlap (AS5).

Proof of Proposition 1: For symmetrically positioned products, solution to (P2) establishes that the overlapping domain solution (S3) is always dominated by (S2), when the condition for (S3) to be realizable holds. The comparison is thus between separable domain (S1) and barely touching (S2). Solution to (P1) establishes the retail added value regions 1 and 2 in the proposition statement and determines when (S1) dominates (S2) and vice versa.

For asymmetrically positioned products, solution to (P5) establishes that (AS5), the case where product domains overlap on both ends, is always dominated by (AS4). Therefore, we need only compare (AS1) through (AS4). According to the solution to (P3) and when (11) is satisfied, only (AS1) and (AS2) are realizable situations. Furthermore, (11) is the condition that establishes (AS1)'s dominance over (AS2), corresponding to retail added value region 3 in the proposition statement.

Conversely when (13) is satisfied, (AS1) is dominated by (AS2) and can be removed from further consideration. Condition (13) is the link between the solutions to (P3) and (P4). The various cases in the solution to (P4) correspond exactly to retail added value regions 4, 5, and 6 in the proposition statement and determines when (AS2), (AS3), or (AS4) is the optimal solution in each region.

A.2. Suppliers' Game

Solution to (SP13): The objective functions are concave. The first order conditions yield the following:

$$w_1^* = \frac{v + c_1}{2};\ w_2^* = \frac{v + c_2}{2}. \tag{25}$$

And therefore,

$$w_1^* + w_2^* = v + \frac{c_1 + c_2}{2}; \tag{26}$$
$$\pi_1^* = \frac{(v - c_1)^2}{4\theta};\ \pi_2^* = \frac{(v - c_2)^2}{4\theta}. \tag{27}$$

Solution to (SP26): The objective functions are concave. The first order conditions yield the following:

$$w_1^* = \frac{\theta + c_1 + \bar{w}_2}{2};\ w_2^* = \frac{\theta + c_2 + \bar{w}_1}{2}.$$

Solving jointly by letting $\bar{w}_i = w_i^*$,

$$w_1^* = \theta + \frac{2}{3}c_1 + \frac{1}{3}c_2;\ w_2^* = \theta + \frac{1}{3}c_1 + \frac{2}{3}c_2; \tag{28}$$
$$\Rightarrow\quad w_1^* + w_2^* = 2\theta + c_1 + c_2; \tag{29}$$
$$\pi_1^* = 2\theta\left(\frac{1}{2} + \frac{c_2 - c_1}{6\theta}\right)^2;\ \pi_2^* = 2\theta\left(\frac{1}{2} + \frac{c_1 - c_2}{6\theta}\right)^2. \tag{30}$$

Solution to (SP4): The objective functions are concave. The first order conditions yield the following:

$$w_1^* = \frac{2\theta|l_1 - l_2| + c_1 + \bar{w}_2}{2};\ w_2^* = \frac{2\theta|l_1 - l_2| + c_2 + \bar{w}_1}{2}.$$

Solving jointly by letting $\bar{w}_i = w_i^*$,

$$w_1^* = 2\theta|l_1 - l_2| + \frac{2}{3}c_1 + \frac{1}{3}c_2;\ w_2^* = 2\theta|l_1 - l_2| + \frac{1}{3}c_1 + \frac{2}{3}c_2; \tag{31}$$
$$\Rightarrow\quad w_1^* + w_2^* = 4\theta|l_1 - l_2| + c_1 + c_2; \tag{32}$$
$$\pi_1^* = 2\theta\left(|l_1 - l_2| + \frac{c_2 - c_1}{6\theta}\right)^2;\ \pi_2^* = 2\theta\left(|l_1 - l_2| + \frac{c_1 - c_2}{6\theta}\right)^2. \tag{33}$$

Solution to (SP5): The objective functions are concave. First order conditions yield the following:

$$w_1^* = \frac{\theta|l_1 - l_2| + 2v + \bar{w}_2 + 3c_1}{6};\ w_2^* = \frac{\theta|l_1 - l_2| + 2v + \bar{w}_1 + 3c_2}{6}.$$

Solving jointly by letting $\bar{w}_i = w_i^*$,

$$w_1^* = \frac{1}{5}\theta|l_1 - l_2| + \frac{2}{5}v + \frac{18}{35}c_1 + \frac{3}{35}c_2;\ w_2^* = \frac{1}{5}\theta|l_1 - l_2| + \frac{2}{5}v + \frac{18}{35}c_2 + \frac{3}{35}c_1; \tag{34}$$
$$\Rightarrow\quad w_1^* + w_2^* = \frac{2}{5}\theta|l_1 - l_2| + \frac{4}{5}v + \frac{3}{5}(c_1 + c_2); \tag{35}$$
$$\pi_1^* = \frac{3}{4900\theta}(7\theta|l_1 - l_2| + 14v - 17c_1 + 3c_2)^2;\ \pi_2^* = \frac{3}{4900\theta}(7\theta|l_1 - l_2| + 14v - 17c_2 + 3c_1)^2. \tag{36}$$

Proof of Lemma 1: Region 1 ↔ region 2. Consider supplier 1's problem in region 2. Rewrite the right boundary of region 2 as $w_2 = 2v - \theta - w_1$. Plugging it into the objective function in region 2 (SP26) yields the objective function in region 1 (SP13). Thus the transition from region 1 to region 2 is continuous. A similar argument holds for supplier 2.

Region 3 ↔ region 4. Consider supplier 1's problem in region 4. Rewrite the right boundary of region 4 as $w_2 = 2v - 2\theta|l_1 - l_2| - w_1$. Plugging it into the objective function in region 4 (SP4) yields the objective function in region 3 (SP13). Thus the transition from region 3 to region 4 is continuous. A similar argument holds for supplier 2.

Region 4 ↔ region 5 ↔ region 6. Consider supplier 1's problem in regions 4, 5 and 6. Rewrite the left boundary of region 4 or right boundary of region 5 as $w_2 = 2v - 3\theta|l_1 - l_2| - w_1$. Plugging it into the objective function in region 4 (SP4) and the objective function in region 5 (SP5) yields the same objective value. Rewrite the right boundary of region 6 or left boundary of region 5 as $w_2 = 2v - 2\theta + 2\theta|l_1 - l_2| - w_1$. Plugging it into the objective function in region 5 (SP5) and the objective function in region 6 (SP26) also yields the same objective value. Thus the transition from region 4 to region 5 and then to region 6 is continuous. A similar argument holds for supplier 2.

Proof of Lemma 2: Region 4's optimal objective functions for the suppliers are stated in (33). Both objective functions increase in $|l_1 - l_2|$ if

$$\begin{cases} |l_1 - l_2| \geq \frac{c_1 - c_2}{6\theta} \\ |l_1 - l_2| \geq \frac{c_2 - c_1}{6\theta} \end{cases} \Rightarrow |c_1 - c_2| \leq 6\theta|l_1 - l_2|,$$

which is always satisfied because it is exactly suppliers' participating constraints in region 4[10].

Region 5's optimal objective function for each supplier is described respectively by (36). Both objective functions increase in $|l_1 - l_2|$ if

$$\begin{cases} 7\theta|l_1 - l_2| \geq -14v + 17c_1 - 3c_2 \\ 7\theta|l_1 - l_2| \geq -14v + 17c_2 - 3c_1 \end{cases} \Rightarrow \begin{cases} 17c_1 - 3c_2 \leq 7\theta|l_1 - l_2| + 14v; \\ 17c_2 - 3c_1 \leq 7\theta|l_1 - l_2| + 14v. \end{cases}$$

The above is always satisfied because it is exactly suppliers' participating constraints in region 5[11]. Again there is incentive to increase $|l_1 - l_2|$ as much as possible.

Proof of Theorem 1: First we establish a few inequalities on the channel added value $v - \frac{c_1+c_2}{2}$ to characterize the 6 solution regions (refer to Figures 5 and 6, which are based on retail added value thresholds $v - \frac{w_1+w_2}{2}$). By backward induction, we want to show that the suppliers have the incentives to choose the solution regions that the retailer will also choose, thereby leading to equilibrium results. These inequalities, summarized in Table 5 below, are both sufficient and necessary. We observe that the only two channel added value thresholds absent of $|l_1 - l_2|$ are θ and $\frac{3}{2}\theta$, which give rise to the following segments: $v - \frac{c_1+c_2}{2} \geq \frac{3}{2}\theta$, $\theta < v - \frac{c_1+c_2}{2} < \frac{3}{2}\theta$, and $v - \frac{c_1+c_2}{2} \leq \theta$.

[10] Suppliers will participate if they can make a positive profit by having the product on the market, i.e. $w_i \geq c_i$ for each i. The suppliers' participating constraints in region 4 are obtained by plugging (31), respectively for each supplier, into the previously stated inequality.

[11] The suppliers' participating constraints in region 5 are obtained by plugging (34), respectively for each supplier, into the inequality $w_i \geq c_i$.

Regions	Characterization of Solution Regions	Representation with Inequalities
1, 3	Reg. 1 is not in eqm. ⇒ Reg. 2 may be; Reg. 3 is not in eqm. ⇒ Reg. 4, 5, 6 may be	$v-\frac{w_1^*+w_2^*}{2} > \frac{\theta}{2},\ v-\frac{w_1^*+w_2^*}{2} > \theta\lvert l_1-l_2\rvert \overset{(26)}{\Longrightarrow} v-\frac{c_1+c_2}{2} > \theta$
	Both regions are in eqm.	$v-\frac{w_1^*+w_2^*}{2} \le \frac{\theta}{2},\ v-\frac{w_1^*+w_2^*}{2} \le \theta\lvert l_1-l_2\rvert \overset{(26)}{\Longrightarrow} v-\frac{c_1+c_2}{2} \le 2\theta\lvert l_1-l_2\rvert$
	Reg. 1 is in eqm.; Reg. 3 is not in eqm. ⇒ Reg. 4, 5, 6 may be	$2\theta\lvert l_1-l_2\rvert < v-\frac{c_1+c_2}{2} \le \theta$
2, 6	Reg. 2 is not in eqm. ⇒ Reg. 1 may be; Reg. 6 is not in eqm. ⇒ Reg. 3, 4, 5 may be;	$v-\frac{w_1^*+w_2^*}{2} < \frac{\theta}{2},\ v-\frac{w_1^*+w_2^*}{2} < \theta-\frac{\theta}{2}\lvert l_1-l_2\rvert \overset{(29)}{\Longrightarrow} v-\frac{c_1+c_2}{2} < \frac{3}{2}\theta$
	Both regions are in eqm.	$v-\frac{w_1^*+w_2^*}{2} \ge \frac{\theta}{2},\ v-\frac{w_1^*+w_2^*}{2} \ge \theta-\frac{\theta}{2}\lvert l_1-l_2\rvert \overset{(29)}{\Longrightarrow} v-\frac{c_1+c_2}{2} \ge 2\theta-\frac{\theta}{2}\lvert l_1-l_2\rvert$
	Reg. 2 is in eqm.; Reg. 6 is not in eqm. ⇒ Reg. 3, 4, 5 may be	$\frac{3}{2}\theta \le v-\frac{c_1+c_2}{2} < 2\theta-\frac{\theta}{2}\lvert l_1-l_2\rvert$
4	Reg. 4 is not in eqm. ⇒ Reg. 5, 6 may be	$v-\frac{w_1^*+w_2^*}{2} > \frac{3}{2}\theta\lvert l_1-l_2\rvert \overset{(32)}{\Longrightarrow} v-\frac{c_1+c_2}{2} > \frac{7}{2}\theta\lvert l_1-l_2\rvert$
	Reg. 4 is not in eqm. ⇒ Reg. 3 may be	$v-\frac{w_1^*+w_2^*}{2} < \theta\lvert l_1-l_2\rvert \overset{(32)}{\Longrightarrow} v-\frac{c_1+c_2}{2} < 3\theta\lvert l_1-l_2\rvert$
	Reg. 4 is not in eqm. by Lemma 2	$3\theta\lvert l_1-l_2\rvert \le v-\frac{c_1+c_2}{2} \le \frac{7}{2}\theta\lvert l_1-l_2\rvert$
5	Reg. 5 is not in eqm. ⇒ Reg. 6 may be	$v-\frac{w_1^*+w_2^*}{2} > \theta-\frac{\theta}{2}\lvert l_1-l_2\rvert \overset{(35)}{\Longrightarrow} v-\frac{c_1+c_2}{2} > \frac{5}{3}\theta-\frac{\theta}{2}\lvert l_1-l_2\rvert$
	Reg. 5 is not in eqm. ⇒ Reg. 3, 4 may be	$v-\frac{w_1^*+w_2^*}{2} < \frac{3}{2}\theta\lvert l_1-l_2\rvert \overset{(35)}{\Longrightarrow} v-\frac{c_1+c_2}{2} < \frac{17}{6}\theta\lvert l_1-l_2\rvert$
	Reg. 5 is not in eqm. by Lemma 2	$\frac{17}{6}\theta\lvert l_1-l_2\rvert \le v-\frac{c_1+c_2}{2} \le \frac{5}{3}\theta-\frac{\theta}{2}\lvert l_1-l_2\rvert$

Table 5 Characterization of Solution Regions

Case I: When $v-\frac{c_1+c_2}{2} \ge \frac{3}{2}\theta$, we compare the lower bound $\frac{3}{2}\theta$ for this case with the inequalities in Table 5. For the two regions (region 1 and 2) with symmetric product positioning, we observe that the suppliers' optimal solution to region 2 is in equilibrium, but the suppliers' optimal solution to region 1 is not. We know that when region 1 is not in equilibrium, the suppliers' equilibrium solution may be in region 2, by Lemma 1 and Table 5—this turns out to be the case. For the suppliers' solutions to the regions with asymmetric product positioning, we consider the following cases on region 6:

1. When $|l_1-l_2| \in \left[\frac{c_1+c_2-2v+4\theta}{\theta}, \frac{1}{2}\right]$ or equivalently $v-\frac{c_1+c_2}{2} \ge 2\theta-\frac{\theta}{2}|l_1-l_2|$, the suppliers' solution to region 6, equivalent in objective value of region 2 by (SP26), is in equilibrium according to Table 5. The suppliers' solution to region 3 is not in equilibrium—the equilibrium may instead lie in regions 4, 5, or 6, by Lemma 1 and Table 5. The fact that only $v-\frac{c_1+c_2}{2} > \frac{7}{2}\theta|l_1-l_2|$ for region 4 has a non-empty intersection with $v-\frac{c_1+c_2}{2} \ge 2\theta-\frac{\theta}{2}|l_1-l_2|$ means region 4 is also not in equilibrium, leaving only regions 5 and 6 as candidates for an equilibrium. Likewise, we argue that region 5 is not in equilibrium, because only $v-\frac{c_1+c_2}{2} > \frac{5}{3}\theta-\frac{\theta}{2}|l_1-l_2|$ for region 5 has a non-empty intersection with $v-\frac{c_1+c_2}{2} \ge 2\theta-\frac{\theta}{2}|l_1-l_2|$. We can conclude that the suppliers' solutions in regions 2 and 6 are the only possible equilibria.

2. When $|l_1-l_2| \in \left(0, \frac{c_1+c_2-2v+4\theta}{\theta}\right)$ or equivalently $\frac{3\theta}{2} \le v-\frac{c_1+c_2}{2} < 2\theta-\frac{\theta}{2}|l_1-l_2|$, the suppliers' solution to region 6 is not in equilibrium—the equilibrium may instead lie in regions 3, 4, or 5, by Lemma 1 and Table 5. We also know that the suppliers' solution to region 3 under the condition imposed on the channel added value is not in equilibrium, leaving regions 4, 5, or 6 as candidates for an equilibrium. Combining the two observations, we need only consider regions 4 and 5. Now by Lemma 2, neither region 4 nor region 5 can form an equilibrium, because there is incentive to maximally differentiate the product positions. Therefore, the only equilibrium lies in region 2.

Furthermore, the minimal set for region 6 to be in equilibrium requires $v - \frac{c_1+c_2}{2} \geq 2\theta$ (by setting $|l_1 - l_2| \downarrow 0$ in the channel added value condition that defines sub-case 1), while the minimal set for region 6 to not be in equilibrium requires $\frac{3\theta}{2} \leq v - \frac{c_1+c_2}{2} < \frac{7\theta}{4}$ (by setting $|l_1 - l_2| = \frac{1}{2}$ in the channel added value condition that defines sub-case 2). Observe that in these two sets, the choice of $|l_1 - l_2|$ can be arbitrary without affecting the solution characteristics (in or not in equilibrium) of region 6. Lastly for the segment $\frac{7\theta}{4} \leq v - \frac{c_1+c_2}{2} < 2\theta$, region 6 can be in or not in equilibrium depending on the two cases on $|l_1 - l_2|$ that we previously examined.

Case II: When $v - \frac{c_1+c_2}{2} \leq \theta$, we compare the upper bound θ for this case with the inequalities in Table 5. For the two regions (region 1 and 2) with symmetric product positioning, we observe that the suppliers' solution to region 1 is in equilibrium, but region 2 is not in equilibrium. We know that when region 2 is not in equilibrium, the suppliers' equilibrium solution may be in region 1, by Lemma 1 and Table 5—this turns out to be the case. For the suppliers' solutions to the regions with asymmetric product positioning, we consider the following cases on region 3:

1. When $|l_1 - l_2| \in \left[\frac{2v-c_1-c_2}{4\theta}, \frac{1}{2}\right]$ or equivalently $v - \frac{c_1+c_2}{2} \leq 2\theta|l_1 - l_2|$, the suppliers' solution to region 3, equivalent in objective value of region 1 by (SP13), is in equilibrium according to Table 5. The suppliers' solution to region 6 is not in equilibrium—the equilibrium may instead lie in regions 3, 4, or 5, by Lemma 1 and Table 5. The fact that only $v - \frac{c_1+c_2}{2} < \frac{17}{6}\theta|l_1 - l_2|$ for region 5 has a non-empty intersection with $v - \frac{c_1+c_2}{2} \leq 2\theta|l_1 - l_2|$ means region 5 is also not in equilibrium, leaving only regions 3 and 4 as possible candidates for an equilibrium. Likewise, we argue that region 4 is not in equilibrium, because only $v - \frac{c_1+c_2}{2} < 3\theta|l_1 - l_2|$ for region 4 has a non-empty intersection with $v - \frac{c_1+c_2}{2} \leq 2\theta|l_1 - l_2|$. We can conclude that the suppliers' solutions to regions 1 and 3 are the only possible equilibria.

2. When $|l_1 - l_2| \in \left(0, \frac{2v-c_1-c_2}{4\theta}\right)$ or equivalently $v - \frac{c_1+c_2}{2} > 2\theta|l_1 - l_2|$, the suppliers' solution to region 3 is not in equilibrium—the equilibrium may instead lie in regions 4, 5, or 6, by Lemma 1 and Table 5. We also know that the suppliers' solution to region 6 under the condition imposed on the channel added value is not in equilibrium, leaving regions 3, 4, or 5 as candidates for an equilibrium. Combining the two observations, we need only consider regions 4 and 5. Now by Lemma 2, neither region 4 nor region 5 can form an equilibrium, because there is incentive to maximally differentiate the product positions. Therefore, the only equilibrium lies in region 1.

Case III: When $\theta < v - \frac{c_1+c_2}{2} < \frac{3}{2}\theta$, neither region 2 nor region 6 is in equilibrium according to Table 5. Similarly neither region 1 nor region 3 is in equilibrium. Lastly by Lemma 2, neither region 4 nor region 5 can form an equilibrium. Therefore, an equilibrium does not exist in this case.

Appendix B: Technical Details of Benchmark Systems

Proof of Theorem 2: First we establish a side proposition.

Proposition 2. *When product positions are symmetric, the retailer's optimal solution can be segmented as the following:*

- *Region 1 ($v - \frac{c_1+c_2}{2} < \frac{\theta}{2}$) - Separable solution is optimal;*
- *Region 2 ($v - \frac{c_1+c_2}{2} > \frac{\theta}{2}$) - Barely touching solution is optimal.*

When product positions are asymmetric, the retailer's optimal solution can be segmented as the following:

- *Region 3* ($v - \frac{c_1+c_2}{2} < \theta|l_1 - l_2|$) *- Separable solution is optimal;*
- *Region 4* ($\theta|l_1 - l_2| < v - \frac{c_1+c_2}{2} < \frac{3}{2}\theta|l_1 - l_2|$) *- The solution where one end barely touches and the other end does not touch is optimal;*
- *Region 5* ($\frac{3}{2}\theta|l_1 - l_2| < v - \frac{c_1+c_2}{2} < \theta - \frac{1}{2}\theta|l_1 - l_2|$) *- The solution where one end overlaps and the other end does not touch is optimal;*
- *Region 6* ($v - \frac{c_1+c_2}{2} > \theta - \frac{1}{2}\theta|l_1 - l_2|$) *- The solution where one end overlaps and the other end barely touches is optimal.*

The solution to each region is summarized in Table 6.

Proof: We simply replace w_i in Proposition 1 with c_i. Q.E.D.

Regions	p_1^*	p_2^*	π^*
1, 3	$\frac{v+c_1}{2}$	$\frac{v+c_2}{2}$	$\frac{(v-c_1)^2}{2\theta} + \frac{(v-c_2)^2}{2\theta}$
2	$v + \frac{c_1}{4} - \frac{c_2}{4} - \frac{\theta}{4}$	$v + \frac{c_2}{4} - \frac{c_1}{4} - \frac{\theta}{4}$	$v + \frac{(c_1-c_2)^2}{4\theta} - \frac{c_1+c_2}{2} - \frac{\theta}{4}$
4	$v + \frac{c_1-c_2}{4} - \frac{\theta}{2}\|l_1 - l_2\|$	$v - \frac{c_1-c_2}{4} - \frac{\theta}{2}\|l_1 - l_2\|$	$(2v - c_1 - c_2)\|l_1 - l_2\| - \theta\|l_1 - l_2\|^2 + \frac{(c_1-c_2)^2}{4\theta}$
5	$\frac{v+c_1}{2} + \frac{\theta}{4}\|l_1 - l_2\|$	$\frac{v+c_2}{2} + \frac{\theta}{4}\|l_1 - l_2\|$	$\frac{1}{4\theta}\left(v - c_1 + \frac{\theta}{2}\|l_1 - l_2\|\right)^2 + \frac{1}{4\theta}\left(v - c_2 + \frac{\theta}{2}\|l_1 - l_2\|\right)^2 + \frac{1}{8\theta}(c_1 - c_2)^2$
6	$\frac{c_1-c_2}{4} + v + \frac{\theta}{2}\|l_1 - l_2\| - \frac{\theta}{2}$	$\frac{c_2-c_1}{4} + v + \frac{\theta}{2}\|l_1 - l_2\| - \frac{\theta}{2}$	$v - \frac{\theta}{2} + \frac{\theta}{2}\|l_1 - l_2\| - \frac{c_1+c_2}{2} + \frac{(c_1-c_2)^2}{4\theta}$

Table 6 Summary of Results for Retailer's Problem in Fully Centralized System

We can now show that the incentives exist for maximum product differentiation in regions 4, 5 and 6, if the respective solutions lie in the interior of the said regions. In other words, it is optimal for the monopolist player to maximally differentiate the two products, i.e. $|l_1 - l_2| = \frac{1}{2}$. The optimal profit for each region can be found under the column π^* in Table 6.

Region 4's optimal profit increases in $|l_1 - l_2|$ if

$$\frac{\partial \pi^*}{\partial |l_1 - l_2|} > 0 \quad \Rightarrow \quad v - \frac{c_1 + c_2}{2} > \theta|l_1 - l_2|,$$

which turns out to be the left boundary of region 4 (see Figure 6 with w_i replaced with c_i) and is thus always satisfied. When $|l_1 - l_2| \uparrow \frac{1}{2}$, region 4's optimal profit becomes that of region 2. We can therefore conclude that region 4's solution is always dominated by region 2.

Region 5's optimal profit increases in $|l_1 - l_2|$ if

$$\frac{\partial \pi^*}{\partial |l_1 - l_2|} > 0 \quad \Rightarrow \quad v - \frac{c_1 + c_2}{2} > -\frac{1}{2}\theta|l_1 - l_2|,$$

which is always satisfied because $c_i \leq v$. Under $|l_1 - l_2| \uparrow \frac{1}{2}$, region 5 becomes degenerate in the limit, i.e.

$$c_2 = 2v - \frac{3\theta}{2} - c_1. \tag{37}$$

Region 5's optimal profit then becomes $\frac{(v-c_1)^2}{\theta} - \frac{3}{2}(v - c_1) + \frac{17\theta}{16}$, which is the same as region 2's optimal profit with (37) plugged in. Therefore region 5's solution is always dominated by region 2.

Region 6's optimal profit is linear in $|l_1 - l_2|$. The coefficient in front of $|l_1 - l_2|$ is $\frac{\theta}{2} > 0$. Thus, region 6's optimal profit always increases in $|l_1 - l_2|$. When $|l_1 - l_2| \uparrow \frac{1}{2}$, region 6's optimal profit becomes that of region 2 in the limit. We can therefore conclude that region 6's solution is always dominated by region 2.

When $|l_1 - l_2| = \frac{1}{2}$, we have established previously that regions 4, 5, and 6 effectively consolidate into a region 2 (barely touching on both ends) solution, which is governed by the channel added value condition $v - \frac{c_1+c_2}{2} > \frac{\theta}{2}$. On the other hand, when $v - \frac{c_1+c_2}{2} < \frac{\theta}{2}$, we end up with a solution that corresponds to either region 1 or region 3 (separable product domains)[12]. We can therefore conclude with the two segments defined in the theorem statement.

Solution to (SP13′): The objective function is concave, and the constraint is linear. Therefore, the KKT conditions are both sufficient and necessary. The Lagrangian is:

$$\mathcal{L}(\mathbf{w}, \lambda) = (w_1 - c_1)\frac{v - w_1}{\theta} + (w_2 - c_2)\frac{v - w_2}{\theta} + \lambda(w_1 - w_2 - \alpha).$$

The KKT conditions $\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}, \lambda) = 0$ yield the following:

$$w_1 = \frac{v + c_1 + \lambda\theta}{2};\ w_2 = \frac{v + c_2 + \lambda\theta}{2}. \tag{38}$$

Assuming that strict complementarity holds, we have two cases to consider:

1.$\lambda = 0 \Leftrightarrow$ interior solution is optimal.

$$\begin{aligned}
(38) \Rightarrow\quad & w_1^* = \frac{v + c_1}{2};\ w_2^* = \frac{v + c_2}{2}; && (39)\\
\Rightarrow\quad & w_1^* + w_2^* = v + \frac{c_1 + c_2}{2} > \alpha; &&\\
\Rightarrow\quad & \text{For region 1, } v - \frac{c_1 + c_2}{2} < \theta;\ \text{for region 3, } v - \frac{c_1 + c_2}{2} < 2\theta|l_1 - l_2|; &&\\
\Rightarrow\quad & \pi^* = \frac{(v - c_1)^2}{4\theta} + \frac{(v - c_2)^2}{4\theta}. && (40)
\end{aligned}$$

2.$\lambda > 0 \Leftrightarrow$ boundary solution is optimal.

$$\begin{aligned}
(38) \Rightarrow\quad & w_1 + w_2 = v + \lambda\theta + \frac{c_1 + c_2}{2} = \alpha;\\
\Rightarrow\quad & \lambda = \frac{\alpha - v}{\theta} - \frac{c_1 + c_2}{2\theta} > 0;\\
\Rightarrow\quad & \text{For region 1, } v - \frac{c_1 + c_2}{2} > \theta;\ \text{for region 3, } v - \frac{c_1 + c_2}{2} > 2\theta|l_1 - l_2|.
\end{aligned}$$

The optimal boundary solution for region 1 is, therefore,

$$\begin{aligned}
& w_1^* = v - \frac{\theta}{2} + \frac{c_1 - c_2}{4};\ w_2^* = v - \frac{\theta}{2} + \frac{c_2 - c_1}{4};\\
& \pi^* = v - \frac{\theta}{2} - \frac{c_1 + c_2}{2} + \frac{(c_1 - c_2)^2}{8\theta}.
\end{aligned} \tag{41}$$

The optimal boundary solution for region 3 is, therefore,

$$\begin{aligned}
& w_1^* = v - \theta|l_1 - l_2| + \frac{c_1 - c_2}{4};\ w_2^* = v - \theta|l_1 - l_2| + \frac{c_2 - c_1}{4};\\
& \pi^* = 2|l_1 - l_2|v - 2\theta|l_1 - l_2|^2 - (c_1 + c_2)|l_1 - l_2| + \frac{(c_1 - c_2)^2}{8\theta}.
\end{aligned} \tag{42}$$

Observe that the optimal profit for the boundary solution of region 3 increases in $|l_1 - l_2|$, because

$$\frac{\partial \pi^*}{\partial |l_1 - l_2|} > 0 \quad \Rightarrow \quad v - \frac{c_1 + c_2}{2} > 2\theta|l_1 - l_2|,$$

which is exactly the condition under which region 3 has an optimal boundary solution.

[12] The upper bound that defines region 3, i.e., $\theta|l_1 - l_2|$, becomes $\frac{\theta}{2}$, when $|l_1 - l_2| = \frac{1}{2}$.

Solution to (SP26′): The Lagrangian is:

$$\mathcal{L}(\mathbf{w},\lambda) = (w_1 - c_1)\left(\frac{1}{2} - \frac{w_1}{2\theta} + \frac{w_2}{2\theta}\right) + (w_2 - c_2)\left(\frac{1}{2} - \frac{w_2}{2\theta} + \frac{w_1}{2\theta}\right) + \lambda(\alpha - w_1 - w_2).$$

The KKT conditions $\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w},\lambda) = 0$ yield the following:

$$w_1 - w_2 = \frac{c_1 - c_2}{2} + \frac{\theta}{2} - \theta\lambda;\ w_1 - w_2 = \frac{c_1 - c_2}{2} - \frac{\theta}{2} + \theta\lambda.$$

Clearly, we have that

$$\frac{\theta}{2} - \theta\lambda = -\frac{\theta}{2} + \theta\lambda \Rightarrow \lambda = \frac{1}{2} > 0.$$

Therefore, the constraint is always active, and the optimal solution for region 2 is

$$w_1^* = v - \frac{\theta}{2} + \frac{c_1 - c_2}{4};\ w_2^* = v - \frac{\theta}{2} + \frac{c_2 - c_1}{4}; \tag{43}$$

$$\pi^* = v - \frac{\theta}{2} - \frac{c_1 + c_2}{2} + \frac{(c_1 - c_2)^2}{8\theta}. \tag{44}$$

Observe that (44) is the same as (41), the optimal boundary solution of region 1, since they share the same boundary. Similarly, the optimal solution for region 6 is

$$w_1^* = v - \theta + \frac{\theta}{2}|l_1 - l_2| + \frac{c_1 - c_2}{4};\ w_2^* = v - \theta + \frac{\theta}{2}|l_1 - l_2| + \frac{c_2 - c_1}{4};$$

$$\pi^* = v - \theta + \frac{\theta}{2}|l_1 - l_2| - \frac{c_1 + c_2}{2} + \frac{(c_1 - c_2)^2}{8\theta}. \tag{45}$$

Since (44) $\geq$ (45), the optimal profit π^* for region 2 always dominates that of region 6 for the monopolist supplier. It is also evident that the optimal profit π^* for region 6 increases in $|l_1 - l_2|$, because $\theta > 0$.

Solution to (SP4′): The Lagrangian is:

$$\begin{aligned}\mathcal{L}(\mathbf{w},\mu,\lambda) =& (w_1 - c_1)\left(|l_1 - l_2| - \frac{w_1}{2\theta} + \frac{w_2}{2\theta}\right) + (w_2 - c_2)\left(|l_1 - l_2| - \frac{w_2}{2\theta} + \frac{w_1}{2\theta}\right).\\ &+ \mu(w_1 + w_2 - 2v + 3\theta|l_1 - l_2|) + \lambda(2v - 2\theta|l_1 - l_2| - w_1 - w_2)\end{aligned}$$

The KKT conditions $\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w},\mu,\lambda) = 0$ yield the following:

$$w_1 - w_2 = \frac{c_1 - c_2}{2} + \theta|l_1 - l_2| + \theta(\mu - \lambda);\ w_1 - w_2 = \frac{c_1 - c_2}{2} - \theta|l_1 - l_2| - \theta(\mu - \lambda).$$

Clearly, we have that

$$\theta|l_1 - l_2| + \theta(\mu - \lambda) = -\theta|l_1 - l_2| - \theta(\mu - \lambda) \Rightarrow \mu - \lambda = -|l_1 - l_2|.$$

Recall that we have assumed $|l_1 - l_2|$ to be strictly greater than 0. μ and λ cannot both be positive, since the upper and lower bounds cannot both be active at the same time. μ and λ cannot both be 0, since $0 \neq -|l_1 - l_2|$. If $\mu > 0$ and $\lambda = 0$, then we have $\mu = -|l_1 - l_2| < 0$, a contradiction to condition imposed on the Lagrange multiplier $\mu \geq 0$.

The only possibility is therefore $\mu = 0$ and $\lambda > 0$, which leads to $\lambda = |l_1 - l_2| > 0$. This implies that region 4′s optimal solution for the monopolist supplier always lies at the bound, where $w_1 + w_2 = 2v - 2\theta|l_1 - l_2|$, or equivalently where $v - \frac{w_1 + w_2}{2} = \theta|l_1 - l_2|$. Solving the following system jointly yields the optimal solution:

$$\begin{cases} w_1 - w_2 = \frac{c_1 - c_2}{2}; \\ w_1 + w_2 = 2v - 2\theta|l_1 - l_2|; \end{cases}$$

$$\Rightarrow\quad w_1^* = v - \theta|l_1 - l_2| + \frac{c_1 - c_2}{4};\ w_2^* = v - \theta|l_1 - l_2| + \frac{c_2 - c_1}{4};$$

$$\pi^* = 2|l_1 - l_2|v - 2\theta|l_1 - l_2|^2 - (c_1 + c_2)|l_1 - l_2| + \frac{(c_1 - c_2)^2}{8\theta}. \tag{46}$$

Observe that (46) is equivalent to (42), the optimal boundary solution of region 3, as these two regions share the same boundary.

Solution to (SP5′): The objective function is concave, and the constraint is linear. Therefore, the KKT conditions are both sufficient and necessary. The Lagrangian is:

$$\begin{aligned}\mathcal{L}(\mathbf{w},\mu,\lambda) =& (w_1-c_1)\left(\frac{1}{4}|l_1-l_2|+\frac{v}{2\theta}+\frac{w_2}{4\theta}-\frac{3w_1}{4\theta}\right)+(w_2-c_2)\left(\frac{1}{4}|l_1-l_2|+\frac{v}{2\theta}+\frac{w_1}{4\theta}-\frac{3w_2}{4\theta}\right)\\ &+\mu(w_1+w_2-(2v-2\theta+\theta|l_1-l_2|))+\lambda(2v-3\theta|l_1-l_2|-w_1-w_2).\end{aligned}$$

The KKT conditions $\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w},\mu,\lambda)=0$ yield the following:

$$\begin{aligned}&3w_1-w_2=v+\frac{\theta}{2}|l_1-l_2|+\frac{3c_1-c_2}{2}+2\theta(\mu-\lambda);\ 3w_2-w_1=v+\frac{\theta}{2}|l_1-l_2|+\frac{3c_2-c_1}{2}+2\theta(\mu-\lambda);\\ \Rightarrow\quad &w_1=\frac{v}{2}+\frac{\theta}{4}|l_1-l_2|+\frac{c_1}{2}+\theta(\mu-\lambda);\ w_2=\frac{v}{2}+\frac{\theta}{4}|l_1-l_2|+\frac{c_2}{2}+\theta(\mu-\lambda);\\ &w_1+w_2=v+\frac{\theta}{2}|l_1-l_2|+\frac{c_1+c_2}{2}+2\theta(\mu-\lambda).\end{aligned}\tag{47}$$

There are 3 cases to consider.

1.$\mu=0,\lambda>0\Leftrightarrow$ the upper bound on w_1+w_2 is active.

$$\begin{aligned}(47)\Rightarrow\quad &w_1^*=v-\frac{3}{2}\theta|l_1-l_2|+\frac{c_1-c_2}{4};\ w_2^*=v-\frac{3}{2}\theta|l_1-l_2|+\frac{c_2-c_1}{4};\\ &\pi^*=2|l_1-l_2|v-3\theta|l_1-l_2|^2-(c_1+c_2)|l_1-l_2|+\frac{(c_1-c_2)^2}{8\theta}.\end{aligned}\tag{48}$$

By $\lambda>0$, we have that

$$v-\frac{c_1+c_2}{2}<\frac{7}{2}\theta|l_1-l_2|.$$

2.$\mu>0,\lambda=0\Leftrightarrow$ the lower bound on w_1+w_2 is active.

$$\begin{aligned}(47)\Rightarrow\quad &w_1^*=v+\frac{\theta}{2}|l_1-l_2|-\theta+\frac{c_1-c_2}{4};\ w_2^*=v+\frac{\theta}{2}|l_1-l_2|-\theta+\frac{c_2-c_1}{4};\\ &\pi^*=v+\frac{\theta}{2}|l_1-l_2|-\theta-\frac{c_1+c_2}{2}+\frac{(c_1-c_2)^2}{8\theta}.\end{aligned}\tag{49}$$

By $\mu>0$, we have that

$$v-\frac{c_1+c_2}{2}>2\theta-\frac{1}{2}\theta|l_1-l_2|.$$

Observe that (49) is equivalent to (45), the optimal boundary solution of region 6, since these two regions share the same boundary. π^* increases in $|l_1-l_2|$, because $\theta>0$.

3.$\mu=0,\lambda=0\Leftrightarrow$ region 5 has an interior solution.

$$\begin{aligned}(47)\Rightarrow\quad &w_1^*=\frac{v}{2}+\frac{\theta}{4}|l_1-l_2|+\frac{c_1}{2};\ w_2^*=\frac{v}{2}+\frac{\theta}{4}|l_1-l_2|+\frac{c_2}{2}.\\ &\pi^*=\frac{(v-c_1)^2}{8\theta}+\frac{(v-c_2)^2}{8\theta}+\frac{(c_1-c_2)^2}{16\theta}+\frac{|l_1-l_2|}{4}v+\frac{|l_1-l_2|^2}{16}\theta-\frac{|l_1-l_2|}{8}(c_1+c_2)\end{aligned}\tag{50}$$

Plugging $w_1^*+w_2^*$ into the constraints of (SP5′) yields

$$\frac{7}{2}\theta|l_1-l_2|<v-\frac{c_1+c_2}{2}<2\theta-\frac{1}{2}\theta|l_1-l_2|.$$

Observe that (50) increases in $|l_1-l_2|$ if

$$\frac{\partial\pi^*}{\partial|l_1-l_2|}>0\quad\Rightarrow\quad v-\frac{c_1+c_2}{2}>-\frac{1}{2}\theta|l_1-l_2|,$$

which is always satisfied since $c_i\le v$.

Proof of Theorem 3: We want to show that only regions 1, 2, and 3 have viable solutions that result in an equilibrium for the monopolist supplier—in other words, he will initiate the equilibrium that the retailer will also regard as optimal. For all other regions, we will show that the monopolist supplier has incentives to maximally differentiate the products to improve category profit. In fact, the solutions associated with those regions will either converge to the optimal solution for region 2, or be strictly dominated by it.

From the solution to (SP13′), we know that (42), the boundary solution for region 3, increases in $|l_1 - l_2|$. When $|l_1 - l_2| \uparrow \frac{1}{2}$, (42) approaches (44), the optimal solution for region 2, from below. The same asymptotic argument can be made for (46), the solution to region 4 derived from (SP4′), since it shares the same boundary as region 3.

From the solution to (SP26′), we know that (45), the solution to region 6, also increases in $|l_1 - l_2|$. When $|l_1 - l_2| \uparrow \frac{1}{2}$,

$$(45) = v - \frac{3}{4}\theta - \frac{c_1 + c_2}{2} + \frac{(c_1 - c_2)^2}{8\theta} \leq v - \frac{\theta}{2} - \frac{c_1 + c_2}{2} + \frac{(c_1 - c_2)^2}{8\theta} = (44).$$

We can make the same argument for (49), the boundary solution that region 5 shares with region 6, according to the solution to (SP5′).

We observe that (48), the other boundary solution of region 5, is less than (46), the solution of region 4, under any $|l_1 - l_2| \in (0, \frac{1}{2}]$, and therefore cannot be an equilibrium for the monopolist supplier. Lastly, we know that (50), the interior solution to (SP5′), increases in $|l_1 - l_2|$. When $|l_1 - l_2| \uparrow \frac{1}{2}$, the domain of this solution degenerates into a single point, i.e.,

$$v - \frac{c_1 + c_2}{2} = \frac{7}{4}\theta \quad \Rightarrow \quad c_2 = 2v - c_1 - \frac{7}{2}\theta.$$

Substituting the expression on c_2 from above into (50) gives a supplier profit of

$$\frac{(v - c_1)^2}{2\theta} + \frac{81\theta}{32} - \frac{7}{4}(v - c_1),$$

which is less than

$$\frac{(v - c_1)^2}{2\theta} + \frac{89\theta}{32} - \frac{7}{4}(v - c_1),$$

i.e., region 2′s optimal solution with the expression above on c_2 substituted into (44).

In summary, we see that the monopolist supplier is better off with a separable domain solution (as in regions 1 or 3), when the channel added value is below θ. Conversely when the channel added value is above θ, he is better off with a solution that allow the domains of the maximally differentiated products extend the entire market (as in region 2).

Appendix C: Technical Details of Model Comparison

Proof of Theorem 4: By Observation 1, we consider the following cases:

- $0 \leq v - \frac{c_1+c_2}{2} < \frac{\theta}{2}$: All three systems have optimal solutions from regions 1 or 3. For the centralized system, the total channel profit corresponding to regions 1 or 3 is shown in Table 6 to be

$$\Pi_C^{13} = \frac{(v - c_1)^2}{2\theta} + \frac{(v - c_2)^2}{2\theta}.$$

For the decentralized system, the total channel profit is the sum of π_1^* and π_2^* in (27) and the retailer's profit in regions 1 or 3 from Table 1 ($\frac{(v-w_1)^2}{2\theta} + \frac{(v-w_2)^2}{2\theta}$) with the optimal wholesale prices in (26) plugged in:

$$\Pi_D^{13} = \frac{3(v-c_1)^2}{8\theta} + \frac{3(v-c_2)^2}{8\theta}.$$

For the monopolist supplier system, the total channel profit is the sum of π^* in (40) and the retailer's profit in regions 1 or 3 from Table 1 ($\frac{(v-w_1)^2}{2\theta} + \frac{(v-w_2)^2}{2\theta}$) with the optimal wholesale prices in (39) plugged in:

$$\Pi_{MS}^{13} = \frac{3(v-c_1)^2}{8\theta} + \frac{3(v-c_2)^2}{8\theta}.$$

It is clear that

$$\Pi_C^{13} \geq \Pi_D^{13} = \Pi_{MS}^{13}.$$

- $\frac{\theta}{2} \leq v - \frac{c_1+c_2}{2} \leq \theta$: The decentralized system and the monopolist supplier system still have optimal solutions from regions 1 or 3, expressed as Π_D^{13} and Π_{MS}^{13}. Both are less efficient than the centralized system in terms of channel profit.
- $\frac{3}{2}\theta \leq v - \frac{c_1+c_2}{2} < \frac{7}{4}\theta$: All three systems have optimal solutions from region 2. For the fully centralized system, the total channel profit (same as retailer profit) corresponding to region 2 is shown in Table 6 to be

$$\Pi_C^2 = v + \frac{(c_1-c_2)^2}{4\theta} - \frac{c_1+c_2}{2} - \frac{\theta}{4}.$$

For the decentralized system, the total channel profit is the sum of π_1^* and π_2^* in (30) and the retailer's profit in region 2 from Table 1 ($v + \frac{(w_1-w_2)^2}{4\theta} - \frac{w_1+w_2}{2} - \frac{\theta}{4}$) with the optimal wholesale prices in (28) plugged in, i.e.,

$$\Pi_D^2 = v + \frac{5(c_1-c_2)^2}{36\theta} - \frac{c_1+c_2}{2} - \frac{\theta}{4}.$$

For the monopolist supplier system, the total channel profit is the sum of π^* in (44) and the retailer's profit in region 2 from Table 1 ($v + \frac{(w_1-w_2)^2}{4\theta} - \frac{w_1+w_2}{2} - \frac{\theta}{4}$) with the optimal wholesale prices in (43) plugged in

$$\Pi_{MS}^2 = v + \frac{3(c_1-c_2)^2}{16\theta} - \frac{c_1+c_2}{2} - \frac{\theta}{4}.$$

It is clear that

$$\Pi_C^2 \geq \Pi_{MS}^2 \geq \Pi_D^2.$$

- $v - \frac{c_1+c_2}{2} \geq \frac{7}{4}\theta$: The centralized and monopolist supplier system still have optimal solutions from region 2, expressed as Π_C^2 and Π_{MS}^2, while the decentralized system has an optimal solution from region 6. For the decentralized system, the total channel profit is the sum of π_1^* and π_2^* in (30) and the retailer's profit in region 6 from Table 1 ($v + \frac{(w_1-w_2)^2}{4\theta} - \frac{w_1+w_2}{2} - \frac{\theta}{2} + \frac{\theta}{2}|l_1 - l_2|$) with the optimal wholesale prices in (28) plugged in, i.e.,

$$\Pi_D^6 = v + \frac{5(c_1-c_2)^2}{36\theta} - \frac{c_1+c_2}{2} - \frac{\theta}{2} + \frac{\theta}{2}|l_1 - l_2|.$$

Since $0 < |l_1 - l_2| \leq \frac{1}{2}$, and therefore

$$\Pi_D^2 \geq \Pi_D^6,$$

it is clear that

$$\Pi_C^2 \geq \Pi_{MS}^2 \geq \Pi_D^2 \geq \Pi_D^6.$$

Positioning Distortion: Recall that positioning distortion occurs when $v - \frac{c_1+c_2}{2} \geq \frac{7}{4}\theta$, where region 6 solutions are optimal. In addition, we note that region 2 solutions are in fact borderline cases for region 6 solutions. From the proof of Theorem 4, we know that

$$\Pi_D^2 - \Pi_D^6 = \frac{\theta}{4} - \frac{\theta}{2}|l_1 - l_2|,$$

which we define as the loss in channel profit due to positioning distortion.

Cost Difference Distortion: We demonstrate algebraically the elimination of channel inefficiency when we set $c_1 = c_2$ by comparing the results from the decentralized system and the monopolist supplier system against the fully centralized system. Recall that cost difference distortion occurs when $v - \frac{c_1+c_2}{2} > \theta$. Furthermore, we only consider cases where the product positions are maximally differentiated and product domains cover the entire market, i.e., region 2 solutions.

From the proof of Theorem 4, and letting $c_1 = c_2 = c$, it is clear that

$$\Pi_C^2 = \Pi_{MS}^2 = \Pi_D^2 = v - c - \frac{\theta}{4},$$

implying that there is no loss in channel inefficiency when cost difference is removed.

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