# Violation of the Law of Demand

Yakar Kannai • Larry Selden

Received: date / Accepted: date

Abstract Following the classic work of Mitjuschin, Polterovich and Milleron, necessary and sufficient as well as sufficient conditions have been developed for when the multicommodity Law of Demand holds. We show when the widely cited Mitjuschin and Polterovich sufficient condition also becomes necessary. Using this result, violation regions for the very popular Modified Bergson (or HARA (hyperbolic absolute risk aversion)) class of utility functions, are fully characterized in terms of preference parameters. For a natural extension of the CES member of the Modified Bergson family that is neither homothetic nor quasihomothetic, we create the first simple, explicit example of which we are aware that (i) fully characterizes violation regions in both the preference parameter and commodity spaces and (ii) analyzes the range of relative income and price changes within which violations occur.

**Keywords** Law of demand  $\cdot$  Monotonicity  $\cdot$  Violation region  $\cdot$  Minimum concavity point

JEL Classification D01.D11

Yakar Kannai

Larry Selden

Graduate School of Business, Columbia University, New York, NY 10027, USA E-mail: larry@larryselden.com

We would like to thank Michael Jerison, John Quah and Herakles Polemarchakis for their insightful comments. We are also grateful to an anonymous referee and the editor for their very helpful suggestions. Earlier versions of this paper were presented at seminars at the Instsitute for Mathematical Economics, University of Bielefeld, Bielefeld, Germany, the Economics Department, University of Queensland, Queensland, Australia and the Economics Department of SUNY, Albany. We benefited from the thoughtful comments of the participants. We also want to thank O. Alper, M. Kang, A. Kiro and especially X. Wei for their research assistance. Of course responsibility for any errors remains with the authors.

Department of Mathematics, Weizmann Institute of Science, Rehovot, ISRAEL E-mail: yakar.kannai@weizmann.ac.il

### 1 Introduction

The question of when the Law of Demand for multiple goods holds has been studied for more than thirty years starting with Milleron (1974) and Mitjuschin and Polterovich (1978) followed by Kannai (1989), Mas-Colell (1991) and Quah (2000, 2003). Given that these conditions are stated locally, they may not always be easily applied to the entire commodity space. One known case where the Law of demand holds globally, as observed by Milleron (1974) and Mas-Colell (1991), is when preferences are homothetic. For general preference relations Mitjuschin and Polterovich (1978)<sup>1</sup> and for separable preferences Quah (2003) have proposed sufficient conditions for the Law of Demand to hold that are far simpler to verify.

Despite these numerous contributions, relatively little is known about (i) when the Law is violated for commonly used forms of utility, (ii) where the violations occur in the commodity space and (iii) the properties of the violations in terms of prices and incomes. The goal of this paper is to contribute to filling these voids in the literature. Moreover, we create the first simple, explicit example of which we are aware that characterizes the violation regions and properties.

To address these questions, the existing sufficient conditions for the Law to hold although simpler in form are of little value unless they are also necessary. Given the observation of Mas-Colell (1991) that the least concave representation of preferences<sup>2</sup> is the "best candidate" to use when applying the widely cited Mitjuschin and Polterovich sufficient condition, it is natural to wonder whether its use ensures that the sufficient condition becomes necessary locally as well, and in particular in the entire commodity space. While this is not the case in general, we show that it is true for the widely cited Modified Bergson family (Pollak 1970, Section 1).<sup>3</sup> In this case, the simplified Mitjuschin and Polterovich sufficient condition can be used to derive a very clear characterization of when violations occur in term of preference parameters for each member of the family.

To more generally investigate violations, we introduce a natural extension of the CES member of the Modified Bergson family. Although for this utility we cannot use the Mitjuschin and Polterovich sufficient condition, we are nevertheless able to derive a clear specification of violation regions in both the preference parameter and commodity spaces. In the preference parameter space, we contrast the violation regions obtained from the necessary and sufficient and the sufficient conditions. In terms of the violation

 $<sup>^{1}</sup>$  Although the necessary and sufficient and simplified sufficient conditions are generally associated with Mitjuschin and Polterovich (1978), the results were derived independently by Milleron (1974).

<sup>&</sup>lt;sup>2</sup> A concave function U representing the preference relation  $\succeq$  is least concave if and only if every concave utility function representing the preference relation  $\succeq$  is given by  $T \circ U$ , where T is a strictly increasing, concave function of a single variable (see Debreu 1976). Whenever we refer to a least concave utility, it is understood that the representation is only defined up a positive affine transformation.

<sup>&</sup>lt;sup>3</sup> This class includes the homothetic CES (constant elasticity of substitution) utility and quasihomothetic translated CES, negative exponential and quadratic utilities as well as other non-standard forms (see, for instance, Pollak 1970). If one assumes an uncertainty contingent claim setting where the appropriate NM (von Neumann-Morgenstern) axioms hold, then the Modified Bergson family corresponds to the HARA (hyperbolic absolute risk aversion) class of NM indices (Gollier 2001 and Rubinstein 1976). Although because of its popularity in financial economics, the HARA terminology is more widely used, we generally use the Modified Bergson terminology, since the new utility introduced in Section 3 relates more naturally to the certainty Bergson family.

regions in the commodity space, for a fixed price ratio we derive specific bounds in terms of income levels. Also for the set of points along the expansion path between the income bounds where the Law of Demand is violated, we characterize the set of corresponding price change ratios.

In the next Section, we first show when the simplified Mitjuschin and Polterovich sufficient condition becomes necessary and then use this result to analyze the Modified Bergson family. In Section 3, the new extension of the CES utility is introduced and used to analyze violation regions and properties. Finally, Section 4 contains the concluding comments.

# 2 A Simplified Approach

Mitjuschin and Polterovich (1978) propose both a necessary and sufficient condition and a sufficient condition for characterizing when the Law of Demand holds. Because the latter is simpler, it is more widely referenced. However since it is only a sufficient condition for the Law of Demand to hold, it cannot be used directly to characterize when the Law is violated. In this Section after reviewing both conditions, we demonstrate when the more widely cited sufficient condition also becomes necessary and can be used to characterize violations. The sufficient condition is then used for the popular Modified Bergson class of preferences to derive very simple restrictions in terms of the underlying preference parameters for when violations occur.

Let  $\leq$  be a strictly convex and concavifiable preference ordering (defined on a convex subset  $\Omega$  of the space of n commodities  $\mathbf{c} =_{def} (c_1, ..., c_n)$  where typically it is assumed that  $\Omega = \mathbb{R}^n_+$ ) which is represented by a  $C^3$ , strictly monotone, concave utility function  $U(\mathbf{c})$ . Denote by  $\partial U(\mathbf{c})$  the vector of partial derivatives  $\left(\frac{\partial U}{\partial c_1}, \frac{\partial U}{\partial c_2}, ..., \frac{\partial U}{\partial c_n}\right)$  and by  $\partial^2 U(\mathbf{c})$  the matrix  $\left(\frac{\partial^2 U}{\partial c_i \partial c_j}\right)_{1 \leq i \leq n, 1 \leq j \leq n}$ . Let the preference ordering  $\leq$  generate a differentiable demand function  $\mathbf{h}(\mathbf{p})$ , where  $\mathbf{p}$  denotes the price vector and we normalize the income by setting  $(\mathbf{h}(\mathbf{p}), \mathbf{p}) = 1$ . The demand is *monotone* or the Law of Demand holds at  $\mathbf{p}$  if the inequality

$$\sum_{i,j}^{n} \frac{\partial h_i}{\partial p_j} x_i x_j < 0 \tag{1}$$

holds for every nonzero price change vector  $\mathbf{x} \in \mathbb{R}^n$ , where  $\mathbf{h} = (h_1, \cdots, h_n)^4$ .

A well known result due to Milleron (1974) and Mitjuschin and Polterovich (1978) is that  $\mathbf{h}$  is monotone at  $\mathbf{p}$  if and only if the inequality<sup>5</sup>

$$\frac{(\mathbf{c}, \partial U(\mathbf{c}))}{(\partial^2 U(\mathbf{c})^{-1} \partial U(\mathbf{c}), \partial U(\mathbf{c}))} - \frac{(\partial^2 U(\mathbf{c})\mathbf{c}, \mathbf{c})}{(\mathbf{c}, \partial U(\mathbf{c}))} < 4$$
(2)

<sup>&</sup>lt;sup>4</sup> The traditional statement of the Law of Demand is  $(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{h} (\mathbf{p}') - \mathbf{h} (\mathbf{p})) < 0$  for any  $\mathbf{p} \neq \mathbf{p}'$ . Taking  $\mathbf{x} = \Delta \mathbf{p}$  and noticing that  $\Delta \mathbf{h} = \frac{\partial \mathbf{h}}{\partial \mathbf{p}} \cdot \Delta \mathbf{p}$ , it can be seen that the definition in eqn. (1) is the differentiable form of the Law of Demand (see Section 4.C in Mas-Colell, Whinston and Green 1995).

<sup>&</sup>lt;sup>5</sup> It should be noted that Milleron (1974), Mas-Colell (1991), p. 282 and Quah (2003) define the monotonicity condition eqn. (1) as a strict inequality. Mas-Colell for instance refers to this as strict monotonicity. Alternatively, Mitjuschin and Polterovich (1978) and Kannai (1989) define monotonicity as a weak inequality. For the latter h is monotone if and only if the left hand side of eqn. (2) is  $\leq 4$ .

holds for  $\mathbf{c} = \mathbf{h}(\mathbf{p})$ .<sup>6</sup> The term on the left hand side of (2) will be referred to as the Mitjuschin-Polterovich coefficient and denoted by M.<sup>7</sup> Our primary interest in this paper will be in violations of the Law of Demand. And when we say the Law of Demand or monotonicity is violated, we will mean that there exists at least one point  $\mathbf{c}$  in the commodity space such that the sign of the inequality (1) reverses for at least one price change vector  $\mathbf{x}$ . This is equivalent to M > 4 at that point.

It was observed in Mitjuschin and Polterovich (1978) that by the concavity of U, the first term on the left hand side of (2) is non-positive, so that a *sufficient* condition for monotonicity of **h** is

$$\frac{(\partial^2 U(\mathbf{c})\mathbf{c}, \mathbf{c})}{(\mathbf{c}, \partial U(\mathbf{c}))} < 4$$
(3)

and the left hand side of (3) will be referred to as the modified Mitjuschin-Polterovich coefficient, which will be denoted as  $\mathcal{M}_U$ . (See also Milleron 1974). As noted above, given its simplicity this form is more widely cited than (2). In this paper when monotonicity is said to hold without specifying **p**, it should be understood to hold for the full commodity space.

In order to use it to characterize violations, we next answer the natural question of when (3) also becomes necessary.

**Proposition 1** The first term on the left hand side of (2) vanishes if and only if the Hessian matrix of U is singular, i.e., when the Hessian determinant  $H_U(\mathbf{c})$  vanishes.

*Proof* First prove sufficiency. It follows from footnote 6 that if H = 0, then the first term of eqn. (2) vanishes. Next prove necessity. Assume that  $H \neq 0$ . Note the identity in Mitjuschin and Polterovich (1978)  $1/(\partial^2 U(\mathbf{c})^{-1}\partial U(\mathbf{c}), \partial U(\mathbf{c})) = \sup (\partial^2 U(\mathbf{c})\mathbf{y}, \mathbf{y})$  where the supremum is over the set of  $\mathbf{y}$  for which  $(\mathbf{y}, \partial U(\mathbf{c})) = 1$ . Next we will show that

$$\left(\partial^2 U(\mathbf{c})^{-1} \partial U(\mathbf{c}), \partial U(\mathbf{c})\right) = -\frac{B_H}{H},\tag{4}$$

where  $B_H$  is the bordered Hessian determinant of U. Diagonalize the Hessian matrix  $\partial^2 U$ , so that in the appropriate coordinate system this matrix assumes the form

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \\ & & & \ddots & \\ & & & & \lambda_n \end{pmatrix}$$
(5)

and the Hessian determinant is given by  $H = \prod_{i=1}^{n} \lambda_i$ . Then the bordered Hessian  $B_H$  of U is the determinant of the matrix

$$\begin{pmatrix} \lambda_1 & U_1 \\ \lambda_2 & U_2 \\ & \ddots & \\ & & \\ & & \lambda_n \ U_n \\ U_1 \ U_2 \ U_n & 0 \end{pmatrix}, \tag{6}$$

 $<sup>^{6}</sup>$  It should be noted that if the Hessian matrix is singular, then its inverse matrix is understood as having infinite eigenvalues corresponding to zero eigenvalues of the Hessian, and if they appear in the denominator of the first term of eqn. (2), then this term is 0.

 $<sup>^{7}\,</sup>$  It should be noted that M is invariant under monotone increasing transformations of the utility representation.

where  $\partial U = (U_1, U_2, ..., U_n)$ . Expanding the determinant of (6) by the last row (or column), we get the well-known formula

$$B_H = -\sum_{i=1}^n U_i^2 \prod_{j \neq i} \lambda_j \tag{7}$$

so that the ratio of the bordered Hessian determinant to the Hessian determinant is given by

$$\frac{B_H}{H} = -\sum_{i=1}^n \frac{U_i^2}{\lambda_i}.$$
(8)

The right hand side of (8) is obviously equal to  $\left(\partial^2 U(\mathbf{c})^{-1} \partial U(\mathbf{c}), \partial U(\mathbf{c})\right)$ . It follows that the first term of (2) equals  $\frac{(\mathbf{c}, \partial U(\mathbf{c}))H}{B_H}$ . The assumptions on U imply that  $(\mathbf{c}, \partial U(\mathbf{c})) > 0$  and that  $B_H$  is nonzero, hence the Proposition.

We call any point **c** where the corresponding  $H_U(\mathbf{c}) = 0$  a minimum concavity point (based on U)<sup>8,9</sup> and next show that at such a point the simplified Mitjuschin and Polterovich sufficient condition becomes necessary (also see Kannai 1989). Note that this set of points is maximal if preferences are represented by a least concave utility representation.

**Proposition 2** The sufficient condition (3) is necessary and sufficient if and only if  $\mathbf{c} = \mathbf{h}(\mathbf{p})$  is a minimum concavity point. Consequently, the sufficient condition also becomes necessary for all of  $\Omega$  if and only if every point  $\mathbf{c} \in \Omega$  is a minimum concavity point (based on U).

*Proof* Proposition 2 follows at once from the definition of minimum concavity points and Proposition 1.

Note that the less concave U is, the larger the set of minimum concavity points related to it. (For general U, minimum concavity points may not exist at all.) Thus it is natural to consider minimum concavity points based on least concave utility representations (as defined above in footnote 2). It is well known (Kannai 1985) that if  $\Omega$  is compact, then every indifference surface of  $\leq$  contains at least one minimum concavity point. Hence it is natural when analyzing monotonicity, to consider least concave utility functions u representing  $\leq$ . We use  $\mathcal{M}_u$  to denote the modified Mitjuschin-Polterovich coefficient (the left hand side of eqn. (3)) based on the least concave form u. It should be stressed that the first term on the left hand side of eqn. (2) vanishes only at minimum concavity points and there M becomes equal to the modified Mitjuschin-Polterovich coefficient  $\mathcal{M}_u$ . Given our interest in violations we look for points where M > 4, or alternatively, when monotonicity holds, given that this condition is necessary and sufficient.

<sup>&</sup>lt;sup>8</sup> A point **c** is a point of minimum concavity for the function U if and only if there exists a neighborhood of **c** such that U is least concave there. Equivalently, the graph of U, which is defined by  $\{(\mathbf{c}, t) : U(\mathbf{c}) = t\}$ , has a contact of second order (at least at one direction) at **c** with the tangent (hyper)plane of the graph. In other words, at least one principal curvature of the graph vanishes at **c**.

<sup>&</sup>lt;sup>9</sup> Given that a minimum concavity point may not be in the commodity space  $\Omega$ , it is advantageous for certain forms of U to compactify  $\Omega$  and to define minimum concavity at a boundary point **c** by the asymptotic vanishing of the ratio  $\frac{H}{B_H}$  as points of  $\Omega$  approach **c**.

The set of minimum concavity points based on u corresponds to all of  $\Omega$  for the widely referenced classes of homothetic and quasihomothetic preference relations, where the terms homothetic and quasihomothetic are defined as customary (see Deaton 1980, pp. 143-5).

**Proposition 3** If  $\leq$  is defined on all of  $\Omega$  and is homothetic or quasihomothetic,<sup>10</sup> then every point is a minimum concavity point for the least concave u representing  $\leq$ . Hence, the Mitjuschin-Polterovich sufficient condition (3) becomes necessary as well for all of  $\Omega$ .

*Proof* A homothetic  $\leq$  defined on all of  $\Omega$  may be represented by a utility function u such that u is homogeneous of degree 1 (for example,  $u(\mathbf{c}) = |\mathbf{c}'|$ , where  $\mathbf{c}' \sim \mathbf{c}$  and  $\mathbf{c}'$  is on the principal diagonal of  $\Omega$ ). See also Kannai (1970) and Hosoya (2011) for the construction of u. The first derivatives of u are homogeneous of degree 0, so that by Euler's Theorem

$$\sum_{j=1}^{n} c_j \frac{\partial^2 u}{\partial c_i \partial c_j} = 0, \quad i = 1, 2, ..., n \tag{9}$$

i.e., **c** is an eigenvector of the Hessian matrix  $\partial^2 u(\mathbf{c})$  with a zero eigenvalue. Hence the Hessian determinant varnishes everywhere. A similar argument implies that the Hessian of the corresponding utility representation of quasihomothetic preferences vanishes as well.

A stronger result can be obtained if we add the assumption that  $\leq$  is representable by an additively separable concave U, i.e.,

$$U(\mathbf{c}) = \sum_{i=1}^{n} U_i(c_i). \tag{10}$$

**Proposition 4** Assume  $\leq$  is defined on  $\Omega$  and is representable by an additively separable concave U. Then the set of minimum concavity points based on u is the whole space  $\Omega$  if and only if U is homothetic or quasihomothetic.

*Proof* We have to show that if a set of minimum concavity points (based on the least concave representation)  $\mathbf{c}^* = (c_1^*, ..., c_n^*)$  is the whole space  $\Omega$  and the preference relation  $\leq$  is representable by an additively separable U, then  $\leq$  is homothetic or quasihomothetic. It may be shown by an elementary computation that if U is additively separable, then at a minimum concavity point (based on the least concave representation) the function

$$a(U, \mathbf{c}) = \left[\sum_{i=1}^{n} \frac{(U_i')^2}{U_i''}\right]^{-1}$$
(11)

has a maximal value relative to the indifference surface U = const. (These points may be found even if we don't know explicitly the least concave u corresponding to  $\preceq$ ). Application of the Lagrange multiplier method leads to the system

$$\frac{\partial a}{\partial c_i} = \lambda \frac{\partial U}{\partial c_i}, \ i = 1, \cdots, n$$

<sup>&</sup>lt;sup>10</sup> It should be noted that although  $\Omega = \mathbb{R}^n_+$  for homothetic preferences as is standard, for quasihomothetic preferences, it may correspond to a subset of  $\mathbb{R}^n_+$  due to the translated origin.

or

$$-\frac{1}{K^2}\frac{\partial}{\partial c_i}\frac{(U_i')^2}{U_i''} = \lambda U_i', i = 1, \cdots, n,$$
(12)

where

$$K = -\frac{B_H}{H} = \sum_{i=1}^n \frac{(U_i')^2}{U_i''}.$$
(13)

Evaluating (12), we get the system

$$-\frac{1}{K^2} \left[ 2U'_i - \frac{U''_i(U'_i)^2}{(U''_i)^2} \right] = \lambda U'_i, i = 1, \cdots, n$$

or, equivalently,

$$2 - \frac{U_i''U_i'}{(U_i'')^2} = -\lambda K^2, i = 1, \cdots, n.$$
(14)

We may eliminate  $\lambda K^2$  from the system (14) of n equations so as to get a system of n-1 equations in n unknowns. Such a system represents, generically, a curve. It suffices for our purposes to consider the case n = 2. We are then led to the equation

$$2 - \left(\frac{U_1'''U_1}{(U_1'')^2}\right)(c_1^*) = 2 - \left(\frac{U_2'''U_2}{(U_2'')^2}\right)(c_2^*),\tag{15}$$

which is valid whenever  $(c_1^*, c_2^*)$  is a point of least concavity. Setting

$$2 - \frac{U_i''U_i'}{(U_i'')^2}(c_i) \equiv a_i(c_i), i = 1, 2,$$
(16)

we see that the set  $\{(c_1^*, c_2^*)\}$  of points of least concavity coincides with the set of solutions of the equation  $a_1(c_1) = a_2(c_2)$ , or is a subset of the set of solutions. In our setting, the set of minimum concavity points coincides with the full set  $\Omega$ . This may happen if and only if  $a_1 = a_2 = \text{const.}$  Consider the ordinary differential equation

$$2 - \frac{U_i''U_i'}{(U_i'')^2}(x) \equiv a,$$
(17)

where a is a constant and  $U_i$  is the unknown function. Setting  $v = U'_i$  we obtain the ordinary differential equation

$$\frac{v''v}{(v')^2}(x) \equiv 2 - a.$$
 (18)

The solution of (18) is well known, see Kamke (1959), item 6.125, p. 573. The solution is given by the representation

$$v(x) = \begin{cases} |C_1 x + C_2|^{\frac{1}{a-1}} & (a \neq 1) \\ C_1 e^{C_2 x} & (a = 1) \end{cases},$$
(19)

where  $C_1$  and  $C_2$  are constants. We distinguish the following sub-cases: a)  $a \equiv 1$ . Then  $U_i(x) = C'_1 e^{C_2 x}$ ; b)  $a \equiv 0$ . Then  $U_i(x) = C'_1 \ln |C_1 x + C_2|$ ;

c) a is a constant, different from 0,1. Then

$$U_i(x) = \pm \left| C_1' x + C_2' \right|^{\frac{1}{a-1}+1},\tag{20}$$

where  $C'_1$  and  $C'_2$  are constants. All of these cases are homothetic or quasihomothetic. (Having  $U_i(x)$  monotone and concave in all of  $\mathbb{R}_+$  imposes restrictions on  $a, C_1, C'_1$ ,  $C_2$  and  $C'_2$ . In some other cases, one gets monotonicity and concavity on subsets of  $\mathbb{R}_+$ ).

It follows from cases (a)-(c) in the proof of Proposition 4 that an additively separable utility function (10) which represents homothetic or quasi preferences corresponds exactly to the Modified Bergson Family.<sup>11</sup> (Although Pollak 1971 never considers the question of minimum concavity points, he shows that the class of additively separable, homothetic or quasihomothetic utility functions generate demand functions that are locally linear in income and include the Bergson and modified Bergson families.)

- (i) CES form:  $U_i(x) = -x^{-\delta}/\delta, \delta > -1;$ (ii) CES form with a negative translated origin:  $U_i(x) = -(x+q_i)^{-\delta}/\delta$ , where  $q_i > 0$  $0, \delta > -1;$
- (iii) CES form with a positive translated origin:  $U_i(x) = -(x-q_i)^{-\delta}/\delta$ , where  $q_i > 0$  $0, \delta > -1, \, x > q_i;$
- (iv) Negative exponential form:  $U_i(x) = -\exp(-\alpha_i x) / \alpha_i, \alpha_i > 0$ ; and
- (v) Other general forms:  $U_i(x) = -(q_i x)^{\delta}$ , where  $q_i > 0, \delta > 1$  and  $x < q_i$ .

Next we show that for each of these utilities, the necessary and sufficient condition for monotonicity or the Law of Demand to be violated can be stated very simply as follows.

**Proposition 5** For the Modified Bergson family members (i)-(v), the necessary and sufficient condition for the Law of Demand to be violated is characterized by:

- 1. Type (i) utility: Law of Demand is never violated for any  $\mathbf{c} \in \Omega$ ;
- 2. Type (ii) utility: Law of Demand is violated for some  $\mathbf{c} \in \Omega$  if and only if  $\delta > 3$ ;
- 3. Types (iii)-(v): Law of Demand is always violated independently of the preference parameters for some  $\mathbf{c} \in \Omega$ .

*Proof* For Type (i) utility, since preferences are homothetic, it follows directly from Proposition 6 below that M = 0 and hence monotonicity always holds. In the following proof, for simplicity, we assume that there are only two commodities. The multiple commodity case can be discussed similarly. For Type (ii) utility, we have

$$U(c_1, c_2) = -\frac{(c_1 + q_1)^{-\delta}}{\delta} - \frac{(c_2 + q_2)^{-\delta}}{\delta},$$
(21)

where  $q_1, q_2 > 0$  and  $\delta > -1$ . The set of the minimum concavity points is again the whole space and so we can use the original Mitjuschin-Polterovich sufficient condition (3) in terms of the least concave form as a necessary and sufficient condition for monotonicity. The least concave form is given by

$$u(c_1, c_2) = \left( (c_1 + q_1)^{-\delta} + (c_2 + q_2)^{-\delta} \right)^{-1/\delta}.$$
 (22)

<sup>&</sup>lt;sup>11</sup> For the first three classes, when  $\delta \rightarrow 0$ , the power utility function becomes the logarithmic utility function. For simplicity, a positive coefficient in front of each  $U_i$  is ignored. It can be easily verified that Proposition 5 also holds for  $\lambda_i U_i$ , where  $\lambda_i > 0$ .

It can be verified that

$$\mathcal{M}_{u} = -\frac{\left(\partial^{2} u(\mathbf{c})\mathbf{c}, \mathbf{c}\right)}{\left(\mathbf{c}, \partial u(\mathbf{c})\right)} = \frac{\left(\frac{(1+\delta)c_{1}}{c_{1}+q_{1}} - \frac{(1+\delta)c_{2}}{c_{2}+q_{2}}\right)^{2}}{(1+\delta)AB},\tag{23}$$

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where

$$A = (c_1 + q_1)^{\delta} + (c_2 + q_2)^{\delta}, \qquad (24)$$

$$B = (c_1 + q_1)^{-1-\delta} c_1 + (c_2 + q_2)^{-1-\delta} c_2.$$
 (25)

If the preference ordering is defined on  $(0,\infty) \times (0,\infty)$ , then

$$\mathcal{M}_{u} \leq (1+\delta) \frac{\left(\frac{c_{1}}{c_{1}+q_{1}} - \frac{c_{2}}{c_{2}+q_{2}}\right)^{2}}{\frac{c_{1}}{c_{1}+q_{1}} + \frac{c_{2}}{c_{2}+q_{2}}} \leq (1+\delta) \left|\frac{c_{1}}{c_{1}+q_{1}} - \frac{c_{2}}{c_{2}+q_{2}}\right| \leq 1+\delta,$$
(26)

where the equal sign can be obtained if and only if  $(c_1, c_2) = (0, \infty)$  or  $(\infty, 0)$ . For quasihomothetic preferences, the necessary and sufficient condition for the Law of Demand to be violated can be stated as  $\mathcal{M}_u > 4$ . Noting that the maximum value of  $\mathcal{M}_u$ can be reached in the limit, the necessary and sufficient condition for monotonicity to be violated is

$$\mathcal{M}_u > 4 \Leftrightarrow \delta > 3. \tag{27}$$

For Type (iii) utility, we have

$$U(c_1, c_2) = -\frac{(c_1 - q_1)^{-\delta}}{\delta} - \frac{(c_2 - q_2)^{-\delta}}{\delta},$$
(28)

where  $q_1, q_2 > 0$  and  $\delta > -1$ . It can be verified that

$$\mathcal{M}_{u} = -\frac{\left(\partial^{2} u(\mathbf{c})\mathbf{c}, \mathbf{c}\right)}{\left(\mathbf{c}, \partial u(\mathbf{c})\right)} = \frac{\left(\frac{(1+\delta)c_{1}}{c_{1}-q_{1}} - \frac{(1+\delta)c_{2}}{c_{2}-q_{2}}\right)^{2}}{(1+\delta)AB},\tag{29}$$

where

$$A = (c_1 - q_1)^{\delta} + (c_2 - q_2)^{\delta}, \qquad (30)$$

$$B = (c_1 - q_1)^{-1-\delta} c_1 + (c_2 - q_2)^{-1-\delta} c_2.$$
(31)

We next show that monotonicity of demand fails for some **c** if  $q_1 = q_2 = q$ , and the preference ordering is defined on  $(q, \infty) \times (q, \infty)$ . The case when  $q_1 \neq q_2$  can be discussed similarly. Assuming that  $c_2 = 2c_1 - q$ , we will have

$$\mathcal{M}_{u} = \frac{(1+\delta) \left(\frac{q}{2(c_{1}-q)}\right)^{2}}{\frac{c_{1}}{c_{1}-q} + \frac{2^{\delta}c_{1}}{c_{1}-q} + \frac{2c_{1}-q}{2^{\delta}(c_{1}-q)} + \frac{2c_{1}-q}{2(c_{1}-q)}} = \frac{(1+\delta) q^{2}}{4 (c_{1}-q) \left((1+2^{\delta}) c_{1} + (1+2^{-\delta}) (2c_{1}-q)\right)}.$$
(32)

Letting  $c_1 \to q$ , we will have

$$\lim_{c_1 \to q} \mathcal{M}_u = \infty > 4. \tag{33}$$

Therefore, the monotonicity condition is always violated somewhere. For Type (iv) utility, we have

$$U(c_1, c_2) = -\frac{\exp(-\alpha_1 c_1)}{\alpha_1} - \frac{\exp(-\alpha_2 c_2)}{\alpha_2},$$
 (34)

where  $\alpha_1, \alpha_2 > 0$ . It can be verified that

$$\mathcal{M}_{u} = \frac{\left(\alpha_{1}c_{1} - \alpha_{2}c_{2}\right)^{2}\exp\left(\alpha_{1}c_{1} + \alpha_{2}c_{2}\right)}{\left(\alpha_{1}\exp\left(\alpha_{1}c_{1}\right) + \alpha_{2}\exp\left(\alpha_{2}c_{2}\right)\right)\left(c_{2}\exp\left(\alpha_{1}c_{1}\right) + c_{1}\exp\left(\alpha_{2}c_{2}\right)\right)}.$$
(35)

Letting  $c_1 \to 0$ , we will have

$$\lim_{c_1 \to 0} \mathcal{M}_u = \frac{\alpha_2^2 c_2 \exp\left(\alpha_2 c_2\right)}{\alpha_1 + \alpha_2 \exp\left(\alpha_2 c_2\right)}.$$
(36)

Then letting  $c_2 \to \infty$ , we will have

$$\lim_{c_1 \to 0, c_2 \to \infty} \mathcal{M}_u \to \alpha_2 c_2 \to \infty.$$
(37)

Therefore, monotonicity is always violated. It should be noted that due to symmetry, when  $c_1 \to \infty$  and  $c_2 \to 0$ , we also have  $\mathcal{M}_u \to \infty$ . For Type (v) utility, we have

$$U(c_1, c_2) = -(q_1 - c_1)^{\delta} - (q_2 - c_2)^{\delta}, \qquad (38)$$

where  $\delta > 1$ ,  $q_1, q_2 > 0$  and  $q_1 - c_1, q_2 - c_2 > 0$ . It can be verified that

$$\mathcal{M}_{u} = \frac{\left(\delta - 1\right)\left(q_{1} - c_{1}\right)^{\delta - 1}\left(q_{2} - c_{2}\right)^{\delta - 1}\left(q_{1}c_{2} - q_{2}c_{1}\right)^{2}}{\left(\left(q_{1} - c_{1}\right)^{\delta} + \left(q_{2} - c_{2}\right)^{\delta}\right)\left(c_{1}\left(q_{1} - c_{1}\right)^{\delta}\left(q_{2} - c_{2}\right) + c_{2}\left(q_{2} - c_{2}\right)^{\delta}\left(q_{1} - c_{1}\right)\right)}.$$
(39)

Letting  $c_1 \to 0$ , we will have

$$\lim_{c_1 \to 0} \mathcal{M}_u = \frac{(\delta - 1) c_2 q_1^{\delta}}{(q_2 - c_2) \left( q_1^{\delta} + (q_2 - c_2)^{\delta} \right)}.$$
 (40)

Then letting  $c_2 \rightarrow q_2$ , we will have

$$\lim_{c_1 \to 0, c_2 \to q_2} \mathcal{M}_u \to \infty.$$
(41)

Therefore, monotonicity is always violated. It should be noted that due to symmetry, when  $c_1 \to q_1$  and  $c_2 \to 0$ , we also have  $\mathcal{M}_u \to \infty$ .

The Modified Bergson utilities (i)-(v) are quite special in that for four of the five members, monotonicity is either never violated or always violated no matter what preference parameters are assumed. Although it is possible to show when violations occur for the utilities (i)-(v), it is not straightforward to characterize analytically the resulting violation regions in the commodity space. Using numerical analysis, we plot the violation regions of the commodity space for the Type (iii) utility in terms of contours in Figures 1(a) and (b). We indicate values of  $\mathcal{M}_u$ , the modified Mitjuschin-Polterovich coefficient evaluated in terms of u, corresponding to different regions of violations in commodity space for the case  $q_1 = q_2 = 1$  and for the two different preference parameter values  $\delta = 1$  and  $\delta = 2$ . A similar, but messier, computation shows that for arbitrary nonnegative  $q_1, q_2$ , not both zero, monotonicity is violated for the commodity bundles **c** satisfying  $c_2 - q_2 = 2(c_1 - q_1)$  with  $c_1$  close to  $q_1$ .



Fig. 1 Violation regions in commodity space

It is natural to ask why the violations are centered at the boundaries of the commodity space. In fact it is clear from the argument in the proof of Proposition 5 that the maximum value of  $\mathcal{M}_u$  is always reached at a boundary point. Although we cannot give a direct economic interpretation of why violations occur at the boundaries, it is possible to provide some interesting intuition in the classic contingent claim – financial asset asset model. In Appendix A utilizing Hurwicz, Jordan and Kannai (1987) and Kubler, Selden and Wei (2013), we provide a simple example based on Type (iii) utility where the Law of Demand is violated for contingent claims and corresponding to a natural transformation of variables, the risk free asset can become a Giffen good. The region of asset space where the risk free asset is a Giffen good is seen to occur at a boundary which is associated with low levels of income and has a natural economic interpretation. Moreover this interpretation readily carries over to the violation region in contingent claim space.<sup>12</sup>

Finally as noted in Section 1, Milleron (1974) and Mas-Colell (1991) observed that the left hand side of the Mitjuschin-Polterovich sufficient condition (3) is identically 0 if  $\leq$  is homothetic and the representation U is homogeneous of degree 1. In fact, the converse of this observation is also true.

**Proposition 6** A strictly convex preference ordering, which is represented by a concave utility function U the Hessian determinant of which never vanishes, is homothetic if and only if M vanishes.

**Proof** It follows from (3) in Theorem 2.1 of Kannai (1989) that M vanishes if and only if the consumption vector  $\mathbf{c}$  is proportional to the tangent vector of the income expansion path corresponding to  $\leq$  at this  $\mathbf{c}$ . This implies that the income expansion path coincides with a line through the origin. This is equivalent to homotheticity. An elementary proof not using the results of Kannai (1989) may also be given.

 $<sup>^{12}\,</sup>$  We thank an anonymous referee for drawing our attention to the possibility of using a transformation of variables to gain additional economic insight.

### **3** Violation Region and Properties: A Canonical Example

In this Section, we seek to overcome the difficulty that for Modified Bergson preferences it is not straightforward to analytically characterize either the violation region in the commodity space or the dependence of violations on prices and income by introducing a new form of preferences.

#### 3.1 WAES Utility

Assume that for the two commodity case, preferences are represented by

$$U(c_1, c_2) = -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{c_2^{-\delta_2}}{\delta_2},$$
(42)

where  $\delta_1, \delta_2 > -1$ .<sup>13</sup> It should be noted that for this utility, straightforward computation of the standard elasticity of substitution  $\eta$  yields

$$\frac{1}{\eta} =_{def} - \frac{d\ln\frac{c_2}{c_1}}{d\ln(\frac{U_2}{U_1})} = \frac{c_1^{\delta_1}}{c_1^{\delta_1} + c_2^{\delta_2}} \frac{1}{\eta_1} + \frac{c_2^{\delta_2}}{c_1^{\delta_1} + c_2^{\delta_2}} \frac{1}{\eta_2},\tag{43}$$

where

$$\eta_i = \frac{1}{\delta_i + 1} \quad (i = 1, 2).$$
(44)

It is clear that in this case, the resulting elasticity of substitution is simply the weighted harmonic average of the elasticities of substitutions of the CES forms corresponding to  $\delta = \delta_1$  and  $\delta = \delta_2$ . It is for this reason that we refer to (42) as the weighted average elasticity of substitution (WAES) utility.

To see why the WAES utility can be viewed as a natural extension of the CES utility, define  $\delta = (\delta_1 + \delta_2)/2$ . Then  $\delta_1$  and  $\delta_2$  can be thought of as perturbations of  $\delta$  where  $\delta_1 = \delta + e$  and  $\delta_2 = \delta - e$  (i.e.,  $e = (\delta_1 - \delta_2)/2$ ). Rewriting (42), yields

$$U(c_1, c_2) = -\frac{c_1^{-(\delta+e)}}{\delta+e} - \frac{c_2^{-(\delta-e)}}{\delta-e}.$$
(45)

It is natural to ask how, for a given  $\delta$  associated with the homothetic special case where  $\delta_1 = \delta_1 = \delta$ , perturbing e affects the shape of the WAES indifference curves. Figure 2 considers the case of Cobb-Douglas preferences where  $\delta \to 0$  along with perturbations associated with e = 0.5 and 1.0. The base Cobb-Douglas indifference curves associated with e = 0 are clearly symmetric around the 45° ray. Increasing efrom 0 to 0.5 and 1.0 represents a type of rotation of the indifference curves resulting in increasing asymmetry. And when  $\delta > 1$ , sufficient levels of asymmetry will be seen in the next Subsection to give rise to violations in the monotonicity of demand (see footnote 15).

 $<sup>^{13}~</sup>$  If either  $\delta_1$  or  $\delta_2$  but not both equals -1, all our results below still apply unless otherwise stated.



Fig. 2 WAES indifference curves for  $\delta \rightarrow 0$  case

#### 3.2 Violation Region

Since preferences represented by the WAES utility (42) are neither homothetic nor quasihomothetic, it follows from Proposition 4 that the set of minimum concavity points (based on u) is not the entire commodity space and hence we cannot use the sufficient condition (3) to discuss restrictions on preference parameters corresponding to violations.<sup>14</sup> Nevertheless as we next show, it is possible to use the necessary and sufficient form (2) to determine when violations occur in terms of the preference parameters  $\delta_1$  and  $\delta_2$  and where violations occur in the commodity space.

Straightforward computation of M yields

$$M = \frac{(\delta_1 - \delta_2)^2 c_1^{\delta_1} c_2^{\delta_2}}{(c_1^{\delta_1} + c_2^{\delta_2})((1+\delta_1)c_1^{\delta_1} + (1+\delta_2)c_2^{\delta_2})}.$$
(46)

To find out when M < 4, define  $t = c_1^{\delta_1}/c_2^{\delta_2}$ . Then

$$M = \frac{(\delta_1 - \delta_2)^2 t}{(1+t)((1+\delta_2) + (1+\delta_1)t)}.$$
(47)

Solving the inequality M < 4 and investigating the roots of M = 4, we find that if  $(\delta_1 - \delta_2)^2 < 8(\delta_1 + \delta_2)$  then (2) holds for all positive t, i.e., demand is monotone over all of the positive orthant. If, however,  $(\delta_1 - \delta_2)^2 > 8(\delta_1 + \delta_2)$ , then elementary calculation shows that the roots  $t_1, t_2$  of the quadratic equation M = 4 are real. If, in

<sup>&</sup>lt;sup>14</sup> Actually for this form of utility, it can be shown that no finite **c** in the commodity space is a minimum concavity point no matter which representation one chooses. For example, when  $\delta_1 > \delta_2 > 0$ , the set of minimum concavity points is a limit line  $\{(c_1, \infty) \mid c_1 \in C_1\}$ .



Fig. 3 WAES violation boundaries in commodity space

addition,  $\delta_1 + \delta_2 > 2$ , then the roots are strictly positive and monotonicity is violated for positive  $(c_1, c_2)$ -pairs – namely, for those consumption pairs for which

$$t_1 < \frac{c_1^{\delta_1}}{c_2^{\delta_2}} < t_2 \tag{48}$$

holds. Summarizing, the necessary and sufficient condition for monotonicity to hold everywhere in  $\Omega$  is<sup>15</sup>

$$(\delta_1 - \delta_2)^2 < 8(\delta_1 + \delta_2) \quad \text{or} \quad \delta_1 + \delta_2 \le 2.$$

$$(49)$$

As shown in Appendix B, this result can be extended to the case with more than two commodities.

The resulting violation region defined by (48) corresponds to a subset of the consumption space bounded by two curves (generalized parabolas or parabolas), which are defined by

$$c_2 = g(c_1) = t_1^{\frac{-1}{2}} c_1^{\frac{-1}{2}} \qquad \text{and} \qquad c_2 = k(c_1) = t_2^{\frac{-1}{2}} c_1^{\frac{-1}{2}}.$$
(50)

See Figure 3.<sup>16</sup> It should be noted that for WAES utility, if the necessary and sufficient condition for monotonicity is violated then there exists a curve in the commodity space defined by  $c_1^{\delta_1} = tc_2^{\delta_2}$  where t > 0 and each point along this curve has the same M value greater than 4. Note that each of these curves which lies between the M = 4 boundary curves in Figure 3, intersects every vertical and horizontal line in  $\Omega$ .

<sup>&</sup>lt;sup>15</sup> It should be noted that the necessary and sufficient condition (49) for monotonicity to hold can be also written as:  $|e| < 2\sqrt{\delta}$  or  $\delta \leq 1$ . The comparison between e and  $\delta$  shows how much the utility departs from the CES case.

<sup>&</sup>lt;sup>16</sup> Clearly, the violation region in Figure 3 differs from that of the translated origin Modified Bergson example in Figure 1, where violations are centered at the boundaries of the commodity space. However if one considers an analogous translation of the origin for the WAES case, then the resulting pattern of violations is also centered at the boundaries.



Fig. 4 Characterizing monotonicity in WAES parameter space

Figure 4(a) illustrates in terms of the preference parameter space defined by  $\delta_1$  and  $\delta_2$ , where monotonicity holds and fails. Note that if  $\delta_1 = \delta_2$ , then M = 0 implying that demand is monotone. If the values of  $\delta_1$  and  $\delta_2$  lie in the southwest triangle of Figure 4(a) formed by the points (-1, 3), (-1, -1) and (3, -1), then  $\delta_1 + \delta_2 < 2$  and demand will be monotone.

Finally, it should be stressed that when preferences are neither homothetic nor quasihomothetic, using the least concave utility u in the sufficient condition (3) does not yield the necessary and sufficient condition in general. To see this, assume  $\delta_1 > \delta_2 > 0$ , resulting in  $u = (-\delta_2 U)^{-\frac{1}{\delta_1}}$ . Then  $-\frac{(\partial^2 u(\mathbf{c})\mathbf{c},\mathbf{c})}{(\mathbf{c},\partial u(\mathbf{c}))} < 4$  if and only if

$$\delta_1 + \delta_2 \le 3 \tag{51}$$

or

$$\delta_1 + \delta_2 > 3$$
 and  $(\delta_1 + \delta_2 - 3)^2 - 4\delta_2 (1 + \delta_1) < 0,$  (52)

which clearly differs from (49) and hence should not be used to characterize the violation region in the parameter space.

Remark 1 It is interesting to observe that applying the sufficient condition given in Quah (2003), p. 719, to the WAES case, one obtains

$$\left|\delta_1 - \delta_2\right| < 4,\tag{53}$$

which is clearly different from the necessary and sufficient condition given in (49). This is illustrated in Figure 4(b), where the inequality (53) defines the region between two  $45^{\circ}$  rays, which start from the points (-1, 3) and (3, -1). These two rays divide the  $M_2$  region in Figure 4(a) for monotonicity to hold into three parts. The area labeled  $M_{22}$  corresponding to (53) is clearly smaller than the region given by the necessary and sufficient condition (49), which is  $M_{21} + M_{22} + M_{23}(= M_2)$  in Figure 4(b).



Fig. 5 Normalized income bounds for violations

1

#### 3.3 Violation Properties

Building on the characterization of the violation region in the commodity space discussed above and illustrated in Figure 3, we next consider the subset of violation points corresponding to a given price ratio  $p_1/p_2$ . In Figure 5, we plot the expansion path  $c_2 = f(c_1; p_1/p_2)$  associated with  $p_1/p_2 = 3$ . The expansion path always intersects both violation boundaries. The two parallel budget constraints going through the upper and lower intersection points are denoted  $I^{(1)}/p_2$  and  $I^{(2)}/p_2$ , respectively. For each normalized income value  $I/p_2$  between  $I^{(1)}/p_2$  and  $I^{(2)}/p_2$ , monotonicity will be violated for every  $(c_1, c_2)$  point along the expansion path. Following the calculations in the previous Section, it is straightforward to identify the upper and lower bounds for  $I/p_2$  given a  $p_1/p_2$  ratio. (Without loss of generality, we will always assume that  $\delta_1 > \delta_2$  in the discussion below.)

**Result 1** Assume the WAES utility (42). For a fixed  $p_1/p_2$ , the Law of Demand is violated if and only if<sup>47</sup>

$$\frac{I^{(2)}}{p_2} < \frac{I}{p_2} < \frac{I^{(1)}}{p_2} \tag{54}$$

where

$$\frac{d^{(1)}}{p_2} = \left(t_1^{\frac{1+\delta_1}{\delta_1 - \delta_2}} + t_1^{\frac{1+\delta_2}{\delta_1 - \delta_2}}\right) \left(\frac{p_1}{p_2}\right)^{\frac{\delta_1}{\delta_1 - \delta_2}},$$
(55)

$$\frac{I^{(2)}}{p_2} = \left(t_2^{\frac{1+\delta_1}{\delta_1 - \delta_2}} + t_2^{\frac{1+\delta_2}{\delta_1 - \delta_2}}\right) \left(\frac{p_1}{p_2}\right)^{\frac{\delta_1}{\delta_1 - \delta_2}},\tag{56}$$

$$t_1 = \frac{(\delta_1 - \delta_2)^2 - 4(\delta_1 + \delta_2 + 2) + (\delta_1 - \delta_2)\sqrt{(\delta_1 - \delta_2)^2 - 8(\delta_1 + \delta_2)}}{8(1 + \delta_1)}$$
(57)

<sup>17</sup> It should be noted that using the convention  $0^0 = 1$ , Result 1 also holds when  $\delta_2 = -1$ .

$$t_2 = \frac{\left(\delta_1 - \delta_2\right)^2 - 4\left(\delta_1 + \delta_2 + 2\right) - \left(\delta_1 - \delta_2\right)\sqrt{\left(\delta_1 - \delta_2\right)^2 - 8\left(\delta_1 + \delta_2\right)}}{8\left(1 + \delta_1\right)}.$$
 (58)

Given that the Law of Demand is violated for each of the points along the expansion path  $c_2 = f(c_1; p_1/p_2)$  between  $I^{(1)}/p_2$  and  $I^{(2)}/p_2$ , it is natural to wonder if one can characterize in some way the set of corresponding price change ratios  $x_1/x_2$  resulting in the left hand side of the inequality (1) being strictly positive.<sup>18</sup>

**Result 2** Assume the WAES utility (42). Given a price ratio  $p_1/p_2$ , for any point in the violation region along the corresponding expansion path  $f(c_1; p_1/p_2)$ , the price change ratio  $x_1/x_2$  that makes the left hand side of (1) strictly positive must be (i) negative and (ii) in the range

$$A_1 \frac{p_1}{p_2} < \frac{x_1}{x_2} < A_2 \frac{p_1}{p_2},\tag{59}$$

where

$$A_1 = \sqrt{(1+\delta_1)(1+\delta_2)} - \frac{1}{2}(\delta_1 + \delta_2 + Q), \qquad (60)$$

$$A_2 = \sqrt{(1+\delta_1)(1+\delta_2)} - \frac{1}{2}(\delta_1 + \delta_2 - Q)$$
(61)

and

$$Q = \sqrt{\left(\delta_1 + \delta_2 - 2\sqrt{(1+\delta_1)(1+\delta_2)}\right)^2 - 4.}$$
(62)

*Proof* Denote the optimal demand functions as  $h_1$  and  $h_2$  respectively. Implicitly differentiating  $h_1$  and  $h_2$  and defining

$$z = \frac{\delta_2 + 1}{\delta_1 + 1} \left(\frac{p_1}{p_2}\right)^{\frac{\delta_2}{\delta_2 + 1}} h_1^{\frac{\delta_2 - \delta_1}{\delta_2 + 1}},\tag{63}$$

we obtain the following simple price derivatives<sup>19</sup>

$$\frac{\partial h_1}{\partial p_1} = \frac{-h_1((1+\delta_1)z+1)}{p_1(1+\delta_1)(1+z)} < 0, \tag{64}$$

$$\frac{\partial h_1}{\partial p_2} = \frac{-h_1 \delta_2}{p_2 (1+\delta_1)(1+z)},$$
(65)

$$\frac{\partial h_2}{\partial p_1} = \frac{-h_1\delta_1}{p_2(1+\delta_1)(1+z)},\tag{66}$$

and

$$\frac{\partial h_2}{\partial p_2} = \frac{-p_1 h_1 (1+z+\delta_2)}{p_2^2 (1+\delta_1)(1+z)z} < 0.$$
(67)

Therefore, we have

$$(x_1, x_2) \begin{pmatrix} \frac{\partial h_1}{\partial p_1} & \frac{\partial h_1}{\partial p_2} \\ \frac{\partial h_2}{\partial p_1} & \frac{\partial h_2}{\partial p_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = L_1 + L_2 + L_3,$$
(68)

 $^{18}\,$  We thank Michael Jerison for emphasizing this point.

<sup>19</sup> It should be noted that if  $\delta_1 = -1$  or  $\delta_2 = -1$ , although the derivatives below are not welldefined, it can be verified that the relation between the sign of  $\partial h_i / \partial p_j$  and  $\delta_k$   $(i, j, k \in \{1, 2\})$ still holds. where

$$L_1 = \frac{-h_1}{(1+\delta_1)(1+z)} \left(\sqrt{\frac{1}{p_1}}x_1 - \sqrt{\frac{p_1}{p_2^2}}x_2\right)^2 \le 0,\tag{69}$$

$$L_2 = \frac{-h_1}{(1+\delta_1)(1+z)} \left( \sqrt{\frac{(1+\delta_1)z}{p_1}} x_1 + \sqrt{\frac{p_1(1+\delta_2)}{p_2^2 z}} x_2 \right)^2 \le 0$$
(70)

and

L

$$_{3} = \frac{-2x_{1}x_{2}h_{1}\left(\sqrt{\left(1+\frac{\delta_{1}+\delta_{2}}{2}\right)^{2}} - \sqrt{(1+\delta_{1})\left(1+\delta_{2}\right)}\right)}{(1+\delta_{1})(1+z)p_{2}}.$$
(71)

Since  $L_1, L_2 \leq 0$ , for a violation to occur we must have  $L_3 > 0$ . But from (71) this can happen only if  $x_1x_2 < 0$ , because

$$\sqrt{\left(1 + \frac{\delta_1 + \delta_2}{2}\right)^2} - \sqrt{(1 + \delta_1)(1 + \delta_2)} > 0, \tag{72}$$

thus verifying (i) in the Result. From (70) because  $\sqrt{\frac{(1+\delta_1)z}{p_1}}, \sqrt{\frac{p_1(1+\delta_2)}{p_2^2z}} > 0$ , one can always select price changes  $x_1$  and  $x_2$  such that  $L_2 = 0.20$  Combining (69) and (71) and simplifying yields

$$L_1 + L_3 = \frac{-h_1 x_2^2}{(1+\delta_1)(1+z)} \left(\frac{1}{p_1} \frac{x_1^2}{x_2^2} + \frac{(\delta_1 + \delta_2 - 2\sqrt{(1+\delta_1)(1+\delta_2)})}{p_2} \frac{x_1}{x_2} + \frac{p_1}{p_2^2}\right).$$
(73)

For the violation to occur, we require that  $L_1 + L_3 > 0$ , implying that

$$A_1 \frac{p_1}{p_2} < \frac{x_1}{x_2} < A_2 \frac{p_1}{p_2},\tag{74}$$

where  $A_1$  and  $A_2$  are defined by (60) and (61).

To gain additional insight into Result 2, note that if one assumes  $\delta_2 = -1$ , it follows from the definition  $\delta = (\delta_1 + \delta_2)/2$  that  $\delta_1 = 1 + 2\delta$  and the expression (59) simplifies to

$$-\left(\delta + \sqrt{\delta^2 - 1}\right)\frac{p_1}{p_2} < \frac{x_1}{x_2} < -\left(\delta - \sqrt{\delta^2 - 1}\right)\frac{p_1}{p_2}.$$
(75)

In this case, if we choose the price change ratio  $\frac{x_1}{x_2} = -\frac{p_1}{p_2}$ , we know that there exists at least one point on the corresponding expansion path for which the left hand side of (1) is strictly positive. To see this, remember that for a violation we must have  $\delta > 1$ , and it follows that

$$\delta + \sqrt{\delta^2 - 1} > 1$$
 and  $\delta - \sqrt{\delta^2 - 1} = \frac{1}{\delta + \sqrt{\delta^2 - 1}} < 1.$  (76)

Since we don't specify  $I/p_2$  in Result 2, the price change ratio range we give above is the union of the set of ranges for all points along a given expansion path in the violation region. Therefore, this range should be always larger than the price ratio range corresponding to a specific  $I/p_2$ . To illustrate this point more clearly, define

<sup>&</sup>lt;sup>20</sup> Alternatively  $L_2$  can be set equal to zero at any point along an expansion path defined by a given price ratio, assuming a specific price change  $x_1x_2 < 0$ , by selecting an income *I*.



Fig. 6 WAES perturbations and price change ratio for violations

the angle between the price change vector  $(x_1, x_2)$  and the commodity change vector  $(\Delta c_1, \Delta c_2)$  which is obtained by applying the  $\frac{\partial \mathbf{h}}{\partial \mathbf{p}}$  to  $(x_1, x_2)$  as  $\theta$ . Then it can be easily seen that

$$\sum_{i,j}^{2} \frac{\partial h_i}{\partial p_j} x_i x_j \stackrel{\geq}{=} 0 \Leftrightarrow \theta \stackrel{\leq}{=} 90^{\circ}.$$
(77)

In the case of a single good, a violation can only occur when demand and price move in the same direction (the Giffen good effect); for the WAES case both goods are normal and yet the angle  $\theta$  can be less than 90° (but is bounded away from 0). For a given  $p_1/p_2$  and  $I/p_2$ , Figure 6 illustrates (i) the range of the price change ratio when the angle is less than 90° and (ii) the minimal value of the angle. The price change ratio is plotted versus the angle for three different ( $\delta_1, \delta_2$ ) pairs (in order to emphasize the role of the comparison between  $\delta$  and e for a violation to occur, we show ( $\delta, e$ ) values instead of ( $\delta_1, \delta_2$ ) in Figure 6). The bottom, middle and top curves correspond, respectively, to a constant M value greater than, equal to and less than 4. Given the range of  $x_1/x_2$ values when  $\theta < 90^\circ$  in Figure 6, it can be verified by straightforward calculations that this range is less than that derived from Result 2.

To view the range in Result 2 directly, see Figure 7, where  $\delta_1 = 16$ ,  $\delta_2 = 2$  and  $p_1/p_2 = 3$ . The angle  $\theta$  is plotted in the space corresponding to  $x_1/x_2$  and  $I/p_2$ . Inside the oval labeled 90°, the angle  $\theta < 90^{\circ}$ , implying that  $\sum_{i,j}^{2} \frac{\partial h_i}{\partial p_j} x_i x_j > 0$ . The leftmost and rightmost  $x_1/x_2$  values on the oval correspond to the boundary values given in Result 2. It should be noted that in this Figure it is also possible to obtain the  $x_1/x_2$  range with the angle  $\theta$  less than 90° for a fixed  $I/p_2$  as in Figure 6 by drawing a horizontal ray corresponding to the given  $I/p_2$ . The two intersection points between this ray and the 90° oval determine the associated  $x_1/x_2$  range.



Fig. 7 Violation region in price change ratio - normalized income space

### 4 Concluding Comments

Following the initial contributions of Mitjuschin, Polterovich and Milleron, research on the Law of Demand has focused predominantly on when the monotonicity condition holds. In this paper, we take the first steps in examining when the converse is true not just in general terms but in terms of specific restrictions on preference parameters and regions in the commodity space. Using the simplified Mitjuschin and Polterovich sufficient condition, we are able to characterize violations in terms specific preference parameter restrictions for members of the widely used Modified Bergson (or HARA) family. In contrast to the known result that for the homothetic CES member where violations never occur at any point in the commodity space no matter what preference parameter is assumed, we show that for three of the remaining four members of the family violations always occur independent of the preference parameters assumed for some point in the commodity space. To provide sharper focus on the behavior of violations in the commodity space, we introduce a natural generalization of the Modified Bergson family where each commodity is still a normal good. For this nonhomothetic utility we are able to fully characterize the region of the commodity space in which violations are possible and derive for points in this region the bounds on possible income levels and price change ratios.

This research would seem to raise a number of potentially interesting questions such as the following. Is there any simple economic intuition to explain the location of the violation region in the commodity space? (In Appendix A, we provide some indirect insight for the case where an economically meaningful transformation of variables can be applied.) Moreover, at a specific point where a violation occurs for a given budget line and price change ratio, is it possible to explain intuitively what is required in terms of the shape of indifference curves that results in the violation?

## Appendix

# A Location of Violation Region: Intuition

As noted in the Section 2 discussion of the Type (iii) Modified Bergson utility, violations of monotonicity occur at the boundaries of the commodity space (Figure 1). Some interesting, albeit indirect, intuition for why this is the case can be obtained from using the result that if the commodities violate the Law of Demand, then there is a corresponding set of composite commodities, at least one of which is a Giffen good. The quantity demanded of each composite commodity is a fixed linear combination of the quantities demanded of the original goods.<sup>21</sup>

In this appendix, we illustrate the regions of violations of the Law of Demand and regions of a Giffen transformed good in a simple example paralleling the case considered in Figure 1(a). The original goods can be interpreted as contingent claims and the transformed composite commodities are two financial assets, one risky, the other risk free. The latter is the Giffen good and some insight can be gained into why the violation regions for both the contingent claims and financial assets occur where they do.

Consider the following specific form of the Type (iii) Modified Bergson utility

$$U(c_1, c_2) = -(c_1 - 1)^{-1} - (c_2 - 1)^{-1}.$$
(78)

It follows from Proposition 5(3) that for this utility, the Law of Demand will be violated for some  $(c_1, c_2) \in \Omega$ . Consider the following transformation

$$z_1 = \frac{2(c_1 - c_2)}{3}$$
 and  $z_2 = \frac{4c_2 - c_1}{3}$ , (79)

which will be seen below to result in the transformed variable  $z_2$  being a Giffen good. The utility function (78) becomes

$$U(z_1, z_2) = -(2z_1 + z_2 - 1)^{-1} - (0.5z_1 + z_2 - 1)^{-1}.$$
(80)

In order to compare the violation regions for the original and transformed variables, we plot contours corresponding to Mitjuschin-Polterovich coefficient values for  $(c_1, c_2)$  and  $(z_1, z_2)$  respectively in Figures 8(a) and (b), where Figure 8(a) is the same as Figure 1(a).<sup>22</sup> The

$$\sum_{i,j=1}^{2} \frac{\partial c_i}{\partial p_j} x_i x_j > 0$$

Consider the following transformation

$$z_1 = -x_2c_1 + x_1c_2$$
 and  $z_2 = x_1c_1 + x_2c_2$ .

Denoting the prices for  $(c_1, c_2)$  as  $(p_1, p_2)$  and the prices for  $(z_1, z_2)$  as  $(P_1, P_2)$ , it can be easily verified that

$$p_1 = -x_2P_1 + x_1P_2$$
 and  $p_2 = x_1P_1 + x_2P_2$ ,

implying that

$$\begin{aligned} \frac{\partial z_2}{\partial P_2} &= x_1 \frac{\partial c_1}{\partial P_2} + x_2 \frac{\partial c_2}{\partial P_2} \\ &= x_1 \left( \frac{\partial c_1}{\partial p_1} \frac{\partial p_1}{\partial P_2} + \frac{\partial c_1}{\partial p_2} \frac{\partial p_2}{\partial P_2} \right) + x_2 \left( \frac{\partial c_2}{\partial p_1} \frac{\partial p_1}{\partial P_2} + \frac{\partial c_2}{\partial p_2} \frac{\partial p_2}{\partial P_2} \right) \\ &= \sum_{i,j=1}^2 \frac{\partial c_i}{\partial p_j} x_i x_j > 0 \end{aligned}$$

and hence  $z_2$  is a Giffen good.

 $^{22}$  The dented boundaries in Figures 8(b) and (c) are due to limitations in simulation accuracy and can be ignored. Also note that the different regions inside the ovals in Figures 8(b) and

<sup>&</sup>lt;sup>21</sup> The proof of this result can be outlined as follows using the analysis in Hurwicz, Jordan and Kannai [5]. For simplicity, we only consider the two good case. When a violation occurs, there exists a price change vector  $(x_1, x_2)$  such that



Fig. 8 Commodity and transformed commodity comparisons

lighter color areas in both Figures correspond to the violation region where the Mitjuschin-Polterovich coefficient is greater than 4.

Next we want to show that  $z_2$  is a Giffen good. To see this, denote the prices for  $(c_1, c_2)$  as  $(p_1, p_2)$  and the prices for  $(z_1, z_2)$  as  $(P_1, P_2)$ , and it can be easily verified that

$$P_1 = 2p_1 + 0.5p_2$$
 and  $P_2 = p_1 + p_2.$  (81)

In Figure 8(c), the lighter color oval contains the set of  $(z_1, z_2)$  pairs where  $\partial z_2/\partial P_2 > 0$  assuming  $P_1 = 1$ , defining the Giffen good region. Comparing the ovals in Figures 8(b) and (c), it is clear as expected that the Giffen good region is a subset of the violation region.

Now in general, when performing this variable transformation, the new composite commodity that exhibits Giffen good behavior may not have a clear economic meaning. However in the classic contingent claim – financial asset setting, a natural interpretation can be given to the transformed variables and intuition given for the location of the Giffen good region as well as for the contingent claim violation region.

Suppose  $c_1$  and  $c_2$  denote quantities of contingent claim commodities in states one and two, where the states are equiprobable. Then the transformed variables can be viewed as asset holdings, where  $z_1$  denotes units of a risky asset which pays 2 in state one and 0.5 in state two and  $z_2$  denotes units of a risk free asset which pays 1 in both states. The contingent claim quantities are given by

$$c_1 = 2z_1 + z_2$$
 and  $c_2 = 0.5z_1 + z_2$ . (82)

The utility function (80) is affinely equivalent to

$$U(z_1, z_2) = -\frac{1}{2} \left( 2z_1 + z_2 - 1 \right)^{-1} - \frac{1}{2} \left( 0.5z_1 + z_2 - 1 \right)^{-1},$$
(83)

where  $\frac{1}{2}$  can be viewed as the state probabilities, consistent with our equiprobable assumption. The utility (83) can be viewed as a standard Expected Utility representation with the NM (von Neumann-Morgenstern) index defined on consumption c

$$W(c) = -(c-1)^{-1}.$$
(84)

The intuition for why at low levels of income the risk free asset can always become a Giffen good for an NM index that takes the Type (iii) form is discussed in Kubler, Selden and Wei (2013). The argument parallels the famous story of potatoes being a Giffen good. When the consumer's income is small, most of her income is invested in the risk free asset  $z_2$  rather than

<sup>(</sup>c), correspond respectively to different Mitjuschin-Polterovich values and different values of  $\partial z_2/\partial P_2$  as defined below.

the risky asset  $z_1$  and the lower state  $c_2$  is close to the subsistence level. In this case, safety from starvation is provided by  $z_2$  rather than potatoes. When the price of the risk free asset  $z_2$  increases, if the consumer reduces her  $z_2$  holdings, consumption in the worse outcome state  $c_2$  will decrease and the consumer faces a greater risk of starvation. Therefore, the associated income effect outweighs the substitution effect leading the consumer to actually increase her demand for  $z_2$ . The risk free asset is a Giffen good in this region of low income, where demand for the risky asset is small. Since the violation of the Law of Demand in assets occurs at low income, one would expect the violation region in  $z_1 - z_2$  space to also correspond to low income levels, where  $z_2$  is large and  $z_1$  is small.<sup>23</sup> Similarly, one would expect the contingent claim violation region also to occur at lower levels of income or near the boundary as in Figure 8(a).

#### **B** Multiple Commodity WAES Case

We show that it is possible to extend the characterization of violations of the Law of Demand for the WAES preference case derived for two commodities to the case of multiple commodities. The resulting restriction on preference parameters for violations to occur is analogous to the two commodity case. However, the discussion about the violation boundary and the relation between income, price ratio and price change ratio cannot be extended to the multiple commodity case.

For the WAES utility function  $^{24}$ 

$$U = -\sum_{i=1}^{n} \frac{c_i^{-\delta_i}}{\delta_i},\tag{85}$$

assume without loss of generality that  $\delta_1 \geq \delta_2 \geq ... \geq \delta_n > -1$ . The necessary and sufficient condition for monotonicity to hold is

$$(\delta_1 - \delta_n)^2 < 8(\delta_1 + \delta_n) \quad \text{or} \quad \delta_1 + \delta_n \le 2.$$
(86)

The reason can be seen as follows. It can be verified that

$$M = \frac{\sum_{i=1}^{n} c_i^{-\delta_i} (1+\delta_i)}{\sum_{i=1}^{n} c_i^{-\delta_i}} - \frac{\sum_{i=1}^{n} c_i^{-\delta_i}}{\sum_{i=1}^{n} c_i^{-\delta_i} / (1+\delta_i)}.$$
(87)

If we denote

$$\pi_{i} = \frac{c_{i}^{-\delta_{i}}}{\sum_{i=1}^{n} c_{i}^{-\delta_{i}}},$$
(88)

then we can rewrite M as

$$M = \sum_{i=1}^{n} \pi_i (1+\delta_i) - \frac{1}{\sum_{i=1}^{n} \frac{\pi_i}{1+\delta_i}},$$
(89)

which is a difference between the weighted arithmetic and geometric means of  $1 + \delta_i$  with weights  $\pi_i$ . Using the inequality provided in Shisha and Mond (1967), the necessary and sufficient condition for monotonicity to hold is

$$M \le M^{\max} = \left(\sqrt{1+\delta_1} - \sqrt{1+\delta_n}\right)^2 < 4,\tag{90}$$

which is equivalent to

$$(\delta_1 - \delta_n)^2 < 8(\delta_1 + \delta_n) \quad \text{or} \quad \delta_1 + \delta_n \le 2.$$
 (91)

 $^{23}$  It should be noted that since the violation region is larger than the Giffen good region, the violation may occur even when  $z_2$  is not a Giffen good.

<sup>24</sup> If we assume the utility function has a more general form  $U = -\sum_{i=1}^{n} \frac{\lambda_i c_i^{-\delta_i}}{\delta_i}$ , the conclusions below continue to hold.

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